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REMARKS ON SYMMETRIZATION OF $2 \times 2$ SYSTEMS AND THE CHARACTERISTIC MANIFOLDS

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1. Introduction

Let $L$ be a first order differential operator on $C^\infty(\Omega, \mathbb{C}^N)$ where $\Omega$ is an open set in $\mathbb{R}^{n+1}$ with coordinates $x = (x_0, x_1, \ldots, x_n) = (x', x^0)$. We say that $L$ is strongly hyperbolic at $\dot{x} \in \Omega$ with respect to $x_0$ if the Cauchy problem for $L + Q$ is $C^\infty$ well posed near $\dot{x}$ for every $Q \in C^\infty(\Omega, M_2(\mathbb{C}))$ with respect to $x_0$, that is there are a neighborhood $\omega \subset \Omega$ of $\dot{x}$ and a positive number $\varepsilon$ such that $L + Q$ is an isomorphism on $\{u \in C^\infty(\omega, \mathbb{C}^N); u = 0 \text{ in } \Omega \setminus U \}$ for every $|\tau| < \varepsilon$ (for more details, see [2], [3]). Choosing a local coordinates $(x, \xi) = (x', x^0, \xi_0, \xi')$ in the cotangent bundle $T^*\Omega$ and a basis for $\mathbb{C}^N$ let

$$L(x, \xi) = L_1(x, \xi) + L_0(x)$$

be the complete symbol of $L$, $L_1$ being the principal symbol. Let $h(x, \xi)$ denote the determinant of $L_1(x, \xi)$ which is in $C^\infty(T^*\Omega)$.

If the Cauchy problem for $L + Q$ is $C^\infty$ well posed near $\dot{x}$ with respect to $x_0$, it follows from the Lax-Mizohata theorem that $h(x, \xi + \tau dx_0) = 0$ admits only real zeros $\tau$ for every $\xi \in T^*_\Omega \setminus 0$, $x$ close to $\dot{x}$. Therefore we are always assuming $h$ to be hyperbolic in this sense.

If $L$ is strongly hyperbolic at $\dot{x} \in \Omega$ with respect to $x_0$ then one can find a neighborhood $U$ of $\dot{x}$ such that either $h$ is effectively hyperbolic or the rank of $L_1$ is at most $N-2$ in every multiple characteristic on $T^*_\Omega$, $x \in U$ (see [2], [3]). Since the situation for effectively hyperbolic determinants has already been elucidated (cf. [1]) it is natural to see what happens when the rank of $L_1$ falls to $N-2$ or less at a multiple characteristic $\rho \in T^*_\Omega \setminus 0$. In particular, this condition turns out to be $L_0(\rho) = O(2 \times 2 \text{ zero matrix})$ when $N=2$. The hyperbolicity and $h$ being the determinant of $L_1$ with $L_1(\rho) = O$ give a bound for the rank of the Hessian of $h$ at $\rho$. Indeed, denoting by Hess $h(\rho)$ the Hessian of $h$ at $\rho$, we have

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Lemma 1.1. (Lemma 4.1 in [2]) Let \( N = 2 \) and \( L_1(\rho) = 0 \). Then \( \text{rank Hess } h(\rho) \leq 4 \). In particular, if \( L_1 \) is real then \( \text{rank Hess } h(\rho) \leq 3 \).

We say that Hess \( h(\rho) \) has maximal rank if \( \text{rank Hess } h(\rho) = 4 \) (resp. if \( \text{rank Hess } h(\rho) = 3 \) when \( L_1 \) is real). Our partial converse result is then the following.

Theorem 1.1. Let \( N = 2 \). Assume that \( L_1(\rho) = 0 \) and Hess \( h(\rho) \) has maximal rank at every double characteristic \( \rho \in T^*_x \Omega \setminus 0 \). Then \( L \) is strongly hyperbolic at \( x \) with respect to \( x_0 \).

The assumption of Theorem 1.1 implies that the doubly characteristic set \( \Sigma = \{ (x, \xi) ; h(x, \xi) = dh(x, \xi) = 0 \} \) of \( h \) is a manifold. Indeed

Proposition 1.1. Let \( N = 2 \). Assume that \( L_1(\rho) = 0 \) and Hess \( h(\rho) \) is of maximal rank at a double characteristic \( \rho \). Then \( \Sigma \) is a \( C^\infty \) manifold near \( \rho \) with codim \( \Sigma = \text{rank Hess } h(\rho) \) on which \( L_1 \) vanishes.

Theorem 1.1 will be proved constructing a suitable symmetrizer for \( L_1 \), more precisely we have

Proposition 1.2. Let \( N = 2 \). Assume that \( L_1(\rho) = 0 \) and \( \Sigma \) is a \( C^\infty \) manifold near \( \rho = (x, \xi) \in T^*_x \Omega \setminus 0 \) with codim \( \Sigma = \text{rank Hess } h(\rho) \) on which \( L_1 \) vanishes. Then \( L_1 \) has a symmetrizer near \( \rho \), that is, there is a \( 2 \times 2 \) matrix valued symbol \( S(x, \xi') \) defined near \( \rho' = (x, \xi') \), homogeneous of degree 0 in \( \xi' \) such that

\[
S^*(x, \xi') = S(x, \xi') \quad \text{and} \quad S(x, \xi') \quad \text{is positive definite},
\]

\[
S(x, \xi') L_1(x, \xi) = L^*_1(x, \xi) S(x, \xi'),
\]

where \( L^*_1 \) denotes the adjoint matrix of \( L_1 \).

Remark 1.1. The assertion of Proposition 1.1 is equivalent to: \( S = \{(x, \xi); L_1(x, \xi) = 0\} \) is a \( C^\infty \) manifold with codim \( S = \text{rank Hess } h(\rho) \).

When \( L_1(\rho) = 0 \) and \( L_1 \) is real, hence the maximal rank of Hess \( h(\rho) \) is 3, Proposition 1.1 can be easily seen. Since Proposition 1.2 was proved in [2] when \( \text{rank Hess } h(\rho) \leq 3 \) and Theorem 1.1 is an immediate consequence of Proposition 1.2, it will be enough to prove Propositions 1.1 and 1.2 assuming \( \text{rank Hess } h(\rho) = 4 \).

Here we note that the result can be easily generalized to \( N \times N \) system such that all characteristics of \( h \) are at most double. The theorems below follow easily from Propositions 1.1, 1.2 and the same arguments proving Theorem 2.3 in [2].

Theorem 1.2. Assume that every multiple characteristic on \( T^*_x \Omega \setminus 0 \) is at most double. Suppose that either \( h \) is effectively hyperbolic or \( \text{rank } L_1 \leq N - 2 \) and
Hess $h$ has maximal rank in every double characteristic on $T^*_x\Omega\setminus 0$. Then $L$ is strongly hyperbolic at $x$ with respect to $x_0$.

**Theorem 1.3.** (cf. Theorem 2.3 in [2]) Assume that every multiple characteristic on $T^*_x\Omega\setminus 0$ is at most double and one of the following conditions is satisfied in every double characteristic $p \in T^*_x\Omega\setminus 0$:

1. $h$ is effectively hyperbolic at $p$,
2. the doubly characteristic set $\Sigma$ of $h$ is a $C^\infty$ manifold near $p$ on which rank $L \leq N-2$.

Then $L$ is strongly hyperbolic at $x$ with respect to $x_0$.

**2. Proof of Lemma and Propositions**

We first note that we may assume that $L_1(0, 1, \ldots, 0) = -I_2$, the identity matrix of order 2, so that

$$L_1(x, \xi) = -\xi_0 I_2 + A'(x, \xi'), \quad A'(x, \xi') = \sum_{j=1}^N A_j(x) \xi_j, \quad A_j(x) \in C^\infty(\Omega, M_2)$$

which is also written

$$L_1(x, \xi) = -(\xi_0 - \frac{1}{2} \text{Tr} \ A'(x, \xi')) I_2 + A(x, \xi'), \quad \text{Tr} \ A(x, \xi') = 0.$$ 

Here $g(x, \xi') = \text{the determinant of } A(x, \xi') = \det A(x, \xi') \leq 0$ and $\text{Tr} A'(x, \xi') = \text{the trace of } A'(x, \xi')$ is real which follow from the hyperbolicity of $h$. Let us denote

$$A(x, \xi') = \begin{bmatrix} a(x, \xi') & b(x, \xi') \\ c(x, \xi') & -a(x, \xi') \end{bmatrix}, \quad \rho = (\hat{x}, \hat{\xi}), \quad \rho' = (\hat{x}, \hat{\xi}').$$

We first show Lemma 1.1. Note that

$$\text{Hess } h(\rho) = d\gamma d\eta - (da \circ da + db \circ dc)$$

where $\gamma = \xi_0 - 2^{-1} \text{Tr} A'$ and $db \circ dc$ denotes the symmetric tensor product of $db$ and $dc$. Then it is enough to show that the rank of the quadratic form $Q = da \circ da + db \circ dc$ (at $\rho'$), which is non negative definite, is at most 3 (resp. at most 2 when $a, b, c$ are real). Here we recall that a real quadratic form $Q(X)$ in $T^*_x(T^*_\Omega)$ which is non negative definite cannot vanish on a linear subspace $V \subset T^*_x(T^*_\Omega)$ unless codim $V \geq \text{rank } Q$.

Denoting by $R$ and $I$ the real part and the imaginary part of $a$ respectively we see that

$$0 \leq Q = dR \circ dR - dI \circ dI + dR \circ dR - dR \circ dR$$

$$\leq dR \circ dR + dR \circ dR - dI \circ dI.$$
It is clear that $Q$ vanishes on $\{X; dR(a(X))=d\mathcal{R}b(X)=d\mathcal{Z}b(X)=0\}$ which shows that rank $Q \leq 3$. The same argument shows that rank $Q \leq 2$ if $a, b, c$ are real.

We turn to the proofs of Propositions 1.1 and 1.2. As noted in Introduction it is enough to prove these propositions assuming rank $\text{Hess} h(\rho) = 4$. Since the hypothesis rank $\text{Hess} h(\rho) = 4$ reduces to rank $\text{Hess} g(\rho') = 3$ we may assume that $Q$ is non negative definite and has rank 3.

We first remark that $dR(a(\rho')) \neq 0$. If it were not true we would have

$$0 \leq Q = -d\mathcal{Z}a \circ d\mathcal{Z}a + d\mathcal{R}b \circ d\mathcal{R}c - d\mathcal{Z}b \circ d\mathcal{Z}c \leq d\mathcal{R}b \circ d\mathcal{R}c - d\mathcal{Z}b \circ d\mathcal{Z}c.$$ It is clear that there is a linear subspace $V(\subset T_{\rho}(T^*\Omega))$ with codim $V \leq 2$ on which $Q$ vanishes and hence rank $Q \leq 2$. This contradicts the assumption.

Set $\varphi = R a$ and denote by $b|_{\varphi=0}$ the restriction of $b$ to the surface $\{\varphi=0\}$.

**Lemma 2.1.** Let $b = b \varphi + \bar{b}, c = c \varphi + \bar{c}$ with $b = b|_{\varphi=0} = \bar{b}_1 + i\bar{b}_2, \bar{c} = e|_{\varphi=0} = \bar{c}_1 + i\bar{c}_2$ where $\bar{b}_i, \bar{c}_i$ are real. Then we have

$$d\bar{b}_i \neq 0, \quad d\bar{c}_i \neq 0 \text{ at } \rho', \quad i = 1, 2.$$

Proof. Denoting $\mathcal{Z}a = \alpha \varphi + \bar{a}, \bar{a} = a|_{\varphi=0}$, one can write

$$A(x, \xi') = \varphi \begin{bmatrix} (1+i\alpha) & \beta \\ \gamma & -(1+i\alpha) \end{bmatrix} + \begin{bmatrix} i\bar{a} & \bar{b}_1 + i\bar{b}_2 \\ \bar{c}_1 + i\bar{c}_2 & -i\bar{a} \end{bmatrix}.$$ From the non-positivity of $g$ on $\{\varphi=0\}$ it follows that

(2.1) $\bar{b}_1 \bar{c}_1 - \bar{b}_2 \bar{c}_2 - \bar{a}^2 \geq 0$,

(2.2) $\bar{b}_1 \bar{c}_2 + \bar{b}_2 \bar{c}_1 = 0$

near $\rho'$. Suppose, for instance, that $d\bar{b}_1(\rho') = 0$ and hence $d\bar{b}_2 = 0$ or $d\bar{c}_1 = 0$ (at $\rho'$) by (2.2). If $d\bar{b}_2 = 0$ then $d\bar{c}_1 = 0$ by (2.1) and then $Q$ vanishes on $\{X; d\varphi(X) = 0\}$ because $da = (1+i\alpha) d\varphi$ at $\rho'$. This is a contradiction. The other cases will be proved similarly.

**Lemma 2.2.** $d\bar{b}_1$ is not proportional to $d\bar{b}_2$ at $\rho'$. There is a positive function $m(x, \xi')$ defined near $\rho'$, homogeneous of degree 0 in $\xi'$ such that

$$\bar{c}_1(x, \xi') = m(x, \xi') \bar{b}_1(x, \xi'), \quad \bar{c}_2(x, \xi') = -m(x, \xi') \bar{b}_2(x, \xi').$$

Proof. Suppose that $d\bar{b}_2 = k d\bar{b}_1$ at $\rho'$ with some $k \in R$ and hence $d\bar{c}_2 = -kd\bar{c}_1$, by (2.2) at $\rho'$. Since (by (2.1))

$$d\bar{b}_1 \circ d\bar{c}_1 - d\bar{b}_2 \circ d\bar{c}_2 - d\bar{a} \circ d\bar{a} = (1+k^2) d\bar{b}_1 \circ d\bar{c}_1 - d\bar{a} \circ d\bar{a} \geq 0,$$

$d\bar{b}_1$ and $d\bar{c}_1$ must be proportional to $d\bar{a}$ at $\rho'$ if $d\bar{a} \neq 0$. Then it is clear that $Q$ vanishes on $\{X; d\bar{a}(X) = d\varphi(X) = 0\}$ which is a contradiction. If $d\bar{a} = 0$ (at $\rho'$)
then \(Q\) vanishes on \(\{X; d\varphi(X)=d\varepsilon_1(X)=0\}\) which also gives a contradiction. This proves the first assertion. The second assertion easily follows from the first one and (2.1), (2.2).

The following lemma proves Proposition 1.1.

**Lemma 2.3.** Let \(\beta=\beta_1+i\beta_2, \gamma=\gamma_1+i\gamma_2, \beta_1, \gamma_1 \text{ real.} \) Set \(\psi_i=\bar{b}_i+\beta_i\varphi\) \((i=1, 2)\), \(B=\gamma_2+m\beta_2, C=\gamma_1-m\beta_1. \) Then we have

\[
A = \varphi \begin{pmatrix}
1 & 0 \\
C+iB & -1
\end{pmatrix} + \psi_1 \begin{pmatrix}
-iB/2 & 1 \\
m & iB/2
\end{pmatrix} + \psi_2 \begin{pmatrix}
-iC/2 & i \\
-m & iC/2
\end{pmatrix}.
\]

Moreover \(d\varphi, d\psi_i\) are linearly independent at \(p'\) and the set \(\{(x, \xi'); A(x, \xi')=0\}\) is given by

\[
S = \{(x, \xi'); \varphi(x, \xi') = \psi_1(x, \xi') = \psi_2(x, \xi') = 0\}.
\]

Proof. Recall that

\[
A = \varphi \begin{pmatrix}
1+i\alpha & \beta \\
\gamma & -(1+i\alpha)
\end{pmatrix} + \begin{pmatrix}
i\bar{a} & \bar{b}_1+i\bar{b}_2 \\
m(\bar{b}_1-i\bar{b}_2) & -i\bar{a}
\end{pmatrix}.
\]

We observe the imaginary part of \(g:\)

\[
\Im g = 2\alpha \varphi^2 + 2\bar{a} \varphi + \Im(\beta \gamma) \varphi \gamma + \Im(\gamma + \beta m) \varphi \bar{b}_1 + \Re(\gamma - \beta m) \varphi \bar{b}_2.
\]

Since \(\Im g=0\) near \(p'\) and \(d\varphi=0\) at \(p'\) it follows that

\[
2\alpha \varphi + 2\bar{a} + \Im(\beta \gamma) \varphi + \Im(\gamma + \beta m) \bar{b}_1 + \Re(\gamma - \beta m) \bar{b}_2 = 0
\]

near \(p'\). Now we set

\[
D = \Im(\beta \gamma), B = \Im(\gamma + \beta m) = \gamma_2 + \beta_2 m, C = \Re(\gamma - \beta m) = \gamma_1 - \beta_1 m.
\]

Noticing \(D=\beta_1 B + \beta_2 C\) it follows from (2.3) that

\[
(\alpha \varphi + \bar{a}) = -\frac{1}{2} (\psi_1 B + \psi_2 C)
\]

which shows that \(a=(1+i\alpha) \varphi + i\bar{a}=\varphi - i(\psi_1 B + \psi_2 C)/2. \) On the other hand it is easy to see

\[
m(\bar{b}_1-i\bar{b}_2)+\gamma \varphi=(C+iB) \varphi + m(\psi_1-i\psi_2), \quad \bar{b}_1+i\bar{b}_2+\beta \varphi=\psi_1+i\psi_2
\]

since \(\gamma_1=C+m\beta_1 \) and \(\gamma_2=B-m\beta_2. \) These prove the first part. The rest of the assertion is obvious.

**Lemma 2.4.**

\[
4m-\left(B^2+C^2\right)>0 \text{ at } p'.
\]
Proof. Let us set \( \tilde{B} = B|_{\varphi = 0}, \tilde{C} = C|_{\varphi = 0} \). From (2.4) it follows that
\[
\tilde{a} = - (\tilde{B} \tilde{b}_1 + \tilde{C} \tilde{b}_2)/2.
\]
On the other hand (2.1) and Lemma 2.2 give that
\[
m(\tilde{b}_1^2 + \tilde{b}_2^2) - \tilde{a}^2 \geq 0 \text{ near } \rho'.
\]
Since the quadratic form
\[
m(d_b \circ d_{\tilde{b}_1} + d_b \circ d_{\tilde{b}_2}) - (\tilde{B} \tilde{d}_{\tilde{b}_1} + \tilde{C} \tilde{d}_{\tilde{b}_2}) \circ (\tilde{B} \tilde{d}_{\tilde{b}_1} + \tilde{C} \tilde{d}_{\tilde{b}_2})/4
\]
is the restriction of \( Q \) to \( \{X; d\varphi(X) = 0\} \), this must have rank 2 and then positive definite. This shows that \( 4m - (\tilde{B}^2 + \tilde{C}^2) > 0 \) at \( \rho' \) and hence the result.

To finish the proof of Proposition 1.2 we give a required symmetrizer for \( L_1 \):
\[
S(x, \xi') = \begin{bmatrix} 2m(x, \xi') & -C(x, \xi') + iB(x, \xi') \\ -(C(x, \xi') + iB(x, \xi')) & 2 \end{bmatrix}
\]
which satisfies \( S(x, \xi') = S^*(x, \xi') \) clearly. Using Lemma 2.3 it is easy to check that \( S(x, \xi') A(x, \xi') = A^*(x, \xi') S(x, \xi') \) and hence \( S(x, \xi') L_1(x, \xi) = L_1^*(x, \xi) S(x, \xi') \). The positivity of \( S \) follows from Lemma 2.4.

Added in proof: after submitted the paper we knew that L. Hörmander has obtained a strong stability property of double characteristics of maximal rank, including Proposition 1.1 in “Hyperbolic systems with double characteristics”, 1990, preprint.

References

