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ON QF-3 AND 1-GORENSTEIN RINGS

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Let R be a ring with identity, E the injective hull of R_R and Q the maximal right quotient ring of R . Consider the following conditions.

A_1 : E is projective.

A_2 : R has a minimal faithful right R -module.

A_3 : R has a faithful injective right ideal.

A_4 : E is torsionless.

A_5 : Any finitely generated submodule of E is torsionless.

A_6 : Any finitely generated submodule of Q is torsionless.

For arbitrary ring R , $A_2 \Rightarrow A_3 \Rightarrow A_4 \Rightarrow A_5 \Rightarrow A_6$ hold (see [14, p, 47]). On the other hand if R is a left perfect ring, A_2 , A_3 and A_4 are equivalent (see [4, Theorem 3.2], [1, Theorem 2] and [13, Proposition 3.1]). Any commutative integral domain which is not a field satisfies A_5 , but does not satisfy A_4 . However, Rutter [8, Corollary 3] proved that for any right or left artinian ring R A_5 implies A_2 . In § 2 we shall define a new ring, called a right DWA ring, which is a generalization of a right semi-artinian ring, and we shall show that for any right DWA and right perfect ring R A_5 implies A_3 . As a consequence, for any perfect ring R A_5 implies A_2 . We owe the method of the proof essentially to Rutter [8, Lemma 2] and the proofs of Colby and Rutter [1, Theorem 2] and Jans [4, Theorem 3.2]. We call a ring R right QF-3 if R satisfies A_5 and R right QF-3' if R satisfies A_6 .

By Wu, Mochizuki and Jans [16] (or Colby and Rutter [1, Theorem 1] and Kato [6, Remark, p. 236]), a characterization of right artinian rings satisfying A_1 (or rings satisfying A_4) was given. In § 3 we shall give an analogous result for right QF-3 rings. In § 4 we shall show that any right QF-3' and right 1-Gorenstein ring is QF-3 (see § 4 for the definition of a 1-Gorenstein ring). Moreover using this result, we shall show that any noetherian right QF-3' ring such that $E \oplus E/R$ is an injective cogenerator is right and left 1-Gorenstein.

1. Definitions and notations

Throughout this note we assume that R is a ring with identity and all R -modules are unitary. We denote by E the injective hull of R_R and by Q the

maximal right quotient ring of R . For an R -module M we denote by M^* the dual module $\text{Hom}_R(M, R)$. We assume that "torsion theory" means the Lambek torsion theory. We call R *right QF-3* if any finitely generated submodule of E is torsionless. (Note that we call such a ring right QF-3' in [11] and [12]). On the other hand we call R *right QF-3'* if any finitely generated submodule of Q is torsionless.

Let M be a right R -module. For a submodule N of M , we say that N is saturated in M if M/N is torsion-free. (In [12] we call such a submodule closed). See [10] for saturated submodules. We say that M is *D-noetherian* (resp. *D-artinian*) if M satisfies the ascending (resp. descending) chain condition on saturated submodules of M . (The letter D comes from the topology consisting of dense right ideals of R .) A ring R is called *right D-noetherian* (resp. *right D-artinian*) if a right R -module R_R is D-noetherian (resp. D-artinian). There exists a maximal chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_n$$

of saturated submodules M_i of M if and only if M is D-noetherian and D-artinian. If M has a maximal chain as above, we say that M is *finite Lambek-dimensional* (or simply *finite dimensional*) and denote it by $\dim M = n$. Otherwise we define it by $\dim M = \infty$. We call a module M *strongly uniform* (simply *s-uniform*) if M is torsion-free and $\dim M = 1$. We say that M is *D-weakly artinian* (or simply *DWA*) if any submodule N of M with $\dim N \geq 1$ has a submodule L with $\dim L = 1$ and R is *right DWA* if R_R is DWA. Note that any right semi-artinian (in particular left perfect) ring and any right D-artinian ring is right DWA

2. Perfect QF-3 rings

We can obtain the following theorem and corollaries (in particular Corollary 1) by Rutter [8, Lemma 2] and a slight modification of the proofs of Colby and Rutter [1, Theorem 1.2] and Jans [4, Theorem 3.2].

Theorem 1. *Let R be a right DWA and right perfect ring. Then the following conditions are equivalent:*

- (1) R is right QF-3.
- (2) R has a faithful injective right ideal.

Proof. It always holds that (2) implies (1). Assume (1). Let I be an s-uniform right ideal of R . $E(I)$ is clearly imbedded in E , where $E(I)$ is the injective hull of I . If M is a non-zero finitely generated submodule of $E(I)$, M is torsionless, i.e. $\bigcap_{f \in M^*} \text{Ker } f = 0$. Hence there exists a map $f \in M^*$ such that $I \cap M \not\subseteq \text{Ker } f$, since $I \cap M \neq 0$. Suppose $\text{Ker } f \neq 0$. Then $I \cap \text{Ker } f$ is a non-

zero proper submodule of $I \cap M$. But $I \cap M / I \cap \text{Ker } f$ is imbedded in a torsion-free R -module $M / \text{Ker } f$. Since $\dim I \cap M = 1$, we have $I \cap M / I \cap \text{Ker } f = 0$ or equivalently $I \cap M \subset \text{Ker } f$. This is a contradiction. Thus we have $\text{Ker } f = 0$, so M is imbedded in R (see Proposition 1). Therefore by [8, Lemma 2] $E(I)$ is flat and so is projective. Since $E(I)$ is clearly indecomposable, $E(I)$ is isomorphic to a right ideal eR of R , where e is a primitive idempotent of R . Let e_1R, \dots, e_iR be a complete set of pairwise non-isomorphic injective indecomposable right ideals of R and put $P = e_1R + \dots + e_iR$, where the sum is clearly direct. We show P is faithful. Let $0 \neq a \in R$. Then there exists an s-uniform right ideal I contained in aR . Since we have a monomorphism $I \rightarrow e_iR$ for some i and e_iR is injective, there exists $x \in e_iR$ such that $xI (\cong I) \neq 0$. Thus we have $xa \neq 0$ and this shows P is faithful.

Corollary 1. *Let R be a perfect ring. Then the following conditions are equivalent:*

- (1) R is right QF-3.
- (2) R has a minimal faithful right R -module.

Corollary 2. *Let R be a semi-perfect right DWA ring. Then the following conditions are equivalent:*

- (1) E is torsionless, where E is the injective hull of R_R .
- (2) R has a faithful injective right ideal.

We call a module M *essentially finite dimensional* if there exists a finite dimensional submodule N of M such that N is essential in M . Then the property that the torsionless module M in the proof of Theorem 1 is imbedded in R is extended as follows:

Proposition 1. *Let M be an essentially finite dimensional right R -module. If M is torsionless, then M can be imbedded in a finitely generated free right R -module.*

Proof. Let N be a finite dimensional and essential submodule of M . Since M is torsionless, we have a monomorphism $f: M \rightarrow \prod_{\alpha \in A} R_\alpha$ with R_α copies of R_R . Then by [12, Lemma 1'] there exists a finite subset B of A such that $pf g: N \rightarrow \prod_{\beta \in B} R_\beta$ is monomorphic, where $p: \prod_{\alpha \in A} R_\alpha \rightarrow \prod_{\beta \in B} R_\beta$ and $g: N \rightarrow M$ are the canonical projection and the canonical injection, respectively. This shows $N \cap \text{Ker } pf = 0$. Since N is essential in M , $\text{Ker } pf = 0$, and so M is imbedded in a finitely generated free right R -module.

The following holds on essentially finite dimensional R -modules.

Proposition 2. *Let R be a ring. Then we have*

(1) *Any torsion-free and essentially finite dimensional right R -module is DWA and finite Goldie-dimensional.*

(2) *Any DWA and finite Goldie-dimensional right R -module is essentially finite dimensional.*

(3) *Any D -artinian right R -module is essentially finite dimensional.*

Proof. (1) and (2) are immediate consequences.

(3) Let M be a D -artinian right R -module and $t(M)$ the torsion submodule of M . Then $M/t(M)$ is DWA and finite Goldie-dimensional by [12, Lemma 2]. Hence by (2) there exists an essential and finite dimensional submodule $N/t(M)$ of $M/t(M)$. Then N is clearly an essential and finite dimensional submodule of M .

3. QF-3 rings

Wu, Mochizuki and Jans [16] gave a characterization of right artinian rings satisfying the condition A_1 (in the introduction). Moreover this characterization is also that of rings satisfying A_4 (see Colby and Rutter [1, Theorem 1] and Kato [6, Remark, p. 236]). We shall show that an analogous characterization holds for right QF-3 rings. A right R -module M is called *finitely imbedded* (simply FI) if M is imbedded in some finitely generated right R -module. Note that any submodule and any factor module of FI modules is also FI, and any FI torsion-free right R -module is isomorphic to a submodule of some finitely generated torsion-free right R -module.

Theorem 2. *Let R be a ring. Then the following conditions (1) and (2) are equivalent:*

(1) *R is right QF-3.*

(2) (a) *If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence such that M is FI (or finitely generated), and L and N are torsionless, then M is torsionless.*

(b) *For every FI right R -module M , $M^* = 0$ if and only if M is a torsion module.*

Proof. (1) \Rightarrow (2). (a) is clear since any torsionless module is torsion-free. (b) follows from the definition of FI-modules.

(2) \Rightarrow (1). The proof is due to Wu, Mochizuki and Jans [16]. Let M be an FI torsion-free right R -module and K the kernel of a canonical map $M \rightarrow M^{**}$. As is well-known, K is the smallest submodule of M such that M/K is torsionless. Moreover let K' be the kernel of a canonical map $K \rightarrow K^{**}$. Then we have an exact sequence $0 \rightarrow K/K' \rightarrow M/K' \rightarrow M/K \rightarrow 0$. Since M/K and K/K' are torsionless, so is M/K' by (a). Hence $K=K'$, so $K^*=0$ and K is a torsion module by (b). Consequently $K=0$ since K is torsion-free. Thus M is torsionless.

Let \mathfrak{M} be the category of right R -modules and let \mathfrak{M}_f be the full subcategory of \mathfrak{M} whose objects are FI R -modules. An R -module M is torsionless, i.e., M can be imbedded in a products of copies of R if and only if for any non-zero submodule N of M a canonical map $M^* \rightarrow N^*$ is non-zero. Now we adopt the latter condition as the definition of torsionless R -modules in \mathfrak{M}_f . We denote by \mathfrak{L} (resp. \mathfrak{L}_f) the class of torsionless modules in \mathfrak{M} (resp. \mathfrak{M}_f), and denote by \mathfrak{X} (resp. \mathfrak{X}_f) the class of modules M in \mathfrak{M} (resp. \mathfrak{M}_f) such that $M^* (= \text{Hom}_R(M, R)) = 0$. Then it is clear that R is right QF-3 if and only if \mathfrak{L}_f is closed under taking essential extensions in \mathfrak{M}_f . On the other hand since M is a torsion module if and only if $N^* = 0$ for any submodule N of M , (2)-(b) in Theorem 2 is equivalent to the condition that \mathfrak{X}_f is closed under taking submodules (see [15, Proposition 1]). Thus Theorem 2 shows that we can replace \mathfrak{L} , \mathfrak{X} and \mathfrak{M} with \mathfrak{L}_f , \mathfrak{X}_f and \mathfrak{M}_f , respectively, in [1, Theorem 1] (if we consider right R -modules instead of left R -modules in it). That is,

Corollary 3. *Let R be a ring. Then the following conditions (1) and (2) are equivalent:*

- (1) \mathfrak{L}_f is closed under taking essential extensions in \mathfrak{M}_f .
- (2) (a) \mathfrak{L}_f is closed under taking extensions in \mathfrak{M}_f by elements of \mathfrak{L}_f .
 (b) \mathfrak{X}_f is closed under taking submodules.

4. QF-3 and 1-Gorenstein rings

We shall give a sufficient condition for a right QF-3' ring to be right QF-3. We call a ring R a *right n -Gorenstein ring* if R is (right and left) noetherian and R has the right self-injective dimension $\leq n$.

Lemma 1. *Let M be a D -noetherian R -module which is not a torsion module. Then there exists a submodule N of M such that M/N is s -uniform.*

Proof. If N is a maximal element in the set consisting of saturated submodules of M , N clearly satisfies the assertion.

Lemma 2. *Let M be an s -uniform right R -module. Then M can be imbedded into Q , where Q is the maximal right quotient ring of R .*

Proof. Since M is torsion-free, we have a non-zero map $M \rightarrow E$, where E denotes the injective hull of R_R . Then by [11, Lemma 1] this map is monomorphic. Thus we may assume that M is contained in E . Consider an exact sequence $0 \rightarrow M \cap Q \rightarrow M \rightarrow M/M \cap Q \rightarrow 0$. Since $M \cap Q \neq 0$, $M/M \cap Q$ is a torsion module. On the other hand, since $M/M \cap Q \cong M+Q/Q \subset E/Q$, $M/M \cap Q$ is torsion-free, so $M/M \cap Q = 0$. Therefore we have $M \subset Q$.

REMARK. The proof of the above lemma shows that any s -uniform submodule of E is contained in Q if we regard Q as a submodule of E .

Proposition 3. *Let R be a right D-noetherian and right QF-3' ring. Then for every finitely generated right R -module M , $M^*=0$ if and only if M is a torsion module.*

Proof. Let M be a finitely generated right R -module. If M is a torsion module, $M^*=0$ is clear. Assume M is not a torsion module. Since M is D-noetherian by [12, Lemma 2], there exists a submodule N of M such that M/N is s-uniform by Lemma 1. M/N can be imbedded in Q by Lemma 2 and so M/N is torsionless. Now we have an exact sequence $0 \rightarrow (M/N)^* \rightarrow M^*$. Thus we have $M^* \neq 0$ since $(M/N)^* \neq 0$.

Lemma 3. *Let R be a ring with right self-injective dimension ≤ 1 . If $0 \rightarrow L_R \rightarrow M_R \rightarrow N_R \rightarrow 0$ is an exact sequence such that L is torsionless and N is a submodule of a finitely generated free right R -module, then M is torsionless.*

Proof. Since R has the right self-injective dimension ≤ 1 , we have an exact sequence $0 \rightarrow N^* \rightarrow M^* \rightarrow L^* \rightarrow 0$. Hence we have a following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L^{**} & \longrightarrow & M^{**} & \longrightarrow & N^{**} \end{array}$$

It follows that M is torsionless.

From Theorem 2, Proposition 3, Lemma 3 and [9, Theorem 1.1], the following theorem is immediate.

Theorem 3. *Let R be a right QF-3' and right 1-Gorenstein ring. Then R is a QF-3 ring with finite dimension on both sides.*

The following theorem is a slight extension of Theorem 3.

Theorem 4. *Let R be a right noetherian right QF-3' ring satisfying the descending chain condition on annihilator right ideals and assume R has the right self-injective dimension ≤ 1 . Then R is right finite dimensional and right QF-3.*

Proof. Let M be a finitely generated torsion-free right R -module with finite dimension. Put $\dim M = n$. First we show that such a module M is torsionless by induction on n . When $n=1$, the result holds by Lemma 2. Assume $n > 1$. Then there exists a submodule N of M such that M/N is s-uniform. By inductual assumption N and M/N are torsionless. In particular M/N can be imbedded in R_R since M/N is s-uniform and $(M/N)^* \neq 0$. Hence by Lemma 3 M is torsionless.

Next we show that any finitely generated right R -module is finite dimen-

sional. By using Lemma 1 inductively we have a following descending chain of right ideals I_i of R such that I_{i-1}/I_i is an s -uniform right R -module:

$$R = I_0 \supset I_1 \supset I_2 \supset \dots .$$

Then R/I_i is a torsion-free module with dimension i for each i . Therefore R/I_i is torsionless and I_i is an annihilator right ideal. Hence the above chain terminates. Thus a right R -module R_R and consequently any finitely generated right R -module is finite dimensional.

Combining the above facts, we conclude that any finitely generated torsion-free right R -module is torsionless.

EXAMPLE. We give a noetherian QF-3' ring which is not right (and left) finite dimensional. Let S be a commutative noetherian ring and put $R = S[X, Y]/(XY, Y^2)$, where $S[X, Y]$ is a polynomial ring with variables X and Y . Then R is a commutative (hence QF-3') noetherian ring. Let x and y be the image of X and Y by the canonical map $S[X, Y] \rightarrow R$, respectively. Put $I_0 = R, I_1 = xR + yR, I_i = x^{i-1}R (i \geq 2)$ and consider a following chain of ideals I_i of R :

$$R = I_0 \supset I_1 \supset I_2 \supset \dots .$$

Since each factor I_i/I_{i+1} is isomorphic to an ideal yR, R is not finite dimensional. Hence in particular R is not QF-3 by [11. Proposition 1].

Recently Iwanaga [3, Theorem 1] showed that if R is a (right and left) 1-Gorenstein ring, $E \oplus E/R$ is an injective cogenerator. Here we show that the converse holds for noetherian right QF-3' ring.

Lemma 4. *Let R be a right 1-Gorenstein ring. If $0 \rightarrow M_R \rightarrow F_R \rightarrow L_R \rightarrow 0$ is an exact sequence where F is a finitely generated free module and L is torsionless, then M is reflexive.*

Proof. This is immediate from a following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & L \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M^{**} & \longrightarrow & F^{**} & \longrightarrow & L^{**} . \end{array}$$

Theorem 5. *Let R be a noetherian right QF-3' ring and E the injective hull of R_R . If $E \oplus E/R$ is an injective cogenerator, then R is a 1-Gorenstein ring.*

Proof. Let Q be the maximal right quotient ring. If S is a simple torsion right R -module, S is imbedded in Q/R since Q/R is the torsion submodule of a cogenerator $E \oplus E/R$. On the other hand we have an isomorphism $\text{Hom}_R(S, Q/R) \rightarrow \text{Ext}_R^1(S, R)$. Therefore $\text{Ext}_R^1(S, R) \neq 0$.

Now we show that every finitely generated torsionless right R -module is reflexive, which implies R is a left 1-Gorenstein ring by [5, Corollary 1.3]. Suppose that there exists a finitely generated torsionless right R -module M which is not reflexive. Then M_R can be imbedded in a finitely generated free module F_R . Since F is noetherian, there is a maximal element in the set consisting of submodules N of F such that N is not reflexive. We may assume that M is such a maximal element. By Theorem 3 R is OF-3. Hence, if F/M is torsion-free, F/M is torsionless and so M is reflexive by Lemma 4. This is a contradiction. Therefore the torsion submodule L/M of F/M is non-zero. If $L \supset L_1 \supset L_2 \supset \dots \supsetneq M$ is a descending chain of submodules L_i of L containing M , we have an ascending chain $L^* \subset L_1^* \subset L_2^* \subset \dots \subset M^*$. Then there is an integer n such that $L_i^* = L_{i+1}^*$ for each $i \geq n$ since M^* is noetherian. But by the maximality of M , L_i is reflexive and hence we have $L_i = L_{i+1}$ for each $i \geq n$. This shows L/M is artinian. Let P/M is a simple submodule of L/M and consider an exact sequence $0 \rightarrow M \rightarrow P \rightarrow P/M \rightarrow 0$. Then we have an exact sequence $0 \rightarrow P^* \rightarrow M^* \rightarrow \text{Ext}_R^1(P/M, R) \rightarrow 0$. Since M^* and P^* are reflexive and $\text{Ext}_R^1(P/M, R) \neq 0$, a derived map $M^{**} \rightarrow P^{**}$ is not isomorphic. By [7, Proposition 1.2] and [12, Proposition 6] $\text{Ext}_R^1(P/M, R)$ is a torsion module. Therefore we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & P/M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M^{**} & \longrightarrow & P^{**} & & \end{array}$$

Since P is reflexive and P/M is simple, M is reflexive. This is a contradiction. Thus R is a 1-Gorenstein ring.

For any (right and left) noetherian ring R , the socle of R_R is zero if and only if so is that of ${}_R R$ (see [2, Theorem 5]). Therefore, by Theorem 5 and [3, Corollary 2], we have:

Corollary 4. *Let R be a noetherian right QF-3' ring. If $E|R$ is an injective cogenerator, then R is a 1-Gorenstein ring with zero socle (on both sides), where E is the injective hull of R_R .*

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