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Decision analysis using set of fuzzy priority weight vectors estimated from a fuzzy pairwise comparison matrix

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Abstract

The fuzzy numbers have been introduced to the analytic hierarchy process (AHP) to reflect the vagueness of the decision maker's judgments. In fuzzy AHP (FAHP), a normalized fuzzy priority weight vector is estimated from a fuzzy pairwise comparison matrix (FPCM). Because the FPCM components are supposed to show the ratios of fuzzy priority weights, the deviations between them are considered natural criteria. Thus, if a normalized fuzzy priority weight vector has the same deviations as a solution to the estimation problem, it can be considered another solution. We may find such solutions, and the estimation problem can have many solutions. In this paper, we propose an FAHP approach to decision analysis using a set of solutions to the estimation problem under an FPCM. First, we study the estimation problem of the normalized fuzzy priority weight vector under a given FPCM and review a conventional approach. Minimizing the deviations between the FPCM components and the ratios of fuzzy priority weights becomes more complex than the conventional approach. We adopt a solution of the conventional approach. We extend it to a set of solutions because we can find other normalized fuzzy priority weight vectors having the same deviations as the solution. A decision analysis is proposed using all of these normalized fuzzy priority weight vectors. In numerical examples, we demonstrate a detailed decision analysis from multiple perspectives, considering all potential orders of alternatives. Therefore, the decision maker may select the final solution from several recommended orders of alternatives in various ideas according to her/his consent.

Keywords Fuzzy AHP · Multiple criteria decision making · Linear programming · Centroid defuzzification · Non-uniqueness

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1 Introduction

The analytic hierarchy process (AHP) Saaty (1980) is one of the most widely used methods for multiple criteria decision analysis. It is applied to many decision making problems in various fields. In AHP, the decision maker (DM) makes pairwise comparisons between items such as alternatives and criteria. The DM evaluates the relative importance between items in each pairwise comparison. The conventional AHP requires the DM to give precise evaluations of the relative importance. However, it would be difficult for the DM to give consistent evaluations because human evaluations are often vague and imprecise. To overcome this difficulty, a method of representing relative importance by intervals (Sugihara et al., 2004; Wang & Elhag, 2007; Mikhailov, 2004), fuzzy numbers (Buckley, 1985; Laarhoven & Pedrycz, 1983) and twofold intervals (Inuiguchi & Innan, 2022) has been proposed. Using such a method, we eventually obtain a pairwise comparison matrix (PCM) with intervals, fuzzy numbers, or twofold intervals.

In this paper, we focus on the case where the components of a PCM are fuzzy numbers, i.e., a fuzzy AHP, or more simply, an FAHP (Buckley, 1985). The FAHP has been widely applied in various fields, as reported in the literature. In the FAHP, several methods have been proposed to estimate priority weights from a PCM with fuzzy components. A PCM with fuzzy components is called simply an FPCM, as an abbreviation of a fuzzy PCM, in what follows. One of the earliest FAHP methods was proposed by Buckley (1985), who introduced the use of fuzzy positive reciprocal matrices and derived a fuzzy priority weight vector using the geometric mean method. His approach laid the foundation for many subsequent methods. The method of extent analysis proposed by Chang (1996), which derives a fuzzy priority weight vector from synthetic extent values and fuzzy dominance comparisons. This method has been adopted in FAHP applications, mainly because of its computational and implementation simplicity. The estimated priority weights are crisp in some methods and fuzzy in others. In Mikhailov (2003), an estimation method of crisp priority weights from an FPCM has been proposed based on fuzzy preference programming. There are many approaches to estimating a normalized fuzzy priority weight vector from an FPCM. In Wang et al. (2006a), a normalized fuzzy priority weight vector is estimated by the logarithmic least squares method. The linear goal programming approach is proposed for estimating a normalized fuzzy priority weight vector in Wang and Chin (2008). Similar to the eigenvalue method in the classical AHP, the fuzzy maximal eigenvector of an FPCM is defined and used for the estimation of fuzzy priority weights (Krejci, 2017). On the other hand, the geometric mean method in the classical AHP is extended to the case of PCM with triangular fuzzy numbers so that three parameters of fuzzy priority weights expressed by triangular fuzzy numbers are obtained by applying the geometric mean method to lower-bound, middle and upper-bound PCMs (Liu et al., 2017; Ramík & Korviny, 2010). Moreover, a heuristic-based approach is proposed for obtaining fuzzy priority weights from a PCM with triangular fuzzy numbers (Wang, 2019).

The FPCM components are supposed to show the ratios of fuzzy priority weights. From this fact, the deviations between them are considered natural criteria for evaluating solutions of the estimation problem. If a normalized fuzzy priority weight vector has the same deviations as a solution to the estimation problem, it can be another solution. Such solutions can be found, and the estimation problem can have non-unique solutions, as demonstrated in Inuiguchi (2016) in the interval case. However, most existing FAHP methods estimate a unique fuzzy priority weight vector. Moreover, minimizing the deviations between the FPCM components and ratios of fuzzy priority weights becomes more complex than the previous estimation methods because of the multiplicity of component-wise deviations and the nonlinearity. Then, we adopt a solution of a conventional method and extend it to a set of solutions by adding normalized fuzzy priority vectors that have the same deviations between FPCM components and ratios of fuzzy priority weights as the adopted solution. Obtaining the set of solutions enables us to analyze the decision problem more deeply, as we know the DM's hesitation among potential solutions.

In this paper, we investigate the decision analysis using the set of solutions to the problem of estimating the normalized fuzzy priority weight vector from an FPCM. We treat the case where all fuzzy components of both the pairwise comparison matrix and the fuzzy priority weight vector are given by triangular or trapezoidal fuzzy numbers. Namely, we treat the case where the DM expresses her/his evaluation of the relative importance of the *i*-th item to the *j*-th one by a triangular or trapezoidal fuzzy number for all pairs of items. Triangular fuzzy numbers are used when the DM evaluates the relative importance by a plausible value and the range of all possible values. On the other hand, trapezoidal fuzzy numbers are used when the DM evaluates the relative importance by a range of most plausible values and a range of all possible values. Those cases would be more often than the general FPCMs as human evaluations, because giving general fuzzy numbers is not an easy task for humans. For the sake of simplicity, we call a PCM with triangular fuzzy numbers a triangular fuzzy pairwise comparison matrix (TFPCM) and a PCM with trapezoidal fuzzy numbers a trapezoidal fuzzy pairwise comparison matrix (TZFPCM). The fuzzy priority weights are assumed to be triangular fuzzy numbers for a TFPCM and trapezoidal fuzzy numbers for a TZFPCM. An estimation method for a normalized triangular or trapezoidal fuzzy priority weight vector has already been proposed (Wang & Chin, 2008). In the paper (Wang & Chin, 2008), a sound and simple method for estimating fuzzy priority weights from an FPCM is proposed. The fuzzy priority weights are obtained simply by solving a linear goal programming (LGP) problem. It is demonstrated that the fuzzy priority weights obtained by the LGP method are more reasonable than those obtained by the extent analysis method (Chang, 1996). The LGP method is considered one of the reasonable and well-investigated approaches to the estimation of normalized fuzzy priority weights. Therefore, we adopt a solution of the LGP method (LGP solution) and extend it to a set of solutions having the same deviations between FPCM components and ratios of fuzzy priority weights. Because scalar multiplications of the upper and lower bases of the LGP solution by any positive numbers preserve the deviations of ratios of fuzzy priority weights from FPCM components, we easily obtain the required set of solutions simply by taking care of the normality conditions. Owing to this property, the set of solutions is obtained easily as a line segment in the TFPCM case and a polygon in the TZFPCM case. As we obtain the set of solutions, the decision analysis using the LGP solution is extended to the analysis with a set of solutions. Throughout this paper, we demonstrate that the introduction of the non-uniqueness of the solution to the estimation problem of normalized fuzzy priority weights enables us to obtain all potential preference orders of alternatives that the DM may agree on, and to analyze the DM's preference in more detail. The DM can select the final solution from several recommended orders of alternatives according to her/his consent.

This paper is organized as follows. In the next section, we briefly introduce the linear goal programming (LGP) method (Wang & Chin, 2008) for estimating a normalized triangular/trapezoidal fuzzy priority weight vector from a given triangular/trapezoidal fuzzy pairwise comparison matrix. In Section 3, we explain that the problem of estimating a triangular/trapezoidal normalized fuzzy priority weight vector can have a non-unique solution. A simple method for obtaining a solution set from a normalized fuzzy priority weight vector obtained by the LGP method is given. The LGP method is extended by introducing the solution set. In Section 4, we describe the calculation of the total utility value of each alternative as a fuzzy number. The ordering methods based on the total utility value are shown based on Wang (2009); Wang et al. (2006b). In Section 5, a numerical example is given to demonstrate the usefulness and advantages of the proposed modification. In Section 6, the concluding remarks are given.

2 Linear goal programming method in fuzzy AHP

In the AHP, first, the criteria and alternatives involved in a multiple criteria decision making problem are arranged in a hierarchy. Then, the criteria and alternatives are evaluated at each level of the hierarchy. In the AHP, priority weights of criteria and alternatives for each criterion are estimated from PCMs given by the DM. However, the normalization of priority weights of alternatives for each criterion can be controversial (Belton, 1986). Then, in this paper, we assume that the priority weights (marginal utility values) of alternatives for each criterion are given in some way by the decision maker or by experts knowing alternatives well, to avoid the discussion of the adequateness of their normalization. Accordingly, the sum of the marginal utility values of all alternatives for each criterion is not always one, i.e., the normalization is not assumed for the marginal utility values. On the other hand, we estimate the priority weights of the criteria through pairwise comparisons evaluated by the DM because the normalization of priority weights does not change the preference order of alternatives.

In the conventional AHP, a priority weight vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ for criteria is estimated from a PCM $A = (a_{ij})_{n\times n}$. In the conventional AHP, (*i*,*j*)-th component a_{ij} of PCM A shows the relative importance of the *i*-th criterion over the *j*-th criterion. It assumes that a_{ij} is equal to w_i/w_j, $i, j \in N = \{1, 2, \ldots, n\}$, if human judgements are precise. However, due to the vagueness of the DM's judgments, we may assume only $a_{ij} \approx w_i/w_j$, $i, j \in N$. Then, priority weights $w_i, i \in N$ are estimated by the eigenvalue method (Saaty, 1980) or the geometric mean method (Crawford, 1987). Both methods minimize the sum of deviations between a_{ij} and w_i/w_j, $i, j \in N$ (Innan & Inuiguchi, 2024), where the deviations are defined differently.

On the other hand, in a type of FAHP (Wang & Chin, 2008), to reflect the vagueness of the DM's judgments in a PCM, a FPCM is considered by representing the vagueness of the components a_{ij}, $i, j \in N$ $(i \neq j)$, by fuzzy numbers. Accordingly, fuzzy priority weight vector $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^{\mathrm{T}}$ is estimated from a FPCM $\tilde{A}$ and used for comparing alternatives.

In this paper, we consider the cases where components a_{ij}, $i, j \in N$ $(i \neq j)$ of FPCM $\tilde{A}$ are specified by triangular fuzzy numbers, and by trapezoidal fuzzy numbers as shown in Fig. 1. As marginal utility values of alternatives for each criterion are assumed to be given in some way by the DM or by experts knowing the alternatives well, alternatives are compared using the estimated fuzzy priority weights for criteria and the marginal utility values.

We describe the method when a TFPCM $\tilde{A}$, i.e., each component $\tilde{a}_{ij}$ is a triangular fuzzy number $(a^{\mathrm{L}}_{ij}, a^{\mathrm{M}}_{ij}, a^{\mathrm{U}}_{ij})$, where a^{L}_{ij}, a^{M}_{ij} and a^{U}_{ij} are the lower bound, the most plausible value and the upper bound for conceivable values for the relative importance of the *i*-th criterion over the *j*-th criterion. Accordingly, the FPCM is represented by

$$\tilde{A} = (\tilde{a}_{ij})_{n \times n} = \begin{pmatrix} 1 & (a^{\mathrm{L}}_{12}, a^{\mathrm{M}}_{12}, a^{\mathrm{U}}_{12}) & \cdots & (a^{\mathrm{L}}_{1n}, a^{\mathrm{M}}_{1n}, a^{\mathrm{U}}_{1n}) \\ (a^{\mathrm{L}}_{21}, a^{\mathrm{M}}_{21}, a^{\mathrm{U}}_{21}) & 1 & \cdots & (a^{\mathrm{L}}_{2n}, a^{\mathrm{M}}_{2n}, a^{\mathrm{U}}_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (a^{\mathrm{L}}_{n1}, a^{\mathrm{M}}_{n1}, a^{\mathrm{U}}_{n1}) & (a^{\mathrm{L}}_{n2}, a^{\mathrm{M}}_{n2}, a^{\mathrm{U}}_{n2}) & \cdots & 1 \end{pmatrix}, \quad (1)$$

where $a^{\mathrm{L}}_{ij} \leq a^{\mathrm{M}}_{ij} \leq a^{\mathrm{U}}_{ij}$, $a^{\mathrm{L}}_{ij} = 1/a^{\mathrm{U}}_{ji}$, $a^{\mathrm{M}}_{ij} = 1/a^{\mathrm{M}}_{ji}$, $i, j \in N$, but $i \neq j$, and $\tilde{a}_{ii} = 1, i \in N$. The triangular fuzzy number $(a^{\mathrm{L}}_{ij}, a^{\mathrm{M}}_{ij}, a^{\mathrm{U}}_{ij})$ representing the (*i*, *j*)-th component of $\tilde{A}$ has a membership function shown on the left side of Fig. 1. Then, the TFPCM $\tilde{A}$ can be split into the following three crisp matrices:

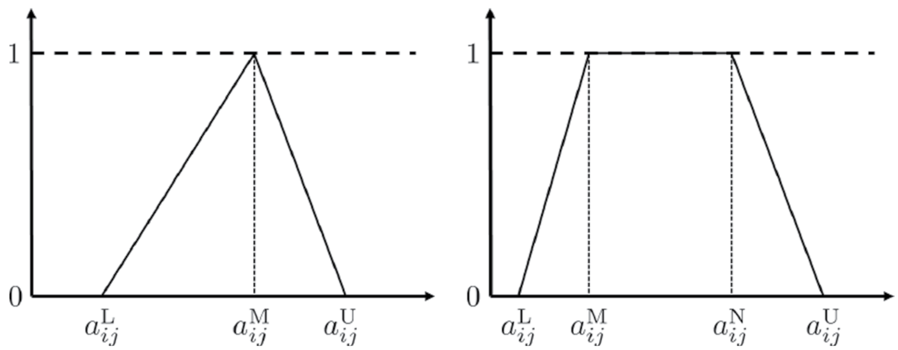

Fig. 1 Triangular and trapezoidal fuzzy numbers representing $\tilde{a}_{ij}$.

$$A_{\mathrm{L}} = (a_{ij}^{\mathrm{L}}) = \begin{pmatrix} 1 & \cdots & a_{1n}^{\mathrm{L}} \\ \vdots & \ddots & \vdots \\ a_{n1}^{\mathrm{L}} & \cdots & 1 \end{pmatrix}, \quad A_{\mathrm{M}} = (a_{ij}^{\mathrm{M}}) = \begin{pmatrix} 1 & \cdots & a_{1n}^{\mathrm{M}} \\ \vdots & \ddots & \vdots \\ a_{n1}^{\mathrm{M}} & \cdots & 1 \end{pmatrix},$$
$$A_{\mathrm{U}} = (a_{ij}^{\mathrm{U}}) = \begin{pmatrix} 1 & \cdots & a_{1n}^{\mathrm{U}} \\ \vdots & \ddots & \vdots \\ a_{n1}^{\mathrm{U}} & \cdots & 1 \end{pmatrix}. \tag{2}$$

Corresponding to TFPCM $\tilde{A}$, we consider triangular fuzzy numbers $\tilde{w}_i = (w_i^{\mathrm{L}}, w_i^{\mathrm{M}}, w_i^{\mathrm{U}})$, $i \in N$ for the fuzzy priority weights to be estimated. When there exist triangular fuzzy numbers $\tilde{w}_i = (w_i^{\mathrm{L}}, w_i^{\mathrm{M}}, w_i^{\mathrm{U}})$, $i \in N$ satisfying $\tilde{a}_{ij} = (a_{ij}^{\mathrm{L}}, a_{ij}^{\mathrm{M}}, a_{ij}^{\mathrm{U}}) = \tilde{w}_i / \tilde{w}_j = (w_i^{\mathrm{L}}/w_j^{\mathrm{U}}, w_i^{\mathrm{M}}/w_j^{\mathrm{M}}, w_i^{\mathrm{U}}/w_j^{\mathrm{L}}), i, j \in N \ (i \neq j)$, the given TFPCM $\tilde{A}$ is considered consistent[1]. From these equations, we obtain the following equations (Wang & Chin, 2008) when the TFPCM $\tilde{A}$ is consistent:

$$A_{\mathrm{L}} W_{\mathrm{U}} = W_{\mathrm{U}} + (n-1) W_{\mathrm{L}}, \tag{3}$$

$$A_{\mathrm{U}} W_{\mathrm{L}} = W_{\mathrm{L}} + (n-1) W_{\mathrm{U}}, \tag{4}$$

$$A_{\mathrm{M}} W_{\mathrm{M}} = n W_{\mathrm{M}}, \tag{5}$$

where $W_{\mathrm{L}} = (w_1^{\mathrm{L}}, \ldots, w_n^{\mathrm{L}})^{\mathrm{T}}$, $W_{\mathrm{M}} = (w_1^{\mathrm{M}}, \ldots, w_n^{\mathrm{M}})^{\mathrm{T}}$, $W_{\mathrm{U}} = (w_1^{\mathrm{U}}, \ldots, w_n^{\mathrm{U}})^{\mathrm{T}}$. As with crisp PCM in the conventional AHP, it is not usual to obtain a consistent TFPCM $\tilde{A}$ due to the vagueness of the DM's judgment. In other words, in real-world applications, we cannot expect that equations (3)–(5) hold. Therefore, deviational variable vector for equations (3)–(5) are defined as follows:

$$E^{+} - E^{-} = (A_{\mathrm{L}} - I) W_{\mathrm{U}} - (n-1) W_{\mathrm{L}}, \tag{6}$$

$$\Gamma^{+} - \Gamma^{-} = (A_{\mathrm{U}} - I) W_{\mathrm{L}} - (n-1) W_{\mathrm{U}}, \tag{7}$$

$$\Delta = (A_{\mathrm{M}} - nI) W_{\mathrm{M}}, \tag{8}$$

where $E^{+} = (\varepsilon_1^{+}, \ldots, \varepsilon_n^{+})^{\mathrm{T}}$, $E^{-} = (\varepsilon_1^{-}, \ldots, \varepsilon_n^{-})^{\mathrm{T}}$, $\Gamma^{+} = (\gamma_1^{+}, \ldots, \gamma_n^{+})^{\mathrm{T}}$, $\Gamma^{-} = (\gamma_1^{-}, \ldots, \gamma_n^{-})^{\mathrm{T}}$, $\Delta = (\delta_1, \ldots, \delta_n)^{\mathrm{T}}$ are deviational variable vectors, and I is an $n \times n$ identity matrix. Components $\varepsilon_i^{+}, \varepsilon_i^{-}, \gamma_i^{+}, \gamma_i^{-}, \delta_i \geq 0$, $i \in N$ are deviational variables satisfying $\varepsilon_i^{+} \cdot \varepsilon_i^{-} = 0$, $\gamma_i^{+} \cdot \gamma_i^{-} = 0$, $i \in N$. Subsequently, the triangular fuzzy priority weight vector $\tilde{W}$ is estimated from the TFPCM $\tilde{A}$ by minimizing the sum of the deviational variables in equations (6)–(8).

As priority weights are frequently normalized in the conventional AHP, the normalization condition for the fuzzy priority weight vector is required. The normaliza-

[1] In Wang and Chin (2008), it is called "precise". However, "precise"is confusable with a case where all triangular fuzzy numbers in the given FPCM are reduced to real numbers. To avoid this confusion, we call it "consistent".

tion conditions (Wang & Elhag, 2006) of the triangular fuzzy priority weight vector $\tilde{W}$ are expressed as

$$\sum_{i\in N\backslash j} w_i^{\mathrm{U}} + w_j^{\mathrm{L}} \geq 1,\ \sum_{i\in N\backslash j} w_i^{\mathrm{L}} + w_j^{\mathrm{U}} \leq 1,\ j\in N,\ \sum_{i\in N} w_i^{\mathrm{M}} = 1. \tag{9}$$

Then, a triangular fuzzy priority vector $\tilde{W}$ is estimated from a TFPCM $\tilde{A}$ by minimizing the sum of deviational variables under the constraints of equations (6)–(9). The resulting problem is the following linear goal programming (LGP) problem: (Wang & Chin, 2008)

$$\begin{aligned}
&\text{minimize} \quad e^{\mathrm{T}}(E^{+}+E^{-}+\Gamma^{+}+\Gamma^{-}+\Delta)\\
&\text{subject to} \ (A_{\mathrm{L}}-I)W_{\mathrm{U}}-(n-1)W_{\mathrm{L}}-E^{+}+E^{-}=0,\\
&\qquad (A_{\mathrm{U}}-I)W_{\mathrm{L}}-(n-1)W_{\mathrm{U}}-\Gamma^{+}+\Gamma^{-}=0,\\
&\qquad (A_{\mathrm{M}}-nI)W_{\mathrm{M}}-\Delta=0,\\
&\qquad \sum_{i\in N\backslash j} w_i^{\mathrm{U}} + w_j^{\mathrm{L}} \geq 1,\ \sum_{i\in N\backslash j} w_i^{\mathrm{L}} + w_j^{\mathrm{U}} \leq 1,\ j\in N, \sum_{i\in N} w_i^{\mathrm{M}} = 1,\\
&\qquad w_i^{\mathrm{U}} \geq w_i^{\mathrm{M}} \geq w_i^{\mathrm{L}} \geq \epsilon,\ i\in N,\\
&\qquad E^{+},E^{-},\Gamma^{+},\Gamma^{-},\Delta \geq 0,
\end{aligned} \tag{10}$$

where $e=(1,1,\ldots,1)^{\mathrm{T}}$ and ϵ is a sufficiently small positive number, employed to treat $w_i^{\mathrm{L}}>0, i\in N$, approximately. This model is called ”the LGP model"proposed by Wang and Chin (2008).

Next, we describe the LGP model when the components of FPCM are trapezoidal fuzzy numbers. In this case, a TZFPCM is given by the DM:

$$\tilde{A}=(\tilde{a}_{ij})_{n\times n}=\begin{pmatrix} 1 & (a_{12}^{\mathrm{L}},\ a_{12}^{\mathrm{M}},\ a_{12}^{\mathrm{N}},\ a_{12}^{\mathrm{U}}) & \cdots & (a_{1n}^{\mathrm{L}},\ a_{1n}^{\mathrm{M}},\ a_{1n}^{\mathrm{N}},\ a_{1n}^{\mathrm{U}}) \\ (a_{21}^{\mathrm{L}},\ a_{21}^{\mathrm{M}},\ a_{21}^{\mathrm{N}},\ a_{21}^{\mathrm{U}}) & 1 & \cdots & (a_{2n}^{\mathrm{L}},\ a_{2n}^{\mathrm{M}},\ a_{2n}^{\mathrm{N}},\ a_{2n}^{\mathrm{U}}) \\ \vdots & \vdots & \ddots & \vdots \\ (a_{n1}^{\mathrm{L}},\ a_{n1}^{\mathrm{M}},\ a_{n1}^{\mathrm{N}},\ a_{n1}^{\mathrm{U}}) & (a_{n2}^{\mathrm{L}},\ a_{n2}^{\mathrm{M}},\ a_{n2}^{\mathrm{N}},\ a_{n2}^{\mathrm{U}}) & \cdots & 1 \end{pmatrix}, \tag{11}$$

where, $a_{ij}^{\mathrm{L}} \leq a_{ij}^{\mathrm{M}} \leq a_{ij}^{\mathrm{N}} \leq a_{ij}^{\mathrm{U}}$, $a_{ij}^{\mathrm{L}}=1/a_{ji}^{\mathrm{U}}$, $a_{ij}^{\mathrm{M}}=1/a_{ji}^{\mathrm{N}}, i,j\in N \ (i\neq j)$, and $\tilde{a}_{ii}=1, i\in N$. The trapezoidal fuzzy number $(a_{ij}^{\mathrm{L}},a_{ij}^{\mathrm{M}},a_{ij}^{\mathrm{N}},a_{ij}^{\mathrm{U}})$ representing the (i,j)-th component of $\tilde{A}$ has a membership function shown on the right side of Fig. 1. The TZFPCM (11) is reduced to a TFPCM (1) when $a_{ij}^{\mathrm{M}}=a_{ij}^{\mathrm{N}}$ for $i,j\in N$.

Similar to TFPCM, the TZFPCM $\tilde{A}$ can be split into four crisp matrices, A_{L}, A_{M}, A_{U} of equation (2) with a_{ij}^{L}, a_{ij}^{M}, a_{ij}^{U} of trapezoidal fuzzy numbers $\tilde{a}_{ij}$ and A_{N} with a_{ij}^{N} defined by:

$$A_{\mathrm{N}}=(a_{ij}^{\mathrm{N}})=\begin{pmatrix} 1 & \cdots & a_{1n}^{\mathrm{N}} \\ \vdots & \ddots & \vdots \\ a_{n1}^{\mathrm{N}} & \cdots & 1 \end{pmatrix}. \tag{12}$$

We consider trapezoidal fuzzy numbers $\tilde{w}_i = (w_i^{\mathrm{L}}, w_i^{\mathrm{M}}, w_i^{\mathrm{N}}, w_i^{\mathrm{U}}), i \in N$ for the fuzzy priority weights corresponding to the TZFPCM $\tilde{A}$. If there exist trapezoidal fuzzy numbers $\tilde{w}_i = (w_i^{\mathrm{L}}, w_i^{\mathrm{M}}, w_i^{\mathrm{N}}, w_i^{\mathrm{U}}), i \in N$ satisfying $\tilde{a}_{ij} = (a_{ij}^{\mathrm{L}}, a_{ij}^{\mathrm{M}}, a_{ij}^{\mathrm{N}}, a_{ij}^{\mathrm{U}}) = \tilde{w}_i/\tilde{w}_j = (w_i^{\mathrm{L}}/w_j^{\mathrm{U}}, w_i^{\mathrm{M}}/w_j^{\mathrm{N}}, w_i^{\mathrm{N}}/w_j^{\mathrm{M}}, w_i^{\mathrm{U}}/w_j^{\mathrm{L}})$, $i, j \in N$ $(i \neq j)$, the TZFPCM is consistent. Then if the TZFPCM is consistent, we obtain four equations, (3), (4) with A_{L} and A_{U} of TZFPCM $\tilde{A}$ and the following two equations:

$$A_{\mathrm{M}} W_{\mathrm{N}} = W_{\mathrm{N}} + (n-1) W_{\mathrm{M}}, \tag{13}$$

$$A_{\mathrm{N}} W_{\mathrm{M}} = W_{\mathrm{M}} + (n-1) W_{\mathrm{N}}, \tag{14}$$

where $W_{\mathrm{N}} = (w_1^{\mathrm{N}}, \ldots, w_n^{\mathrm{N}})^{\mathrm{T}}$.

Due to the vagueness of the DM's judgments, equations (3), (4), (13), (14) do not frequently hold. Then we introduce deviational variable vectors E^+, E^-, Γ^+, Γ^- of equations (6) and (7) with A^{L} and A^{U} of TZFPCM $\tilde{A}$, and Δ^+, Δ^-, Λ^+ and Λ^- defined by the following equations with A^{M} and A^{N} of TZFPCM $\tilde{A}$:

$$\Delta^+ - \Delta^- = (A_{\mathrm{M}} - I) W_{\mathrm{N}} - (n-1) W_{\mathrm{M}}, \tag{15}$$

$$\Lambda^+ - \Lambda^- = (A_{\mathrm{N}} - I) W_{\mathrm{M}} - (n-1) W_{\mathrm{N}}, \tag{16}$$

where $\Delta^+ = (\delta_1^+, \ldots, \delta_n^+)^{\mathrm{T}}$, $\Delta^- = (\delta_1^-, \ldots, \delta_n^-)^{\mathrm{T}}$, $\Lambda^+ = (\lambda_1^+, \ldots, \lambda_n^+)^{\mathrm{T}}$, and $\Lambda^- = (\lambda_1^-, \ldots, \lambda_n^-)^{\mathrm{T}}$ are deviational variable vectors. Components $\delta_i^+, \delta_i^-, \lambda_i^+, \lambda_i^- \geq 0$, $i \in N$ are deviational variables satisfying $\delta_i^+ \cdot \delta_i^- = 0$, $\lambda_i^+ \cdot \lambda_i^- = 0$, $i \in N$.

The normalization condition of the trapezoidal fuzzy priority weight vector composed of $\tilde{w}_i = (w_i^{\mathrm{L}}, w_i^{\mathrm{M}}, w_i^{\mathrm{N}}, w_i^{\mathrm{U}})$, $i \in N$ is expressed as

$$\begin{aligned} &\sum_{i \in N \setminus j} w_i^{\mathrm{U}} + w_j^{\mathrm{L}} \geq 1, \ \sum_{i \in N \setminus j} w_i^{\mathrm{L}} + w_j^{\mathrm{U}} \leq 1, \ j \in N, \\ &\sum_{i \in N \setminus j} w_i^{\mathrm{N}} + w_j^{\mathrm{M}} \geq 1, \ \sum_{i \in N \setminus j} w_i^{\mathrm{M}} + w_j^{\mathrm{N}} \leq 1, \ j \in N. \end{aligned} \tag{17}$$

The trapezoidal fuzzy priority weight vector $\tilde{W}$ is estimated from the TZFPCM $\tilde{A}$ by minimizing the sum of the deviation variables of E^+, E^-, Γ^+, Γ^-, Δ^+, Δ^-, Λ^+ and Λ^- under the constraints of equations (6), (7), (15) and (16) with trapezoidal fuzzy numbers. This estimation problem is formulated as the following LGP problem, similar to the problem from TFPCM $\tilde{A}$:

$$
\begin{array}{ll}
\text{minimize} & e^{\mathrm{T}}(E^{+}+E^{-}+\varGamma^{+}+\varGamma^{-}+\varDelta^{+}+\varDelta^{-}+\varLambda^{+}+\varLambda^{-}) \\
\text{subject to} & (A_{\mathrm{L}}-I)W_{\mathrm{U}}-(n-1)W_{\mathrm{L}}-E^{+}+E^{-}=0 \\
& (A_{\mathrm{U}}-I)W_{\mathrm{L}}-(n-1)W_{\mathrm{U}}-\varGamma^{+}+\varGamma^{-}=0 \\
& (A_{\mathrm{M}}-I)W_{\mathrm{N}}-(n-1)W_{\mathrm{M}}-\varDelta^{+}+\varDelta^{-}=0 \\
& (A_{\mathrm{N}}-I)W_{\mathrm{M}}-(n-1)W_{\mathrm{N}}-\varLambda^{+}+\varLambda^{-}=0 \\
& \sum_{i\in N\setminus j} w_i^{\mathrm{U}}+w_j^{\mathrm{L}}\geq 1,\ \sum_{i\in N\setminus j} w_i^{\mathrm{L}}+w_j^{\mathrm{U}}\leq 1,\ j\in N \\
& \sum_{i\in N\setminus j} w_i^{\mathrm{N}}+w_j^{\mathrm{M}}\geq 1,\ \sum_{i\in N\setminus j} w_i^{\mathrm{M}}+w_j^{\mathrm{N}}\leq 1,\ j\in N \\
& w_i^{\mathrm{U}}\geq w_i^{\mathrm{N}}\geq w_i^{\mathrm{M}}\geq w_i^{\mathrm{L}}\geq \epsilon,\ i\in N \\
& E^{+},E^{-},\varGamma^{+},\varGamma^{-},\varDelta^{+},\varDelta^{-},\varLambda^{+},\varLambda^{-}\geq 0.
\end{array} \tag{18}
$$

This model is also "an LGP model". While problem (10) estimates a triangular fuzzy priority weight vector from a TFPCM, problem (18) estimates a trapezoidal fuzzy priority weight vector from a TZFPCM.

Remark 1 We may see a triangular fuzzy number $\tilde{a}=(a^{\mathrm{L}},a^{\mathrm{M}},a^{\mathrm{U}})$ as a trapezoidal fuzzy number $\tilde{a}'=(a^{\mathrm{L}},a^{\mathrm{M}},a^{\mathrm{M}},a^{\mathrm{U}})$ where its core (the upper base) degenerates a point. In this way, we may see a TFPCM $\tilde{A}=(\tilde{a}_{ij})$ with $\tilde{a}_{ij}=(a_{ij}^{\mathrm{L}},a_{ij}^{\mathrm{M}},a_{ij}^{\mathrm{U}})$ as a TZFPCM $\tilde{A}'=(\tilde{a}'_{ij})$ with $\tilde{a}'_{ij}=(a_{ij}^{\mathrm{L}},a_{ij}^{\mathrm{M}},a_{ij}^{\mathrm{M}},a_{ij}^{\mathrm{U}})$. However, the solution to Problem (10) with TFPCM $\tilde{A}$ will be different from that of Problem (18) with TZFPCM $\tilde{A}'$ because the objective functions are different. Using E^{-}, E^{+}, $\varGamma^{+}$, $\varGamma^{-}$ and $\varDelta$ in Problem (10), the objective function of Problem (18) with TZFPCM $\tilde{A}'$ becomes $e^{\mathrm{T}}(E^{+}+E^{-}+\varGamma^{+}+\varGamma^{-}+2\varDelta)$, which is different from the objective function $e^{\mathrm{T}}(E^{+}+E^{-}+\varGamma^{+}+\varGamma^{-}+\varDelta)$ of Problem (10) with TFPCM $\tilde{A}$. □

As described earlier, we assume that the marginal utility values $u_i(o_j)$, $j\in M$ of alternatives o_j, $j\in M=\{1,2,\dots,m\}$ for each criterion c_i, $i\in N$ is given in some way. Estimated fuzzy priority weights $\tilde{w}_i$, $i\in N$ of the criteria c_i, $i\in N$, we obtain the total utility values of alternatives o_j, $j\in M$ as fuzzy numbers $\tilde{U}(o_j)$, $j\in M$ is obtained by

$$
\mu_{\tilde{U}(o_j)}(r)=\sup\left\{\min_{i\in N}\mu_{\tilde{w}_i}(w_i)\;\middle|\; r=\sum_{i\in N}w_iu_i(o_j),\ \sum_{i\in N}w_i=1\right\},\ j\in M, \tag{19}
$$

where $\mu_{\tilde{U}(o_j)}$ and $\mu_{\tilde{w}_i}$ the membership functions of $\tilde{U}(o_j)$ ($j\in M$) and $\tilde{w}_i$, respectively.

The fuzzy total utility values of alternatives $\tilde{U}(o_j)$, $j\in M$ are defuzzified by the centroid defuzzification method, i.e.,

$$U^{\mathrm{C}}(o_j) = \frac{\int_{\mathbb{R}} r\mu_{\tilde{U}(o_j)}(r)dr}{\int_{\mathbb{R}} \mu_{\tilde{U}(o_j)}(r)dr}, \ j \in M. \tag{20}$$

Then the alternatives o_j, $j \in M$ are ordered in the descending order of $U^{\mathrm{C}}(o_j)$, $j \in M$. This ordering method is investigated further in Section 4.

3 The solution sets under given TFPCM and TZFPCM

The previous section reviewed the LGP method Wang and Chin (2008) for estimating triangular fuzzy priority weights from a TFPCM and trapezoidal fuzzy priority weights from a TZFPCM. Those estimation problems are reduced to linear programming problems. The LGP method (Wang & Chin, 2008) is useful because fuzzy priority weights are obtained simply by solving a linear programming problem. The fuzzy priority weights obtained by solving the linear programming problem are used for the decision analysis, such as ordering alternatives.

Because components of a given PCM are supposed to show the ratios of the priority weights, the deviations between them are natural criteria for the evaluation of an estimated normalized priority weight vector. Then, if a normalized priority vector has the same deviations as the solution to the estimation problem, it is another solution. Recently, from this point of view, it has been shown that there are non-unique solutions to the estimation problem of a normalized interval priority weight vector in interval AHP (Inuiguchi, 2016). Many normalized interval priority weight vectors have the same deviations between the ratios of the estimated normalized interval priority weights and components of a given crisp/interval PCM. Thus, we have non-unique solutions. A few investigations taking care of the non-uniqueness have been done in the interval AHP (Inuiguchi et al., 2022).

The estimation problem of a normalized fuzzy priority weight vector can also be analyzed in the same way. If a normalized fuzzy priority weight vector has the same deviations as the solution to the estimation problem, it is another solution. When we have many normalized fuzzy priority vectors having the same deviations, we would also have non-unique solutions to the estimation problem of a normalized fuzzy priority weight vector. In this section, we demonstrate the non-uniqueness of the solution having the same deviations from a given TFPCM/TZFPCM as a given solution. More concretely, we show the set of normalized fuzzy priority weight vectors $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^{\mathrm{T}}$ such that the deviations between $\tilde{w}_i/\tilde{w}_j$ and $\tilde{a}_{ij}$, $i, j \in N$ $(i \neq j)$ are the same as a given solution.

We define the deviation vectors between the given FPCM components and the ratios of fuzzy priority weights. A deviation vector between a TFPCM/TZFPCM component and the corresponding ratio between triangular/trapezoidal fuzzy priority weights is defined as follows:

Definition 1 The deviation vector between a TFPCM component and the corresponding ratio of fuzzy priority weights is defined as follows:

$$Dev\left(\tilde{a}_{ij}, \frac{\tilde{w}_i}{\tilde{w}_j}\right) = \left(\left| a_{ij}^{\mathrm{L}} - \frac{w_i^{\mathrm{L}}}{w_j^{\mathrm{U}}} \right|, \left| a_{ij}^{\mathrm{M}} - \frac{w_i^{\mathrm{M}}}{w_j^{\mathrm{M}}} \right|, \left| a_{ij}^{\mathrm{U}} - \frac{w_i^{\mathrm{U}}}{w_j^{\mathrm{L}}} \right| \right), \; i, j \in N \; (i \neq j), \tag{21}$$

where $\tilde{a}_{ij} = (a_{ij}^{\mathrm{L}}, a_{ij}^{\mathrm{M}}, a_{ij}^{\mathrm{U}})$ and $\tilde{w}_i = (w_i^{\mathrm{L}}, w_i^{\mathrm{M}}, w_i^{\mathrm{U}})$, $i \in N$.

For TZFPCM, it is defined in the same way as

$$Dev\left(\tilde{a}_{ij}, \frac{\tilde{w}_i}{\tilde{w}_j}\right) = \left(\left| a_{ij}^{\mathrm{L}} - \frac{w_i^{\mathrm{L}}}{w_j^{\mathrm{U}}} \right|, \left| a_{ij}^{\mathrm{M}} - \frac{w_i^{\mathrm{M}}}{w_j^{\mathrm{N}}} \right|, \left| a_{ij}^{\mathrm{N}} - \frac{w_i^{\mathrm{N}}}{w_j^{\mathrm{M}}} \right|, \left| a_{ij}^{\mathrm{U}} - \frac{w_i^{\mathrm{U}}}{w_j^{\mathrm{L}}} \right| \right), \quad i, j \in N \; (i \neq j), \tag{22}$$

where $\tilde{a}_{ij} = (a_{ij}^{\mathrm{L}}, a_{ij}^{\mathrm{M}}, a_{ij}^{\mathrm{N}}, a_{ij}^{\mathrm{U}})$ and $\tilde{w}_i = (w_i^{\mathrm{L}}, w_i^{\mathrm{M}}, w_i^{\mathrm{N}}, w_i^{\mathrm{U}})$, $i \in N$.

Note that we use the same function name *Dev* for TFPCM and TZPCM as we suppose that no confusion occurs.

As the deviation vectors between given FPCM components and the ratios of normalized fuzzy priority weights are defined, we can reasonably formulate the estimation problem by minimizing these deviation vectors. However, this problem becomes a nonlinear multiple objective programming problem. Then the formulated problem is less tractable than the LGP method. By the tractability of the formulated problem and its multiple research outcomes, we adopt the LGP solution and extend it to a set of solutions in this paper.

As shown in the following example in the case of a TZFPCM, we can extend the LGP solution to a set of normalized fuzzy priority weight vectors using the deviation of (22).

Example 1 Consider a TZFPCM $\tilde{A}$ defined by

$$\tilde{A} = \begin{pmatrix} (1,1,1,1) & (\frac{2}{3}, \frac{7}{8}, \frac{3}{2}, 2) & (\frac{6}{7}, \frac{7}{6}, \frac{9}{4}, \frac{10}{3}) \\ (\frac{1}{2}, \frac{2}{3}, \frac{8}{7}, \frac{3}{2}) & (1,1,1,1) & (\frac{5}{7}, 1, 2, 3) \\ (\frac{3}{10}, \frac{4}{9}, \frac{6}{7}, \frac{7}{6}) & (\frac{1}{3}, \frac{1}{2}, 1, \frac{7}{5}) & (1,1,1,1) \end{pmatrix}.$$

$\tilde{A}$ is a consistent FPCM as it satisfies (3), (4), (13), and (14) with the following normalized fuzzy priority weight vector $\tilde{W}^p = (\tilde{w}_1^p, \tilde{w}_2^p, \tilde{w}_3^p)^{\mathrm{T}}$:

$$\tilde{W}^p = \begin{pmatrix} (0.30, 0.35, 0.45, 0.50) \\ (0.25, 0.30, 0.40, 0.45) \\ (0.15, 0.20, 0.30, 0.35) \end{pmatrix}.$$

Consider another normalized fuzzy priority weight vector $\tilde{W}^q = (\tilde{w}_1^q, \tilde{w}_2^q, \tilde{w}_3^q)^{\mathrm{T}}$ defined by

$$\tilde{W}^q = \begin{pmatrix} (0.273, 0.336, 0.432, 0.455) \\ (0.2275, 0.288, 0.384, 0.4095) \\ (0.1365, 0.192, 0.288, 0.3185) \end{pmatrix}.$$

Then, we have

$$Dev(\tilde{a}_{ij}, \tilde{w}_i^p/\tilde{w}_j^p) = Dev(\tilde{a}_{ij}, \tilde{w}_i^q/\tilde{w}_j^q) = (0,\ 0,\ 0)\,,\ i, j \in N\ (i \neq j).$$

Therefore, the normalized fuzzy priority weight vector corresponding to a consistent FPCM $\tilde{A}$ is not always unique.

Example 1 demonstrates the non-uniqueness of the normalized fuzzy priority weights corresponding to a consistent FPCM. From this example, we know the potential non-uniqueness of the solution to the problem for estimating the normalized fuzzy priority weight vector $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^{\mathrm{T}}$ from a given FPCM $\tilde{A}$ because a consistent FPCM $A_{\tilde{W}} = (\tilde{w}_i/\tilde{w}_j)$ is built from a fuzzy priority weight vector $\tilde{W}$.

As described above, we adopt the LGP solution, and extend it to a set of normalized fuzzy priority weight vectors having the same deviations. We show a simple method for obtaining all solutions $\tilde{W}$ satisfying $Dev(\tilde{a}_{ij}, \tilde{w}_i/\tilde{w}_j) = Dev(\tilde{a}_{ij}, \tilde{w}_i^*/\tilde{w}_j^*)$, $i, j \in N$ $(i \neq j)$ from a normalized fuzzy priority weight vector $\tilde{W}^* = (\tilde{w}_1^*, \tilde{w}_2^*, \ldots, \tilde{w}_n^*)$ obtained by solving Problems (10) and (18). Let $A_{\tilde{W}}$ and $A_{\tilde{W}^*}$ be FPCMs associated with normalized fuzzy priority weight vectors $\tilde{W}$ and $\tilde{W}^*$, respectively. The equality relations $Dev(\tilde{a}_{ij}, \tilde{w}_i/\tilde{w}_j) = Dev(\tilde{a}_{ij}, \tilde{w}_i^*/\tilde{w}_j^*)$ for all $(i, j) \in N \times N$ such that $i \neq j$ imply $A_{\tilde{W}} = A_{\tilde{W}^*}$. Therefore, FPCMs $A_{\tilde{W}}$ and $A_{\tilde{W}^*}$ have the same deviations from the given FPCM $\tilde{A}$. All normalized fuzzy priority weight vectors $\tilde{W}$ are reasonably considered solutions to the estimation problem if the FPCMs associated with them have the same deviations from the given FPCM $\tilde{A}$ as the FPCM associated with the LGP solution $\tilde{W}^*$.

3.1 The solution set under a given TFPCM $\tilde{A}$

We describe a simple way for obtaining the set of normalized triangular fuzzy priority weight vectors $\tilde{W}$ such that the ratios of their components $\tilde{w}_i/\tilde{w}_j$, $i, j \in N$ $(i \neq j)$ are the same deviations from components $\tilde{a}_{ij}$, $i, j \in N$ $(i \neq j)$ as $\tilde{W}^*$, where $\tilde{W}^*$ is a normalized triangular fuzzy priority weight vector corresponding to an optimal solution to Problem (10) with TFPCM $\tilde{A}$. For the sake of convenience, $\tilde{W}^*$ is called a TF-solution of the LGP method by solving Problem (10), or simply a TF-solution of the LGP method.

From the constraints of Problem (10), the solution $\tilde{W}^* = (\tilde{w}_1^*, \tilde{w}_2^*, \ldots, \tilde{w}_n^*)^{\mathrm{T}}$ composed of $\tilde{w}_i^* = (w_i^{\mathrm{L}*}, w_i^{\mathrm{M}*}, w_i^{\mathrm{U}*})$, $i \in N$ satisfies

$$\begin{aligned} &\sum_{i \in N \setminus j} w_i^{\mathrm{U}*} + w_j^{\mathrm{L}*} \geq 1,\ \sum_{i \in N \setminus j} w_i^{\mathrm{L}*} + w_j^{\mathrm{U}*} \leq 1,\ j \in N, \sum_{i \in N} w_i^{\mathrm{M}*} = 1, \\ &w_i^{\mathrm{U}*} \geq w_i^{\mathrm{M}*} \geq w_i^{\mathrm{L}*} > 0,\ i \in N. \end{aligned} \tag{23}$$

The other constraints of Problem (10) evaluate the values of deviational variables E^+, E^-, Γ^+, Γ^- and Δ. Therefore, we are interested in $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^{\mathrm{T}}$ satisfying

(23) with replacement of $\tilde{w}_i^*$ by $\tilde{w}_i$, and $Dev(\tilde{a}_{ij}, \tilde{w}_i/\tilde{w}_j) = Dev(\tilde{a}_{ij}, \tilde{w}_i^*/\tilde{w}_j)$, $i, j \in N$ $(i \neq j)$, where $\tilde{w}_i = (w_i^{\mathrm{L}}, w_i^{\mathrm{M}}, w_i^{\mathrm{U}})$, $i \in N$. W composed of $\tilde{w}_i^{\mathrm{L}} = t\tilde{w}_i^{\mathrm{L}*}$, $\tilde{w}_i^{\mathrm{U}} = t\tilde{w}_i^{\mathrm{U}*}$, $\tilde{w}_i^{\mathrm{M}} = t'\tilde{w}_i^{\mathrm{M}*}$, $i \in N$ with $t, t' > 0$ satisfies $Dev(\tilde{a}_{ij}, \tilde{w}_i/\tilde{w}_j) = Dev(\tilde{a}_{ij}, \tilde{w}_i^*/\tilde{w}_j^*)$ and thus we should find $t > 0$ satisfying

$$\begin{aligned} &\sum_{i \in N \setminus j} tw_i^{\mathrm{U}*} + tw_j^{\mathrm{L}*} \geq 1, \ \sum_{i \in N \setminus j} tw_i^{\mathrm{L}*} + tw_j^{\mathrm{U}*} \leq 1, \ j \in N, \sum_{i \in N} t' w_i^{\mathrm{M}*} = 1, \\ &tw_i^{\mathrm{U}*} \geq t' w_i^{\mathrm{M}*} \geq tw_i^{\mathrm{L}*} > 0, \ i \in N. \end{aligned} \tag{24}$$

From $\sum_{i \in N} w_i^{\mathrm{M}*} = 1$ in (23), $\sum_{i \in N} t' w_i^{\mathrm{M}*} = 1$ implies $t' = 1$. From the first two constraints of (24), we obtain

$$t \cdot \min_{j \in N} \left(\sum_{i \in N \setminus j} w_i^{\mathrm{U}*} + w_j^{\mathrm{L}*} \right) \geq 1, \quad t \cdot \max_{j \in N} \left(\sum_{i \in N \setminus j} w_i^{\mathrm{L}*} + w_j^{\mathrm{U}*} \right) \leq 1, \tag{25}$$

and from the last equation, we obtain

$$t \cdot \min_{k \in N} \left(\frac{w_k^{\mathrm{U}*}}{w_k^{\mathrm{M}*}} \right) \geq 1, \quad t \cdot \max_{k \in N} \left(\frac{w_k^{\mathrm{L}*}}{w_k^{\mathrm{M}*}} \right) \leq 1. \tag{26}$$

Then, eventually, we have the range of t as an interval $[t^{\mathrm{L}}, t^{\mathrm{U}}]$ defined by

$$t^{\mathrm{L}} = \max \left\{ \max_{k \in N} \frac{w_k^{\mathrm{M}*}}{w_k^{\mathrm{U}*}}, \ \frac{1}{\min_{i \in N} \left(w_i^{\mathrm{L}*} + \sum_{j \in N \setminus i} w_j^{\mathrm{U}*} \right)} \right\}, \tag{27}$$

$$t^{\mathrm{U}} = \min \left\{ \min_{k \in N} \frac{w_k^{\mathrm{M}*}}{w_k^{\mathrm{L}*}}, \ \frac{1}{\max_{i \in N} \left(w_i^{\mathrm{U}*} + \sum_{j \in N \setminus i} w_j^{\mathrm{L}*} \right)} \right\}. \tag{28}$$

In summary, a set $\mathcal{W}_{\mathrm{T}}$ of solutions $\tilde{W}$ satisfying $Dev(\tilde{a}_{ij}, \tilde{w}_i/\tilde{w}_j) = Dev(\tilde{a}_{ij}, \tilde{w}_i^*/\tilde{w}_j^*)$, $i, j \in N$ $(i \neq j)$ is obtained as

$$\mathcal{W}_{\mathrm{T}} = \left\{ \tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^{\mathrm{T}} \,|\, \tilde{w}_i = (tw_i^{\mathrm{L}*}, w_i^{\mathrm{M}*}, tw_i^{\mathrm{U}*}), \ i \in N, \ t \in [t^{\mathrm{L}}, t^{\mathrm{U}}] \right\}. \tag{29}$$

The solution set $\mathcal{W}_{\mathrm{T}}$ is uniquely obtained for Problem (10) as far as $\tilde{W}^*$ is a unique TF-solution of the LGP method.

Any solution $\bar{W}$ in $\mathcal{W}_{\rm T}$, composed of $\tilde{w}_i^* = (tw_i^{\rm L*}, w_i^{\rm M*}, tw_i^{\rm U*}), i \in N, t \in [t^{\rm L}, t^{\rm U}]$ is a feasible solution to Problem (10) together with tE^{+^*}, tE^{-^*}, $t\Gamma^{+^*}$, $t\Gamma^{-^*}$ and Δ^*, where E^{+^*}, E^{-^*}, Γ^{+^*}, Γ^{-^*} and Δ^* are values of deviational variables E^+, E^-, Γ^+, Γ^- and Δ at the optimal solution to Problem (10). Namely, we have

$$\begin{array}{l}
(A_{\rm L} - I)(tW_{\rm U}^*) - (n-1)(tW_{\rm L}^*) - tE^{+^*} + tE^{-^*} = 0, \\
(A_{\rm U} - I)(tW_{\rm L}^*) - (n-1)(tW_{\rm U}^*) - t\Gamma^{+^*} + t\Gamma^{-^*} = 0, \\
(A_{\rm M} - nI)(tW_{\rm M}^*) - t\Delta^* = 0, \\
(A_{\rm L} - I)tW_{\rm U} - (n-1)tW_{\rm L} - tE^{+} + tE^{-} = 0, \\
(A_{\rm U} - I)tW_{\rm L} - (n-1)tW_{\rm U} - t\Gamma^{+} + t\Gamma^{-} = 0, \\
(A_{\rm M} - nI)W_{\rm M} - \Delta = 0, \\
\displaystyle\sum_{i \in N \setminus j} tw_i^{\rm U*} + tw_j^{\rm L*} \geq 1, \ \sum_{i \in N \setminus j} tw_i^{\rm L*} + tw_j^{\rm U*} \leq 1, \ j \in N, \\
\displaystyle\sum_{i \in N} w_i^{\rm M*} = 1, \\
tw_i^{\rm U*} \geq w_i^{\rm M*} \geq tw_i^{\rm L*} > 0, \ i \in N, \\
tE^{+^*}, tE^{-^*}, t\Gamma^{+^*}, t\Gamma^{-^*}, \Delta^* \geq 0.
\end{array} \tag{30}$$

The objective function value of this feasible solution to Problem (10) becomes

$$e^{\rm T}(tE^{+^*} + tE^{-^*} + t\Gamma^{+^*} + t\Gamma^{-^*} + \Delta^*) = te^{\rm T}(E^{+^*} + E^{-^*} + \Gamma^{+^*} + \Gamma^{-^*}) + e^{\rm T}\Delta^*. \tag{31}$$

Because $E^{+^*} + E^{-^*} + \Gamma^{+^*} + \Gamma^{-^*}$ is non-negative, the minimum objective function value among the feasible solutions corresponding to $\tilde{W} \in \mathcal{W}_{\rm T}$ is obtained for $t = t^{\rm L}$. From $t = 1 \in [t^{\rm L}, t^{\rm U}]$, we obtain

$$t^L e^{\rm T}(E^{+^*} + E^{-^*} + \Gamma^{+^*} + \Gamma^{-^*}) + e^{\rm T}\Delta^* \leq e^{\rm T}(E^{+^*} + E^{-^*} + \Gamma^{+^*} + \Gamma^{-^*} + \Delta^*). \tag{32}$$

The right-hand side value of (32) is the optimal value of Problem (10), i.e., the minimum objective function value of feasible solutions to Problem (10). This implied that equality should hold in (32). Therefore, we obtain $t^{\rm L} = 1$.

3.2 The solution set under a given TZFPCM $\tilde{A}$

Let us discuss the solution set in the case where TZFPCM $\tilde{A}$ is given. In this case, the normalized trapezoidal fuzzy priority weight vector $\tilde{W}^* = (\tilde{w}_1^*, \tilde{w}_2^*, \ldots, \tilde{w}_n^*)^{\rm T}$ obtained by solving Problem (18) satisfies

$$\begin{array}{l}
\displaystyle\sum_{i \in N \setminus j} w_i^{\rm U*} + w_j^{\rm L*} \geq 1, \ \sum_{i \in N \setminus j} w_i^{\rm L*} + w_j^{\rm U*} \leq 1, \ j \in N, \\
\displaystyle\sum_{i \in N \setminus j} w_i^{\rm N*} + w_j^{\rm M*} \geq 1, \ \sum_{i \in N \setminus j} w_i^{\rm M*} + w_j^{\rm N*} \leq 1, \ j \in N, \\
w_i^{\rm U*} \geq w_i^{\rm N*} \geq w_i^{\rm M*} \geq w_i^{\rm L*} > 0, \ i \in N,
\end{array} \tag{33}$$

where $\tilde{w}_i^* = (w_i^{\rm L*}, w_i^{\rm M*}, w_i^{\rm N*}, w_i^{\rm U*})$. For the sake of convenience, $\tilde{W}^*$ is called a solution of the LGP method by solving Problem (18), or simply a TZF-solution of the LGP method. In the same way as the case where TFPCM $\tilde{A}$ is given, the other constraints of Problem (18) evaluate the values of deviational variables $E^+, E^-, \Gamma^+, \Gamma^-, \Delta^+, \Delta^-, \Lambda^+$ and Λ^-. Therefore, we are interested in $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^{\mathrm{T}}$ satisfying (33) with replacement of $\tilde{w}_i^*$ by $\tilde{w}_i$, and $Dev(\tilde{a}_{ij}, \tilde{w}_i/\tilde{w}_j) = Dev(\tilde{a}_{ij}, \tilde{w}_i^*/\tilde{w}_j^*)$, $i, j \in N$ $(i \neq j)$, where $\tilde{w}_i = (w_i^{\mathrm{L}}, w_i^{\mathrm{M}}, w_i^{\mathrm{N}}, w_i^{\mathrm{U}})$, $i \in N$. The fuzzy priority weight vector $\tilde{W}$ composed of $\tilde{w}_i^{\mathrm{L}} = t_1\tilde{w}_i^{\mathrm{L}*}$, $\tilde{w}_i^{\mathrm{U}} = t_1\tilde{w}_i^{\mathrm{U}*}$, $\tilde{w}_i^{\mathrm{M}} = t_2\tilde{w}_i^{\mathrm{M}*}$, $\tilde{w}_i^{\mathrm{N}} = t_2\tilde{w}_i^{\mathrm{N}*}$, $i \in N$ with $t_1, t_2 > 0$ satisfies $Dev(\tilde{a}_{ij}, \tilde{w}_i/\tilde{w}_j) = Dev(\tilde{a}_{ij}, \tilde{w}_i^*/\tilde{w}_j^*)$. Thus, we should find $t_1, t_2 > 0$ satisfying

$$\begin{aligned}
&\sum_{i\in N\setminus j} t_1 w_i^{\mathrm{U}*} + t_1 w_j^{\mathrm{L}*} \geq 1,\ \sum_{i\in N\setminus j} t_1 w_i^{\mathrm{L}*} + t_1 w_j^{\mathrm{U}*} \leq 1,\ j \in N,\\
&\sum_{i\in N\setminus j} t_2 w_i^{\mathrm{N}*} + t_2 w_j^{\mathrm{M}*} \geq 1,\ \sum_{i\in N\setminus j} t_2 w_i^{\mathrm{M}*} + t_2 w_j^{\mathrm{N}*} \leq 1,\ j \in N,\\
&t_1 w_i^{\mathrm{U}*} \geq t_2 w_i^{\mathrm{N}*} \geq t_2 w_i^{\mathrm{M}*} \geq t_1 w_i^{\mathrm{L}*} > 0,\ i \in N.
\end{aligned} \tag{34}$$

From the first four constraints of (34), we obtain

$$t_1 \cdot \min_{j\in N}\left(\sum_{i\in N\setminus j} w_i^{\mathrm{U}*} + w_j^{\mathrm{L}*}\right) \geq 1, t_1 \cdot \max_{j\in N}\left(\sum_{i\in N\setminus j} w_i^{\mathrm{L}*} + w_j^{\mathrm{U}*}\right) \leq 1, \tag{35}$$

$$t_2 \cdot \min_{j\in N}\left(\sum_{i\in N\setminus j} w_i^{\mathrm{N}*} + w_j^{\mathrm{M}*}\right) \geq 1, t_2 \cdot \max_{j\in N}\left(\sum_{i\in N\setminus j} w_i^{\mathrm{M}*} + w_j^{\mathrm{N}*}\right) \leq 1. \tag{36}$$

From the fifth constraint of (34), we obtain

$$\frac{t_2}{t_1} \in \left[\max_{i\in N} \frac{w_i^{\mathrm{L}*}}{w_i^{\mathrm{M}*}}, \min_{i\in N} \frac{w_i^{\mathrm{U}*}}{w_i^{\mathrm{N}*}}\right], \quad \text{or equivalently,} \quad \frac{t_1}{t_2} \in \left[\max_{i\in N} \frac{w_i^{\mathrm{N}*}}{w_i^{\mathrm{U}*}}, \min_{i\in N} \frac{w_i^{\mathrm{M}*}}{w_i^{\mathrm{L}*}}\right]. \tag{37}$$

Thus, from (35) and (36), we define

$$t_1^{\mathrm{L}} = \frac{1}{\min\limits_{i\in N}\left(w_i^{\mathrm{L}*} + \sum\limits_{j\in N\setminus i} w_j^{\mathrm{U}*}\right)}, t_1^{\mathrm{U}} = \frac{1}{\max\limits_{i\in N}\left(w_i^{\mathrm{U}*} + \sum\limits_{j\in N\setminus i} w_j^{\mathrm{L}*}\right)}, \tag{38}$$

$$t_2^{\mathrm{L}} = \frac{1}{\min\limits_{i\in N}\left(w_i^{\mathrm{M}*} + \sum\limits_{j\in N\setminus i} w_j^{\mathrm{N}*}\right)}, t_2^{\mathrm{U}} = \frac{1}{\max\limits_{i\in N}\left(w_i^{\mathrm{N}*} + \sum\limits_{j\in N\setminus i} w_j^{\mathrm{M}*}\right)}. \tag{39}$$

Finally, introducing (37), the ranges of t_1 and t_2 are obtained as

$$t_1 \in \left[\max\left(t_1^{\rm L}, t_2^{\rm L} \max_{i\in N} \frac{w_i^{\rm N*}}{w_i^{\rm U*}}\right), \min\left(t_1^{\rm U}, t_2^{\rm U} \min_{i\in N} \frac{w_i^{\rm M*}}{w_i^{\rm L*}}\right)\right], \tag{40}$$

$$t_2 \in \left[\max\left(t_2^{\rm L}, t_1 \max_{i\in N} \frac{w_i^{\rm L*}}{w_i^{\rm M*}}\right), \min\left(t_2^{\rm U}, t_1 \min_{i\in N} \frac{w_i^{\rm U*}}{w_i^{\rm N*}}\right)\right], \tag{41}$$

where we note that the range of t_2 depends on variable t_1. For the sake of simplicity, the lower and upper bounds of those ranges are given by $\tau_1^{\rm L}$, $\tau_1^{\rm U}$, $\tau_2^{\rm L}(t_1)$ and $\tau_2^{\rm U}(t_1)$ so that we simply write the ranges as $t_1 \in \left[\tau_1^{\rm L}, \tau_1^{\rm U}\right]$ and $t_2 \in \left[\tau_2^{\rm L}(t_1), \tau_2^{\rm U}(t_1)\right]$. Namely, we define

$$\tau_1^{\rm L} = \max\left(t_1^{\rm L}, t_2^{\rm L} \max_{i\in N} \frac{w_i^{\rm N*}}{w_i^{\rm U*}}\right), \qquad \tau_1^{\rm U} = \min\left(t_1^{\rm U}, t_2^{\rm U} \min_{i\in N} \frac{w_i^{\rm M*}}{w_i^{\rm L*}}\right), \tag{42}$$

$$\tau_2^{\rm L}(t_1) = \max\left(t_2^{\rm L}, t_1 \max_{i\in N} \frac{w_i^{\rm L*}}{w_i^{\rm M*}}\right), \qquad \tau_2^{\rm U}(t_1) = \min\left(t_2^{\rm U}, t_1 \min_{i\in N} \frac{w_i^{\rm U*}}{w_i^{\rm N*}}\right), \tag{43}$$

where we write again that $w_i^{\rm L*}$, $w_i^{\rm U*}$, $w_i^{\rm M*}$ and $w_i^{\rm N*}$, $i \in N$ are obtained in the optimal solutions to Problem (18). Namely, $t_1^{\rm L}$, $t_1^{\rm U}$, $t_2^{\rm L}$, $t_2^{\rm U}$, $\tau_1^{\rm L}$ and $\tau_1^{\rm U}$ are obtained as real numbers while $\tau_2^{\rm L}(t_1)$ and $\tau_2^{\rm L}(t_1)$ are obtained as functions of a real number t_1.

In summary, when TZFPCM $\tilde{A}$ is given, a set $\mathcal{W}_{\rm TZ}$ of solutions $\tilde{W}$ satisfying $Dev(\tilde{a}_{ij}, \tilde{w}_i/\tilde{w}_j) = Dev(\tilde{a}_{ij}, \tilde{w}_i^*/\tilde{w}_j^*)$, $i, j \in N$ $(i \neq j)$ is obtained as

$$\mathcal{W}_{\rm TZ} = \Big\{\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^{\rm T} \,|\, \tilde{w}_i = (t_1 w_i^{\rm L*}, t_2 w_i^{\rm M*}, t_2 w_i^{\rm N*}, t_1 w_i^{\rm U*}),\ i \in N, \\ t_2 \in \left[\tau_2^{\rm L}(t_1), \tau_2^{\rm U}(t_1)\right],\ t_1 \in \left[\tau_1^{\rm L}, \tau_1^{\rm U}\right]\Big\}. \tag{44}$$

The solution set $\mathcal{W}_{\rm TZ}$ is uniquely obtained for Problem (18) as far as $\tilde{W}^*$ is a unique TZF-solution of the LGP method.

Remark 2 The ranges given by (41) are the t_1-oriented expressions. We also have the t_2-oriented expressions as

$$t_2 \in \left[\max\left(t_2^{\rm L}, t_1^{\rm L} \max_{i\in N} \frac{w_i^{\rm L*}}{w_i^{\rm M*}}\right), \min\left(t_2^{\rm U}, t_1^{\rm U} \min_{i\in N} \frac{w_i^{\rm R*}}{w_i^{\rm N*}}\right)\right], \tag{45}$$

$$t_1 \in \left[\max\left(t_1^{\rm L}, t_2 \max_{i\in N} \frac{w_i^{\rm N*}}{w_i^{\rm U*}}\right), \min\left(t_1^{\rm U}, t_2 \min_{i\in N} \frac{w_i^{\rm M*}}{w_i^{\rm L*}}\right)\right], \tag{46}$$

where the range of t_1 depends on variable t_2. □

Similar to the case where a TFPCM is given, any solution $\tilde{W}[t_1, t_2] \in \mathcal{W}_{\rm TZ}$, $t_2 \in [\tau_2^{\rm L}(t_1), \tau_2^{\rm U}(t_1)]$, $t_1 \in [\tau_1^{\rm L}, \tau_1^{\rm U}]$ composed of $\tilde{w}[t_1, t_2]_i^* = (t_1 w_i^{\rm L*}, t_2 w_i^{\rm M*}, t_2 w_i^{\rm N*}, t_1 w_i^{\rm U*})$, $i \in N$ is a feasible solution to Problem (18) together with $t_1 E^{+*}$, $t_1 E^{-*}$, $t_1 \varGamma^{+*}$, $t_1 \varGamma^{-*}$, $t_2 \varDelta^{+*}$, $t_2 \varDelta^{-*}$, $t_2 \varLambda^{+*}$ and $t_2 \varLambda^{-*}$, where $E^{+*}, E^{-*}, \Gamma^{+*}, \Gamma^{-*}, \Delta^{+*}, \Delta^{-*}, \Lambda^{+*}$ and Λ^{-*} are values of deviational variables $E^{+}, E^{-}, \Gamma^{+}, \Gamma^{-}, \Delta^{+}, \Delta^{-}, \Lambda^{+}$ and Λ^{-} at the optimal solution to Problem (18). The objective function value of the solution corresponding to $\tilde{W}[t_1, t_2]$ is obtained as $e^{\mathrm{T}}(t_1 E^{+*} + t_1 E^{-*} + t_1 \Gamma^{+*} + t_1 \Gamma^{-*} + t_2 \Delta^{+*} + t_2 \Delta^{-*} + t_2 \Lambda^{+*} + t_2 \Lambda^{-*}) = t_1 e^{\mathrm{T}}(E^{+*} + E^{-*} + \Gamma^{+*} + \Gamma^{-*}) + t_2 e^{\mathrm{T}}(\Delta^{+*} + \Delta^{-*} + \Lambda^{+*} + \Lambda^{-*})$. As $E^{+*} + E^{-*} + \Gamma^{+*} + \Gamma^{-*}$ and $\Delta^{+*} + \Delta^{-*} + \Lambda^{+*} + \Lambda^{-*}$ are nonnegative, among feasible solutions corresponding to $\tilde{W}[t_1, t_2]$, the solution $\tilde{W}[\tau_1^{\mathrm{L}}, \tau_2^{\mathrm{L}}(\tau_1^{\mathrm{L}})]$, i.e., the solution $\tilde{W}[t_1, t_2]$ with $t_1 = \tau_1^{\mathrm{L}}$ and $t_2 = \tau_2^{\mathrm{L}}(\tau_1^{\mathrm{L}})$ minimizes the objective function of Problem (18). On the other hand, this solution corresponds to $\tilde{W}^*$ and thus, it is optimal for Problem (18). This implies $\tau_1^{\mathrm{L}} = 1$ and $\tau_2^{\mathrm{L}}(1) = 1$.

A numerical example illustrating the calculation of the ranges of t_1 and t_2 and the decision analysis with solution set $\mathcal{W}_{\mathrm{TZ}}$ is given in Section 5.2.

4 Decision analysis using the set of fuzzy priority weight vectors

This paper adopts the fuzzy total utility values (20) for evaluating alternatives. As described in Section 2, we assume that marginal utility values $u_i(o_j), j \in M$ of alternatives $o_j, j \in M = \{1, 2, \ldots, m\}$ for each criterion $c_i, i \in N$ is given. As described in the previous section, we obtain a set of fuzzy priority weight vectors from the solution obtained by the LGP method. Since we assume that the DM has a flexible mind from a wide perspective and evaluates vaguely, we regard each fuzzy priority weight vector in the set as an acceptable opinion for the DM. Namely, the DM may accept each fuzzy priority weight vector in the set. Therefore, we calculate the fuzzy total utility values of alternatives for each fuzzy priority weight vector in the set. Utilizing the centroid method for the defuzzification of fuzzy total utility values of alternatives, we obtain a preference order among alternatives for each fuzzy priority weight vector in the set. As the fuzzy priority weight vector varies with parameter *t* or parameters t_1 and t_2, we reveal the DM's flexible thinking by showing the variations of defuzzified fuzzy total utility values of alternatives with parameter *t* under a given TFPCM, and by a map of the preference orders of alternatives in the t_1-t_2 parameter space under a given TZFPCM. Those are shown in the next section.

Let $u_i(o_j), i \in N$ be the marginal utility value of the *j*-th alternative o_j for the *i*-th criterion. Given a fuzzy priority weight vector $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^{\mathrm{T}}$, the fuzzy total utility value $\tilde{U}(o_j)$ and the defuzzified total utility value $U^{\mathrm{C}}(o_j)$ are calculated by (19) and (20), respectively, where the calculation of $\tilde{U}(o_j)$ is shown by its membership function. In the remaining of this section, we show the simple calculation of the defuzzified total utility value when $\tilde{w}_i$, $i \in N$ are triangular fuzzy priority weights $(w_i^{\mathrm{L}}, w_i^{\mathrm{M}}, w_i^{\mathrm{U}})$, $i \in N$.

Let $[\tilde{U}(o_j)]_\alpha$ and $[\tilde{w}_i]_\alpha$ be the α-level sets of $\tilde{U}(o_j)$ and $\tilde{w}_i$ for $\alpha \in (0, 1]$, respectively. From (19), we obtain

$$[\tilde{U}(o_j)]_\alpha = \left\{ r | \mu_{\tilde{U}(o_j)}(r) \geq \alpha \right\} = \left\{ r \;\middle|\; r = \sum_{i \in N} w_i u_i(o_j),\ \sum_{i \in N} w_i = 1,\ w_i \in [\tilde{w}_i]_\alpha \right\}, \tag{47}$$

where $[\tilde{w}_i]_\alpha = \{r|_{\tilde{w}_i}(r) \geq \alpha\} = [\alpha w_i^{\mathrm{M}} + (1-\alpha) w_i^{\mathrm{L}}, \alpha w_i^{\mathrm{M}} + (1-\alpha) w_i^{\mathrm{U}}]$.

The lower and upper bounds of $\tilde{U}(o_j)_\alpha$ are obtained respectively by

$$[\tilde{U}(o_j)]_\alpha^{\mathrm{L}} = \min \left\{ \sum_{i \in N} w_i u_i(o_j) \;\middle|\; \sum_{i \in N} w_i = 1, w_i \in [\tilde{w}_i]_\alpha,\ i \in N \right\}, \tag{48}$$

$$[\tilde{U}(o_j)]_\alpha{}^{\mathrm{U}} = \max \left\{ \sum_{i \in N} w_i u_i(o_j) \;\middle|\; \sum_{i \in N} w_i = 1, w_i \in [\tilde{w}_i]_\alpha,\ i \in N \right\}. \tag{49}$$

These equations show that the lower and upper bounds of $[\tilde{U}(o_j)]_\alpha$ are obtained by solving continuous knapsack problems.

For the sake of simplicity, we define $w_i^{\mathrm{L}}(\alpha) = \alpha w_i^{\mathrm{M}} + (1-\alpha) w_i^{\mathrm{L}}$ and $w_i^{\mathrm{U}}(\alpha) = \alpha w_i^{\mathrm{M}} + (1-\alpha) w_i^{\mathrm{U}}$. We obtain $[\tilde{U}(o_j)]_\alpha^{\mathrm{L}} = \sum_{i \in N} w_i^{\mathrm{M}} u_i(o_j)$ for $\alpha = 1$. For the calculation of $[\tilde{U}(o_j)]_\alpha^{\mathrm{L}}$ for $\alpha \in (0, 1)$, we consider a permutation $\pi : N \to N$ satisfying $u_{\pi(1)}(o_j) \leq u_{\pi(2)}(o_j) \leq \cdots \leq u_{\pi(n)}(o_j)$ and $\pi(s) < \pi(s+1)$ if $u_{\pi(s)}(o_j) = u_{\pi(s+1)}(o_j)$, $s \leq n-1$. Then a priority weight vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ satisfying $w_{\pi(i)} = w_{\pi(i)}^{\mathrm{U}}(\alpha)$, for $i < k$, $w_{\pi(i)} = w_{\pi(i)}^{\mathrm{L}}(\alpha)$, for $i > k$, and $w_{\pi(k)} = 1 - \sum_{i \neq k} w_{\pi(i)}$, where integer $k \in N$ satisfies

$$\sum_{i \leq k} w_{\pi(i)}^{\mathrm{U}}(\alpha) + \sum_{i > k} w_{\pi(i)}^{\mathrm{L}}(\alpha) > 1 > \sum_{i < k} w_{\pi(i)}^{\mathrm{U}}(\alpha) + \sum_{i \geq k} w_{\pi(i)}^{\mathrm{L}}(\alpha). \tag{50}$$

From the definitions of $w_i^{\mathrm{L}}(\alpha)$ and $w_i^{\mathrm{U}}(\alpha)$ and $\sum_{i \in N} w_{\pi(i)}^{\mathrm{M}} = \sum_{j \in N} w_j^{\mathrm{M}} = 1$, for any $\alpha \in (0, 1)$, this equation is rewritten as

$$\sum_{i \leq k} w_{\pi(i)}^{\mathrm{U}} + \sum_{i > k} w_{\pi(i)}^{\mathrm{L}} > 1 > \sum_{i < k} w_{\pi(i)}^{\mathrm{U}} + \sum_{i \geq k} w_{\pi(i)}^{\mathrm{L}}. \tag{51}$$

This equation implies that *k* is the same for any $\alpha \in (0, 1)$.

As the continuous knapsack problem in (48) can be solved by the greedy method, $\boldsymbol{w}$ defined above is the optimal solution. Eventually, for any $\alpha \in (0, 1]$, $[\tilde{U}(o_j)]_\alpha^{\mathrm{L}}$ is obtained as

$$[\tilde{U}(o_j)]_\alpha^{\mathrm{L}} = \sum_{i < k} w_{\pi(i)}^{\mathrm{U}}(\alpha) u_{\pi(i)} + \sum_{i > k} w_{\pi(i)}^{\mathrm{L}}(\alpha) u_{\pi(i)} + \left(1 - \sum_{i < k} w_{\pi(i)}^{\mathrm{U}}(\alpha) - \sum_{i > k} w_{\pi(i)}^{\mathrm{L}}(\alpha) \right) u_{\pi(k)}, \tag{52}$$

where $\alpha = 1$ is included because the right-hand side value of (52) equals to $\sum_{i\in N} w_i^{\mathrm{M}} u_i(o_j)$.

In the same discussion, we obtain

$$[\tilde{U}(o_j)]_\alpha^{\mathrm{U}} = \sum_{i<l} w_{\nu(i)}^{\mathrm{U}}(\alpha) u_{\nu(i)} + \sum_{i>l} w_{\nu(i)}^{\mathrm{L}}(\alpha) u_{\nu(i)} + \left(1 - \sum_{i<l} w_{\nu(i)}^{\mathrm{U}}(\alpha) - \sum_{i>l} w_{\nu(i)}^{\mathrm{L}}(\alpha)\right) u_{\nu(k)}, \tag{53}$$

where $\nu : N \to N$ is a permutation satisfying $u_{\nu(1)}(o_j) \geq u_{\nu(2)}(o_j) \geq \cdots \geq u_{\nu(n)}(o_j)$ and and $\nu(s) < \nu(s+1)$ if $u_{\nu(s)}(o_j) = u_{\nu(s+1)}(o_j)$, $s \leq n-1$. Integer $l \in N$ satisfies

$$\sum_{i\leq l} w_{\nu(i)}^{\mathrm{U}}(\alpha) + \sum_{i>l} w_{\nu(i)}^{\mathrm{L}}(\alpha) > 1 > \sum_{i<l} w_{\nu(i)}^{\mathrm{U}}(\alpha) + \sum_{i\geq l} w_{\nu(i)}^{\mathrm{L}}(\alpha). \tag{54}$$

As shown in (52) and (53), $[\tilde{U}(o_j)]_\alpha^{\mathrm{L}}$ and $[\tilde{U}(o_j)]_\alpha^{\mathrm{U}}$ are linear with respect to α. Then the fuzzy total utility value $\tilde{U}(o_j)$ becomes a triangular fuzzy number $\tilde{U}(o_j) = (U^{\mathrm{L}}(o_j), U^{\mathrm{M}}(o_j), U^{\mathrm{U}}(o_j))$, where

$$U^{\mathrm{L}}(o_j) = \sum_{i<k} w_{\pi(i)}^{\mathrm{U}} u_{\pi(i)} + \sum_{i>k} w_{\pi(i)}^{\mathrm{L}} u_{\pi(i)} + \left(1 - \sum_{i<k} w_{\pi(i)}^{\mathrm{U}} - \sum_{i>k} w_{\pi(i)}^{\mathrm{L}}\right) u_{\pi(k)}, \tag{55}$$

$$U^{\mathrm{M}}(o_j) = \sum_{i\in N} w_i^{\mathrm{M}} u_i(o_j), \tag{56}$$

$$U^{\mathrm{L}}(o_j) = \sum_{i<l} w_{\nu(i)}^{\mathrm{U}} u_{\nu(i)} + \sum_{i>l} w_{\nu(i)}^{\mathrm{L}} u_{\nu(i)} + \left(1 - \sum_{i<l} w_{\nu(i)}^{\mathrm{U}} - \sum_{i>l} w_{\nu(i)}^{\mathrm{L}}\right) u_{\nu(k)}, \tag{57}$$

where integers *k* and *l* are defined by (50) and (54).

The result of the centroid defuzzification (Wang, 2009) of the triangular fuzzy number $\tilde{U}(o_j) = (U^{\mathrm{L}}(o_j), U^{\mathrm{M}}(o_j), U^{\mathrm{U}}(o_j))$ is obtained as

$$U^{\mathrm{C}}(o_j) = \frac{\int_{\mathbb{R}} r\mu_{\tilde{U}(o_j)}(r)dr}{\int_{\mathbb{R}} \mu_{\tilde{U}(o_j)}(r)dr} = \frac{1}{3}(U^{\mathrm{L}}(o_j) + U^{\mathrm{M}}(o_j) + U^{\mathrm{U}}(o_j)). \tag{58}$$

Therefore, under triangular fuzzy priority weights $(w_i^{\mathrm{L}}, w_i^{\mathrm{M}}, w_i^{\mathrm{U}})$, $i \in N$, fuzzy total utility values $U^{\mathrm{C}}(o_j)$, $j \in M$ are easily calculated, and the alternatives o_j, $j \in M$ are ordered by $U^{\mathrm{C}}(o_j)$, $j \in M$. On the other hand, under trapezoidal fuzzy priority weights $(w_i^{\mathrm{L}}, w_i^{\mathrm{M}}, w_i^{\mathrm{N}}, w_i^{\mathrm{U}})$, $i \in N$, fuzzy total utility values $U^{\mathrm{C}}(o_j)$, $j \in M$ are obtained by numerical calculations of (20).

5 Numerical example

In this section, we show two examples of multiple criteria decision problems to illustrate the decision analysis based on the FAHP when marginal utility values of alternatives are given for each criterion. One of the two examples treats a TFPCM, and the other treats a TZFPCM. In these examples, we demonstrate the decision analysis taking care of the non-uniqueness of the solution to the estimation problem of a normalized fuzzy priority weight vector.

5.1 Decision Analysis under a given triangular FPCM

We consider a hypothetical multiple criteria decision problem with five criteria $c_i, i \in N = \{1, 2, \ldots, 5\}$ and three alternatives o_1, o_2, o_3. The problem is given abstractly, but, for example, we may imagine a problem where c_i, $i = 1, 2, \ldots, 5$ are subjects such as 'mathematics', 'physics', 'foreign language', 'chemistry' and 'literature', and $o_j, j = 1, 2, 3$ are students. In this imagination, the problem is to rank these three students based on the given scores of the subjects, i.e., marginal utility values.

The marginal utility values of the alternatives for each of the five criteria are given in Table 1. The DM gives the following TFPCM for showing the relative importance between criteria:

$$\tilde{A} = \begin{pmatrix} (1,1,1) & (\frac{3}{2},2,\frac{5}{2}) & (\frac{5}{2},3,\frac{7}{2}) & (\frac{5}{2},3,\frac{7}{2}) & (\frac{7}{2},4,\frac{9}{2}) \\ (\frac{2}{5},\frac{1}{2},\frac{2}{3}) & (1,1,1) & (\frac{3}{2},2,\frac{5}{2}) & (\frac{3}{2},2,\frac{5}{2}) & (\frac{5}{2},3,\frac{7}{2}) \\ (\frac{2}{7},\frac{1}{3},\frac{2}{5}) & (\frac{2}{3},\frac{1}{2},\frac{2}{5}) & (1,1,1) & (\frac{2}{3},1,\frac{3}{2}) & (\frac{3}{2},2,\frac{5}{2}) \\ (\frac{2}{7},\frac{1}{3},\frac{2}{5}) & (\frac{2}{3},\frac{1}{2},\frac{2}{5}) & (\frac{2}{3},1,\frac{3}{2}) & (1,1,1) & (\frac{3}{2},2,\frac{5}{2}) \\ (\frac{2}{9},\frac{1}{4},\frac{2}{7}) & (\frac{2}{5},\frac{1}{3},\frac{2}{7}) & (\frac{2}{3},\frac{1}{2},\frac{2}{5}) & (\frac{2}{5},\frac{1}{2},\frac{2}{3}) & (1,1,1) \end{pmatrix}.$$

Solving Problem (10), we obtain the following normalized triangular fuzzy priority weight vector:

$$\tilde{W} = \begin{pmatrix} (w_1^{\mathrm{L}}, w_1^{\mathrm{M}}, w_1^{\mathrm{U}}) \\ (w_2^{\mathrm{L}}, w_2^{\mathrm{M}}, w_2^{\mathrm{U}}) \\ (w_3^{\mathrm{L}}, w_3^{\mathrm{M}}, w_3^{\mathrm{U}}) \\ (w_4^{\mathrm{L}}, w_4^{\mathrm{M}}, w_4^{\mathrm{U}}) \\ (w_5^{\mathrm{L}}, w_5^{\mathrm{M}}, w_5^{\mathrm{U}}) \end{pmatrix} = \begin{pmatrix} (0.3672,\ 0.4045,\ 0.4045) \\ (0.2106,\ 0.2450,\ 0.2609) \\ (0.1128,\ 0.1369,\ 0.1561) \\ (0.1128,\ 0.1369,\ 0.1561) \\ (0.0671,\ 0.0767,\ 0.0849) \end{pmatrix}.$$

Applying (27) and (28), we obtain $t^{\mathrm{L}} = 1$ and $t^{\mathrm{U}} = 1.0859$. Then the solution set is obtained as

$$\mathcal{W}_{\mathrm{T}} = \left\{ \tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^{\mathrm{T}} \;\middle|\; (tw_i^{\mathrm{L}}, w_i^{\mathrm{M}}, tw_i^{\mathrm{U}}),\ i \in N,\ t \in [1, 1.0859] \right\}. \quad (59)$$

Table 1 The marginal utility values of alternatives for each criterion

	c_1	c_2	c_3	c_4	c_5
o_1	0.24	0.23	0.08	0.23	0.22
o_2	0.12	0.46	0.21	0.10	0.11
o_3	0.22	0.19	0.45	0.06	0.08

For each $\tilde{W} \in \mathcal{W}_{\mathrm{T}}$, we calculate $\tilde{U}(o_j) = (U^{\mathrm{L}}(o_j), U^{\mathrm{M}}(o_j), U^{\mathrm{U}}(o_j))$, $j = 1, 2, 3$ by (55), (56) and (57). For example, for $t = t^{\mathrm{L}}$, we obtain $k(o_1) = 4$, $l(o_1) = 3$, $k(o_2) = 4$, $l(o_2) = 3$, $k(o_3) = 4$ and $l(o_3) = 3$ as integers k and l satisfying (50) and (54) for each object. Then we obtain the following fuzzy total utility values:

$$\tilde{U}_{|t=t^{\mathrm{L}}} = \begin{pmatrix} (0.2094, 0.2127, 0.2164) \\ (0.2006, 0.2121, 0.2198) \\ (0.2013, 0.2115, 0.2207) \end{pmatrix}.$$

From (58), we have $U^{\mathrm{C}}(o_1) = 0.2129$, $U^{\mathrm{C}}(o_2) = 0.2108$ and $U^{\mathrm{C}}(o_3) = 0.2114$. Therefore, we obtain $o_1 \succ o_3 \succ o_2$ from $U^{\mathrm{C}}(o_1) > U^{\mathrm{C}}(o_3) > U^{\mathrm{C}}(o_2)$.

Similarly, for $t = t^{\mathrm{U}}$, we obtain we obtain $k(o_1) = 2$, $l(o_1) = 2$, $k(o_2) = 2$, $l(o_2) = 2$, $k(o_3) = 2$ and $l(o_3) = 2$ as integers k and l satisfying (50) and (54) for each object. Then we obtain the following fuzzy total utility values:

$$\tilde{U}_{|t=t^{\mathrm{U}}} = \begin{pmatrix} (0.2078, 0.2127, 0.2153) \\ (0.2046, 0.2121, 0.2242) \\ (0.2029, 0.2115, 0.2223) \end{pmatrix}.$$

From (58), we have $U^{\mathrm{C}}(o_1) = 0.2119$, $U^{\mathrm{C}}(o_2) = 0.2136$ and $U^{\mathrm{C}}(o_3) = 0.2122$. Therefore, we obtain $o_2 \succ o_3 \succ o_1$ from $U^{\mathrm{C}}(o_2) > U^{\mathrm{C}}(o_3) > U^{\mathrm{C}}(o_1)$. The order of the alternatives changed by changing t from t^{L} to t^{U}.

The variation in the order of the alternatives with t changing from t^{L} to t^{U} is shown in Fig. 2. Figure 2 is useful to understand intuitively all possible orders of alternatives and their situation derived from the given TZFPCM. We can also see the changes of $U^{\mathrm{C}}(o_j)$, $j = 1, 2, 3$ with t in Fig. 2.

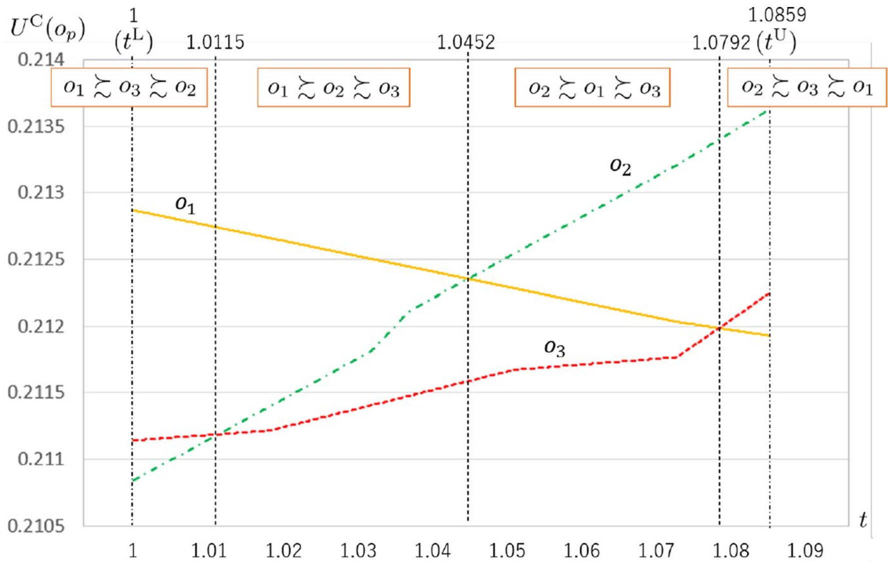

Fig. 2 Transitions of the defuzzified total fuzzy utility values $U^{\mathrm{C}}(o_i)$, $i = 1, 2, 3$.

To see Fig. 2, we observe that $U^{\mathrm{C}}(o_2)$ and $U^{\mathrm{C}}(o_3)$ increase with t while $U^{\mathrm{C}}(o_1)$ decreases. $U^{\mathrm{C}}(o_2)$ increases more than $U^{\mathrm{C}}(o_3)$. As each two of $U^{\mathrm{C}}(o_j)$, $j = 1, 2, 3$ have an intersection, the order of the alternatives changes at three values, i.e., where t is around 1.01, around 1.045, and around 1.08. Therefore, we understand that there are four possible orders of the alternatives. In the conventional LGP method, only $o_1 \succ o_3 \succ o_2$ was obtained, and this order has been considered the DM's preference. However, by considering the non-uniqueness of the solution of fuzzy priority weights, we found that the DM may consider those four possible orders, and s/he may hesitate to choose one of the four orders. However, we understand that o_3 is never the most preferred alternative. The length of the range of t where o_1 is better than o_2 is $1.0452 - 1 = 0.452$, while that where o_2 is better than o_1 is $1.0859 - 1.0452 = 0.407$. If the DM agrees that the difference between 0.452 and 0.407 is significant, o_1 would be better than o_2 for the DM. Otherwise, the DM is asked to provide additional preference data or employ some other techniques for decision making.

5.2 Example of trapezoidal FPCM

We consider a decision making problem in which a faculty member must select the most suitable student for a part-time research assistant position. Three students o_j, $j = 1, 2, 3$ have applied. As the characters of those students are good for the research assistant position, the decision is to be made based on their academic performance in five subjects: c_1: mathematics (MA), c_2: physics (PH), c_3: chemistry (CH), c_4: computer programming (PR), and c_5: English (EN). The scores, i.e., marginal utility values, of those students in the five subjects are shown in Table 2. Student o_1 takes the best scores in MA, PR, and EN, but the differences from the second scores are not very large. Student o_2 takes by far the best score in PH than others, but the worst score in MA. Student o_3 takes by far the best score in CH, but the worst score in PH, PR, and EN. Each student has her/his merit and demerit. The importance of these subjects varies depending on the role of the position in the research project. A proper analysis is necessary for a reasonable solution. Then, the DM required to make pairwise comparisons among the five subjects for obtaining a PCM. We assume that the DM gives the following 5×5 TZFPCM:

$$\tilde{A} = \begin{pmatrix} (1,1,1,1) & (\frac{5}{6},\frac{29}{26},\frac{31}{24},\frac{7}{4}) & (1,\frac{29}{21},\frac{31}{19},\frac{7}{3}) & (\frac{5}{4},\frac{29}{16},\frac{31}{14},\frac{7}{2}) & (\frac{5}{3},\frac{29}{11},\frac{31}{9},7) \\ (\frac{4}{7},\frac{24}{31},\frac{26}{29},\frac{6}{5}) & (1,1,1,1) & (\frac{4}{5},\frac{8}{7},\frac{26}{19},2) & (1,\frac{3}{2},\frac{13}{7},3) & (\frac{4}{3},\frac{24}{11},\frac{26}{9},6) \\ (\frac{3}{7},\frac{19}{31},\frac{21}{29},1) & (\frac{1}{2},\frac{19}{26},\frac{7}{8},\frac{5}{4}) & (1,1,1,1) & (\frac{3}{4},\frac{19}{16},\frac{3}{2},\frac{5}{2}) & (1,\frac{19}{11},\frac{7}{3},5) \\ (\frac{2}{7},\frac{14}{31},\frac{16}{29},\frac{4}{5}) & (\frac{1}{3},\frac{7}{13},\frac{2}{3},1) & (\frac{2}{5},\frac{2}{3},\frac{16}{19},\frac{4}{3}) & (1,1,1,1) & (\frac{2}{3},\frac{14}{11},\frac{16}{9},4) \\ (\frac{1}{7},\frac{9}{31},\frac{11}{29},\frac{3}{5}) & (\frac{1}{6},\frac{9}{26},\frac{11}{24},\frac{3}{4}) & (\frac{1}{5},\frac{3}{7},\frac{11}{19},1) & (\frac{1}{4},\frac{9}{16},\frac{11}{14},\frac{3}{2}) & (1,1,1,1) \end{pmatrix}. \quad (60)$$

Table 2 The marginal utility values of students in each subject

Students	MA	PH	CH	PR	EN
o_1	79	79	57	70	65
o_2	62	98	68	62	60
o_3	71	68	97	57	57

Each component of $\tilde{A}$ shows the fuzzy evaluation of the relative importance between subjects c_i and c_j.

By solving the trapezoidal fuzzy priority weight estimation model (18) for the above trapezoidal fuzzy comparison matrix, we obtain the following normalized trapezoidal fuzzy priority weight vector:

$$\tilde{W} = \begin{pmatrix} (0.2174,\ 0.2816,\ 0.3010,\ 0.3044) \\ (0.1739,\ 0.2330,\ 0.2524,\ 0.2609) \\ (0.1304,\ 0.1845,\ 0.2039,\ 0.2174) \\ (0.0870,\ 0.1359,\ 0.1553,\ 0.1739) \\ (0.0435,\ 0.0874,\ 0.1068,\ 0.1304) \end{pmatrix}. \tag{61}$$

We know that MA is the most important subject, and the importance decreases in the following order: MA, PH, CH, PR, and EN.

From this estimated trapezoidal fuzzy priority weight vector $\tilde{W}$, we obtain the set of all trapezoidal fuzzy priority weight vectors having the same distance between $\tilde{a}_{ij}$ and $\tilde{w}_i/\tilde{w}_j$, $i, j \in N$, $i \neq j$ as

$$\begin{aligned} \mathcal{W}_{TZ} = \Big\{ & \tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^{\mathrm{T}} \,|\tilde{w}_i = (t_1 w_i^{\mathrm{L}}, t_2 w_i^{\mathrm{M}}, t_2 w_i^{\mathrm{N}}, t_1 w_i^{\mathrm{U}}),\ i \in N, \\ & t_2 \in \left[\tau_2^{\mathrm{L}}(t_1), \tau_2^{\mathrm{U}}(t_1)\right],\ t_1 \in \left[\tau_1^{\mathrm{L}}, \tau_1^{\mathrm{U}}\right] \Big\}, \end{aligned} \tag{62}$$

where τ_1^{L}, τ_1^{U}, $\tau_2(t_1)^{\mathrm{L}}$ and $\tau_2(t_1)^{\mathrm{U}}$ are obtained by (42) and (43) with t_1^{L}, t_1^{U}, t_2^{L} and t_2^{U} calculated by (38) and (39). Indeed, from $t_1^{\mathrm{L}} = 1$, $t_1^{\mathrm{U}} = 1.3528$, $t_2^{\mathrm{L}} = 1$ and $t_2^{\mathrm{U}} = 1.0618$, we obtain

$$\tau_1^{\mathrm{L}} = 1,\ \tau_1^{\mathrm{U}} = 1.3528,\ \tau_2^{\mathrm{L}}(t_1) = \max(1, 0.7721 t_1),\ \tau_2^{\mathrm{U}}(t_1) = \min(1.0618, 1.0112 t_1). \tag{63}$$

The alternatives (students) o_i, $i = 1, 2, 3$ are ordered by using the estimated fuzzy priority weight vectors of the criteria $(t_1 w_i^{\mathrm{L}}, t_2 w_i^{\mathrm{M}}, t_2 w_i^{\mathrm{N}}, t_1 w_i^{\mathrm{U}})$, $i \in N$, $t_2 \in [t_{2(t_1)}^{\mathrm{L}}, t_{2(t_1)}^{\mathrm{U}}]$, $t_1 \in [t_1^{\mathrm{L}}, t_1^{\mathrm{U}}]$. We calculate the fuzzy total utility values of the three alternatives using α-cuts of the estimated trapezoidal fuzzy priority weights. From these results, the vector of centroids of the fuzzy total utility values, i.e., $U^{\mathrm{C}} = [U^{\mathrm{C}}(o_1), U^{\mathrm{C}}(o_2), U^{\mathrm{C}}(o_3)]^{\mathrm{T}}$, is calculated by (20). In this example, we use the α-cuts with $\alpha = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,$ and 1.

By changing the values of t_1 and t_2 in the range $t_2 \in [\tau_2^{\mathrm{L}}(t_1), \tau_2^{\mathrm{U}}(t_1)]$ and $t_1 \in [\tau_1^{\mathrm{L}}, \tau_1^{\mathrm{U}}]$, we obtain different U^{C}. For example, we show U^{C} for a few settings of t_1 and t_2. Wen $t_1 = 1$ and $t_2 = 1$, we obtain

$$U^{\mathrm{C}} = (71.33, 70.79, 70.87)^{\mathrm{T}}. \tag{64}$$

Therefore, $U^{\mathrm{C}}(o_1) > U^{\mathrm{C}}(o_3) > U^{\mathrm{C}}(o_2)$, which implies $o_1 \succsim o_3 \succsim o_2$. This result is obtained for the conventional LGP method with the TZFPCM of (60). When $t = 1.15$ and $t_2 = 1.03$, we obtain

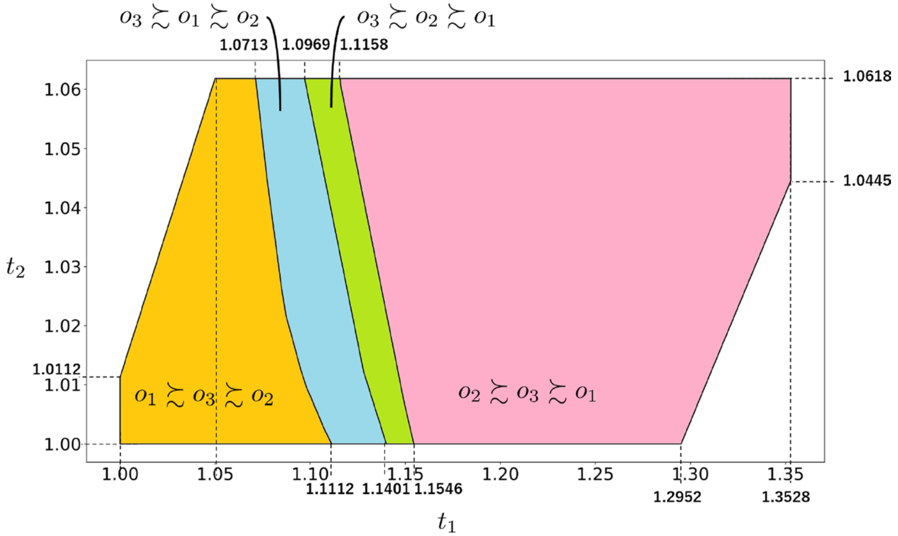

Fig. 3 Variety of possible orders of alternatives and their areas in t_1-t_2 coordinate.

$$U^{\mathrm{C}} = (71.40, 71.54, 71.50)^{\mathrm{T}}. \tag{65}$$

Although the centroids of fuzzy total utility values of alternatives are similar, it shows $U^{\mathrm{C}}(o_2) > U^{\mathrm{C}}(o_3) > U^{\mathrm{C}}(o_1)$. Then we obtain $o_2 \succsim o_3 \succsim o_1$. The orders of alternatives obtained are different between those cases. That is, the order of alternatives depends on the parameter setting. Therefore, the potentially recommendable order of alternatives under the given TZFPCM is not unique, and it will be worthwhile presenting all potentially recommendable orders of alternatives to the DM for making a reasonable decision support.

Exploring all possible orders of alternatives and their areas by changing the parameters t_1 and t_2 within the range defined by $t_2 \in [\tau_2^{\mathrm{L}}(t_1), \tau_2^{\mathrm{U}}(t_1)]$ and $t_1 \in [\tau_1^{\mathrm{L}}, \tau_1^{\mathrm{U}}]$, we obtain Fig. 3. Fig. 3 shows the map of possible orders of alternatives in the range of parameters t_1 and t_2. The orders of alternatives are indicated in the figure. We observe four possible orders of alternatives in Fig. 3. If the possible order is unique, the DM's decision to order alternatives is decisive. However, in this example, we have four. Then, we infer that the DM focuses on the four orders of alternatives. If the DM selects an order intuitively from the figure, the decision problem is solved. Otherwise, we analyze Fig. 3. Alternative o_3 becomes better than o_2 as t_1 decreases. Moreover, o_1 also becomes better as t_1 decreases and it becomes the best when t_1 is small. In this example, parameter t_1 is more important than parameter t_2 because it changes the order of alternatives. However, the selection of parameters t_1 and t_2 is not an easy task, we may select the order of alternatives with the largest area, or the order of alternatives at the gravity center of the range of parameters, as the recommended order of alternatives for the DM. In this example, the area of order $o_2 \succsim o_3 \succsim o_1$ is the largest, and the area includes the gravity center. Indeed, the sizes of the yellow, blue, green and pink areas are 0.004019 (20.87%), 0.001939 (10.07%), 0.001099 (5.71%) and 0.012199 (63.35%), respectively, where the percentage in the parenthesis shows the ratio to the area of the hexagon where parameters t_1 and t_2 take. The pink area is the largest and corresponds to the order $o_2 \gtrsim o_3 \gtrsim o_1$. On the other hand, the gravity center is obtained as $t_1 = 1.176431$ and $t_2 = 1.03105$, included in the pink area. Therefore, in both ways, we may recommend the DM to select the order $o_2 \gtrsim o_3 \gtrsim o_1$, which is very different from the order obtained in the conventional LGP method, in this example.

Another possible way to select a recommendable order is to utilize the Borda count (Emerson, 2013) developed in the collective decision science. Borda count aggregates multiple individual orders of alternatives into a consensus order of alternatives. In the Borda count, alternative o_j ranked the *k*-th in an order of *n* alternatives gets $(n-k)$ points. Then, an alternative gets points from all individual orders. The consensus order of alternatives is obtained as the descending order of the total points of alternatives. In our problem setting, each point (t_1, t_2) can be seen as individuals. However, there are infinitely many (t_1, t_2) in the range $t_2 \in [\tau_2^{\mathrm{L}}(t_1), \tau_2^{\mathrm{U}}(t_1)]$, $t_1 \in [\tau_1^{\mathrm{L}}, \tau_1^{\mathrm{U}}]$. Then we regard the size of the area of each order as the population of individuals supporting the order. As we have shown the sizes of areas of all possible orders, we obtain the total points $B(o_i)$, $i = 1, 2, 3$ of alternatives as

$$
\begin{aligned}
B(o_1) =& 2 \times 0.004019 + 1 \times 0.001939 + 0 \times 0.001099 + 0 \times 0.012199 = 0.009977,\\
B(o_2) =& 0 \times 0.004019 + 0 \times 0.001939 + 1 \times 0.001099 + 2 \times 0.012199 = 0.025497,\\
B(o_3) =& 1 \times 0.004019 + 2 \times 0.001939 + 2 \times 0.001099 + 1 \times 0.012199 = 0.022294.
\end{aligned}
$$

Then, from $B(o_2) > B(o_3) > B(o_1)$, we obtain the order $o_2 \gtrsim o_3 \gtrsim o_1$ as a recommendable order by the Borda count,

The above way to select a recommendable order does not consider the differences between the centroids of the fuzzy total utility values of alternatives but only their orders. The other possible recommendation is based simply on the average of the centroids of the fuzzy total utility values $U^{\mathrm{C}}(o_i)$. Calculating them, we obtain $\bar{U}^{\mathrm{C}}(o_1) = 71.4002$, $\bar{U}^{\mathrm{C}}(o_2) = 71.6716$ and $\bar{U}^{\mathrm{C}}(o_3) = 71.5767$, where $\bar{U}^{\mathrm{C}}(o_i)$ shows the average of the centroids of the fuzzy total utility values of o_i, $i = 1, 2, 3$. Therefore, although the differences are small, we obtain $\bar{U}^{\mathrm{C}}(o_2) \geq \bar{U}^{\mathrm{C}}(o_3) \geq \bar{U}^{\mathrm{C}}(o_1)$ which implies $o_2 \gtrsim o_3 \gtrsim o_1$.

In this example, $o_2 \gtrsim o_3 \gtrsim o_1$ is recommended from multiple perspectives derived from the set of normalized fuzzy priority weight vectors. Namely, student o_2 taking a good score in PH is recommended as the best solution, and student o_1 with no very remarkable subject scores is ranked at the last position. The result is very different from that of the LGP solution, which recommends $o_1 \gtrsim o_3 \gtrsim o_3$ in this example, although the fuzzy total utility values are similar between the three students. Normalized fuzzy priority vectors having the same deviations between the FPCM components and ratios of fuzzy priority weights as the LGP solution are reasonably considered other solutions to the problem of estimating a normalized fuzzy priority weight vector, because FPCM components are supposed to show ratios of fuzzy priority weights. The result $o_2 \gtrsim o_3 \gtrsim o_1$ is supported by multiple perspectives, taking into account the set of normalized fuzzy priority weight vectors. Therefore, the DM can agree to adopt this order as the final selection.

Finally, we note that the result of the decision analysis with the set of solutions is not always different from that of the LGP method. When both results coincide, we can confirm that the result is robust.

6 Conclusion

In this paper, the non-uniqueness of the solution to the estimation problem of the normalized fuzzy priority weight vector is considered under a given triangular or trapezoidal fuzzy pairwise comparison matrix. It is shown that all solutions are easily obtained by parameters under a given solution of the conventional estimation method. In the estimation problem with a triangular fuzzy pairwise comparison matrix, the solution set becomes a line segment as far as the solution of the conventional estimation method is unique. On the other hand, in the estimation problem with a trapezoidal fuzzy pairwise comparison matrix, the solution set usually configures a hexagon, pentagon, or tetragon. We depicted the situations about the orders of alternatives over the set of estimated normalized fuzzy priority weight vectors in the figures. If all estimated normalized fuzzy priority weight vectors suggest an order of alternatives, it is a unique recommendable order, and this order is robust. Similarly, if all estimated normalized fuzzy priority weight vectors suggest an alternative as the best, the alternative is a robust solution to the given multiple criteria decision making problem. The figure of the situation about the orders of alternatives over the set of estimated normalized fuzzy priority weight vectors will be useful for the DM to understand the potential solutions intuitively. Together with the size of the area in the figure as well as the Borda count, it may help the DM to make up her/his mind to select a solution.

Depicting the figure requires heavy calculations of the centroids of the fuzzy total utility values for many t_1 and t_2 in the problem with a trapezoidal fuzzy pairwise comparison matrix. Their simple calculations should be investigated. In this paper, we consider the fuzzy pairwise comparison matrix only in the highest layer of the hierarchy of the decision problem. The introduction of fuzzy pairwise comparison matrices in the other layers of the hierarchy of the decision problem is one of the future topics.

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Declarations

Competing interests The authors declare no competing interests.

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