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# APPLICATIONS OF MALLIAVIN'S CALCULUS TO TIME-DEPENDENT SYSTEMS OF HEAT EQUATIONS

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## 1. Introduction

Recently, various applications of Malliavin's calculus are studied by several authors. In particular, Kusuoka-Stroock applied Malliavin's calculus to the investigation of second order differential operators of Hörmander type ([5]). In fact, they have shown that the semigroup generated by a differential operator $L=\frac{1}{2}\sum_1^r(V_i)^2+V_0$, $V_j$'s being all $C_b^\infty$-vector fields on $R^d$, has a $C^\infty$-kernel if $V_j$'s satisfy the restricted Hörmander condition (cf. [8], [5]). Furthermore, they showed that the above $L$ is hypoelliptic when the general Hörmander condition is satisfied ([5]). Our aim of this paper is to extend their result to a time-dependent system associated with an operator $A$(s): $C^\infty(R^d; R^d)\to C^\infty(R^d; R^d)$, where $C^\infty(R^d; R^d)$ is the space of all $C^\infty$-mappings of $R^d$ into itself. Indeed, suppose that the operator $A$(s) is represented as

$$(1) \qquad (A(s)f)_j(x) = ([\frac{1}{2}\sum_{i=1}^r(V_i(s))^2+V_0(s)]f_j)(x) + \sum_{i=1}^r\sum_{m=1}^d a_j^{im}(s, x)\,(V_i(s)f_m)\,(x)+\sum_{m=1}^d c_j^m(s, x)f_m(x)\,,$$

where $f=(f_1, \cdots, f_d)\in C^\infty(R^d; R^d)$, $V_j(s)$'s are time-dependent $C^\infty$-vector fields on $R^d$ and $a_j^{im}(s)$, $c_j^m(s)\in C^\infty(R^d)$ for every $s\in[0, \infty)$. We will show the fundamental solution $P(s, t)$: $C^\infty(R^d; R^d)\to C^\infty(R^d; R^d)$ for the system of heat equations:

$$(1.2) \qquad \left(\frac{\partial}{\partial s}+A(s)\right)u = 0$$
$$u(t) = g\in C_b^\infty(R^d; R^d)$$

has a $C^\infty$-density function if

(1.3) mappings $(s, x)\mapsto\partial_x^\alpha h(s, x)$, $h\in\{a_j^{im}, c_j^m, V_j^i\}$ are all bounded and con-

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tinuous, where $V_j^i$ is the $i$-th component of $V_j$, i.e. the function $V_j^i(s, x)$ on $[0, \infty)\times R^d$ is defined by $V_j(t)=\sum_{i=1}^d V_j^i(t, \cdot)\frac{\partial}{\partial x_i}$, and

(1.4) either of the following assumptions (A.1) and (A.2) is satisfied:

(A.1) (i) $\mathrm{Lie}(V_1(s), \cdots, V_r(s))\,(x)=R^d$ for all $(s, x)\in[0, \infty)\times R^d$,

(ii) $\partial_x^\alpha V_j^i(s, x)$ is continuously differentiable with respect to $(s, x)$ for each $1\leqq i\leqq d$, $1\leqq j\leqq r$ and multiindex $\alpha$,

(iii) for every multiindex $\alpha$ and $R>0$, there exists a constant $C$ such that

$$\left|\frac{\partial}{\partial s}\partial_x^\alpha V_j^i(s, x)\right|\leqq C,\ (s, x)\in[0, R]\times R^d,\ 1\leqq i\leqq d,\ 1\leqq j\leqq r\,.$$

(A.2) for all $0\leqq a<b$, there exist families $\{f_i\}_0^r$ of Hölder continuous functions on $[a, b]$ and $\{W_i\}_0^r$ of $C^\infty$-vector fields on $R^d$ with the properties that

(i) $V_i(s)=f_i(s)W_i$, $0\leqq i\leqq r$

(ii) there exists a positive number $\delta_0$ such that $\delta_0\leqq|f_i(s)|\leqq\delta_0^{-1}$, $s\in[a, b]$, $0\leqq i\leqq r$,

(iii) $R^d=\mathrm{span}\,\{W(x);\ W=W_{i_0}$ or $[W_{i_q}, \cdots, [W_{i_1}, W_{i_0}]\cdots]$, $1\leqq i_0\leqq r$, $0\leqq i_j\leqq r$, $1\leqq j\leqq q$, $q\in N\}$ (2) for all $x\in R^d$.

Note that Assumption (A.1) appears stronger compared with the time-independent case, where the restricted Hörmander condition assures the existence of $C^\infty$-density function. We will see that the condition analogous to the restricted Hörmander condition is not sufficient for the time-dependent systems (see Remark 2.1.).

To show the above assertion, we apply Malliavin's calculus to the diffusion processes generated by the time-dependent second order differential operator $L(s)=\frac{1}{2}\sum_1^r(V_i(s))^2+V_0(s)$. We will show that the inverse of the Malliavin covariance of such diffusion processes has nice bounds in $L_p$-spaces with respect to the Wiener measure under either of Assumptions (A.1) and (A.2). To obtain the estimations in $L_p$-spaces, we follow the idea of Kusuoka-Stroock used in [8]. Then we will apply the integration by parts formula with respect to the Wiener measure. For this purpose, we will discuss the probabilistic construction of $P(s, t)$ in Section 3.

Section 2 is devoted to the estimations of the Malliavin covariance and the proof of our conclusion will be done in Section 3. We will also give a brief introduction to the probabilistic construction of $P(s, t)$ in Section 3.

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(2) Here and in the sequel, for $C^\infty$-vector fields $V$ and $W$ on $R^d$, $V(x)$ is the tangent vector at $x$ associated with $V$ and $[V, W]$ is the bracket product of $V$ and $W$.

## 2. Estimations of Malliavin covariance

In this section, we will study the boundedness of the inverse of the Malliavin covariance of time-dependent diffusion processes in $L_p$-spaces with respect to the Wiener measure. Let $\Theta$ be the space of $R^r$-valued continuous functions $\theta$ defined on $[0, \infty)$ with $\theta(0)=0$ and $P$ be the Wiener measure on $\Theta$. Throughout this paper, $E$ stands for the expectation with respect to $P$. Let $\{V_i(s); 0\leqq i\leqq r, s\in[0, \infty)\}$ be a family of $C^\infty$-vector fields on $R^d$ such that the mapping $(s, x)\mapsto \partial_x^\alpha V_t(s, x)$ of $[0, \infty)\times R^d$ into $R^d$ is bounded and measurable for each $0\leqq i\leqq r$ and multiindex $\alpha$. Here we used the usual identification of $C^\infty$-vector fields on $R^d$ with $C^\infty$-mappings of $R^d$ into itself and in the following we will use this identification without mentioning it. Given $s\geqq 0$ and $x\in R^d$, we denote by $\{X(x, t; x)\}_{t\geqq s}$ the unique solution to the stochastic integral equation:

$$(2.1)\quad X(s, t; x) = x+\textstyle\sum_1^r\displaystyle\int_s^t V_i(u, X(s, u; x))\, d\theta^i(u)+\int_s^t \tilde{V}_\theta(u, X(s, u; x))\, du\,,$$

where $V_i(u, x)=V_i(u)(x)$, $0\leqq i\leqq r$, $\tilde{V}_0^i(u)=V_0^i(u)+\sum_{k=1}^r \sum_{j=1}^d V_k^j(u)\frac{\partial}{\partial x_j}V_k^i(u)$ and $\theta(u)=(\theta^1(u), \cdots, \theta^r(u))$ is an $r$-dimensional Brownian motion with $\theta(0)=0$ realized on $(\theta, P)$. By Theorem (2.19) in [4], we note that each $i$-th component $X^i(s, t; x)$ of $X(s, t; x)$ is an infinitely differentiable Wiener functional in the sense of Malliavin (for the definition of the infinite differentiability of Wiener functonals, see [1], [4] or [8]). Let $A(s, t; x)$ be the Malliavin covariance of $X(s, t; x)$ (for the definition, see the same articles as above). Then, due to Lemma (2.9) in [4], in exactly the same way as in the case of time-independent stochastic integral equations, we have

$$(2.2)\qquad A(s, t; x) = \textstyle\sum_1^r\displaystyle\int_s^t \{J(s, t; x)J(s, u; x)^{-1} V_i(u, X(s, u; x))\}^{\otimes 2}\, du\,.$$

Here $J(s, t; x)$ is the Jacobian matrix of $X(s, t; x)$ with respect to $x$ and for a $\xi\in R^d$, we define a $d\times d$-matrix $\xi^{\otimes 2}=(\xi_i\,\xi_j)_{1\leqq i,j\leqq d}$. Our goal of this section is the following assertion.

**Theorem 2.1.** *Let $V_i(s)$'s be as above. Suppose that $V_i(s)$'s satisfy either of Assumptions* (A.1) *and* (A.2) *in Section* 1. *Then, given $0\leqq a<b$, a compact set $K$ in $R^d$ and $p\in N$, there exist constnats $C\geqq 0$ and $\mu\geqq 0$ such that*

$$(2.3)\qquad E[\{\det A(s, t; x)\}^{-p}]\leqq C((t-s))\wedge 1)^{-\mu}$$

*for every $x\in K$ and $a\leqq s<t\leqq b$.*

REMARK 2.1. If $V_i(s)$'s are all time-independent $C^\infty$-vector fields $W_i$'s on $R^d$, the restricted Hörmander condition (the condition (iii) in Assumption (A.2)) is sufficient for (2.3) to hold. On the other hand, the condition (i) in

Assumption (A.1) is stronger than the following condition; for all $(s, x)\in[0, \infty)\times R^d$,

$$R^d = \text{span}\,\{V(x);\ V = V_{i_0}(s) \text{ or } [V_{i_q}(s), \cdots, [V_{i_1}(s), V_{i_0}(s)] \cdots],\ 1\leqq i_0\leqq r,\ 0\leqq i_j\leqq r,\ 1\leqq j\leqq q,\ q\in N\}\ . \tag{2.4}$$

Thus, the preceding theorem makes us ask if Condition (2.4), the analogue to the restricted Hörmander condition, leads to the estimation (2.3). In general, the answer is No. For example, let $d=2$, r=1 and

$$V_1(s) = \frac{\partial}{\partial x_1} + 2sx_1 \frac{\partial}{\partial x_2}$$

$$V_0(s) = (x_1)^2 \frac{\partial}{\partial x_2}\ .$$

Then the solution to the stochastic integral equation associated with $V_1(s)$ and $V_0(s)$ is given by

$$X(s, t; x) = (x_1+\theta(t)-\theta(s),\ x_2+(t-s)\,(x_1+\theta(t)-\theta(s))^2)$$

where $\theta(t)$ is a 1-dimensional Brownian motion with $\theta(0)=0$. Therefore,

$$\det A(s, t; x) = \det \begin{pmatrix} t-s & 2(t-s)^2\,(x_1+\theta(t)-\theta(s)) \\ 2(t-s)^2\,(x_1+\theta(t)-\theta(s)) & 4(t-s)^3\,(x_1+\theta(t)-\theta(s))^2 \end{pmatrix} \equiv 0\ .$$

However, it is easily seen that the family $\{V_i(s)\}_0^1$ satisfies Condition (2.4).

Now we proceed to the proof of Theorem (2.1). The proof is separated into two part according to Assumptions under which we work: the first part is devoted to the proof that Assumption (A.1) yields the estimation (2.3) and the second part is devoted to the proof under Assumption (A.2).

*Part 1* Assume that Assumption (A.1) is satisfied. Fix $0\leqq a<b$ and a compact set $K$ in $R^d$. Define

$$\tilde{A}(s, t; x) = \sum_1^r \int_s^t \{J(s, u; x)^{-1}\, V_i(u, X(s, u; x))\}^{\otimes 2}\, du\ . \tag{2.5}$$

Obviously, it holds

$$A(s, t; x) = J(s, t; x)\, \tilde{A}(s, t; x)\, J(s, t; x)^* \tag{2.6}$$

where $X^*$ denotes the adjoint matrix of a $d\times d$-matrix $X$. Moreover, we observe that in order to show (2.3), it suffices to see the existence of constants $C, B, \lambda>0$ and $m_0\in N$ such that

$$P((\eta, \tilde{A}(s, t; x)\,\eta) \leqq 1/N^{m_0}) \leqq B \exp(-\lambda N) \tag{2.7}$$

for every $a\leqq s<t\leqq b$, $x\in K$, $\eta\in S^{d-1}\equiv\{y\in R^d;\ |y|=1\}$ and $N\geqq C((t-s)\wedge 1)^{-1/3}$.

Indeed, note that the inverse matrix of $J(s, t; x)$ satisfies the following:

$$J(s, t; x)^{-1} = I - \sum_1^r \int_s^t J(s, u; x)^{-1} \partial_x V_i(u, X(s, u; x))\, d\theta^i(u) + \int_s^t J(s, u; x)^{-1} [\sum_1^r (\partial_x V_i(u))^2 - \partial_x \tilde{V}_0(u)] (X(s, u; x))\, du ,$$

where $\partial_x V_i(u, x) = \partial_x V_i(u)(x) = \left(\frac{\partial V_i^j(u)}{\partial x_k}(x)\right)_{1\leqq j,k\leqq d}$ (see Lemma (8.2) in [8]). Hence, using the standard argument, we can deduce the following estimate from the boundedness of $\partial_x^\alpha V_i$'s, $|\alpha|\leqq 1$,[3] with respect to $(s, x)$:

(2.8) $$\sup_{s\in[a,b]} E[(\sup_{t\in[s,b]} |J(s, t; x)^{-1}|)^p] < +\infty,\ p>1 .$$[4]

Combining this with (2.8), we note that

$$E[(\det A(s, t; x))^{-p}] \leqq C_p^{(1)} (E[(\det \tilde{A}(s, t; x))^{-2p}])^{1/2},\ a\leqq s\leqq t\leqq b,\ x\in K$$

for some $C_p^{(1)}\geqq 0$. On the other hand, using (2.8) and the same argument as in Lemma 3.5 in [11], we notice that (2.7) implies the existence of constants $C_p^{(2)}$ and $\mu_p$, given $p\in \boldsymbol{N}$, such that

$$E[(\det \tilde{A}(s, t; x))^{-2p}] \leqq C_p^{(2)}((t-s)\wedge 1)^{-\mu_p},\ a\leqq s<t\leqq b,\ x\in K .$$

Combining this with the previous inequality, we see that (2.7) yields (2.3).

Now we turn to the proof of (2.7). To this end, we modify the argument in the proof of Theorem (8.31) in [8]. We first introduce some notations. Let $\mathfrak{A} = \{\phi\} \cup \{(\beta_1, \cdots, \beta_k);\ 1\leqq \beta_i\leqq r,\ 1\leqq i\leqq k,\ k\in \boldsymbol{N}\}$. For a $\beta$ and a family $\{Y_i\}_0^r$ of $C^\infty$-vector fields on $\boldsymbol{R}^d$, we set $|\beta|=0$ or $k$ and $Y_{(\beta)}=0$ or $[Y_{\beta_1}, \cdots, [Y_{\beta_{k-1}}, Y_{\beta_k}]\cdots]$ accordingly as $\beta=\phi$ or $(\beta_1, \cdots, \beta_k)$. Take $R>0$ such that $K\subset B(0, R)\equiv \{y\in \boldsymbol{R}^d;\ |y|<R\}$. It follows from the condition (i) in Assumption (A.1) and the continuity of $\partial_x^\alpha V_i(s)$'s, $1\leqq i\leqq r$, with respect to $(s, x)$ that there are an $\varepsilon>0$ and a $k_0\in \boldsymbol{N}$ such that

(2.9) $$\inf_{\substack{s\in[0,b+1]\\ x\in B(0,R)}} \inf_{\eta\in S^{d-1}} \sum_{|\beta|\leqq k_0} (V_{(\beta)}(s, x), \eta)^2 \geqq 2\varepsilon ,$$

where $(\ ,\ )$ is the inner product in $\boldsymbol{R}^d$. In the remaining of this part, we fix $a\leqq s<t\leqq b$, $x\in K$ and $\eta\in S^{d-1}$. However, we will mention that the constants appearing in the sequel are all independent of a particular choice of $a\leqq s<t\leqq b$, $x\in K$ and $\eta\in S^{d-1}$. Now we define a sequence $\{\sigma_N\}_1^\infty$ of stopping times by

(2.10) $$\sigma_N = \inf\{u\geqq s;\ |X(s, u; x)| \wedge |J(s, u; x)^{-1} - I| > \delta \wedge (\varepsilon/B_0)\} \wedge (s+N^{-3}) ,$$

(3) For a multiindex $\alpha=(\alpha_1, \cdots, \alpha_d)$, $|\alpha|=\alpha_1+\cdots+\alpha_d$.

(4) For a $d\times d$-matrix $X=(x_{ij})_{1\leqq i,j\leqq d}$, $|X|=(\sum_{i,j=1}^d (x_{ij})^2)^{1/2}$.

where $\delta=\text{dist}(K, B(0,R)^c)$ and $B_0=\sup\{|V_{(\beta)}(u,x)|; |\beta|\leqq k_0+1, (u,x)\in[0,\infty)\times R^d\}$. Then, there exist constants $K_1$, $\lambda_1>0$ and $B_1\geqq 1$, depending only on $\varepsilon$, $\delta$, $d$ and $\sup\{|\partial_x^\alpha V_j^i(u,x)|; 1\leqq i\leqq d, 0\leqq j\leqq r, |\alpha|\leqq 2, (u,x)\in[0,\infty)\times R^d\}$, such that

$$(2.11)\qquad P(\sigma_N\neq s+N^{-3})\leqq K_1\exp(-\lambda_1 N^3),\quad N\geqq B\,,$$

(cf. Theorem 4.2.1 in [10]). Moreover, the direct calculation shows that

$$(2.12)\quad \sum_{|\beta|\leqq k_0}\int_s^{\sigma_N}(J(s,u;x)^{-1}V_{(\beta)}(u,X(s,u;x)),\eta)^2\,du\geqq\varepsilon(\sigma_N-s),\ N\geqq 1\,.$$

Define $E_k(N)$, $N\geqq 1$, $0\leqq k\leqq k_0$, by

$$E_k(N)=\{\sum_{|\beta|\leqq k}\int_s^{\sigma_N}(J(s,u;x)^{-1}V_{(\beta)}(u,X(s,u;x)),\eta)^2\,du\leqq r^k/N^{m_k}\}\,,$$

where $m_k=4^{k_0-k}\times 5$. Then, obviously,

$$\{(\eta,\tilde{A}(s,t;x)\eta)\leqq 1/N^{m_0}\}\subset E_0(N)\quad\text{if } N\geqq((t-s)\wedge 1)^{-1/3}\,.$$

Furthermore, noting that $m_{k-1}=4m_k$, we have, on $E_{k-1}(N)\cap E_k(N)^c$,

$$\begin{aligned}&\sum_{|\beta|\leqq k-1}\sum_1^r\int_s^{\sigma_N}(J(s,u;x)^{-1}[V_j(u),V_{(\beta)}(u)](X(s,u;x)),\eta)^2\,du\\ &\geqq\sum_{|\beta|=k}\int_s^{\sigma_N}(J(s,u;x)^{-1}V_{(\beta)}(u,X(s,u;x)),\eta)^2\,du\\ &\geqq(r^k/N^{m_k})-(r^{k-1}/N^{m_{k-1}})\\ &\geqq r^k/2N^{m_k}\end{aligned}$$

if $N\geqq 2$. Then at least one of the above term $\sum_1^r\int_s^{\sigma_N}\cdots$ ds is greater or equal to $r/2k_0N^{m_k}$, because $\#\{\beta; |\beta|\leqq k-1\}=\sum_{j=1}^{k-1}r^j\leqq k_0\,r^{k-1}$. Therefore, noting the inequality $r^{k-1}/N^{m_{k-1}}\leqq 1/N^{4m_k-9}$ if $r^{k_0}\leqq N^9$, we have

$$\begin{aligned}&E_{k-1}(N)\cap E_k(N)^c\\ (2.13)\quad&\subset\bigcup_{|\beta|\leqq k-1}\{\int_s^{\sigma_N}(J(s,u;x)^{-1}V_{(\beta)}(u,X(s,u;x)),\eta)^2\,du\leqq 1/N^{4m_k-9}\,,\\ &\sum_1^r\int_s^{\sigma_N}(J(s,u;x)^{-1}[V_j(u),V_{(\beta)}(u)](X(s,u;x)),\eta)^2\,du\geqq r/2k_0N^{m_k}\}\,.\end{aligned}$$

Also, due to the inequality (2.12), we obtain, if $N^2\geqq r^{k_0}/\varepsilon$,

$$(2.14)\qquad E_k(N)\cap\{\sigma_N=s+N^{-3}\}=\phi\,.$$

Therefore we have

$$(2.15)\quad \{(\eta,\tilde{A}(s,t;x)\eta)\leqq 1/N^{m_0}\}\subset\{\sigma_N\neq s+N^{-3}\}\bigcup_1^{k_0}\bigcup_{|\beta|\leqq k-1}H(\beta,k,N)\,,$$

where $N\geqq((t-s)\wedge 1)^{-1/3}\vee(r^{k_0}\vee 2)^{1/9}\vee(r^{k_0}/\varepsilon)^{1/2}$ and

$$H(\beta, k, N) = \{\int_s^{\sigma_N} (J(s, u; x)^{-1} V_{(\beta)}(u, X(s, u; x)), \eta)^2 du \leqq 1/N^{4m_k-9} ,$$
$$\sum_1^r \int_s^{\sigma_N} (J(s, u; x)^{-1} [V_j(u), V_{(\beta)}(u)] (X(s, u; x)), \eta)^2 du$$
$$\geqq r/2k_0 N^{m_k}, \sigma_N = s+N^{-3}\} .$$

On the other hand, it follows from Ito's formula that for every $C^\infty$-vector field $V=V(s, x)$ on $R^{d+1}$,

$$J(s, t; x)^{-1} V(t, X(s, t; x))$$
$$= V(s, x)+\sum_1^r \int_s^t J(s, u; x)^{-1} [V_i(u), V(u)] (X(s, u; x)) d\theta^i(u)$$
$$+\int_s^t J(s, u; x)^{-1} \left[\frac{\partial}{\partial u} V(u)+\Gamma(u) (V(u))\right] (X(s, u; x)) du ,$$

where $\Gamma(u) (W)=[V_0(u), W]+\frac{1}{2} \sum_1^r [V_k(u), [V_k(u), W]]$ for a $C^\infty$-vector field $W$ on $R^d$. Applying Theorem (8.26) in [8], we can find a constant $\lambda_2>0$, depending only on $B_0$ and $\sup\{|\frac{\partial}{\partial s} V_{(\beta)}(s, x)| ; |\beta| \leqq k_0+1, (s, x)\in[0, b+1]\times R^d\}$, such that

$$P(H(\beta, k, N))\leqq\sqrt{2}N^{5m_k-5} \exp(-\lambda_2 N) \tag{2.16}$$

for all $N\geqq 1$ and $\beta$ with $|\beta| \leqq k_0$. Thus combining (2.15) with (2.11) and (2.16), we obtain the desired estimate (2.7).

*Part 2* Assume that Assumption (A.2) holds. Fix $0\leqq a\leqq b$ and a compact set $K$ in $R^d$. Let $\tilde{A}(s, t; x)$ be as defined by (2.5). Then, similarly to the preceding part, in order to obtain the estimation (2.3) it suffices to show the existence of constants $C$, $B$, $\lambda$, $\mu>0$ and $m_0\in N$ such that

$$P((\eta, \tilde{A}(s, t; x) \eta)\leqq 1/N^{m_0})\leqq B \exp (-\lambda N^{\mu}) , \tag{2.17}$$

for every $a\leqq s<t\leqq b$, $x\in K$, $\eta\in S^{d-1}$ and $N\geqq C((t-s)\wedge 1)^{-1/3}$.

We now proceed to the proof of (2.17). Choose a family $\{f_i\}_0^r$ of Hölder continuous functions on $[0, b+1]$ and a family $\{W_i\}_0^r$ of $C^\infty$-vector fields on $R^d$ as in Assumption (A.2). Define families $C_k$, $k\in \{0\} \cup N$, inductively by

$$C_0 = \{W_1, \cdots, W_r\} \text{ and } C_k = \{[W_k, W]; 0\leqq k\leqq r, W\in C_{k-1}\}, k\geqq 1 .$$

Take an $R>0$ such that $K\subset B(0, R)$. Then by the condition (iii) in Assumption (A.2), there are an $\varepsilon>0$ and a $k_0\in \{0\} \cup N$ such that

$$\inf_{x\in B(0,R)} \inf_{\eta\in S^{d-1}} \sum_{k=1}^{k_0} \sum_{W\in C_k} (W(x), \eta)^2\geqq 2\varepsilon . \tag{2.18}$$

In the following, we fix $a\leqq s<t\leqq b$, $x\in K$ and $\eta\in S^{d-1}$ but we will see that all constants appearing in the following are independent of a particular choice of

$s$, $t$, $x$ and $\eta$. We define a sequence $\{\tau_N\}$ of stopping times by

$$\tau_N = \inf\{u\geqq s;\ |X(s, u; x)-x| \wedge |J(s, u; x)-I| \geqq \delta\wedge(\varepsilon\delta_0^2/B)\} \wedge (s+N^{-3}),$$

where $\delta=\mathrm{dist}\,(K, B(0, R)^c)$ and $B=\max\{|W(x)|;\ W\in C_k,\ 0\leqq k\leqq k_0+2,\ x\in R^d\}$. Then, by Theorem 4.2.1 in [10], there exist constants $K_1$, $\lambda_1>0$ and $N_1\in \boldsymbol{N}$ such that

$$(2.19)\qquad P(\tau_N\neq s+N^{-3})\leqq K_1 \exp(-\lambda_1 N^3) \quad \text{if } N\geqq N_1\,.$$

Also, (2.18) implies the following inequality:

$$(2.20)\qquad \textstyle\sum_{k=0}^{k_0} \sum_{W\in C_k}\displaystyle\int_s^{\tau_N} (J(s, u; x)^{-1} W(X(s, u; x)), \eta)^2\, du \geqq \varepsilon(\tau_N - s)\,.$$

Moreover, the conditions fi) and fii) imply that

$$(2.21)\qquad \delta_0^2([W_k, W](x), \xi)^2 \leqq ([V_k(u), W](x), \xi)^2 \leqq ([W_k, W](x), \xi)^2/\delta_0^2\,,$$

for every $(u, x)\in[0, b+1]\times R^d$, $\xi\in R^d$ and $C^\infty$-vector field $W$ on $R^d$. Hence we have

$$(2.22)\qquad \{(\eta, \tilde{A}(s, t; x)\,\eta)\leqq \delta_0^2/N^m\} \subset \{\textstyle\sum_1^r \displaystyle\int_s^{\tau_N} (J(s, u; x)^{-1} W_i(u, X(s, u; x)), \eta)^2\, du \leqq 1/N^m\},\quad m\in \boldsymbol{N},$$

if $N\geqq (t-s)\wedge 1)^{-1/3}$. Furthei more let

$$H_k(W, N) = \{\int_s^{\tau_N} (J(s, u; x)^{-1} W(X(s, u; x)), \eta)^2\, du \leqq 1/N^{20m_k-9}\,,$$
$$\textstyle\sum_0^r \displaystyle\int_s^{\tau_N} (J(s, u; x)^{-1} [W_j, W](X(s, u; x)), \eta)^2\, du \geqq 5/\delta_0^2\, N^{m_k},$$
$$\tau_N = s+N^{-3}\}\,,$$

where $m_k=20^{k_0-k}\times 6$. Then, using the same argument as in the previous part, we deduce from (2.20) the existence of a number $N_2\in \boldsymbol{N}$, independent of $s$, $t$, $x$ and $\eta$, such that if $N\geqq N_2$, then the right hand side in (2.22) with $m=m_0$ is contained in the set $\{\tau_N\neq s+N^{-3}\} \cup \bigcup_1^{k_0} \bigcup_{W\in C_{k-1}} H_k(W, N)$. Therefore, combining this with (2.22) and (2.19), the proof of estimation (2.17) is completed once we show that there exist constants $K_2$, $\lambda_2$ and $\mu_2>0$, depending only on $B$ in the definition of $\tau_N$, such that

$$(2.23)\qquad P(H_k(W, N))\leqq K_2 \exp(-\lambda_2 N^{\mu_2}),\quad N\geqq 1,\ W\in C_{k-1},\ 1\leqq k\leqq k_0\,.$$

To this end, we note that it follows from (2.21) that for all $u\in[0, b+1]$, $x$, $\xi\in R^d$ and $C^\infty$-vector field $W$ on $R^d$

$$\delta_0^2 \textstyle\sum_0^r ([W_k, W](x), \eta)^2 \leqq \sum_0^r([V_k(u), W](x), \eta)^2$$

$$\leqq \sum_1^r([V_k(u), W(x)], \eta)^2+2(\Gamma(u)(W), \eta)^2$$
$$+\frac{r}{2\delta_0^2}\sum_1^r([V_k(u), [W_k, W]](x), \eta)^2$$

where $\Gamma(u)(W)=[V_0(u), W]+\frac{1}{2}\sum_1^r [V_k(u), [V_k(u), W]]$. Thus, if we fix $1\leqq k\leqq k_0$ and $W\in C_{k-1}$ and set

$$F=\{\int_s^{\tau_N}(J(s, u; x)^{-1}W(X(s, u; x)), \eta)^2\,du\leqq 1/N^{20m_k-9},$$
$$\sum_1^r\int_s^{\tau_N}(J(s, u; x)^{-1}[V_i(u), W](X(s, u; x)), \eta)^2\,du\geqq 1/N^{m_k},$$
$$\tau_N=s+N^{-3}\},$$
$$G=\{\int_s^{\tau_N}(J(s, u; x)^{-1}W(X(s, u; x)), \eta)^2\,du\leqq 1/N^{20m_k-9},$$
$$\int_s^{\tau_N}(J(s, u; x)^{-1}\Gamma(u)(W)(X(s, u; x)), \eta)^2\,du\geqq 1/N^{m_k},$$
$$\tau_N=s+N^{-3}\}$$

and

$$H_j=\{\int_s^{\tau_N}(J(s, u; x)^{-1}W(X(s, u; x)), \eta)^2\,du\leqq 1/N^{20m_k-9},$$
$$\int_s^{\tau_N}(J(s, u; x)^{-1}[V_j(u), [W_j, W]](X(s, u; x)), \eta)^2\,du\geqq 4\delta_0^2/r^2 N^{m_k},$$
$$\tau_N=s+N^{-3}\},$$

then we have

$$H_k(W, N)\subset F\cup G\cup \bigcup_1^r H_j\,.$$

Therefore, noting that

$$H_j\subset\{\int_s^{\tau_N}(J(s, u; x)^{-1}W(X(s, u; x)), \eta)^2\,du\leqq 1/N^{20m_k-9},$$
$$\sum_1^r\int_s^{\tau_N}(J(s, u; x)^{-1}[V_k(u), W](X(s, u; x)), \eta)^2\,du\geqq \delta_0^2/N^{4m_k-9},$$
$$\tau_N=s+N^{-3}\}$$
$$\cup\{\int_s^{\tau_N}(J(s, u; x)^{-1}[W_j, W](X(s, u; x)), \eta)^2\,du\leqq 1/N^{4m_k-9},$$
$$\sum_1^r\int_s^{\tau_N}(J(s, u; x)^{-1}[V_k(u), [W_j, W]](X(s, u; x)), \eta)^2\,du\geqq 4\delta_0^2/r^2N^{m_k},$$
$$\tau_N=s+N^{-3}\}$$

and it holds, for a $C^\infty$-vector field $V$ on $R^d$

$$J(s, t; x)^{-1} V(X(s, t; u)) = V(x)+\sum_1^r\int_s^t J(s, u; x)^{-1} [V_i(u), V](X(s, u; x))\, d\theta^i(u)$$
$$+\int_s^t J(s, u; x)^{-1} \Gamma(u)(V)(X(s, u; x))\, du,$$

we obtain the desired conclusion (2.23) by applying Theorem (8.26) and (8.29) in [8].

The proof is completed.

## 3. Application

In this section, we discuss an application of the previous theorem to the time-dependent system of heat equations.

Let $\{V_i(s);\ 0\leqq i\leqq r,\ s\in[0, \infty)\}$ be a family of $C^\infty$-vector fields on $R^d$ and let $\{a_j^{im}(s);\ 1\leqq i\leqq r,\ 1\leqq m, j\leqq d,\ s\in[0, \infty)\}$ and $\{c_j^m(s);\ 1\leqq m, j\leqq d,\ s\in[0, \infty)\}$ be families of $C_b^\infty$-functions on $R^d$. Throughout this section, we assume that for each multiindex $\alpha$, mapping $(s, x)\mapsto\partial_x^\alpha V_j^i(s, x)$, $(s, x)\mapsto\partial_x^\alpha a_j^{im}(s, x)$ and $(s, x)\mapsto \partial_x^\alpha c_j^m(s, x)$ are all continuous and bounded. Define a time-dependent operator $A(s)$ on $C^\infty(R^d; R^d)$ by

$$(3.1)\qquad (A(s)f)_j = \left[\frac{1}{2}\sum_1^r(V_i(s))^2+V_0(s)\right]f_j$$
$$+\sum_{i=1}^r\sum_{m=1}^d a_j^{im}(s)\, V_i(s) f_m+\sum_1^d c_j^m(s) f_m,\quad 1\leqq j\leqq d,$$

for $f=(f_1, \cdots, f_d)\in C^\infty(R^d; R^d)$.

We first give a brief introduction to the probabilistic construction of the fundamental solution $P(s, t)$ to the time-dependent system of heat equations:

$$(3.2)\qquad \left(\frac{\partial}{\partial s}+A(s)\right) u = 0,\quad 0\leqq s\leqq t$$
$$u(t) = g\in C_b^\infty(R^d; R^d).$$

Let $\{X(s, t; x)\}_{t\geqq s}$ be the unique solution to the stochastic integral equation (2.1) and $\{M(s, t; x)\}_{t\geqq s}=\{(M_j^i(s, t; x))_{1\leqq i,j\leqq d}\}_{t\geqq s}$ be the unique solution to the following stochastic integral equation:

$$(3.3)\quad M_j^i(s, t; x) = \delta_j^i+\sum_{k=1}^r\sum_{m=1}^d\int_s^t a_m^{ki}(u, X(s, u; x))\, M_j^m(s, u; x)\, d\theta^k(u)$$
$$+\sum_{m=1}^d\int_s^t c_m^i(u, X(s, u; x))\, M_j^m(s, u; x)\, du,$$

where $\delta_j^i=1$ or $0$ accordingly as $i=j$ or $i\neq j$. We define an operator $P(s, t)$ by

$$(3.4)\qquad (P(s, t) g)(x) = \left(\sum_1^d E[M_j^i(s, t; x)\, g_i(X(s, t; x))]\right)_{1\leqq j\leqq d},$$
$$g = (g_1, \cdots, g_d)\in C_b^\infty(R^d; R^d).$$

Due to the result in [3] (also see [1]), it is easily seen that $P(s, t)$ maps $C_b(R^d;$

$R^d$) into itself. Moreover, we have the following.

**Proposition 3.1.** *For $g \in C_b^\infty(R^d; R^d)$, define $u(s, x)$ by*

$$u(s, x) = (P(s, t) g)(x).$$

*Then, $u$ is a solution to* (3.3) *with $u(t)=g$.*

Proof. By Ito's formula, we have

$$P(s, t) g = g + \int_s^t P(s, v)(A(v) g)\, dv, \quad g \in C_b^\infty(R^d; R^d).$$

Moreover, because of the multiplicativity of $M(s, t; x)$ and the Markov property of $X(s, t; x)$, we obtain

$$P(s, u) P(u, t) = P(s, t), \quad s \leqq u \leqq t.$$

Applying the standard argument, it follows from these two identities that

$$\frac{\partial}{\partial s} P(s, t) = -A(s) P(s, t).$$

Since $P(t, t)=$identity, this completes the proof.

REMARK 3.1. Under the assumption that $L(s)=\frac{1}{2} \sum_1^r (V_i(s))^2 + V_0(s)$ is elliptic, Stroock showed the result similar to the above proposition ([7]). On the multiplicativity, Pinsky ([6]) studied in more general scheme.

Now we are ready to state our main result in this section. Our conclusion is the following.

**Theorem 3.1.** *Let $P(s, t)$ be as before. Assume that $V_i(s)$'s which appeared in* (3.1) *satisfy either of Assumptions* (A.1) *and* (A.2) *in Section* 1. *Then, there exists a family $\{q_j^i(s, t);\ 1 \leqq i, j \leqq d,\ 0 \leqq s < t < +\infty\}$ of $C^\infty$-functions on $(R^d)^2$ such that*

$$(3.5) \qquad (P(s, t) g)_j (x) = \sum_1^d \int_{R^d} g_i(y)\, q_j^i(s, t; x, y)\, dy, \quad g \in C_b^\infty(E^d; E^d).$$

*In particular, if we define $u(s, x)=(\sum_1^d \int_{R^d} g_i(y)\, q_j^i(s, t; x, y)\, dy)_{1 \leqq j \leqq d}$, then $u$ is a solution to* (3.2) *with $u(t)=g$. Moreover, let $0 \leqq a < b$ and $K$ be a compact set in $E^d$. Then, for each $\delta > 0$ and wultiindeces $\alpha$ and $\beta$, there exist constants $C=C(a, g, K, \alpha, \beta)$, $C_\delta = C(a, b, K, \delta) > 0$, $\varepsilon > 0$ and $\mu \geqq 0$ such that*

$$(3.6) \qquad \| \partial_x^\alpha \partial_y^\beta q_j^i(s, t; x, \cdot) \|_{C_b(R^d)} \leqq C((t-s) \wedge 1)^{-\mu},^{(5)}$$

---

(5) For an $\Omega \subset R^d$ and a continuous function $f$ on $R^d$, $\|f\|_{C_b(\Omega)} = \sup_{x \in \Omega} |f(x)|$.

$$\begin{aligned}(3.7)\qquad &|\partial_x^\alpha \partial_y^\beta q_j^i(s, t; x, \cdot)|_{C_b(R^d\setminus B(x,\delta))}\\ &\leq C((t-s)\wedge 1)^{-\mu} \exp[-C_\delta/((t-s)\varepsilon)],\end{aligned}$$

*for* $a\leqq s<t\leqq b$ *and* $x\in K$, *where* $B(x, \delta)=\{y\in R^d;\ |x-y|<\delta\}$.

Before proceeding to the proof of Theorem (3.7), we prepare a lemma on the existence of a $C^\infty$-kernel function of the finite measure $p(x, dy)$ on $R^d$ parameterized by $x\in R^d$.

**Lemma 3.1.** *Let* $\{p(x, dy)\}_{x\in R^d}$ *be a family of finite measures on* $R^d$ *such that* $\int_{R^d} f(y)\, p(\cdot, dy)\in C^\infty(R^d)$ *for every* $f\in C_b^\infty(R^d)$. *Assume that for each* $R>0$ *and multiindeces* $\alpha$ *and* $\beta$, *there exists a constant* $C_{\alpha,\beta}(R)$ *such that*

$$(3.8)\qquad |\partial_x^\alpha \int_{R^d} \partial^\beta f(y)\, p(x, dy)| \leqq C_{\alpha,\beta}(R)\, |f|_{C_b(R^d)},\ f\in C_b^\infty(R^d),\ x\in B(0, R).$$

*Then there exist a* $p\in C^\infty(R^d\times R^d)$ *such that*

$$(3.9)\qquad p(x, dy) = p(x, y)\, dy.$$

*Moreover, for every* $r<R$ *and* $n\in N$, *there are constants* $C$ *and* $m\in N$, *depending only on* $n$ *and* $r$, *such that*

$$(3.10)\qquad |\partial_x^\alpha \partial_y^\beta p(x, \cdot)|_{C_b(R^d)} \leqq C \sum_{|\tilde\alpha|+|\tilde\beta|\leqq m} C_{\tilde\alpha,\tilde\beta}(R)$$

*holds for every* $x\in B(0, r)$ *and* $\alpha$, $\beta$ *with* $|\alpha|+|\beta|\leqq n$.

Proof. By Lemma (3.1) in [9], for each $x\in R^d$, there exists a $p(x, \cdot)\in C^\infty(R^d)$ satisfying (3.13). Choose $\psi\in C_0^\infty(R^d)$ such that $\psi\equiv 1$ on $B(0, r)$ and $\equiv 0$ outside of $B(0, R)$ and set $\tilde p(x, y)=\psi(x)\, p(x, y)$. Then, by (3.12), for each $k\in N$, there is a $C_k$ such that

$$\begin{aligned}&|\int_{R^d\times R^d} e^{\sqrt{-1}((\xi,x)+(\eta,y))}\, \tilde p(x, y)\, dx\, dy\\ &= \frac{1}{(1+|\xi|^2+|\eta|^2)^k} |\int_{R^d\times R^d} (1+\Delta_x+\Delta_y)^k e^{\sqrt{-1}((\xi,\cdot)(\eta,\cdot))}(x, y)\, \tilde p(x, y)\, dx\, dy|\\ &\leqq C_k(\textstyle\sum_{|\tilde\alpha|+|\tilde\beta|\leqq 4k} C_{\tilde\alpha,\tilde\beta}(R))/(1+|\xi|^2+|\eta|^2)^k,\quad \xi, \eta\in R^d,\end{aligned}$$

where $\Delta$ is the Laplacian on $R^d$. Thus, by Sobolev's inequality ([2]), $\tilde p\in C^\infty(R^d\times R^d)$ and satisfies (3.14). Hence so does $p$.

Now we proceed to the proof of the theorem.

Proof of Theorem 3.1. By Theorem (2.19) in [4], we have

$$\sup_{x\in R^d} \sup_{0\leqq s<t\leqq b+1} E[(|L^k \partial_x^\alpha X^i(s, t; x)| + |L^{k-1} \partial_x^\alpha M_j^i(s, t; x)|)^p] < +\infty,$$

for every multiindex $\alpha$ and $k, p\in N$. Here $L$ is the Ornstein-Uhlenbeck operator on $\Theta$ (cf. [1] or [4]). Hence, using the integration by parts formula with respect to the Wiener measure (cf. [4]), we see that, given multiindeces $\alpha$ and $\beta$, there exists a constant $C$ with the following properties:

$$(3.11)\qquad |\partial_x^\alpha E[M_j^i(s, t; x)(\partial^\beta h)(X(s, t; x))]| \\ \leqq C|h|_{C_b(R^d)} E[(\det A(s, t; x))^{-2(|\alpha|+|\beta|+1)}], \quad h\in C_b^\infty(R^d),$$

$$(3.12)\qquad |\partial_x^\alpha E[M_j^i(s, t; x)(\partial^\beta h)(X((s, t; x))\,\phi(X(s, t; x))]| \\ \leqq C|h|_{C_b(R^d)}(\max\{|\partial^\gamma\phi|_{C_b(R^d)}; |\gamma|\leqq|\alpha|+|\beta|\}) \\ \times(E[\det A(s, t; x))^{-4(|\alpha|+|\beta|+1)}])^{1/2}\times(P(X(s, t; x)\notin B(x, \delta/2)))^{1/2}, \\ h\in C_b^\infty(R^d) \text{ and } \phi\in C_b^\infty(R^d) \text{ such that } \phi\equiv 0 \text{ on } B(0, \delta/2).$$

Moreover, due to Theorem 4.2.1 in [10], for each $\delta>0$, there exist constants $C\geqq 0$, $C_\delta>0$ and $\varepsilon>0$ such that

$$(3.13)\qquad P(X(s, t; x)\notin B(x, \delta/2))\leqq C\exp[-C_\delta/((t-s)\wedge\varepsilon)].$$

Now let $q_j^i(s, t; x, dy)$ be a finite measure on $R^d$ defined by

$$(3.14)\qquad \int_{R^d} f(y)\,q_j^i(s, t; x, dy) = E[M_j^i(s, t; x) f(X(s, t; x))], \quad f\in C_b(R^d).$$

By (3.11) and Theorem 2.1, given multiindeces $\alpha$ and $\beta$ and an $R>0$, we can find constants $C_{\alpha,\beta}(R)$ and $\mu\geqq 0$ such that

$$(3.15)\qquad \left|\partial_x^\alpha\int_{R^d}\partial^\beta f(y)\,q_j^i(s, t; x, dy)\right|\leqq C_{\alpha,\beta}(R)\,((t-s)\wedge 1)^{-\mu}|f|_{C_b(R^d)}, \\ f\in C_b^\infty(R^d),\ x\in B(0, R).$$

Combining this with Lemma 3.1, we see the existence of $q_j^i(s, t)$'s in $C^\infty(R^d\times R^d)$ satisfying (3.5). Moreover, combined with Lemma 3.1, (3.15) yields (3.6). To show (3.7), choose $\psi\in C_b^\infty(R^1)$ such that $\psi\equiv 0$ on $(-1/2, 1/2)$ and $\equiv 1$ outside of $(-1, 1)$ and set $\tilde{q}_j^i(s, t; x, dy)=\psi(|x-y|^2/\delta^2)\,q_j^i(s, t; x, dy)$. Then, by (3.12), (3.13) and Theorem 2.1, given multiindeces $\alpha$ and $\beta$ and an $R>0$, there exist constants $C_{\alpha,\beta}(R)$ and $\mu\geqq 0$ such that

$$(3.16)\qquad \left|\partial_x^\alpha\int_{R^d}\partial^\beta f(y)\,\tilde{q}_j^i(s, t; x, dy)\right|\leqq C_{\alpha,\beta}(R)\,(t-s)\wedge 1)^{-\mu} \\ \times\exp[-C_\delta/2((t-s)\,\varepsilon)]\,|f|_{C_b(R^d)}.$$

Since $q_j^i(s, t; x, dy)=q_j^i(s, t; x, dy)$ outside of $B(x, \delta)$, combining this with Lemma 3.1, we obtain (3.7).

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