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## *Betweenness Geometry*

By Junji HASHIMOTO

### 1. Introduction

Shown by D. Hilbert [2], the postulates on betweenness play important roles in geometry, but the roles seem to be supplementary to the postulates on incidence. Making betweenness play the leading part in the present paper, we intend to construct a geometry upon a system of postulates concerning betweenness only, and investigate how many incidence relations can be deduced from those postulates.

Betweenness is defined in many systems. *Algebraic betweenness* is a ternary relation  $(axb)$  defined in a vector space such that

$$(axb) \Leftrightarrow x = \alpha a + (1-\alpha)b, \quad 0 \leq \alpha \leq 1^{\text{v}}.$$

*Order betweenness* is defined in a partially ordered set as follows:

$$(axb) \Leftrightarrow a \leq x \leq b \quad \text{or} \quad a \geq x \geq b.$$

These relations satisfy many common properties, of which the following five conditions have been proposed by G. Birkhoff [1] as the system of postulates for betweenness:

- (1)  $(axb) \rightarrow (bxa)$ .
- (2)  $(axb), (abx) \rightarrow x = b$ .
- (3)  $(axb), (ayx) \rightarrow (ayb)$ .
- (4)  $(axb), (xby), x \neq b \rightarrow (aby)$ .
- (5)  $(abc), (acd) \rightarrow (bcd)$ .

E. Pitcher and M. F. Smiley [4] have stated the following three conditions moreover.

- (6)  $(abc), (adc), (bxd) \rightarrow (axc)$ .
- (7)  $(abc), (abd), (cxd) \rightarrow (abx)$ .
- (8)  $(abc), (abd), (xbc) \rightarrow (xbd)$ .

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1) By  $\rightarrow$  we denote implication, such as  $A \rightarrow B$  (if  $A$  holds, then  $B$  holds),  $A, B \rightarrow C$  (if  $A$  and  $B$  hold, then  $C$  holds),  $A \rightarrow B$  or  $C$  (if  $A$  holds, then at least one of  $B$  and  $C$  holds).

It seems however that the condition (1)–(8) are not sufficient to characterize usual betweenness. For instance, in a system  $R = \{a, b, c, d, e\}$ , where

$$(xxx), (xxy) \text{ and } (xyy) \text{ for all } x, y \in R, \\ (abc), (abd), (abe), (ced), (cba), (dba), (eba), (dec),$$

the conditions (1)–(8) are satisfied but the condition B5 below does not hold.

Now we consider betweenness spaces or *B-spaces*, applying to all of affine spaces, vector spaces and partially ordered systems, in which (straight) *lines* are defined so that

- (1) a line be determined by any two points on it,
- (2) a line be a partially ordered set,

and  $(axb)$  means that

- (1)  $a, b$  and  $x$  lie on a line  $L$ ,
- (2)  $a \leq x \leq b$  or  $a \geq x \geq b$  under the ordering defined in  $L$ .

Then it is easy to see that in a *B-space* the following conditions are satisfied:

- B1.  $(axb) \rightarrow (aab)$ ,
- B2.  $(axb) \rightarrow (bxa)$ ,
- B3.  $(axb), (abx) \rightarrow x = b$ ,
- B4.  $(axb), (ayx) \rightarrow (yxb)$ ,
- B5.  $(axb), (axc), (byc), x \neq a \rightarrow (abc) \text{ or } (acb)$ ,
- B6.  $(axb), (ayb), (cxd), (cyd), x \neq y \rightarrow (axc) \text{ or } (axd)$ ,
- B7.  $(abx), (aby), (cxd), (cyd), a \neq b, x \neq y \rightarrow (axc) \text{ or } (axd)$ ,

and

- (8)  $(axb), (axc), (bxd), x \neq a, x \neq b \rightarrow (cxd)$ .

By a  $B_0$ -*space* we shall mean below a system  $R$  satisfying B1–B7 and (8).

## 2. Transitivity of betweenness deduced from B1–B6

In this section we shall show that the conditions B1–B6 imply the conditions (1)–(7) of E. Pitcher and M.F. Smiley and some other properties which are useful for our geometry below.

LEMMA 1.  $(axb) \rightarrow (abb)$ .

*Proof.*  $(axb) \rightarrow (bxa) \rightarrow (bba) \rightarrow (abb)$ , by B2 and B1.

LEMMA 2.  $(axb) \rightarrow (axx)$ .

*Proof.*  $(baa)$ ,  $(bxa) \rightarrow (xaa) \rightarrow (xxa) \rightarrow (axx)$ , by B4, B1 and B2.

LEMMA 3.  $(axa) \rightarrow x = a$ .

*Proof.*  $(axa) \rightarrow (axx) \rightarrow (aax)$ , by Lemma 2 and B1.  $(aax)$ ,  $(axa) \rightarrow x = a$ , by B3.

LEMMA 4.  $(axy)$ ,  $(xyb)$ ,  $x \neq y \rightarrow (axb)$ ,  $(ayb)$ .

*Proof.*  $(bxx)$ ,  $(byx)$ ,  $(axy)$ ,  $(ayy)$ ,  $x \neq y \rightarrow (bxa)$  or  $(bxy)$ , by B6. If  $(bxy)$ , then  $(bxy)$ ,  $(byx) \rightarrow x = y$ , which is impossible; hence  $(axb)$ . Similarly  $(ayb)$ .

LEMMA 5.  $(axb)$ ,  $(ayx) \rightarrow (ayb)$ .

*Proof.*  $(axb)$ ,  $(ayx) \rightarrow (yxb)$ , by B4. If  $x \neq y$ , then  $(ayx)$ ,  $(yxb)$ ,  $x \neq y \rightarrow (ayb)$ , by Lemma 4. If  $x = y$ ,  $(axb) \rightarrow (ayb)$ .

By  $(axyb)$  we shall mean that all of  $(axb)$ ,  $(ayb)$ ,  $(axy)$  and  $(xyb)$  are satisfied. Then B4, Lemma 4 and Lemma 5 imply

LEMMA 6.  $(axyb)$  holds in any case of

(1)  $(ayb)$ ,  $(axy)$ , (2)  $(axb)$ ,  $(xyb)$ , (3)  $(axy)$ ,  $(xyb)$ ,  $x \neq y$ .

LEMMA 7.  $(axb)$ ,  $(axc)$ ,  $(byc)$ ,  $x \neq a \rightarrow (axy)$ .

*Proof.*  $(axb)$ ,  $(axc)$ ,  $(byc)$ ,  $x \neq a \rightarrow (abc)$  or  $(acb)$ , by B5. If  $(abc)$ , then  $(abc)$ ,  $(byc) \rightarrow (aby)$ , and  $(aby)$ ,  $(axb) \rightarrow (axy)$ , by Lemma 6. It is similar for  $(acb)$ .

LEMMA 8.  $(axb)$ ,  $(ayb)$ ,  $(xzy) \rightarrow (axyb)$  or  $(ayxb)$ .

*Proof.* If  $x \neq y$ , then  $(yxx)$ ,  $(yyx)$ ,  $(axb)$ ,  $(ayb)$ ,  $x \neq y \rightarrow (yxa)$  or  $(yxb)$ , by B6.  $(axy)$ ,  $(ayb) \rightarrow (axyb)$  and  $(yxb)$ ,  $(ayb) \rightarrow (ayxb)$ . If  $x = y$ , it is evident from Lemma 2.

LEMMA 9.  $(axb)$ ,  $(ayb)$ ,  $(xzy) \rightarrow (azb)$ .

*Proof.* We have  $(axyb)$  or  $(ayxb)$  by Lemma 8. If  $(axyb)$ , then  $(axy)$ ,  $(xzy) \rightarrow (azy)$  and  $(azy)$ ,  $(ayb) \rightarrow (azb)$ . It is similar for  $(ayxb)$ .

LEMMA 10.  $(axyb)$ ,  $(xuyv) \rightarrow (auvb)$ .

*Proof.* It follows from Lemma 9 that  $(axyb)$ ,  $(xuy) \rightarrow (aub)$ . Again  $(xyb)$ ,  $(xvy) \rightarrow (xvb)$  and  $(xvb)$ ,  $(xuv) \rightarrow (uwb)$ . Then  $(aub)$ ,  $(uwb) \rightarrow (auvb)$ , by Lemma 6.

LEMMA 11.  $(xuvv), (auvb), (axb), (ayb), u \neq v \rightarrow (axyb)$ .

*Proof.*  $(axb), (ayb), (xuy) \rightarrow (axyb)$  or  $(ayxb)$ , by Lemma 8. If  $(ayxb)$ , then  $(ayxb), (yvux) \rightarrow (avub)$ , by Lemma 10, and  $(avu), (auv) \rightarrow u = v$ . Hence  $(axyb)$ .

LEMMA 12.  $(axyb), (axz), (zyb), x \neq a, y \neq b \rightarrow (xzy)$ .

*Proof.*  $(axb), (axz), (byz), x \neq a \rightarrow (abz)$  or  $(azb)$ , by B5. If  $(abz)$ , then  $(abz), (ayb) \rightarrow (ybz)$ , which is not compatible with  $(byz)$ , since  $y \neq b$ .  $(azb), (axz) \rightarrow (xzb)$  and  $(xzb), (zyb) \rightarrow (xzy)$ .

LEMMA 13. *If  $(axb), (ayb), (cxd), (cyd), x \neq y$ , then either (1)  $(axc), (ayc), (bxd), (byd)$ , or (2)  $(axd), (ayd), (bxc), (byc)$ .*

*Proof.* Case I:  $(xxy)$  holds. It follows from Lemma 8 that  $(axyb)$  or  $(ayxb)$ , and  $(cxyd)$  or  $(cyxd)$ . Combining those, we have four cases, any of which implies either (1) or (2). For instance,  $(axyb)$  and  $(cxyd)$  imply that  $(axy), (xyd), x \neq y \rightarrow (axyd)$  and  $(cxv), (xyb), x \neq y \rightarrow (cxyb)$ .

Case II:  $(xxy)$  does not hold.  $x = a$  implies  $(xyb)$  and  $(xxy)$ ; hence we may assume  $x \neq a, b, c, d$  and  $y \neq a, b, c, d$ . It follows from B6 that either  $(axc)$  or  $(axd)$ . Suppose  $(axc)$ . Again either  $(ayc)$  or  $(ayd)$  follows from B6. If  $(ayd)$ , then  $(cxa), (cxd), (acd), c \neq x \rightarrow (cxy) \rightarrow (xxy)$ , by Lemma 7 and 2; hence  $(ayc)$ . Further either  $(bxc)$  or  $(bxd)$  follows from B6. If  $(bxc)$ , then  $(cxa), (cxb), (ayb), c \neq x \rightarrow (cxy)$ ; hence  $(bxd)$ . Thus  $(axc)$  implies  $(ayc), (bxd)$  and  $(byd)$ .

Now the conditions (1)–(7) of E. Pitcher and M. F. Smiley correspond with B2, B3, Lemma 5, Lemma 4, B4, Lemma 9 and Lemma 7. It has been proved by K. Morinaga and N. Nishigori [3] that Lemma 12 is independent of the conditions (1)–(8) of E. Pitcher and M. F. Smiley.

### 3. Subspaces

It is natural that by a (linear) *subspace* of  $B_0$ -space  $R$  is meant a subset  $S$  of  $R$  satisfying

- (1)  $a, b \in S, (axb) \rightarrow x \in S,$
- (2)  $a, b \in S, (xay), (xby), a \neq b \rightarrow x \in S,$
- (3)  $a, b \in S, (xya), (xyb), a \neq b, x \neq y \rightarrow x \in S.$

Then it is easy to see that a subspace  $S$  satisfies moreover

- (4)  $a, b \in S, (xab), a \neq b \rightarrow x \in S,$
- (5)  $a, b \in S, (xya), (xyb), a \neq b, x \neq y \rightarrow y \in S.$

The intersection of subspaces is also a subspace. So we may define the subspace  $S(I')$  generated by a subset  $I'$  to mean the intersection of all subspaces containing  $I'$ . We shall call  $x_1, \dots, x_n$  linearly independent if and only if  $x^i \in S(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  for  $i = 1, \dots, n$ .

We have offered to construct a geometry upon the above concepts, but our attempt has not succeeded. In order to obtain some geometrical results our conditions on  $B_0$ -spaces may not yet suffice. So we feel that it needs to add some other conditions. For this purpose we take up the spaces in which every straight line is a doubly directed set. We can show in the following section that such spaces are characterized by the conditions B1-B7 and that

B8. for any points  $x, y \in R$  there exist  $a, b \in R$  such that  $(axb), (ayb)$ .

#### 4. $B_1$ -spaces

By a  $B_1$ -space we shall mean a system  $R$  satisfying that

B1.  $(axb) \rightarrow (aab)$ ,

B2.  $(axb) \rightarrow (bxa)$ ,

B3.  $(axb), (abx) \rightarrow x = b$ ,

B4.  $(axb), (ayx) \rightarrow (yxb)$ ,

B5.  $(axb), (axc), (byc), x \neq a \rightarrow (abc)$  or  $(acb)$ ,

B6.  $(axb), (ayb), (cxd), (cyd), x \neq y \rightarrow (axc)$  or  $(axd)$ ,

B7.  $(abx), (aby), (cxd), (cyd), a \neq b, x \neq y \rightarrow (axc)$  or  $(axd)$ ,

and B8. for any points  $x, y \in R$  there exist  $a, b \in R$  such that  $(axb), (ayb)$ .

In a  $B_1$ -space  $R$  the following lemmas hold.

LEMMA 14.  $(xxx)$  for all  $x \in R$ .

*Proof.* By B8 we can choose  $a, b \in R$  so that  $(axb)$ , whence  $(xxx)$ .

LEMMA 15. If  $(axb), (axc)$  and  $x \neq a$ , then there exists  $u \in R$  such that  $(abu), (acu)$ .

*Proof.* If  $(bbc)$  holds, then it follows from B5 that  $(abc)$  or  $(acb)$ ; hence we have either  $(abc), (acc)$  or  $(abb), (acb)$ . Suppose that  $(bbc)$  does not hold. Then  $b \neq c$ . By B8 we can choose  $u, v \in R$  so that  $(ubv), (ucv)$ . Then  $u \neq b$ , since  $u = b$  implies  $(bcv)$  and  $(bbc)$ . It follows from B7 that  $(axb), (axc), (ubv), (ucv), a \neq x, b \neq c \rightarrow (abu)$  or  $(abv)$ , and similarly  $(acu)$  or  $(acv)$ . If  $(abu)$  and  $(acv)$  hold, then  $(axbu), (axcv)$  and hence it follows from Lemma 12 that  $(axbu), (axv), (vbu), x \neq a, b \neq u \rightarrow (xvb)$ , and  $(xvb), (xcv) \rightarrow (cvb) \rightarrow (cbb)$ . So we have either  $(abu), (acu)$  or  $(abv), (acv)$ .

LEMMA 16. *If  $(axb)$ ,  $(ayb)$ ,  $(cxd)$ ,  $(cyd)$  and  $x \neq y$ , then there exist  $u, v \in R$  such that  $(uav)$ ,  $(ubv)$ ,  $(ucv)$  and  $(udv)$ .*

*Proof.* Referring to Lemma 13, we may assume that  $(axc)$ ,  $(ayc)$ ,  $(bxd)$  and  $(byd)$ . Again since  $x = a$  and  $y = a$  are not compatible, we assume  $x \neq a$ . Then by Lemma 15 we can choose a point  $u \in R$  so that  $(abu)$ ,  $(acu)$ . If  $c \neq x$ , then  $(acu)$ ,  $(axc) \rightarrow (xcu)$  and  $(ucx)$ ,  $(cxd)$ ,  $x \neq c \rightarrow (ucd)$ . It is similar for  $c \neq y$ . So we have  $(uba)$  and  $(ucd)$ . Exchanging  $a, b, c, d$  for  $b, a, d, c$  respectively in the above facts, we have  $(vab)$  and  $(vdc)$  for some  $v \in R$ . Then we conclude that  $(ubav)$  and  $(ucdv)$ , since  $a \neq b$  and  $c \neq d$ .

Now a subspace of a  $B_1$ -space  $R$  is a subset  $S$  of  $R$  satisfying that

$$a, b \in S, a \neq b, (xay), (xby), (xcy) \rightarrow c \in S.$$

In fact, (1) if  $a, b \in S$ ,  $(axb)$ , then  $(aab)$ ,  $(abb)$ ,  $(axb) \rightarrow x \in S$ ; (2) if  $a, b \in S$ ,  $(xay)$ ,  $(xby)$ ,  $a \neq b$ , then  $(xay)$ ,  $(xby)$ ,  $(xxy) \rightarrow x \in S$ ; and (3) if  $a, b \in S$ ,  $(xya)$ ,  $(xyb)$ ,  $a \neq b$ ,  $x \neq y$ , then it follows from Lemma 15 that  $(xau)$ ,  $(xbu)$  for some  $u \in R$  and hence  $x \in S$  follows from (2).

The subspace  $S(a)$  generated by a point  $a$  contains only  $a$ . By a (straight) line we shall mean a subspace  $S(a, b)$  generated by two distinct points  $a, b$ .

THEOREM 1. *Let  $a$  and  $b$  be two distinct points in a  $B_1$ -space  $R$ . Then a point  $x$  is on the line  $S(a, b)$  if and only if  $(uav)$ ,  $(ubv)$  and  $(uxv)$  for some  $u, v \in R$ .*

*Proof.* It is sufficient to prove that the set  $S = \{x; (uav), (ubv), (uxv) \text{ for some } u, v \in R\}$  forms a subspace. Suppose that  $x_1, x_2 \in S$ ,  $x_1 \neq x_2$ ,  $(sx_1t)$ ,  $(sx_2t)$  and  $(sxt)$ . Then we have  $(u_1av_1)$ ,  $(u_1bv_1)$ ,  $(u_1x_1v_1)$ ,  $(u_2av_2)$ ,  $(u_2bv_2)$ ,  $(u_2x_2v_2)$  for some  $u_1, v_1, u_2, v_2 \in R$ . It follows from Lemma 16 that there exist  $p, q \in R$  such that  $(pu_1q)$ ,  $(pv_1q)$ ,  $(pu_2q)$ ,  $(pv_2q)$  and hence  $(paq)$ ,  $(pbq)$ ,  $(px_1q)$ ,  $(px_2q)$ , by Lemma 9. Again using Lemma 16, we get  $(upv)$ ,  $(uqv)$ ,  $(usv)$ ,  $(utv)$  for some  $u, v \in R$ , since  $(px_1q)$ ,  $(px_2q)$ ,  $(sx_1t)$ ,  $(sx_2t)$  and  $x_1 \neq x_2$ . Then  $(uav)$ ,  $(ubv)$  and  $(uxv)$ ; hence  $x \in S$  and thus  $S$  is a subspace.

COROLLARY. *A subset  $S$  of a  $B_1$ -space  $R$  is a subspace if and only if  $a, b \in S$  implies  $S(a, b) \subseteq S$ .*

LEMMA 17. *If  $x_i \in S(a, b)$  for  $i = 1, \dots, n$ , then there exist  $u, v \in S(a, b)$  such that  $(ux_iv)$  for  $i = 1, \dots, n$ .*

*Proof.*  $x \in S(a, b)$  implies that  $(uav)$ ,  $(ubv)$  and  $(uxv)$  for some  $u, v \in R$ .

So we assume that  $(sat)$ ,  $(sbt)$  and  $(sx_i t)$  for  $i = 1, \dots, n-1$ . Since  $x_n \in S(a, b)$ ,  $(paq)$ ,  $(pbq)$  and  $(px_n q)$  for some  $p, q$ . Using Lemma 16, we can choose  $u, v$  so that  $(usv)$ ,  $(utv)$ ,  $(upv)$ ,  $(uqv)$ , whence  $(uav)$ ,  $(ubv)$  and  $(ux_i v)$  for  $i = 1, \dots, n$ . It is obvious that  $u, v \in S(a, b)$ . In the above proof we have assumed  $a \neq b$ , though if  $a = b$  this lemma is evident, since all  $x_i$  coincide with  $a$ .

Now suppose that  $p, q \in S(a, b)$  and  $p \neq q$ . Then  $S(p, q) \subseteq S(a, b)$ . If  $x \in S(a, b)$ , then it follows from Lemma 17 that there exist  $u, v \in R$  such that  $(upv)$ ,  $(uqv)$  and  $(uxv)$ ; hence  $x \in S(p, q)$ . Thus  $S(a, b) \subseteq S(p, q)$  and we have the first main theorem.

**THEOREM 2.** *If  $p, q \in S(a, b)$  and  $p \neq q$ , then  $S(a, b) = S(p, q)$ ; namely, in a  $B_1$ -space any two distinct points  $p, q$  are on one and only one line  $S(p, q)$ .*

Given  $p, q \in R$ , there exist  $a, b \in R$  with  $(apb)$ ,  $(aqb)$  and hence  $S(p, q) = S(a, b)$ . So a line is written in the form  $S(a, b)$  with  $(aab)$ . Next we show

**THEOREM 3.** *In a  $B_1$ -space a line  $L$  is a doubly directed set under an ordering, satisfying that for  $a, b, x \in L$*

$$(axb) \Leftrightarrow a \leq x \leq b \text{ or } a \geq x \geq b.$$

*Proof.* We may write  $L = S(a, b)$  with  $(aab)$  and  $a \neq b$ . Define  $x \leq y$  to mean that  $(uabv)$ ,  $(uxyv)$  for some  $u, v$ . Then for any  $x \in S(a, b)$  we can choose  $u, v$  so that  $(uabv)$ ,  $(uxxv)$ , by using Theorem 1 and Lemma 8; hence  $x \leq x$ . If  $x \leq y$  and  $y \leq z$ , then  $(pabq)$ ,  $(pxyq)$ ,  $(sabt)$ ,  $(syzt)$  for some  $p, q, s, t$ . We can choose by Lemma 16  $u, v$  so that  $(upv)$ ,  $(uqv)$ ,  $(usv)$  and  $(utv)$ . Then it follows that  $(upqv)$  or  $(uqpv)$ . We may assume  $(upqv)$ . Then we get  $(uabv)$  and  $(uxyv)$ , by Lemma 10, and then  $(ustv)$ , by Lemma 11, whence  $(uyzv)$ . We can deduce  $(uxy)$ ,  $(uyz) \rightarrow (uxz)$  and  $(uxz)$ ,  $(uzv) \rightarrow (uxzv)$ ; hence  $x \leq z$ . Again if  $x \leq y$  and  $y \leq x$ , then we get  $(uxyv)$  and  $(uyxv)$  in the same way as above, and  $(uxy)$ ,  $(uyx)$  imply  $x = y$ . Thus  $L$  is a partially ordered set. Now suppose  $x, y, z \in L$  and  $(xyz)$ . It follows from Lemma 17 that  $(uabv)$ ,  $(uxv)$ ,  $(uyv)$ ,  $(uzv)$  for some  $u, v \in L$  and we deduce  $(uxv)$ ,  $(uzv)$ ,  $(xyz) \rightarrow (uxzv)$  or  $(uzxv)$ .  $(uxzv)$  implies  $(uxyv)$  and  $(uyzv)$ ; hence  $x \leq y \leq z$ .  $(uzxv)$  implies similarly  $z \leq v \leq x$ . Conversely if  $x \leq y \leq z$ , then we obtain  $u, v$  so that  $(uxyv)$ ,  $(uyzv)$  and  $(uxzv)$  in the same way as the proof of the transitivity of  $\leq$ ; then it is easy to show  $(xyz)$ . It remains to show that  $L$  is doubly directed. Given  $x, y \in L$ , we can choose  $u, v \in L$  so that  $(uxv)$ ,  $(uyv)$ ,



which imply either  $u \leq x \leq v$ ,  $u \leq y \leq v$ , or  $u \geq x \geq v$ ,  $u \geq y \geq v$ , completing the proof.

By the way, if one rearranges the above proof, it is easy to obtain the characterization of doubly directed sets by betweenness<sup>2)</sup>; namely

**COROLLARY.** *Let  $D$  be any system in which a ternary relation  $(axb)$  is defined so that*

- (1)  $(axb) \rightarrow (aab)$ ,
- (2)  $(axb) \rightarrow (bxa)$ ,
- (3)  $(axb), (abx) \rightarrow x = b$ ,
- (4)  $(axb), (ayx) \rightarrow (yxb)$ ,
- (5)  $(axb), (ayx) \rightarrow (ayb)$ ,
- (6)  $(axb), (ayb), (xzy) \rightarrow (axy)$  or  $(bxy)$ ,
- (7) for any  $x, y, z \in D$  there exist  $a, b \in D$  such that  $(axb), (ayb), (azb)$ .

*Then  $D$  is a doubly directed set under an ordering, satisfying that*

$$(axb) \Leftrightarrow a \leq x \leq b \text{ or } a \geq x \geq b.$$

Now it is natural that by a *plane* is meant a subspace  $S(a, b, c)$ , where  $a, b$  and  $c$  are linearly independent. Then Cor. of Theorem 1 and Theorem 2 imply

**THEOREM 4.** *If two distinct points  $p, q$  lie on a plane  $P$ , then every point on the line passing through  $p, q$  lies on  $P$ .*

In order to consider the plane geometry, however, it needs to admit a condition corresponding to Pasch's axiom<sup>3)</sup>. We shall deal with this for  $B_2$ -spaces which are defined below under stronger conditions than  $B_1$ -spaces.

## 5. $B_2$ -spaces

By a  $B_2$ -space we shall mean a system  $R$  satisfying that

- B1\*.  $(aab)$  for all  $a, b \in R$ ,
- B2.  $(axb) \rightarrow (bxa)$ ,
- B3.  $(axb), (abx) \rightarrow x = b$ ,
- B4.  $(axb), (ayx) \rightarrow (yxb)$ ,
- B5.  $(axb), (axc), (byc), x \neq a \rightarrow (abc)$  or  $(acb)$ ,
- B6.  $(axb), (ayb), (cxd), (cyd), x \neq y \rightarrow (axc)$  or  $(axd)$ .

2) A characterization of partially ordered sets by betweenness is obtained by K. Morinaga and N. Nishigori [3].

3) Axiom der Anordnung II 4. of D. Hilbert [2].

THEOREM 5. *A  $B_2$ -space  $R$  is a  $B_1$ -space.*

*Proof.* It suffices to prove B7 and B8. For any points  $x, y \in R$  we have  $(xxy)$  and  $(xyy)$ ; hence B8 holds. Suppose that  $(abx)$ ,  $(aby)$ ,  $(cxd)$ ,  $(cyd)$ ,  $a \neq b$  and  $x \neq y$ .  $(abx)$ ,  $(aby)$ ,  $(xxy)$ ,  $a \neq b \rightarrow (axy)$  or  $(ayx)$ , by B5; and  $(cxd)$ ,  $(cyd)$ ,  $(xxy) \rightarrow (cxyd)$  or  $(cyxd)$ , by Lemma 8, which is deduced from B1-B6. Then there occur four cases, any of which implies either  $(axc)$  or  $(axd)$ . For instance  $(axy)$  and  $(cxyd)$  with  $x \neq y$  imply  $(axd)$ , by Lemma 4.

THEOREM 6. *In a  $B_2$ -space a line  $L$  is a linearly ordered set under the ordering introduced into  $L$  as a line in a  $B_1$ -space.*

*Proof.* Suppose  $a, b \in L$ . Then  $(aab)$  means that  $a \leq a \leq b$  or  $a \geq a \geq b$  under the ordering mentioned above; hence  $L$  is linearly ordered.

COROLLARY. *If  $x, y$  and  $z$  lie on a line in a  $B_2$ -space, then one of the relations  $(xyz)$ ,  $(yzx)$ ,  $(zxy)$  holds.*

In order to deal with planes in  $B_2$ -spaces we shall assume the following postulate (Pasch's axiom).

(P) *If  $x$  and  $y$  are two distinct points in  $S(a, b, c)$  and  $S(x, y)$  contains a point  $p$  satisfying  $(bpc)$ , then  $S(x, y)$  contains a point  $q$  satisfying either  $(aqb)$  or  $(aqc)$ .*

In a  $B_2$ -space  $R$  satisfying (P) the following lemmas hold.

LEMMA 18. *If  $(abx)$ ,  $a \neq b \neq x$  and  $(bpc)$ , then  $S(x, p)$  contains a point  $q$  such that  $(aqc)$ .*

*Proof.* If  $x = p$ , then  $(abx)$ ,  $(bxc)$ ,  $b \neq x \rightarrow (abxc)$ ; hence  $S(x, p) \ni x$  with  $(axc)$ . If  $x \neq p$ , then it follows from (P) that  $x, p \in S(a, b, c)$ ,  $S(x, p) \ni p$  and  $(bpc)$  imply  $S(x, p) \ni q$  with either  $(aqb)$  or  $(aqc)$ .  $(aqb)$  implies  $(aqbx)$  and  $x \neq q$ ; hence  $S(x, p) = S(x, q) \ni a$  with  $(aac)$ . In any case  $q \in S(x, p)$  exists with  $(aqc)$ .

In the same way as above we can show

LEMMA 19. *If  $(xab)$ ,  $x \neq a \neq b$  and  $(bpc)$ , then  $S(x, p)$  contains a point  $q$  such that  $(aqc)$ .*

LEMMA 20. *Let  $a, b, c, d, e$  and  $f$  be distinct points satisfying  $(bdc)$ ,  $(cea)$  and  $(afb)$ . If  $d, e$  and  $f$  are on a line, then  $a, b$  and  $c$  are also on that line.*

*Proof.* We may assume without loss of generality (*def*). Then using Lemma 18, we obtain  $q \in S(c, e)$  with (*bqf*), since (*bdc*),  $b \neq d \neq c$  and (*def*). It follows from (*cea*) and (*afqb*) that  $S(c, e)$  contains  $q, a, f, b$ .

LEMMA 21. *Let  $a, b$  and  $c$  be linearly independent. If  $x \in S(a, b, c)$  and  $x \bar{\in} S(b, c)$ , then  $S(a, b, c) = S(x, b, c)$ ,*

*Proof.* It is sufficient to show  $a \in S(x, b, c)$ . If  $x \in S(a, b)$ , then it follows from Theorem 2 that  $a \in S(x, b) \subset S(x, b, c)$ ; hence we may assume that  $x \bar{\in} S(a, b)$  and  $x \bar{\in} S(a, c)$ . If any of the lines  $S(b, c)$ ,  $S(c, a)$  and  $S(a, b)$  contains no point other than  $a, b, c$  then three points  $a, b, c$  form a subspace and we have nothing but  $x = a$ . So we may assume that some one of them contains a point  $d$  such that  $d \neq a, b, c$ . We first consider the case that  $S(b, c)$  contains  $d$ .

Case I: (*bdc*). Since  $S(x, d) \ni d$ , (*bdc*) and  $x \neq d$ , we obtain  $q \in S(x, d)$  with either (*aqb*) or (*aqc*). Without loss of generality assume (*aqb*). If  $q = b$ , then  $b \in S(x, d)$  with  $b \neq d$ , whence  $x \in S(b, d) = S(b, c)$ . So  $q \neq b$ . Then  $d \in S(b, c)$ ,  $q \in S(x, d)$  and  $a \in S(b, q)$ ; hence  $a \in S(x, b, c)$ .

Case II: (*bcd*). Since  $c \in S(a, b, d)$ ,  $S(a, b, d) = S(a, b, c) \ni x$  and  $x \bar{\in} S(b, d)$ . Then, exchanging  $c$  for  $d$  in Case I, we get  $S(a, b, d) = S(x, b, d)$  and accordingly  $S(a, b, c) = S(x, b, c)$ .

Next we deal with the case that  $S(a, b)$  contains  $d$ .

Case III: (*adb*). It follows from (P) that  $S(x, d) \ni e$  with either (*cea*) or (*ceb*). If (*ceb*), then we get  $e \in S(b, c)$ ,  $d \in S(x, e)$ ,  $a \in S(b, d)$  and  $a \in S(x, b, c)$ . If  $e = a$ , then  $x \in S(d, e) = S(a, b)$ , contradicting the assumption. So we may assume that (*aec*) with  $e \neq a, b, c$  and look over the three cases (*dex*), (*edx*), (*dxe*). (*bda*),  $b \neq d \neq a$  and (*dex*) imply  $S(a, e) \ni q$  with (*bqx*), by Lemma 18, and hence  $q \neq c$ ,  $q \in S(b, x)$ ,  $a \in (a, e) = S(c, q)$ , so  $a \in S(x, b, c)$ . Similarly (*cea*),  $c \neq e \neq a$  and (*edx*) imply  $a \in S(x, b, c)$ . Further (*adb*),  $a \neq d \neq b$  and (*dxe*) imply  $S(b, x) \ni q$  with (*aqe*), whence also  $q \in S(b, x)$ ,  $a \in S(a, e) = S(c, q)$  and  $a \in S(x, b, c)$ .

Case IV: (*dab*). If  $x \in S(c, d)$ , then it is easy to see  $S(x, b, c) = S(d, b, c) \ni a$ . So we may assume  $x \bar{\in} S(c, d)$ . Then exchanging  $a$  for  $d$  in Case III, we obtain  $S(d, b, c) = S(x, b, c)$ , since  $x \in S(a, b, c) = S(d, b, c)$  and  $x \bar{\in} S(b, c)$ ,  $S(c, d)$ ,  $S(d, b)$ . Consequently  $S(a, b, c) = S(x, b, c)$ .

Case V: (*abd*). Since  $x \in S(a, b, c) = S(a, d, c)$ ,  $S(b, x) \ni b$  and (*abd*), we obtain  $q \in S(b, x)$  with either (*cqa*) or (*cqd*).  $c = q$  implies  $x \in S(b, c)$ ; hence  $c \neq q$ . If (*cqa*), then we infer  $q \in S(x, b)$ ,  $a \in S(c, q)$  and  $a \in S(x, b, c)$ . If (*cqd*), then we infer similarly  $d \in S(x, b, c)$ , whence  $S(x, b, c) = S(d, b, c) = S(a, b, c)$ .

Also for the case  $d \in S(a, c)$  we can show in the same way  $S(a, b, c) = S(x, b, c)$ , completing the proof.

Using these lemmas, we can deduce the fundamental properties of planes.

**THEOREM 7.** *If  $x, y$  and  $z$  are linearly independent points on  $S(a, b, c)$ , then  $S(a, b, c) = S(x, y, z)$ ; namely, in a  $B_2$ -space satisfying (P) any three points not lying on a line are on one and only one plane.*

*Proof.* Some of  $x, y, z$  say,  $x$  is not on  $S(b, c)$ , so it follows from the above lemma that  $S(a, b, c) = S(x, b, c)$ . Either  $y$  or  $z$  is not on  $S(x, c)$ ; hence  $S(x, b, c) = S(x, y, c)$ . Since  $z \in S(x, y)$ ,  $S(x, y, c) = S(x, y, z)$ .

Now all points  $x$  of a plane  $S(a, b, c)$  not lying on the line  $S(b, c)$  are divided into two parts  $P_1, P_2$  as follows: if  $S(b, c)$  contains  $p$  such that  $(apx)$ , then  $x \in P_1$ ; if  $S(b, c)$  does not contain such a point  $p$ , then  $x \in P_2$ . Let  $x, y$  be two points of  $S(a, b, c)$  and  $a, x, y$  be not on a line. If  $x, y \in P_1$ , then  $(xry)$  and  $r \in S(b, c)$  contradict Lemma 20, and hence  $S(b, c)$  cannot contain such a point  $r$  as  $(xry)$ . If  $x, y \in P_2$ , it follows from (P) that  $S(b, c)$  cannot contain  $r$  such that  $(xry)$ , since  $b, c \in S(a, b, c) = S(a, x, y)$ . However if  $x \in P_1$  and  $y \in P_2$ , then (P) implies that  $S(b, c)$  contains  $r$  such that  $(xry)$ . For the case that  $a, x, y$  are on a line it is easy to derive the same results as above. Hence, defining a segment  $ab$  to mean the set of points  $x$  such that  $(axb)$ , we infer

**THEOREM 8.** *A line  $L$ , which lies in a plane  $P$ , divide points of  $P$  not lying on  $L$  into two sides so that: any point  $a$  on one side determine with any point  $b$  on the other side a segment  $ab$ , in which a point of  $L$  lies; any two points  $a, b$  on the same side determine a segment  $ab$ , which contains no point of  $L$ .*

One side of a line, however, may contain no point. The author asks whether Theorems 7 and 8 can be extended for subspaces of higher dimension, though it may be possible if we admit the fact that any two planes through a point  $p$  contain another point  $q$  in common.

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