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# Fluctuations of Eigenvalues of a Polynomial on Haar Unitary and Finite Rank Matrices

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**Abstract**
This paper calculates the fluctuations of eigenvalues of polynomials on large Haar unitaries cut by finite rank deterministic matrices. When the eigenvalues are all simple, we can give a complete algorithm for computing the fluctuations. When multiple eigenvalues are involved, we present several examples suggesting that a general algorithm would be much more complex.

**Keywords** Random matrices · Free probability · Non-commutative probability · BBP phase transition · Cyclic monotone independence

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## 1 Introduction

In random matrix theory, the behavior of large eigenvalues offers valuable insights, particularly regarding their positions and fluctuations. For example, significant statistical results have been derived from studies like [18, 20]. A particularly intriguing phenomenon for researchers is the BBP phase transition, as discussed in works like [4] and [21]. Analyzing this transition poses several challenges, often requiring advanced techniques such as the moment method and complex analysis. While these are standard in the study of random matrix theory, many cases demand intensive calculations for each model [5, 9, 15]. Conversely, non-commutative probability techniques, such as freeness, second order freeness, and infinitesimal freenesses, have proven advantageous for systematically analyzing models based on polynomials of several random matrices [19, 22].

Our previous works [12] and [13] have employed free probability and the moment method to systematically analyze the large eigenvalues of models constructed from polynomials of multiple random matrices, including those of finite rank. Notably, the concept of cyclically monotone independence has been instrumental in computing their moments and revealing their underlying phenomema. This concept was further developed in [10] where they gave a deep explanation of outlier problem with the moment method. Cyclically monotone independence originates from infinitesimal freeness and is connected to recent research on type B freeness, conditional freeness, and cyclic boolean independence [1–3, 10, 11, 17].

Now it is the turn to consider their fluctuation. In free probability, second order freeness gave a systematic way to obtain Gaussian fluctuations for global quantities of polynomials of typical random matrices [19, Chapter 5]. In this paper, we consider random matrices with only discrete eigenvalues in the large $N$ limit as in [12] and present a method for computing the fluctuations of eigenvalues, which provides a deeper understanding of outlier problem. More concretely, the present paper analyzes limiting eigenvalues and their fluctuations of the $N \times N$ random matrix

$$P(A_1U^*, A_2U^*, \dots, A_kU^*, UB_1, UB_2, \dots, UB_\ell) \tag{1.1}$$

in the large $N$ limit, where $P$ is a polynomial in $k+\ell$ noncommuting indeterminates without a constant term, $U \equiv U^{(N)} = (u_{ij})_{i,j\in[N]} \equiv (u_{ij}^{(N)})_{i,j\in[N]}$ is a Haar unitary matrix,

$$A_i = \begin{pmatrix} \widehat{A}_i & 0 \\ 0 & 0 \end{pmatrix} \in M_N(\mathbb{C}) \quad \text{and} \quad B_i = \begin{pmatrix} \widehat{B}_i & 0 \\ 0 & 0 \end{pmatrix} \in M_N(\mathbb{C}) \tag{1.2}$$

with $\widehat{A}_i, \widehat{B}_i \in M_r(\mathbb{C})$. The number $r \in \mathbb{N}$ is fixed, and $N$ is always assumed to satisfy sufficiently large assumed to be sufficiently large. We basically exhibit all possible patterns of the model (1.1) providing methods for calculating limiting eigenvalues and fluctuations, and show that when multiple eigenvalues appear the number of patterns can be very huge.

This model is related to the model in [12] in the sense that both involve Haar unitaries and the limiting eigenvalues are discrete in the large $N$ limit. Although the paper [12] focused only on almost sure convergence of eigenvalues, the present paper is mainly concerned with fluctuations of the eigenvalues.

Let us explain roughly the idea behind the construction proposed in [13]. It relies on the intuition that for any vector subspace $V$ of dimensions $r$ of $\mathbb{C}^N$, if we consider the image $U \cdot V$ of $V$ under the Haar unitary $U$, then $V$ and $U \cdot V$ are almost orthogonal in the sense that the inner product between any normed vector of $V$ and a normed vector of $U \cdot V$ is uniformly close to zero with high probability. This intuition can be lifted at the level of matrices as follows: for the Hilbert-Schmidt norm, any $\tilde{A}_i$ of norm one with the same domain and codomain as a matrix $A_i$ of norm one satisfies the property that $U\tilde{A}_i$ and $\tilde{A}_i U^*$ are almost orthogonal to $A_i$ in a uniform sense. A perfect orthogonality (which, in a sense, occurs when $N \to \infty$) gives naturally rise to the construction of [13]. In a sense, $U\tilde{A}_i$ is obtained from $\tilde{A}_i$ by making a "block row operation" and sending $\tilde{A}_i$ to its almost orthogonal self, whereas $\tilde{A}_i U^*$ is obtained from $\tilde{A}_i$ by making a "block column operation."

For finite $N$, the goal of this paper is to try to view $P(A_1U^*, \dots, A_kU^*, UB_1, \dots, UB_\ell)$ as an $o(N^{-1})$ perturbation of the model of [13], and deduce the fluctuations of the random matrix model from those of the limit model with perturbative methods. In this paper, we obtained the following results on eigenvalue fluctuations based on the above idea.

**Theorem 1.1** *The matrix $P(A_1U^*, \dots, A_kU^*, UB_1, \dots, UB_\ell)$ has $N - 2r$ zero eigenvalues, called the "trivial eigenvalues". The other eigenvalues, called the "non-trivial eigenvalues" (although zeros may be included), converge almost surely to deterministic numbers as $N \to \infty$. See Subsection 3.1 for an algorithm for computing these limits.*

Here, the term "trivial eigenvalues" means that they are always identical to zero independently of the polynomials and the $\widehat{A}_i$, $\widehat{B}_j$'s. Nontrivial eigenvalues may or may not be zero depending on a model.

Let $\{\mu_i^{(N)}\}_{i=1}^{2r}$ be the nontrivial eigenvalues of $P(A_1U^*, \dots, A_kU^*, UB_1, \dots, UB_\ell)$ and $\{\mu_i\}_{i=1}^{2r}$ denote their limits of the eigenvalues as described in Theorem 1.1.

**Theorem 1.2** *In addition, assume that all these values $\{\mu_i\}_{i=1}^{2r}$ appear without multiplicity. Then, for every $i \in [2r]$, the number*

$$\kappa_i := \sup\{\kappa \in \mathbb{R} \mid N^{\frac{\kappa}{2}}(\mu_i^{(N)} - \mu_i) \text{ converges in law to a } \mathbb{C}\text{-valued random variable}\} \quad (1.3)$$

*belongs to the set $\mathbb{N} \cup \{\infty\}$. Here, $\kappa_i = \infty$ means that $N^{\frac{\kappa}{2}}(\mu_i^{(N)} - \mu_i)$ converges in law to 0 for all $\kappa \in \mathbb{R}$, which occurs only when $\mu_i^{(N)} = \mu_i$ a.s. for all sufficiently large $N \in \mathbb{N}$. Moreover, let $I := \{i \in [2r] \mid \kappa_i < \infty\}$. Then the random vectors $(N^{\frac{\kappa_i}{2}}(\mu_i^{(N)} - \mu_i))_{i \in I}$ converge in law to $(P_i)_{i \in I}$ as $N \to \infty$, where $P_i = P_i(x_1, x_2, \dots, x_{2r^2})$ are nonzero homogeneous polynomials of degree $\kappa_i$, $i \in I$, on a standard Gaussian random vector $(x_i)_{i \in [2r^2]}$ on $\mathbb{R}^{2r^2}$.*

The fluctuation limits appearing here can be obtained in principle by specific calculations. We will provide calculations for the two models $UA + AU^*$ and $P(A, UBU^*)$.

Note that the latter model is a special case of (1.1) because $A$ and $UBU^*$ can be expressed e.g. as

$$A = AU^*U \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad UBU^* = UB \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

The fluctuations for $UA + AU^*$ and $P(A, UBU^*)$ are normal distributions and mixtures of exponential distributions, respectively. Remarkably, fluctuations of eigenvalues of $P(A, UBU^*)$ for generic polynomials $P$ can be explicitly calculated in the following way.

**Theorem 1.3** *Let $P(x, y)$ be a polynomial over $\mathbb{C}$ in noncommuting elements $x$ and $y$ such that $P(0, 0) = 0$. Let $P_1(x) := P(x, 0)$ and $Q_1(y) := P(0, y)$. Let*

$$\begin{aligned} A &= \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_r, 0, 0, \ldots, 0), \\ B &= \operatorname{diag}(\beta_1, \beta_2, \ldots, \beta_s, 0, 0, \ldots, 0) \end{aligned}$$

*with $\alpha_i, \beta_j \in \mathbb{C} \setminus \{0\}$ for all $i \in [r]$, $j \in [s]$. Then the $r + s$ nontrivial eigenvalues (the meaning will be made clear in the proof) of the random matrix*

$$P(A, UBU^*) \tag{1.4}$$

*converge to $\{P_1(\alpha_i)\}_{i=1}^r$ and $\{Q_1(\beta_j)\}_{j=1}^s$ a.s. If these $r + s$ limiting values are all distinct, then the nontrivial eigenvalues of (1.4) are of the forms*

$$\begin{aligned} &P_1(\alpha_i) + \frac{1}{N} \sum_{j \in [s]} p_{i,j} \left| \sqrt{N} u_{i,j}^{(N)} \right|^2 + \frac{\xi_i^{(N)}}{N^{\frac{3}{2}}}, \quad i \in [r] \quad \textit{and} \\ &Q_1(\beta_j) + \frac{1}{N} \sum_{i \in [r]} q_{i,j} \left| \sqrt{N} u_{i,j}^{(N)} \right|^2 + \frac{\zeta_j^{(N)}}{N^{\frac{3}{2}}}, \quad j \in [s], \end{aligned}$$

*where $p_{i,j}, q_{i,j}$ are explicit complex constants (shown in the proof, see (3.5) and (3.6)) and $\xi_i^{(N)}, \zeta_j^{(N)}$ denote random variables that converge in law to $\mathbb{C}$-valued random variables. The random variables $\{\left|\sqrt{N} u_{ij}^{(N)}\right|^2 \mid i \in [r], j \in [s]\}$ converge in law to standard exponential iid random variables. See Figures 2 and 3 for simulations.*

**Remark 1.4** Originally the matrices $\widehat{A}_i$ and $\widehat{B}_i$ in (1.2) were assumed to have a common size $r$ and instead were allowed to have zero eigenvalues. For the model $P(A, UBU^*)$ above, however, when $\widehat{A}$ or $\widehat{B}$ (the first $r \times r$ corners of $A$ and $B$) contains zero eigenvalues, the limiting nontrivial eigenvalues of $P(A, UBU^*)$ easily have multiple zero eigenvalues, which violates our assumption of simplicity. Therefore, we assume in Theorem 1.3 that $\widehat{A}$ and $\widehat{B}$ have *only nonzero* eigenvalues and, instead, they are allowed to have different sizes, denoted $r$ and $s$ respectively. Then one sees that the matrix $P(A, UBU^*)$ has $r + s$ "nontrivial eigenvalues".

When multiple eigenvalues appear in the limit, the situation is more complex than for models that have only simple eigenvalues. We will study some typical phenomena through the specific model $A + UBU^*$. Striking features include:

- fluctuations of a multiple eigenvalue may have different orders, see Example 4.4;
- fluctuations can be non-polynomial functions of standard Gaussian random vectors in contrast to the case of no multiplicities, see Examples 4.3, 4.4, 4.5, cf. Theorem 1.2.

This paper is organized as follows. In Section 2, we will present essential lemmas for obtaining fluctuations. In Section 3, we prove the main theorems and provide the aforementioned examples $UA + AU^*$ and $P(A, UBU^*)$. In Section 4, we will examine the model $A + UBU^*$ that has eigenvalues with multiplicities.

## 2 Technical tools

Calculations of the fluctuations of eigenvalues are based on the following two facts. Let $\widehat{U} \equiv \widehat{U}^{(N)} = (u_{ij})_{i,j\in[r]} \equiv (u_{ij}^{(N)})_{i,j\in[r]}$ be the truncation of $U$.

**Lemma 2.1** *(Theorem 4.2.1 and Proposition 4.4.1 in [8]) For $N \geq 2r$, $\widehat{U}^{(N)}$ has the probability density function*

$$c_{N,r} \det(1_r - AA^*)^{N-2r} 1_{\|A\|\leq 1} dA,$$

*where $c_{N,r}$ is a normalization constant and $dA$ is the Lebesgue measure on $M_r(\mathbb{C})$. In particular, as $N$ tends to infinity, the convergence in law*

$$\sqrt{N}\widehat{U}^{(N)} \longrightarrow Z$$

*holds, where $Z = (z_{ij})_{i,j\in[r]}$ is a* standard complex Gaussian random matrix, *i.e., $\{\Re(z_{ij}), \Im(z_{ij}) : i, j \in [r]\}$ are i.i.d. random variables having normal distribution with mean 0 and variance* $1/2$.

**Remark 2.2** According to the Skorohod representation theorem [7, Theorem 6.7], there exist $r \times r$ random matrices $Y, V^{(N)}, N \in \mathbb{N}$ on some probability space such that $Y, V^{(N)}$ have the same distributions as $Z, \widehat{U}^{(N)}$, respectively, and that $\sqrt{N}V^{(N)}$ converges to $Y$ almost surely. Some arguments below (in particular in Section 4) can be simplified by employing $Y$ and $V^{(N)}$.

The previous lemma readily implies that $\widehat{U}^{(N)}$ itself converges to 0 in probability. More strongly, almost sure convergence holds.

**Lemma 2.3** *As $N \to \infty$, $\widehat{U}^{(N)}$ converges to* 0 *a.s.*

***Proof*** It is known that $\mathbb{E}[|u_{ij}^{(N)}|^4] = \frac{2}{N(N+1)}$, see e.g. [14, p. 778]. Taking the sum over $N$ implies that $\sum_{N=1}^{\infty} |u_{ij}^{(N)}|^4$ has finite expectation and hence its value is finite almost surely. □

**Lemma 2.4** *[[6, Chapter 1, Section 4, Problem 1]] Let $X_N, X, Y_N, Y$ be $\mathbb{C}$-valued random variables, $N \in \mathbb{N}$. If $X_N \xrightarrow{law} X$ and $Y_N \xrightarrow{prob} 0$ then $X_N + Y_N \xrightarrow{law} X$ and $X_N Y_N \xrightarrow{prob} 0$.*

Below we denote $\|X\| := \sqrt{\mathrm{Tr}[X^* X]}$ for $X \in M_r(\mathbb{C})$.

**Lemma 2.5** *[Eigenvalues of perturbed matrices] Let $r \geq 2$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r) \in M_r(\mathbb{C})$, where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are distinct complex numbers. Then there exist homogeneous polynomials $\Pi_{p,k}(X)$ ($k \in \mathbb{N}$, $p \in [r]$) of degree $k$ on the $r^2$ complex variables $X = \{x_{ij}\}_{i,j\in[r]}$ and a constant $C > 0$ such that for all $X \in M_r(\mathbb{C})$ with $\|X\| < C$ the eigenvalues of the perturbed matrix $\Lambda + X$ can be expressed as the absolutely convergent series expansions*

$$\lambda_p + \sum_{k=1}^{\infty} \Pi_{p,k}(X), \qquad p \in [r]. \tag{2.1}$$

*In particular, the first two terms $\Pi_{p,1}$ and $\Pi_{p,2}$ are given by*

$$\Pi_{p,1}(X) = x_{pp}, \qquad \Pi_{p,2}(X) = \sum_{i \neq p} \frac{x_{ip} x_{pi}}{\lambda_p - \lambda_i}.$$

***Proof*** The function $f(z, X) := \det(z 1_r - (\Lambda + X))$ is a polynomial of $r^2 + 1$ variables, $f(\lambda_p, 0) = 0$ and $\partial_z f(\lambda_p, 0) \neq 0$; the last condition holds by the assumption of simplicity. By the holomorphic implicit function theorem [16, p. 34], there exist neighborhoods $U_p$ of $\lambda_p \in \mathbb{C}$ and $V_p$ of $0 \in M_r(\mathbb{C})$ and holomorphic function $\mu_p^\Lambda : V_p \to U_p$ such that

$$\{(z, X) \in U_p \times V_p \mid f(z, X) = 0\} = \{(\mu_p^\Lambda(X), X) \mid X \in V_p\}.$$

As being a holomorphic function of several variables, $\mu_p^\Lambda$ has an absolutely convergent series expansion in a neighborhood of 0 and hence is of the form (2.1), as desired.

The formulas for $\Pi_{p,1}$ and $\Pi_{p,2}$ follow from straightforward calculus, i.e., taking partial derivatives in the identity $f(\mu_p^\Lambda(X), X) = 0$ with respect to $x_{ij}$'s and evaluating at $X = 0$ yields formulas for $\partial_{x_{ij}} \mu_p^\Lambda(0)$, $\partial^2_{x_{ij} x_{k\ell}} \mu_p^\Lambda(0)$ for $i, j, k, \ell \in [r]$. □

## 3 Simple eigenvalues: a general algorithm and examples

### 3.1 Algorithm and proofs of Theorems 1.1 and 1.2

The algorithm for computing the fluctuations of eigenvalues of (1.1) is what follows. We specialize in the case $k = \ell = 1$, which lightens the notation but does not decrease the essence. Let $A := A_1$ and $B := B_1$. We first choose the basis (also regarded as an

$N \times N$ matrix)

$$\mathbf{B} := (e_1, e_2, \ldots, e_r, u_1, u_2, \ldots, u_r, e_{r+1}, e_{r+2}, \ldots, e_{N-r}), \tag{3.1}$$

where $u_i$ is the $i$-th column vector of $U^{(N)}$. Note that $\mathbf{B}$ is a basis with probability one since the truncated Haar unitary $\widehat{U}$ has the continuous density in $M_r(\mathbb{C})$ due to Theorem 2.1 and the set of singular matrices is a null set with respect to Lebesgue measure.

The matrix representations of $AU^*$ and $UB$ with respect to the basis $\mathbf{B}$ are given by

$$A' := \begin{pmatrix} \widehat{A}\widehat{U}^* & \widehat{A} & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B' := \begin{pmatrix} 0 & 0 & 0 \\ \widehat{B} & \widehat{B}\widehat{U} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively. Let $P(x, y)$ be a polynomial without a constant term in noncommuting indeterminates $x, y$. Because $P(AU^*, UB) = \mathbf{B}P(A', B')\mathbf{B}^{-1}$, it suffices to compute the eigenvalues of the matrix $P(A', B')$ which is of the form

$$\underbrace{\begin{pmatrix} P_{11}(\widehat{A}, \widehat{B}) & P_{12}(\widehat{A}, \widehat{B}) & * \\ P_{21}(\widehat{A}, \widehat{B}) & P_{22}(\widehat{A}, \widehat{B}) & * \\ 0 & 0 & 0 \end{pmatrix}}_{=:M} + \underbrace{\begin{pmatrix} O(\|\widehat{U}\|) & O(\|\widehat{U}\|) & * \\ O(\|\widehat{U}\|) & O(\|\widehat{U}\|) & * \\ 0 & 0 & 0 \end{pmatrix}}_{=:V}, \tag{3.2}$$

where $P_{ij}(\widehat{A}, \widehat{B})$ $(i, j = 1, 2)$ does not contain $\widehat{U}$. It suffices to work on the submatrix $\tilde{M} + \tilde{V}$ consisting of the first $2r$ row and columns of $M + V$. The eigenvalues of $\tilde{M} + \tilde{V}$ are called the *nontrivial eigenvalues* of $P(AU^*, UB)$.

***Proof of Theorem 1.1*** The entries of the matrix $\tilde{V}$ are polynomials on entries of $\widehat{A}, \widehat{B}, \widehat{U}, \widehat{U}^*$ without a constant term with respect to $\widehat{U}, \widehat{U}^*$, so that, by Lemma 2.3, they converge to 0 almost surely. This implies that the eigenvalues of $\tilde{M} + \tilde{V}$ converge to those of $\tilde{M}$, which can be easily proved by applying the argument principle in complex analysis to the characteristic polynomials (this is a simple case of Lemma 4.1 below where $d_N$ are all equal to $d = 2r$). □

The eigenvalues of $\tilde{M}$ and $\tilde{M} + \tilde{V}$ are denoted by $\{\mu_i\}_{i=1}^{2r}$ and $\{\mu_i^{(N)}\}_{i=1}^{2r}$ respectively according to the notation of Theorem 1.2.

***Proof of Theorem 1.2*** Since $\{\mu_i\}_{i=1}^{2r}$ have no multiplicities by the assumption, there exists an invertible matrix $\tilde{R}$ of size $2r$ such that $\tilde{R}^{-1}\tilde{M}\tilde{R} = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_{2r})$. Apparently the eigenvalues of $\tilde{M} + \tilde{V}$ are exactly those of the matrix

$$\operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_{2r}) + \tilde{R}^{-1}\tilde{V}\tilde{R}.$$

Then Lemmas 2.1 and 2.5 lead to Theorem 1.2 as desired. Indeed, Lemma 2.5 yields

$$\mu_i^{(N)} = \mu_i + \sum_{k=1}^{\infty} \Pi_{i,k}(\tilde{R}^{-1}\tilde{V}\tilde{R}),$$

which absolutely converges for sufficiently large $N$ since $\tilde{V} \to 0$ a.s. The RHS is a power series on variables $\{u_{ij}, \overline{u_{ij}} \mid i, j \in [r]\}$ and can be regrouped into

$$\mu_i^{(N)} = \mu_i + \sum_{k=1}^{\infty} Q_{i,k}(\widehat{U}, \widehat{U}^*),$$

where $Q_{i,k}(X, X^*)$ is a homogeneous polynomial of degree $k$ on commuting indeterminates $X = \{x_{ij}\}_{i,j\in[r]}$ and $X^* = \{\overline{x_{ji}}\}_{i,j\in[r]}$. We set

$$\kappa_i := \inf\{k \in \mathbb{N} \mid Q_{i,k}(X, X^*) \neq 0\}. \tag{3.3}$$

If $\kappa_i = \infty$ then $\mu_i^{(N)} = \mu_i$ a.s. If $\kappa_i < \infty$ then we can easily prove by Lemma 2.4 that $N^{\frac{\kappa_i}{2}}(\mu_i^{(N)} - \mu_i)$ converges in law to $Q_{i,\kappa_i}(Z, Z^*)$, where $Z$ is a standard complex Gaussian matrix. Moreover, this convergence holds jointly for all $i$ for which $\kappa_i < \infty$. Note that $Q_{i,\kappa_i}(Z, Z^*)$ is a nonzero random variable; indeed, because the set $S := \{X \in M_r(\mathbb{C}) \mid Q_{i,\kappa_i}(X, X^*) = 0\}$ is a null set with respect to the Lebesgue measure and $Z$ has a probability density function, the probability of the event $Z \in S$ is zero. This implies that for all $\kappa > \kappa_i$, $N^{\frac{\kappa}{2}}(\mu_i^{(N)} - \mu_i)$ does not converge in law and hence the definitions (1.3) and (3.3) coincide. □

### 3.2 The case *UA* + *AU**

**Proposition 3.1** *The* $2r$ *nontrivial eigenvalues of the matrix*

$$UA + AU^*,$$

*where*

$$A = \begin{pmatrix} \widehat{A} & 0 \\ 0 & 0 \end{pmatrix} \in M_N(\mathbb{C}) \quad \text{and} \quad \widehat{A} = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_r), \quad \alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{C},$$

*converge to* $\{\alpha_i, -\alpha_i\}_{i=1}^r$ *a.s. In addition, if these* $2r$ *limiting numbers are all distinct (which implies that they are nonzero) then the nontrivial eigenvalues of* $UA + AU^*$ *are of the forms*

$$\alpha_i + \frac{\alpha_i}{\sqrt{N}}\Re[\sqrt{N}u_{ii}^{(N)}] + \frac{\omega_{i,+}^{(N)}}{N} \quad \text{and} \quad -\alpha_i + \frac{\alpha_i}{\sqrt{N}}\Re[\sqrt{N}u_{ii}^{(N)}] + \frac{\omega_{i,-}^{(N)}}{N}, \qquad i \in [r],$$

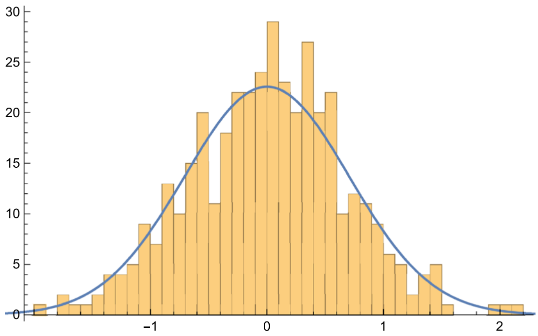


**Fig. 1** A histogram for $\sqrt{N}(\mu_1^{(N)} - 4)/4$ (made of 400 samples), where $\mu_1^{(N)}$ is the eigenvalue near 4 of the matrix model $AU + U^*A$ with $A = \mathrm{diag}(4, 2, 1, 0, 0, \ldots, 0)$ of size $N = 400$. The appended curve is the probability density function of $N(0, 1)$ multiplied by 40

*where $\{\omega_{i,\pm}^{(N)}\}_{i=1}^r$ are random variables converging in law to $\mathbb{C}$-valued random variables. See Figure 1 for a simulation.*

***Proof*** With respect to the basis $\mathbf{B}$ introduced in (3.1), the matrix $UA + AU^*$ has the matrix representation

$$T := \begin{pmatrix} \widehat{A}\widehat{U}^* & \widehat{A} & * \\ \widehat{A} & \widehat{A}\widehat{U} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \widehat{A} & 0 \\ \widehat{A} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \widehat{A}\widehat{U}^* & 0 & * \\ 0 & \widehat{A}\widehat{U} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The main part of $T$ can be diagonalized by the orthogonal matrix

$$R := \begin{pmatrix} \frac{1}{\sqrt{2}}1_r & \frac{1}{\sqrt{2}}1_r & 0 \\ \frac{1}{\sqrt{2}}1_r & -\frac{1}{\sqrt{2}}1_r & 0 \\ 0 & 0 & 1_{N-2r} \end{pmatrix}$$

in such a way that

$$R^{-1}TR = \begin{pmatrix} \widehat{A} & 0 & 0 \\ 0 & -\widehat{A} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \widehat{A}\frac{\widehat{U}+\widehat{U}^*}{2} & \widehat{A}\frac{\widehat{U}^*-\widehat{U}}{2} & * \\ \widehat{A}\frac{\widehat{U}^*-\widehat{U}}{2} & \widehat{A}\frac{\widehat{U}+\widehat{U}^*}{2} & * \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose further that the $2r$ numbers $\pm\alpha_1, \pm\alpha_2, \ldots, \pm\alpha_r$ are all distinct. As a consequence of Lemma 2.5, the nontrivial eigenvalues of $T$, denoted by $\mu_i^{(N)}, \nu_i^{(N)}, i \in [r]$, are of the forms

$$\mu_i^{(N)} = \alpha_i + \frac{1}{2}(\widehat{A}(\widehat{U} + \widehat{U}^*))_{ii} + O(\|\widehat{U}\|^2) = \alpha_i + \alpha_i \Re(u_{ii}) + O(\|\widehat{U}\|^2), \qquad i \in [r],$$
$$\nu_i^{(N)} = -\alpha_i + \alpha_i \Re(u_{ii}) + O(\|\widehat{U}\|^2), \qquad i \in [r].$$

□

**Remark 3.2** With the help of Lemma 2.1 and Lemma 2.4, this proposition implies that, as $N \to \infty$,

$$\sqrt{N}(\mu_i^{(N)} - \alpha_i) \xrightarrow{\text{law}} \frac{\alpha_i}{\sqrt{2}} x_i \quad \text{and} \quad \sqrt{N}(\nu_i^{(N)} + \alpha_i) \xrightarrow{\text{law}} \frac{\alpha_i}{\sqrt{2}} x_i \quad \text{for all} \quad i \in [r],$$

where $\{x_i\}_{i=1}^r$ are iid random variables, each distributed as $N(0, 1)$.

### 3.3 The case $P(A, UBU^*)$ and a proof of Theorem 1.3

The specialized model $P(A, UBU^*)$ is easier to analyze than (1.1) because the main part of the representation matrix is already diagonalized; see (3.4) below. In this subsection we modify the definition of $\widehat{U}$ to the rectangular truncation $(u_{ij})_{i\in[r], j\in[s]}$.

***Proof of Theorem 1.3*** Straightforward calculations yield that, with respect to the modified basis

$$(e_1, e_2, \ldots, e_r, u_1, u_2, \ldots, u_s, e_{r+1}, e_{r+2}, \ldots, e_{N-s}),$$

$A$ and $UBU^*$ have the matrix representations

$$\widetilde{A} := \begin{pmatrix} \widehat{A} & \widehat{A}\widehat{U} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{B} := \begin{pmatrix} 0 & 0 & 0 \\ \widehat{B}\widehat{U}^* & \widehat{B} & * \\ 0 & 0 & 0 \end{pmatrix},$$

respectively, where

$$\widehat{A} := \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_r) \in M_r(\mathbb{C}), \qquad \widehat{B} := \operatorname{diag}(\beta_1, \beta_2, \ldots, \beta_s) \in M_s(\mathbb{C}).$$

From this it is clear that our model $P(A, UBU^*)$ has $N-r-s$ trivial zero eigenvalues.

The polynomial $P(x, y)$ can be decomposed into

$$P(x, y) = P_1(x) + Q_1(y) + \underbrace{\sum_{k,\ell \geq 1} a_{k,\ell} x^k y^\ell}_{=:P_2(x,y)} + \underbrace{\sum_{k,\ell \geq 1} b_{k,\ell} y^\ell x^k}_{=:Q_2(x,y)}$$
$$+ \underbrace{\sum_{k,\ell,m \geq 1} c_{k,\ell,m} x^k y^\ell x^m}_{=:P_3(x,y)} + \underbrace{\sum_{k,\ell,m \geq 1} d_{k,\ell,m} y^\ell x^k y^m}_{=:Q_3(x,y)} + R(x, y),$$

where $b_{k,\ell}, c_{k,\ell,m}, d_{k,\ell,m}$ are complex coefficients, and $R$ is a linear combination of monomials of lengths larger than three (the length of the elements $x^k$ and $y^\ell$ is counted as one). For $k, \ell, m \geq 1$ we have

$$\widetilde{A}^k = \begin{pmatrix} \widehat{A}^k & \widehat{A}^k\widehat{U} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{B}^\ell = \begin{pmatrix} 0 & 0 & 0 \\ \widehat{B}^\ell\widehat{U}^* & \widehat{B}^\ell & * \\ 0 & 0 & 0 \end{pmatrix},$$

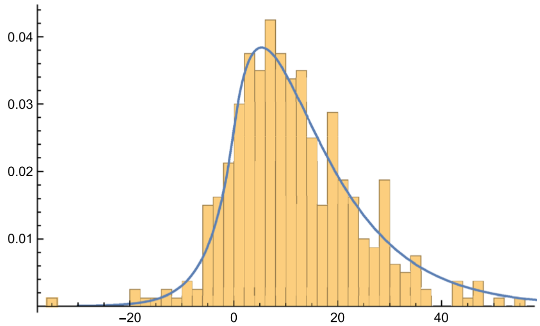


**Fig. 2** A histogram for $N(\mu_2^{(N)}-2)$ (400 samples, normalized to have area 1), where $\mu_2^{(N)}$ is the eigenvalue near 2 of the model $A+UBU^*+AUBU^*A+UBU^*AUBU^*$ with $A=\mathrm{diag}(5,2,1,0,0,\dots,0)$, $B=\mathrm{diag}(4,3,-1,0,0,\dots,0)$ of size $N=400$, together with the theoretical limiting probability density function $\frac{21}{800}e^{3x/14}1_{(-\infty,0)}(x)+\frac{3}{800}(-25e^{-x/6}+32e^{-x/12})1_{[0,\infty)}(x)$

$$\widetilde{A}^k\widetilde{B}^\ell=\begin{pmatrix}\widehat{A}^k\widehat{U}\widehat{B}^\ell\widehat{U}^* & \widehat{A}^k\widehat{U}\widehat{B}^\ell & *\\ 0&0&0\\0&0&0\end{pmatrix},$$
$$\widetilde{B}^\ell\widetilde{A}^k=\begin{pmatrix}0&0&0\\ \widehat{B}^\ell\widehat{U}^*\widehat{A}^k & \widehat{B}^\ell\widehat{U}^*\widehat{A}^k\widehat{U} & 0\\0&0&0\end{pmatrix},$$
$$\widetilde{A}^k\widetilde{B}^\ell\widetilde{A}^m=\begin{pmatrix}\widehat{A}^k\widehat{U}\widehat{B}^\ell\widehat{U}^*\widehat{A}^m & \widehat{A}^k\widehat{U}\widehat{B}^\ell\widehat{U}^*\widehat{A}^m\widehat{U} & 0\\ 0&0&0\\0&0&0\end{pmatrix},$$
$$\widetilde{B}^\ell\widetilde{A}^k\widetilde{B}^m=\begin{pmatrix}0&0&0\\ \widehat{B}^\ell\widehat{U}^*\widehat{A}^k\widehat{U}\widehat{B}^m\widehat{U}^* & \widehat{B}^\ell\widehat{U}^*\widehat{A}^k\widehat{U}\widehat{B}^m & *\\0&0&0\end{pmatrix}.$$

With the convention that $c_{k,\ell,0}=a_{k,\ell}$, $d_{k,\ell,0}=b_{k,\ell}$ and $P_1(x)=\sum_{k\geq 1}a_{k,0}x^k$, $Q_1(y)=\sum_{\ell\geq 1}b_{0,\ell}y^\ell$ we get

$$\begin{aligned}P(\widetilde{A},\widetilde{B})&=\begin{pmatrix}P_1(\widehat{A}) & P_1(\widehat{A})\widehat{U} & 0\\ Q_1(\widehat{B})\widehat{U}^* & Q_1(\widehat{B}) & *\\ 0&0&0\end{pmatrix}+\sum_{k,\ell\geq 1}\begin{pmatrix}a_{k,\ell}\widehat{A}^k\widehat{U}\widehat{B}^\ell\widehat{U}^* & a_{k,\ell}\widehat{A}^k\widehat{U}\widehat{B}^\ell & *\\ b_{k,\ell}\widehat{B}^\ell\widehat{U}^*\widehat{A}^k & b_{k,\ell}\widehat{B}^\ell\widehat{U}^*\widehat{A}^k\widehat{U} & *\\ 0&0&0\end{pmatrix}\\&\quad+\sum_{k,\ell,m\geq 1}\begin{pmatrix}c_{k,\ell,m}\widehat{A}^k\widehat{U}\widehat{B}^\ell\widehat{U}^*\widehat{A}^m & 0 & 0\\ 0 & d_{k,\ell,m}\widehat{B}^\ell\widehat{U}^*\widehat{A}^k\widehat{U}\widehat{B}^m & *\\ 0&0&0\end{pmatrix}\\&\quad+\begin{pmatrix}O(\|\widehat{U}\|^3) & O(\|\widehat{U}\|^2) & *\\ O(\|\widehat{U}\|^2) & O(\|\widehat{U}\|^3) & *\\ 0&0&0\end{pmatrix}\\&=\begin{pmatrix}P_1(\widehat{A}) & 0 & 0\\ 0 & Q_1(\widehat{B}) & *\\ 0&0&0\end{pmatrix}+\sum_{k\geq 1,\ell\geq 0}\begin{pmatrix}0 & a_{k,\ell}\widehat{A}^k\widehat{U}\widehat{B}^\ell & *\\ 0&0&*\\0&0&0\end{pmatrix}\end{aligned}$$

$$+\sum_{k\geq 0,\ell\geq 1}\begin{pmatrix} 0 & 0 & * \\ b_{k,\ell}\widehat{B}^{\ell}\widehat{U}^{*}\widehat{A}^{k} & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$$
$$+\sum_{k,\ell\geq 1,m\geq 0}\begin{pmatrix} c_{k,\ell,m}\widehat{A}^{k}\widehat{U}\widehat{B}^{\ell}\widehat{U}^{*}\widehat{A}^{m} & 0 & 0 \\ 0 & d_{k,\ell,m}\widehat{B}^{\ell}\widehat{U}^{*}\widehat{A}^{k}\widehat{U}\widehat{B}^{m} & * \\ 0 & 0 & 0 \end{pmatrix}$$
$$+\begin{pmatrix} O(\|\widehat{U}\|^{3}) & O(\|\widehat{U}\|^{2}) & * \\ O(\|\widehat{U}\|^{2}) & O(\|\widehat{U}\|^{3}) & * \\ 0 & 0 & 0 \end{pmatrix} \tag{3.4}$$

Suppose that the $r+s$ eigenvalues of the main part

$$\begin{pmatrix} P_1(\widehat{A}) & 0 \\ 0 & Q_1(\widehat{B}) \end{pmatrix}$$

are all distinct. Note that these eigenvalues are $P_1(\alpha_i)$, $i\in[r]$ and $Q_1(\beta_j)$, $j\in[s]$. According to Lemma 2.5 the eigenvalues $\mu_1^{(N)},\mu_2^{(N)},\ldots,\mu_{r+s}^{(N)}$ of the first $(r+s)$-dimensional corner of $P(\widetilde{A},\widetilde{B})$ (called the nontrivial eigenvalues) are of the form

$$\begin{aligned}
\mu_i^{(N)} &= P_1(\alpha_i)+\sum_{k,\ell\geq 1,m\geq 0} c_{k,\ell,m}(\widehat{A}^{k}\widehat{U}\widehat{B}^{\ell}\widehat{U}^{*}\widehat{A}^{m})_{i,i} \\
&\quad +\sum_{j\in[s]}\frac{1}{P_1(\alpha_i)-Q_1(\beta_j)}\sum_{k\geq 1,\ell\geq 0} a_{k,\ell}(\widehat{A}^{k}\widehat{U}\widehat{B}^{\ell})_{i,j} \\
&\quad \sum_{k'\geq 0,\ell'\geq 1} b_{k',\ell'}(\widehat{B}^{\ell'}\widehat{U}^{*}\widehat{A}^{k'})_{j,i}+O(\|\widehat{U}\|^{3}) \\
&= P_1(\alpha_i)+\sum_{k,\ell\geq 1,m\geq 0} c_{k,\ell,m}\sum_{j\in[s]}\alpha_i^{k+m}\beta_j^{\ell}|u_{i,j}|^2 \\
&\quad +\sum_{j\in[s]}\frac{1}{P_1(\alpha_i)-Q_1(\beta_j)}\sum_{k\geq 1,\ell\geq 0} a_{k,\ell}\sum_{k'\geq 0,\ell'\geq 1} b_{k',\ell'}\alpha_i^{k+k'}\beta_j^{\ell+\ell'}|u_{i,j}|^2+O(\|\widehat{U}\|^{3}) \\
&= P_1(\alpha_i)+\sum_{j\in[s]}\Big(P_2(\alpha_i,\beta_j)+P_3(\alpha_i,\beta_j) \\
&\quad +\frac{[P_1(\alpha_i)+P_2(\alpha_i,\beta_j)][Q_1(\beta_j)+Q_2(\alpha_i,\beta_j)]}{P_1(\alpha_i)-Q_1(\beta_j)}\Big)|u_{i,j}|^2 \\
&\quad +O(\|\widehat{U}\|^{3})
\end{aligned} \tag{3.5}$$

for $i\in[r]$, and similarly,

$$\begin{aligned}
\mu_{r+j}^{(N)} &= Q_1(\beta_j)+\sum_{i\in[r]}\Big(Q_2(\alpha_i,\beta_j)+Q_3(\alpha_i,\beta_j) \\
&\qquad -\frac{[P_1(\alpha_i)+P_2(\alpha_i,\beta_j)][Q_1(\beta_j)+Q_2(\alpha_i,\beta_j)]}{P_1(\alpha_i)-Q_1(\beta_j)}\Big)|u_{i,j}|^2
\end{aligned}$$

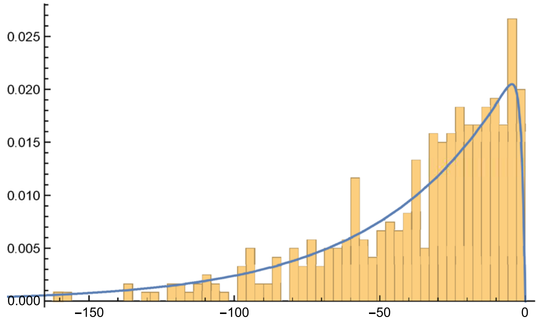


**Fig. 3** A histogram for $N(\mu_1^{(N)}-2)$ (400 samples, normalized to have area 1), where $\mu_1^{(N)}$ is the eigenvalue near 2 of the model $A+UBU^*+AUBU^*+UBU^*A+\frac{1}{2}(AUBU^*A+UBU^*AUBU^*)$ with $A=\mathrm{diag}(2,1,-1,0,0,\ldots,0)$, $B=\mathrm{diag}(4,-0.2,0,0,\ldots,0)$ of size $N=400$, together with the theoretical limiting probability density function $\frac{55}{2352}(e^{x/44}-e^{55x/68})1_{(-\infty,0)}(x)$

$$+\,O(\|\widehat{U}\|^3) \tag{3.6}$$

for $j\in[s]$. The random variables $N|u_{ij}^{(N)}|^2$ converge in law to $(\Re(z_{ij}))^2+(\Im(z_{ij}))^2$ which follows the exponential distribution $e^{-x}\,dx$, $x>0$. □

## 4 Eigenvalues with multiplicities: examples

When the main term of (3.2) has multiple eigenvalues, a general algorithm for computing fluctuations would be complicated (Lemma 2.5 works only for simple eigenvalues). Abandoning the general case, we work with the specific model

$$X=A+UBU^*, \tag{4.1}$$

where

$$\begin{aligned}
&A:=\mathrm{diag}(\alpha_1,\alpha_2,\ldots,\alpha_r,0,0,\ldots,0)\in M_N(\mathbb{R}),\\
&B:=\mathrm{diag}(\beta_1,\beta_2,\ldots,\beta_s,0,0,\ldots,0)\in M_N(\mathbb{R}),\\
&\quad\alpha_i,\beta_j\in\mathbb{R}\setminus\{0\},\ i\in[r],\ j\in[s].
\end{aligned}$$

It is easy to see (e.g., from (4.2)) that the limiting eigenvalues of $X$ are $\alpha_i,\beta_j,i\in[r],\ j\in[s]$ and the others are all zero.

Even for this specific model, a general algorithm for computing fluctuations looks too difficult. We deal with further special cases.

### 4.1 Convergence of polynomials and convergence of roots

Our analysis of the fluctuations of eigenvalues of the sum model (4.1) is based on the characteristic polynomials. The following fact is essential to deal with eigenvalues with multiplicities and is a simple consequence of the argument principle in complex analysis.

**Lemma 4.1** *Let $P(z)$, $P_N(z)$, $N \in \mathbb{N}$ be polynomials with complex coefficients such that $P \not\equiv 0$. Let $d := \deg P(z)$, $d_N := \deg P_N(z)$ and assume that $\sup_{N\in\mathbb{N}} d_N < \infty$. We denote by $\lambda_i$, $i \in [d]$ the roots of $P(z)$ counting multiplicities. Suppose that $P_N$ converges to $P$ pointwisely on $\mathbb{C}$. Then $d_N \geq d$ for sufficiently large $N$ and there is a suitable labeling of the roots of $P_N(z)$ counting multiplicities, denoted by $\lambda_i^{(N)}$, $i \in [d_N]$, such that*

*(i)* $\lim_{N\to\infty} \lambda_i^{(N)} = \lambda_i$ *for all* $i \in [d]$,

*(ii)* $\lim_{N\to\infty} \max_{d+1\leq i\leq d_N} |\lambda_i^{(N)}| = \infty$.

*When $P$ is a nonzero constant, we understand that $d = 0$ and assertion (i) must be deleted. On the other hand, when $d_N = d$ then the number $\max_{d+1\leq i\leq d_N} |\lambda_i^{(N)}|$ is to be interpreted as $\infty$.*

**Example 4.2** Let

$$P_N(x) = \frac{(-1)^N}{N}x^2 + \left(1 - \frac{(-1)^N}{N^2}\right)x - \frac{1}{N} = \frac{1}{N}((-1)^N x + N)\left(x - \frac{1}{N}\right).$$

Then $P_N(x)$ converges to $P(x) = x$ pointwise. The root $x = (-1)^{N-1}N$ tends to $\pm\infty$, while the root $x = \frac{1}{N}$ converges to 0 which is the root of $P(x)$.

## 4.2 The characteristic polynomial

**Notation** For an $m \times n$ matrix $C = (c_{ij})_{i\in[m],j\in[n]}$ and two subsets $I \subseteq [m]$, $J \subseteq [n]$ of the same cardinality we let $[C]_{I,J}$ be the determinant of the submatrix $(c_{ij})_{i\in I,j\in J}$. As convention, we also set $[C]_{\emptyset,\emptyset} := 1$.

According to Example 1.3, with respect to the basis

$$(e_1, e_2, \ldots, e_r, u_1, u_2, \ldots, u_s, e_{r+1}, e_{r+2}, \ldots, e_{N-s})$$

the matrix $X$ in (4.1) has the representation matrix

$$\begin{pmatrix} \widehat{A} & \widehat{A}\widehat{U} & 0 \\ \widehat{B}\widehat{U}^* & \widehat{B} & * \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.2}$$

where $\widehat{A} = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_r)$, $\widehat{B} = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_s)$ and $\widehat{U} = (u_{ij})_{i\in[r],j\in[s]}$. Let $\widetilde{X}$ be the submatrix of (4.2) consisting of the first $(r+s)$ rows and columns. The characteristic polynomial of $\widetilde{X}$ is given by

$$\varphi_N(\lambda) := \sum_{n=0}^{\min\{r,s\}} (-1)^n \sum_{\substack{I\subseteq[r], J\subseteq[s]\\ \#I=\#J=n}} \left\{ \prod_{i\in[r]\setminus I} (\lambda-\alpha_i) \prod_{j\in[s]\setminus J} (\lambda-\beta_j) \prod_{i\in I}\alpha_i \prod_{j\in J}\beta_j \right\} \left|[\widehat{U}]_{I,J}\right|^2,$$

which is a direct consequence of the definition of determinant by prescribing the fixed points of permutations. Investigating this polynomial will reveal fluctuations of the eigenvalues.

### 4.3 Multiplicities within *A* (and/or within *B*)

Suppose first that $\alpha_1 = \alpha_2 = \cdots = \alpha_m$ for some $m \in [r]$ and that none of $\alpha_{m+1}, \alpha_{m+2}, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ equals $\alpha_1$.

**Convergence of rescaled characteristic polynomial.**

By taking the Skorohod representation, we assume for the moment that $\widehat{U}^{(N)}$ converges to $Z$ almost surely, see Remark 2.2. (This replacement will be justified later.) We begin by observing

$$\begin{aligned} N^m \varphi_N\left(\alpha_1 + \frac{1}{N}\tau\right) &= \tau^m \prod_{i=m+1}^{r} (\alpha_1-\alpha_i) \prod_{j=1}^{s} (\alpha_1-\beta_j) \\ &+ \sum_{n=1}^{\min\{m,s\}} (-1)^n \sum_{\substack{I\subseteq[m], J\subseteq[s]\\ \#I=\#J=n}} \tau^{m-n} \\ &\left[ \prod_{i=m+1}^{r} (\alpha_1-\alpha_i) \prod_{j\in[s]\setminus J} (\alpha_1-\beta_j) \right. \\ &\left. \prod_{i\in I}\alpha_i \prod_{j\in J}\beta_j \right] N^n \left|[\widehat{U}]_{I,J}\right|^2 + O\left(\frac{1}{N}\right), \end{aligned}$$

where $O\left(\frac{1}{N}\right)$ is a polynomial on $\tau$ of degree not larger than $r+s$ with coefficients of order $O\left(\frac{1}{N}\right)$ in the usual sense almost surely. Since $N^m \varphi_N\left(\alpha_1 + \frac{1}{N}\tau\right)$ converges almost surely to the polynomial

$$\psi(\tau) := \tau^m + \sum_{n=1}^{\min\{m,s\}} (-1)^n \sum_{\substack{I\subseteq[m], J\subseteq[s]\\ \#I=\#J=n}} \tau^{m-n} \left[ \prod_{j\in J} \frac{\alpha_1\beta_j}{\alpha_1-\beta_j} \right] \left|[Z]_{I,J}\right|^2,$$

by Lemma 4.1, the polynomial $\tau \mapsto \varphi_N\left(\alpha_1 + \frac{1}{N}\tau\right)$ has $m$ consecutive roots $\delta_{11}^{(N)} \geq \delta_{12}^{(N)} \geq \cdots \geq \delta_{1m}^{(N)}$ that converge almost surely to the $m$ real roots of $\psi$.

The roots of $\psi$ can be well described as the eigenvalues of a certain random matrix. Let $\gamma_{1j} := \frac{\alpha_1\beta_j}{\alpha_1-\beta_j}$, $\Gamma_1 := \mathrm{diag}(\gamma_{11}, \gamma_{12}, \dots, \gamma_{1s})$ and $Z_1 = (z_{ij})_{i\in[m], j\in[s]}$ be the truncation of $Z$. Then we have

$$\begin{aligned}\psi(\tau) &= \tau^m + \sum_{n=1}^{\min\{m,s\}} (-1)^n \tau^{m-n} \sum_{\substack{I\subseteq[m],\, J\subseteq[s]\\ \#I=\#J=n}} [Z\Gamma_1]_{I,J}[Z^*]_{J,I} \\ &= \tau^m + \sum_{n=1}^{m} \tau^{m-n}(-1)^n \sum_{\substack{I\subseteq[m]\\ \#I=n}} [Z\Gamma_1 Z^*]_{I,I} \\ &= \tau^m + \sum_{n=1}^{m} \tau^{m-n}(-1)^n \sum_{\substack{I\subseteq[m]\\ \#I=n}} [Z_1\Gamma_1 Z_1^*]_{I,I} \\ &= \det(\tau I_m - Z_1\Gamma_1 Z_1^*).\end{aligned}$$

Note that $[Z\Gamma_1 Z^*]_{I,I} = 0$ if $s < n = \#I$ because the rank of $Z\Gamma_1 Z^*$ is not greater than $s$ and hence we were allowed to replace $\min\{m, s\}$ with $m$.

From the discussions above the random vector $(\delta_{11}^{(N)}, \delta_{12}^{(N)}, \dots, \delta_{1m}^{(N)})$ converges almost surely to the sequence of the eigenvalues of $Z_1\Gamma_1 Z_1^*$ (labeled in the decreasing order), and hence, converges in law.

**Conclusion.**

The convergence in law of the random vector $(\delta_{11}^{(N)}, \delta_{12}^{(N)}, \dots, \delta_{1m}^{(N)})$ also holds for the original random matrix model (without taking the Skorohod representation) because the roots of polynomials can be expressed as measurable (in fact, continuous) functions of the coefficients as a consequence of the argument principle so that each $\delta_{1j}^{(N)}$ is a measurable function of $\widehat{U}^{(N)}$. Convergence in law is a notion completely determined by the law and hence is unchanged by replacing the random variables with other ones with identical laws.

**Example 4.3** The preceding arguments allow us to calculate the joint distribution of fluctuations when the entries of $A$ and $B$ are of the form

$$(\alpha_1, \alpha_2, \dots, \alpha_r) = (\underbrace{\alpha_1', \dots, \alpha_1'}_{m_1 \text{ times}}, \underbrace{\alpha_2', \dots, \alpha_2'}_{m_2 \text{ times}}, \dots, \underbrace{\alpha_p', \dots, \alpha_p'}_{m_p \text{ times}}),$$
$$(\beta_1, \beta_2, \dots, \beta_s) = (\underbrace{\beta_1', \dots, \beta_1'}_{n_1 \text{ times}}, \underbrace{\beta_2', \dots, \beta_2'}_{n_2 \text{ times}}, \dots, \underbrace{\beta_q', \dots, \beta_q'}_{n_q \text{ times}}),$$

where $\alpha_1' \dots, \alpha_p', \beta_1', \dots, \beta_q'$ are distinct. Without loss of generality, we assume that $\alpha_1' > \alpha_2' > \dots > \alpha_p'$ and similarly $\beta_1' > \beta_2' > \dots > \beta_q'$.

Let $\Gamma_k = \mathrm{diag}(\gamma_{k1}, \dots, \gamma_{ks})$ and $H_\ell = \mathrm{diag}(\eta_{1\ell}, \eta_{2\ell}, \dots, \eta_{r\ell})$, where

$$\gamma_{kj} = \frac{\alpha_k'\beta_j}{\alpha_k' - \beta_j} \quad \text{and} \quad \eta_{i\ell} = \frac{\alpha_i\beta_\ell'}{\alpha_i - \beta_\ell'}$$

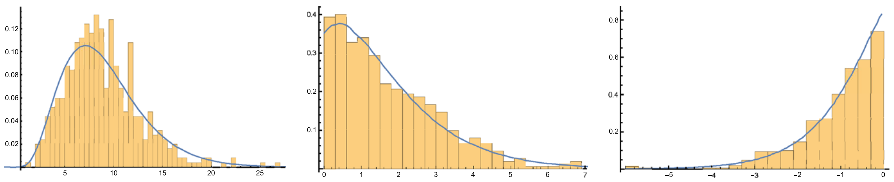

**Fig. 4** Histograms for $\delta_{1,1}^{(N)}$ (left), $\delta_{1,2}^{(N)}$ (middle) and $\delta_{1,3}^{(N)}$ (right) in Example 4.3 for the model $A+UBU^*$ with $A = \mathrm{diag}(2, 2, 2, 0, \dots, 0)$, $B = \mathrm{diag}(1, 1, -1, 0, \dots, 0)$ of size $N = 400$. The histograms are constructed from 500 samples, and the heights are normalized to have area 1. The appended curves are the probability density functions of $\rho_{1,1}$ (left), $\rho_{1,2}$ (middle), $\rho_{1,3}$ (right), drawn by taking $2 \cdot 10^6$ samples and connecting (by line segments) the heights of the histogram

and let $Z_k$, $Y_\ell$ be the $m_k \times s$ and $r \times n_\ell$ submatrices of $Z$, respectively, defined by

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{pmatrix} = (Y_1, Y_2, \dots, Y_q).$$

Let $\{\rho_{k1}, \rho_{k2}, \dots, \rho_{k,m_k}\}$ and $\{\sigma_{1\ell}, \sigma_{2\ell}, \dots, \sigma_{m_\ell,\ell}\}$ be the eigenvalues of $Z_k \Gamma_k Z_k^*$ and of $Y_\ell^* H_\ell Y_\ell$, respectively, labeled in the decreasing order. Then the eigenvalues of $\widetilde{X}$ are of the form $\alpha'_k + \frac{1}{N}\delta_{kj}^{(N)}$ ($j \in [m_k], k \in [p]$) and $\beta'_\ell + \frac{1}{N}\epsilon_{i\ell}^{(N)}$($i \in [n_\ell], \ell \in [q]$) with $\delta_{k1}^{(N)} \geq \delta_{k2}^{(N)} \geq \cdots \geq \delta_{k,m_k}^{(N)}$ and $\epsilon_{1\ell}^{(N)} \geq \epsilon_{2\ell}^{(N)} \geq \cdots \geq \epsilon_{n_\ell,\ell}^{(N)}$ such that

$$\left((\delta_{kj}^{(N)})_{j\in[m_k],k\in[p]}, (\epsilon_{i\ell}^{(N)})_{i\in[n_\ell],\ell\in[q]}\right) \xrightarrow{\text{law}} \left((\rho_{kj})_{j\in[m_k],k\in[p]}, (\sigma_{i\ell})_{i\in[n_\ell],\ell\in[q]}\right)$$

as random vectors on $\mathbb{R}_{\geq}^{m_1} \times \mathbb{R}_{\geq}^{m_2} \times \cdots \times \mathbb{R}_{\geq}^{m_p} \times \mathbb{R}_{\geq}^{n_1} \times \mathbb{R}_{\geq}^{n_2} \times \cdots \times \mathbb{R}_{\geq}^{n_q}$. See Figure 4 for simulations of fluctuations.

### 4.4 Common eigenvalues shared by *A* and *B*

We assume that $r = s = 2$ and discuss the case where some $\alpha_i$ coincides with some $\beta_j$. The fluctuations are more exotic.

**Example 4.4** Suppose that $\alpha_1 = \alpha_2 = \beta_1 \neq \beta_2$. Without loss of generality, we assume that $\alpha_1 < \beta_2$. The characteristic polynomial of $\widetilde{X}$ is explicitly given by

$$\begin{aligned}\varphi_N(\lambda) = (\lambda - \alpha_1)^3(\lambda - \beta_2) - \alpha_1^2(|u_{11}|^2 + |u_{21}|^2)(\lambda - \alpha_1)(\lambda - \beta_2) \\ - \alpha_1\beta_2(|u_{12}|^2 + |u_{22}|^2)(\lambda - \alpha_1)^2 + \alpha_1^3\beta_2|u_{11}u_{22} - u_{12}u_{21}|^2.\end{aligned}$$

Analogously to Section 4.3, we assume for the moment that $\sqrt{N}\widehat{U}^{(N)}$ converges to $Z$ almost surely.

**Fluctuations of $\alpha_1$.** Unexpectedly, there are two different scalings. Observe first that

$$N^{\frac{3}{2}}\varphi_N\left(\alpha_1+\frac{1}{\sqrt{N}}\tau\right)=\tau^3(\alpha_1-\beta_2)-\alpha_1^2N(|u_{11}|^2+|u_{21}|^2)\tau(\alpha_1-\beta_2)+O\left(\frac{1}{\sqrt{N}}\right), \tag{4.3}$$

which reveals that $\varphi_N(\alpha_1+\frac{1}{\sqrt{N}}\tau)$ has two roots $\tau=\delta_1^{(N)},\delta_2^{(N)}$ that respectively converge almost surely to

$$\xi_1:=|\alpha_1|\sqrt{|z_{11}|^2+|z_{21}|^2}\qquad\text{and}\qquad\xi_2:=-|\alpha_1|\sqrt{|z_{11}|^2+|z_{21}|^2}.$$

In addition to $\xi_1,\xi_2$, the limiting polynomial of (4.3) has the root 0 with multiplicity one. This means that the scaling $1/\sqrt{N}$ is irrelevant for the third root near $\alpha_1$. The right scaling is $N^{-1}$ as we see from

$$\begin{aligned}N^2\varphi_N\left(\alpha_1+\frac{1}{N}\tau\right)&=-\alpha_1^2N(|u_{11}|^2+|u_{21}|^2)\tau(\alpha_1-\beta_2)+\alpha_1^3\beta_2N^2\\&\quad|u_{11}u_{22}-u_{12}u_{21}|^2+O\left(\frac{1}{N}\right)\\&\to-\alpha_1^2(|z_{11}|^2+|z_{21}|^2)\tau(\alpha_1-\beta_2)+\alpha_1^3\beta_2|z_{11}z_{22}-z_{12}z_{21}|^2.\end{aligned}$$

Again by Lemma 4.1, the polynomial $\tau\mapsto\varphi_N(\alpha_1+\frac{1}{N}\tau)$ has a root $\tau=\delta_3^{(N)}$ that converges almost surely to the random variable

$$\xi_3:=\frac{\alpha_1\beta_2}{\alpha_1-\beta_2}\cdot\frac{|z_{11}z_{22}-z_{12}z_{21}|^2}{|z_{11}|^2+|z_{21}|^2}.$$

**Fluctuations of $\beta_2$.** A similar analysis yields that $\varphi_N$ has a root of the form $\lambda=\beta_2+\frac{1}{N}\epsilon^{(N)}$ such that $\epsilon^{(N)}$ converges almost surely to

$$\zeta:=\frac{\alpha_1\beta_2}{\beta_2-\alpha_1}(|z_{12}|^2+|z_{22}|^2).$$

**Conclusion.** Considering the scaling and the signs of limiting random variables, for sufficiently large $N$ (depending on samples), we have $\beta_2+\frac{1}{N}\epsilon^{(N)}>\alpha_1+\frac{1}{\sqrt{N}}\delta_1^{(N)}>\alpha_1+\frac{1}{N}\delta_3^{(N)}>\alpha_1+\frac{1}{\sqrt{N}}\delta_2^{(N)}$. Let $\lambda_1^{(N)}\geq\lambda_2^{(N)}\geq\lambda_3^{(N)}\geq\lambda_4^{(N)}$ be the four eigenvalues of $\widetilde{X}$. We conclude that the random vector

$$\left(N(\lambda_1^{(N)}-\beta_2),\sqrt{N}(\lambda_2^{(N)}-\alpha_1),N(\lambda_3^{(N)}-\alpha_1),\sqrt{N}(\lambda_4^{(N)}-\alpha_1)\right)$$

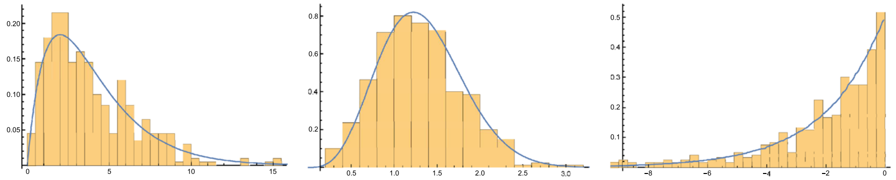


**Fig. 5** Histograms for $N(\lambda_1^{(N)} - 2)$ (left), $\sqrt{N}(\lambda_2^{(N)} - 1)$ (middle), $N(\lambda_3^{(N)} - 1)$ (right) in Example 4.4 for the matrices $A = \mathrm{diag}(1, 1, 0, 0, \ldots, 0)$, $B = \mathrm{diag}(1, 2, 0, 0, \ldots, 0)$ of size $N = 400$. The histograms are constructed from 400 samples, and the heights are normalized to have area 1. The appended curves are the probability density functions of $2(|z_{12}|^2 + |z_{22}|^2)$ (left, $(1/4)xe^{-x/2}$, $x > 0$), $\sqrt{|z_{11}|^2 + |z_{21}|^2}$ (middle, $2x^3e^{-x^2}$, $x > 0$), $\frac{-2|z_{11}z_{22} - z_{12}z_{21}|^2}{|z_{11}|^2 + |z_{21}|^2}$ (right, drawn by taking $2 \cdot 10^6$ samples and connecting the heights of the histogram)

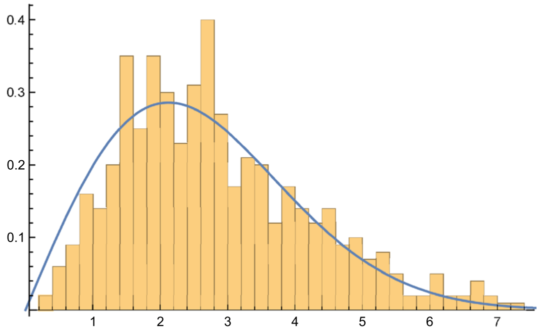


**Fig. 6** A histogram for $\sqrt{N}(\lambda_1^{(N)} - 3)$ in Example 4.5 for the matrices $A = B = \mathrm{diag}(2, 3, 0, 0, \ldots, 0)$ of size $N = 400$, together with the probability density function of $3|z_{22}|$ ($(2/9)xe^{-x^2/9}$, $x > 0$). The histogram is constructed from 500 samples, and the heights are normalized to have area 1

converges in law to

$$\left( \frac{\alpha_1\beta_2}{\beta_2 - \alpha_1}(|z_{12}|^2 + |z_{22}|^2), |\alpha_1|\sqrt{|z_{11}|^2 + |z_{21}|^2}, \frac{-\alpha_1\beta_2|z_{11}z_{22} - z_{12}z_{21}|^2}{(\beta_2 - \alpha_1)(|z_{11}|^2 + |z_{21}|^2)}, \right.$$
$$\left. -|\alpha_1|\sqrt{|z_{11}|^2 + |z_{21}|^2} \right).$$

This also holds without taking the Skorohod representation from the corresponding reasoning in Section 4.3. See Figure 5 for simulations.

**Example 4.5** Suppose that $\alpha_1 = \beta_1 < \alpha_2 = \beta_2$. Let $\lambda_1^{(N)} \geq \lambda_2^{(N)} \geq \lambda_3^{(N)} \geq \lambda_4^{(N)}$ be the four eigenvalues of $\widetilde{X}$. A similar technique reveals that the random vector

$$\left(\sqrt{N}(\lambda_1^{(N)} - \alpha_2), \sqrt{N}(\lambda_2^{(N)} - \alpha_2), \sqrt{N}(\lambda_3^{(N)} - \alpha_1), \sqrt{N}(\lambda_4^{(N)} - \alpha_1)\right)$$

converges in law to

$$(|\alpha_2 z_{22}|, -|\alpha_2 z_{22}|, |\alpha_1 z_{11}|, -|\alpha_1 z_{11}|).$$

See Figure 6 for a simulation.

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**Data Availability** The inserted figures are drawn on Mathematica Version 12.1.1, Wolfram Research, Inc., Champaign, IL. The source supporting the findings of this study are available from the corresponding author upon reasonable request.



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