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Author(s)	Bernardi, Enrico; Nishitani, Tatsuo
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REMARKS ON SYMMETRIZATION OF 2×2 SYSTEMS AND THE CHARACTERISTIC MANIFOLDS

ENRICO BERNARDI* AND TATSUO NISHITANI**

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1. Introduction

Let L be a first order differential operator on $C^\infty(\Omega, \mathbf{C}^N)$ where Ω is an open set in \mathbf{R}^{n+1} with coordinates $x = (x_0, x_1, \dots, x_n) = (x_0, x')$. We say that L is strongly hyperbolic at $\hat{x} \in \Omega$ with respect to x_0 if the Cauchy problem for $L+Q$ is C^∞ well posed near \hat{x} for every $Q \in C^\infty(\Omega, M_N(\mathbf{C}))$ with respect to x_0 , that is there are a neighborhood $\omega \subset \Omega$ of \hat{x} and a positive number ε such that $L+Q$ is an isomorphism on $\{v \in C^\infty(\omega, \mathbf{C}^N); v=0 \text{ in } x_0 < \hat{x}_0 + \tau\}$ for every $|\tau| < \varepsilon$ (for more details, see [2], [3]). Choosing a local coordinates $(x, \xi) = (x_0, x', \xi_0, \xi')$ in the cotangent bundle $T^*\Omega$ and a basis for \mathbf{C}^N let

$$L(x, \xi) = L_1(x, \xi) + L_0(x)$$

be the complete symbol of L , L_1 being the principal symbol. Let $h(x, \xi)$ denote the determinant of $L_1(x, \xi)$ which is in $C^\infty(T^*\Omega)$.

If the Cauchy problem for $L+Q$ is C^∞ well posed near \hat{x} with respect to x_0 , it follows from the Lax-Mizohata theorem that $h(x, \xi + \tau dx_0) = 0$ admits only real zeros τ for every $\xi \in T_x^*\Omega \setminus 0$, x close to \hat{x} . Therefore we are always assuming h to be hyperbolic in this sense.

If L is strongly hyperbolic at $\hat{x} \in \Omega$ with respect to x_0 then one can find a neighborhood U of \hat{x} such that either h is effectively hyperbolic or the rank of L_1 is at most $N-2$ in every multiple characteristic on $T_x^*\Omega$, $x \in U$ (see [2], [3]). Since the situation for effectively hyperbolic determinants has already been elucidated (cf. [1]) it is natural to see what happens when the rank of L_1 falls to $N-2$ or less at a multiple characteristic $\rho \in T_x^*\Omega \setminus 0$. In particular, this condition turns out to be $L_1(\rho) = O(2 \times 2 \text{ zero matrix})$ when $N=2$. The hyperbolicity and h being the determinant of L_1 with $L_1(\rho) = O$ give a bound for the rank of the Hessian of h at ρ . Indeed, denoting by $\text{Hess } h(\rho)$ the Hessian of h at ρ , we have

*Department of Mathematics, University of Bologna

**Department of Mathematics, College of General Education, Osaka University

Lemma 1.1. (Lemma 4.1 in [2]) *Let $N=2$ and $L_1(\rho)=O$. Then $\text{rank Hess } h(\rho) \leq 4$. In particular, if L_1 is real then $\text{rank Hess } h(\rho) \leq 3$.*

We say that $\text{Hess } h(\rho)$ has maximal rank if $\text{rank Hess } h(\rho)=4$ (resp. if $\text{rank Hess } h(\rho)=3$ when L_1 is real). Our partial converse result is then the following.

Theorem 1.1. *Let $N=2$. Assume that $L_1(\rho)=O$ and $\text{Hess } h(\rho)$ has maximal rank at every double characteristic $\rho \in T_{\hat{x}}^*\Omega \setminus 0$. Then L is strongly hyperbolic at \hat{x} with respect to x_0 .*

The assumption of Theorem 1.1 implies that the doubly characteristic set $\Sigma = \{(x, \xi); h(x, \xi) = dh(x, \xi) = 0\}$ of h is a manifold. Indeed

Proposition 1.1. *Let $N=2$. Assume that $L_1(\rho)=O$ and $\text{Hess } h(\rho)$ is of maximal rank at a double characteristic ρ . Then Σ is a C^∞ manifold near ρ with $\text{codim } \Sigma = \text{rank Hess } h(\rho)$ on which L_1 vanishes.*

Theorem 1.1 will be proved constructing a suitable symmetrizer for L_1 , more precisely we have

Proposition 1.2. *Let $N=2$. Assume that $L_1(\rho)=O$ and Σ is a C^∞ manifold near $\rho = (\hat{x}, \hat{\xi}) \in T_{\hat{x}}^*\Omega \setminus 0$ with $\text{codim } \Sigma = \text{rank Hess } h(\rho)$ on which L_1 vanishes. Then L_1 has a symmetrizer near ρ , that is, there is a 2×2 matrix valued symbol $S(x, \xi')$ defined near $\rho' = (\hat{x}, \hat{\xi}')$, homogeneous of degree 0 in ξ' such that*

$$\begin{aligned} S^*(x, \xi') &= S(x, \xi') \quad \text{and} \quad S(x, \xi') \text{ is positive definite,} \\ S(x, \xi') L_1(x, \xi) &= L_1^*(x, \xi) S(x, \xi'), \end{aligned}$$

where L_1^* denotes the adjoint matrix of L_1 .

REMARK 1.1. The assertion of Proposition 1.1 is equivalent to: $S = \{(x, \xi); L_1(x, \xi) = O\}$ is a C^∞ manifold with $\text{codim } S = \text{rank Hess } h(\rho)$.

When $L_1(\rho)=O$ and L_1 is real, hence the maximal rank of $\text{Hess } h(\rho)$ is 3, Proposition 1.1 can be easily seen. Since Proposition 1.2 was proved in [2] when $\text{rank Hess } h(\rho) \leq 3$ and Theorem 1.1 is an immediate consequence of Proposition 1.2, it will be enough to prove Propositions 1.1 and 1.2 assuming $\text{rank Hess } h(\rho)=4$.

Here we note that the result can be easily generalized to $N \times N$ system such that all characteristics of h are at most double. The theorems below follow easily from Propositions 1.1, 1.2 and the same arguments proving Theorem 2.3 in [2].

Theorem 1.2. *Assume that every multiple characteristic on $T_{\hat{x}}^*\Omega \setminus 0$ is at most double. Suppose that either h is effectively hyperbolic or $\text{rank } L_1 \leq N-2$ and*

Hess h has maximal rank in every double characteristic on $T^*_\sharp \Omega \setminus 0$. Then L is strongly hyperbolic at \sharp with respect to x_0 .

Theorem 1.3. (cf. Theorem 2.3 in [2]) *Assume that every multiple characteristic on $T^*_\sharp \Omega \setminus 0$ is at most double and one of the following conditions is satisfied in every double characteristic $\rho \in T^*_\sharp \Omega \setminus 0$:*

- (1) *h is effectively hyperbolic at ρ ,*
- (2) *the doubly characteristic set Σ of h is a C^∞ manifold near ρ on which $\text{rank } L_1 \leq N-2$.*

Then L is strongly hyperbolic at \sharp with respect to x_0 .

2. Proof of Lemma and Propositions

We first note that we may assume that $L_1(0, 1, \dots, 0) = -I_2$, the identity matrix of order 2, so that

$$L_1(x, \xi) = -\xi_0 I_2 + A'(x, \xi'), \quad A'(x, \xi') = \sum_{j=1}^n A_j(x) \xi_j, \quad A_j(x) \in C^\infty(\Omega, M_2(\mathbb{C}))$$

which is also written

$$L_1(x, \xi) = -(\xi_0 - \frac{1}{2} \text{Tr } A'(x, \xi')) I_2 + A(x, \xi'), \quad \text{Tr } A(x, \xi') = 0.$$

Here $g(x, \xi') = \det A(x, \xi') = \det A(x, \xi') \leq 0$ and $\text{Tr } A'(x, \xi') = \text{Tr } A(x, \xi')$ is real which follow from the hyperbolicity of h . Let us denote

$$A(x, \xi') = \begin{bmatrix} a(x, \xi') & b(x, \xi') \\ c(x, \xi') & -a(x, \xi') \end{bmatrix}, \quad \rho = (\sharp, \hat{\xi}), \quad \rho' = (\sharp, \hat{\xi}').$$

We first show Lemma 1.1. Note that

$$\text{Hess } h(\rho) = d\eta \circ d\eta - (da \circ da + db \circ dc)$$

where $\eta = \xi_0 - 2^{-1} \text{Tr } A'$ and $db \circ dc$ denotes the symmetric tensor product of db and dc . Then it is enough to show that the rank of the quadratic form $Q = da \circ da + db \circ dc$ (at ρ'), which is non negative definite, is at most 3 (resp. at most 2 when a, b, c are real). Here we recall that a real quadratic form $Q(X)$ in $T_\rho(T^*\Omega)$ which is non negative definite cannot vanish on a linear subspace $V \subset T_\rho(T^*\Omega)$ unless $\text{codim } V \geq \text{rank } Q$.

Denoting by $\Re a$ and $\Im a$ the real part and the imaginary part of a respectively we see that

$$\begin{aligned} 0 \leq Q &= d\Re a \circ d\Re a - d\Im a \circ d\Im a + d\Re b \circ d\Re c - d\Im b \circ d\Im c \\ &\leq d\Re a \circ d\Re a + d\Re b \circ d\Re c - d\Im b \circ d\Im c. \end{aligned}$$

It is clear that Q vanishes on $\{X; d\Re a(X)=d\Re b(X)=d\Im b(X)=0\}$ which shows that $\text{rank } Q \leq 3$. The same argument shows that $\text{rank } Q \leq 2$ if a, b, c are real.

We turn to the proofs of Propositions 1.1 and 1.2. As noted in Introduction it is enough to prove these propositions assuming $\text{rank Hess } h(\rho)=4$. Since the hypothesis $\text{rank Hess } h(\rho)=4$ reduces to $\text{rank Hess } g(\rho')=3$ we may assume that Q is non negative definite and has rank 3.

We first remark that $d\Re a(\rho') \neq 0$. If it were not true we would have

$$0 \leq Q = -d\Im a \circ d\Im a + d\Re b \circ d\Re c - d\Im b \circ d\Im c \leq d\Re b \circ d\Re c - d\Im b \circ d\Im c.$$

It is clear that there is a linear subspace $V(\subset T_{\rho}(T^*\Omega))$ with $\text{codim } V \leq 2$ on which Q vanishes and hence $\text{rank } Q \leq 2$. This contradicts the assumption.

Set $\varphi = \Re a$ and denote by $b|_{\varphi=0}$ the restriction of b to the surface $\{\varphi=0\}$.

Lemma 2.1. *Let $b = \beta\varphi + \tilde{b}$, $c = \gamma\varphi + \tilde{c}$ with $\tilde{b} = b|_{\varphi=0} = \tilde{b}_1 + i\tilde{b}_2$, $\tilde{c} = c|_{\varphi=0} = \tilde{c}_1 + i\tilde{c}_2$ where \tilde{b}_i, \tilde{c}_i are real. Then we have*

$$d\tilde{b}_i \neq 0, \quad d\tilde{c}_i \neq 0 \text{ at } \rho', \quad i = 1, 2.$$

Proof. Denoting $\Im a = \alpha\varphi + \tilde{a}$, $\tilde{a} = a|_{\varphi=0}$, one can write

$$A(x, \xi') = \varphi \begin{bmatrix} (1+i\alpha) & \beta \\ \gamma & -(1+i\alpha) \end{bmatrix} + \begin{bmatrix} i\tilde{a} & \tilde{b}_1 + i\tilde{b}_2 \\ \tilde{c}_1 + i\tilde{c}_2 & -i\tilde{a} \end{bmatrix}.$$

From the non-positivity of g on $\{\varphi=0\}$ it follows that

$$(2.1) \quad \tilde{b}_1 \tilde{c}_1 - \tilde{b}_2 \tilde{c}_2 - \tilde{a}^2 \geq 0,$$

$$(2.2) \quad \tilde{b}_1 \tilde{c}_2 + \tilde{b}_2 \tilde{c}_1 = 0$$

near ρ' . Suppose, for instance, that $d\tilde{b}_1(\rho')=0$ and hence $d\tilde{b}_2=0$ or $d\tilde{c}_1=0$ (at ρ') by (2.2). If $d\tilde{b}_2=0$ then $d\tilde{a}=0$ by (2.1) and then Q vanishes on $\{X; d\varphi(X)=0\}$ because $da=(1+i\alpha)d\varphi$ at ρ' . This is a contradiction. The other cases will be proved similarly.

Lemma 2.2. *$d\tilde{b}_1$ is not proportional to $d\tilde{b}_2$ at ρ' . There is a positive function $m(x, \xi')$ defined near ρ' , homogeneous of degree 0 in ξ' such that*

$$\tilde{c}_1(x, \xi') = m(x, \xi') \tilde{b}_1(x, \xi'), \quad \tilde{c}_2(x, \xi') = -m(x, \xi') \tilde{b}_2(x, \xi').$$

Proof. Suppose that $d\tilde{b}_2 = kd\tilde{b}_1$ at ρ' with some $k \in \mathbf{R}$ and hence $d\tilde{c}_2 = -kd\tilde{c}_1$ by (2.2) at ρ' . Since (by (2.1))

$$d\tilde{b}_1 \circ d\tilde{c}_1 - d\tilde{b}_2 \circ d\tilde{c}_2 - d\tilde{a} \circ d\tilde{a} = (1+k^2) d\tilde{b}_1 \circ d\tilde{c}_1 - d\tilde{a} \circ d\tilde{a} \geq 0,$$

$d\tilde{b}_1$ and $d\tilde{c}_1$ must be proportional to $d\tilde{a}$ at ρ' if $d\tilde{a} \neq 0$. Then it is clear that Q vanishes on $\{X; d\tilde{a}(X)=d\varphi(X)=0\}$ which is a contradiction. If $d\tilde{a}=0$ (at ρ')

then Q vanishes on $\{X; d\varphi(X)=d\tilde{c}_1(X)=0\}$ which also gives a contradiction. This proves the first assertion. The second assertion easily follows from the first one and (2.1), (2.2).

The following lemma proves Proposition 1.1.

Lemma 2.3. *Let $\beta=\beta_1+i\beta_2, \gamma=\gamma_1+i\gamma_2, \beta_i, \gamma_i$ real. Set $\psi_i=\tilde{b}_i+\beta_i\varphi$ ($i=1, 2$), $B=\gamma_2+m\beta_2, C=\gamma_1-m\beta_1$. Then we have*

$$A = \varphi \begin{bmatrix} 1 & 0 \\ C+iB & -1 \end{bmatrix} + \psi_1 \begin{bmatrix} -iB/2 & 1 \\ m & iB/2 \end{bmatrix} + \psi_2 \begin{bmatrix} -iC/2 & i \\ -im & iC/2 \end{bmatrix}.$$

Moreover $d\varphi, d\psi_i$ are linearly independent at ρ' and the set $\{(x, \xi'); A(x, \xi')=0\}$ is given by

$$S = \{(x, \xi'); \varphi(x, \xi') = \psi_1(x, \xi') = \psi_2(x, \xi') = 0\}.$$

Proof. Recall that

$$A = \varphi \begin{bmatrix} 1+i\alpha & \beta \\ \gamma & -(1+i\alpha) \end{bmatrix} + \begin{bmatrix} i\tilde{a} & \tilde{b}_1+i\tilde{b}_2 \\ m(\tilde{b}_1-i\tilde{b}_2) & -i\tilde{a} \end{bmatrix}.$$

We observe the imaginary part of g :

$$\Im g = 2\alpha\varphi^2 + 2\tilde{a}\varphi + \Im(\beta\gamma)\varphi^2 + \Im(\gamma+\beta m)\varphi\tilde{b}_1 + \Re(\gamma-\beta m)\varphi\tilde{b}_2.$$

Since $\Im g=0$ near ρ' and $d\varphi \neq 0$ at ρ' it follows that

$$(2.3) \quad 2\alpha\varphi + 2\tilde{a} + \Im(\beta\gamma)\varphi + \Im(\gamma+\beta m)\tilde{b}_1 + \Re(\gamma-\beta m)\tilde{b}_2 = 0$$

near ρ' . Now we set

$$D = \Im(\beta\gamma), B = \Im(\gamma+\beta m) = \gamma_2+\beta_2 m, C = \Re(\gamma-\beta m) = \gamma_1-\beta_1 m.$$

Noticing $D=\beta_1 B+\beta_2 C$ it follows from (2.3) that

$$(2.4) \quad (\alpha\varphi + \tilde{a}) = -\frac{1}{2}(\psi_1 B + \psi_2 C)$$

which shows that $a=(1+i\alpha)\varphi + i\tilde{a} = \varphi - i(\psi_1 B + \psi_2 C)/2$. On the other hand it is easy to see

$$m(\tilde{b}_1-i\tilde{b}_2) + \gamma\varphi = (C+iB)\varphi + m(\psi_1-i\psi_2), \quad \tilde{b}_1+i\tilde{b}_2 + \beta\varphi = \psi_1+i\psi_2$$

since $\gamma_1=C+m\beta_1$ and $\gamma_2=B-m\beta_2$. These prove the first part. The rest of the assertion is obvious.

Lemma 2.4.

$$4m-(B^2+C^2) > 0 \text{ at } \rho'.$$

Proof. Let us set $\tilde{B}=B|_{\varphi=0}$, $\tilde{C}=C|_{\varphi=0}$. From (2.4) it follows that

$$\tilde{a} = -(\tilde{B}\tilde{b}_1 + \tilde{C}\tilde{b}_2)/2.$$

On the other hand (2.1) and Lemma 2.2 give that

$$m(\tilde{b}_1^2 + \tilde{b}_2^2) - \tilde{a}^2 \geq 0 \text{ near } \rho'.$$

Since the quadratic form $m(d\tilde{b}_1 \circ d\tilde{b}_1 + d\tilde{b}_2 \circ d\tilde{b}_2) - (\tilde{B} d\tilde{b}_1 + \tilde{C} d\tilde{b}_2) \circ (\tilde{B} d\tilde{b}_1 + \tilde{C} d\tilde{b}_2)/4$ is the restriction of Q to $\{X; d\varphi(X)=0\}$, this must have rank 2 and then positive definite. This shows that $4m - (\tilde{B}^2 + \tilde{C}^2) > 0$ at ρ' and hence the result.

To finish the proof of Proposition 1.2 we give a required symmetrizer for L_1 :

$$S(x, \xi') = \begin{bmatrix} 2m(x, \xi') & -C(x, \xi') + iB(x, \xi') \\ -(C(x, \xi') + iB(x, \xi')) & 2 \end{bmatrix}$$

which satisfies $S(x, \xi') = S^*(x, \xi')$ clearly. Using Lemma 2.3 it is easy to check that $S(x, \xi') A(x, \xi') = A^*(x, \xi') S(x, \xi')$ and hence $S(x, \xi') L_1(x, \xi) = L_1^*(x, \xi) S(x, \xi')$. The positivity of S follows from Lemma 2.4.

Added in proof: after submitted the paper we knew that L. Hörmander has obtained a strong stability property of double characteristics of maximal rank, including Proposition 1.1 in "Hyperbolic systems with double characteristics", 1990, preprint.

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