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ON THE HYPOELLIPTICITY AND THE GLOBAL ANALYTIC-HYPOELLIPTICITY OF PSEUDO-DIFFERENTIAL OPERATORS

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Introduction

In the recent paper [13] Kumano-go and Taniguchi have studied by using oscillatory integrals when pseudo-differential operators in R^n are Fredholm type and examined whether or not the operators $L_k(x, D_x, D_y)=D_x+ix^kD_y$ in Mizohata [15] and $L_{\pm}(x, D_x, D_y)=D_x\pm ixD_y^2$ in Kannai [6] are hypoelliptic by a unified method. In the present paper we shall give the detailed description for results obtained in [13] and study the hypoellipticity for the operator of the form $L=\sum_{|\alpha:\mathfrak{m}|+|\alpha':\mathfrak{m}'|\leqq 1} a_{\alpha\alpha'\gamma\gamma'}x^{\gamma}\tilde{y}^{\gamma'}D_x^{\alpha}D_y^{\alpha'}$ with semi-homogeneity in $(x, \tilde{y}, D_x, D_y)$ by deriving the similar inequality to that of Grushin [4] for the elliptic case. Then we can treat the semi-elliptic case as well as the elliptic case. We shall also give a theorem on the global analytic-hypoellipticity of a non-elliptic operator, and applying it give a necessary and sufficient condition for the operator $L(x, D_x, D_y)$ to be hypoelliptic, when the coefficients of L are independent of $\tilde{y}^{\gamma'}$ (see Theorem 3.1).

In Section 1 we shall describe pseudo-differential operators of class $S^m_{\lambda,\rho,\delta}$ which is defined by using a basic weight function $\lambda=\lambda(x, \xi)$ varying in x and ξ (cf. [13] and also [1]). In Section 2 we shall study the global analytic-hypoellipticity of a non-elliptic pseudo-differential operator and give an example which indicates that the condition (2.3) is necessary in general. In Section 3 we shall consider the local hypoellipticity for the operator L and give some examples.

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1. Algebras and L^2-boundedness

Definition 1.1. For $-\infty<m<\infty$, $0\leqq\delta<1$ and a sequence $\tilde{\tau}$; $0\leqq\tau_0\leqq\tau_1\leqq\cdots$ we define a Fréchet space $\mathcal{A}^m_{\delta,\tilde{\tau}}$ by the set of C^∞-functions $p(\xi, x)$ in $R^{2n}_{\xi,x}$ for which each semi-norm

$$|p|_{\alpha,\beta}^{(m)} = \sup_{x,\xi}\{|p_{(\beta)}^{(\alpha)}(\xi, x)|\langle x\rangle^{-\tau|\beta|}\langle\xi\rangle^{-m-\delta|\beta|}\}$$

is finite, where $p_{(\beta)}^{(\alpha)}=\partial_\xi^\alpha D_x^\beta p$, $D_{x_j}=-i\partial/\partial x_j$, $\partial_{\xi_j}=\partial/\partial\xi_j$, $j=1, \cdots, n$,

$\langle x\rangle=\sqrt{1+|x|^2}$, $\langle\xi\rangle=\sqrt{1+|\xi|^2}$.

We define the oscillatory integral $O_s[p]$ for $p(\xi, x)\in\mathcal{A}_{\delta,\tau}^m$ by

$$O_s[p]\equiv O_s-\iint e^{-ix\cdot\xi}p(\xi, x)dx đ\xi$$
$$=\lim_{\varepsilon\to0}\iint e^{-ix\cdot\xi}\chi_\varepsilon(\xi, x)p(\xi, x)dx đ\xi\,,$$

where $đ\xi=(2\pi)^{-n}d\xi$, $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$ and $\chi_\varepsilon(\xi, x)=\chi(\varepsilon\xi, \varepsilon x)$ $(0<\varepsilon\leqq1)$ for a $\chi(\xi, x)\in\mathcal{S}$ (the class of rapidly decreasing functions of Schwartz) in $R_{\xi,x}^{2n}$ such that $\chi(0, 0)=1$ (cf. ([11], [13]).

REMARK. We can easily obtain the following statements (cf. [11]).
1°) For $p\in\mathcal{A}_{\delta,\tau}^m$ we have

$$O_s[p]=\iint e^{-ix\cdot\xi}\langle x\rangle^{-2l'}\langle D_\xi\rangle^{2l'}\{\langle\xi\rangle^{-2l}\langle D_x\rangle^{2l}p(\xi, x)\}dx đ\xi$$

by taking integers l, l' such that $-2l(1-\delta)+m<-n$ and $-2l'+\tau_{2l}<-n$.
2°) Let $\{p_\varepsilon\}_{0<\varepsilon<1}$ be a bounded set in $\mathcal{A}_{\delta,\tau}^m$ and converges to a $p_0(\xi, x)\in\mathcal{A}_{\delta,\tau}^m$ as $\varepsilon\to0$ uniformly on any compact set of $R_{\xi,x}^{2n}$. Then we have

$$\lim_{\varepsilon\to0}O_s[p_\varepsilon]=O_s[p_0]\,.$$

3°) For $p\in\mathcal{A}_{\delta,\tau}^m$ we have

$$O_s[x^\alpha p]=O_s[D_\xi^\alpha p] \quad\text{and}\quad O_s[\xi^\alpha p]=O_s[D_x^\alpha p]\,.$$

DEFINITION 1.2. We say that a C^∞-function $\lambda(x, \xi)$ in $R_{x,\xi}^{2n}$ is a basic weight function when $\lambda(x, \xi)$ satisfies conditions:

$$A_0^{-1}\langle\xi\rangle^a\leqq\lambda(x, \xi)\leqq A_0(1+|x|^{\tau_0}+|\xi|) \qquad (\tau_0\geqq0,\ a>0)\,, \tag{1.1}$$

$$|\lambda_{(\beta)}^{(\alpha)}(x, \xi)|\leqq A_{\alpha\beta}\lambda(x, \xi)^{1-|\alpha|+\delta|\beta|} \qquad (0\leqq\delta<1)\,, \tag{1.2}$$

$$\lambda(x+y, \xi)\leqq A_1\langle y\rangle^{\tau_1}\lambda(x, \xi) \qquad (\tau_1\geqq0) \tag{1.3}$$

for positive constants A_0, $A_{\alpha\beta}$, A_1.[1)]

DEFINITION 1.3. We say that a C^∞-function $p(x, \xi)$ in $R_{x,\xi}^{2n}$ belongs to $S_{\lambda,\rho,\delta}^m$, $-\infty<m<\infty$, $0\leqq\delta\leqq\rho\leqq1$, $\delta<1$, when for any multi-index α, β

1) For a basic weight function $\lambda(x, \xi)$ satisfying (1.1)–(1.3) we can always find an equivalent basic weight function $\lambda'(x, \xi)$ with $\delta=0$ in (1.2) to $\lambda(x, \xi)$, i.e., $C^{-1}\lambda(x, \xi)\leqq\lambda'(x, \xi)\leqq C\lambda(x, \xi)$.

$$(1.4) \qquad |p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_{\alpha\beta}\lambda(x, \xi)^{m-\rho|\alpha|+\delta|\beta|} .$$

For $p(x, \xi)\in S^m_{\lambda,\rho,\delta}$ we define pseudo-differential operator $P=p(X, D_x)$ with the symbol $\sigma(P)(x, \xi)=p(x, \xi)$ by

$$(1.5) \qquad Pu(x) = \int e^{ix\cdot\xi}p(x, \xi)\hat{u}(\xi)d\xi \qquad \text{for} \quad u\in\mathcal{S},$$

where $\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$ is the Fourier transform of $u\in\mathcal{S}$.

For a $p\in S^m_{\lambda,\rho,\delta}$ we define semi-norms $|p|^{(m)}_{l_1,l_2}$, l_1, $l_2=0, 1, \cdots$ by

$$|p|^{(m)}_{l_1,l_2} = \underset{|\alpha|\leqq l_1,|\beta|\leqq l_2}{\mathrm{Max}} \{\sup_{x,\xi}|p^{(\alpha)}_{(\beta)}(x, \xi)|\lambda(x, \xi)^{-m+\rho|\alpha|-\delta|\beta|}\} .$$

Then $S^m_{\lambda,\rho,\delta}$ makes a Fréchet space.

In what follows we shall only treat the case: $\delta=\rho=0$ or $0=\delta<\rho=1$ since it simplifies the statements below and is sufficient for our aim.

Theorem 1.4. *Let $P_j=p_j(X, D_x)\in S^{m_j}_{\lambda,\rho,0}$, $j=1, 2$. Then $P=P_1P_2$ belongs to $S^{m_1+m_2}_{\lambda,\rho,0}$ and we have for any integer $N>0$*

$$(1.6) \qquad \sigma(P)(x, \xi) \qquad (\textit{denoted also by } p_1\circ p_2(x, \xi))$$
$$= \sum_{|\alpha|<N}\frac{1}{\alpha!}p_\alpha(x, \xi)+N\sum_{|\gamma|=N}\int_0^1\frac{(1-\theta)^{N-1}}{\gamma!}r_{\gamma,\theta}(x, \xi)d\theta$$

where

$$\begin{cases} p_\alpha(x, \xi) = p_1^{(\alpha)}(x, \xi)p_{2(\alpha)}(x, \xi) & (\in S^{m_1+m_2-\rho|\alpha|}_{\lambda,\rho,0}), \\ r_{\gamma,\theta}(x, \xi) = O_s-\iint e^{-iy\cdot\eta}p_1^{(\gamma)}(x, \xi+\theta\eta)p_{2(\gamma)}(x+y, \xi)dyd\eta . \end{cases}$$

The set $\{r_{\gamma,\theta}(x, \xi)\}_{|\theta|\leqq 1}$ is bounded in $S^{m_1+m_2-\rho|\gamma|}_{\lambda,\rho,0}$.

Proof. By the same method of the Theorem 2.5 and 2.6 in [11] we can prove the formula (1.6) if we have only to prove $\{r_{\gamma,\theta}\}$ is a bounded set in $S^{m_1+m_2-\rho|\gamma|}_{\lambda,\rho,0}$. Since $\partial^\alpha_\xi D^\beta_x r_{\gamma,\theta}$ is represented as the linear combination of

$$(1.7) \qquad \iint e^{-iy\cdot\eta}p^{(\alpha_1+\gamma)}_{1(\beta_1)}(x, \xi+\theta\eta)p^{(\alpha_2)}_{2(\beta_2+\gamma)}(x+y, \xi)dyd\eta,$$
$$(\alpha = \alpha_1+\alpha_2,\ \beta = \beta_1+\beta_2)$$

we have only to prove that each term of the form (1.7) is estimated by $C\lambda(x, \xi)^{m_1+m_2-\rho|\gamma|-\rho|\alpha|}$. Here and in what follows we omit the notation O_s-. We have

$$\left|\iint e^{-iy\cdot\eta}p^{(\alpha_1+\gamma)}_{1(\beta_1)}(x, \xi+\theta\eta)p^{(\alpha_2)}_{2(\beta_2+\gamma)}(x+y, \xi)dyd\eta\right|$$
$$= \left|\iint e^{-iy\cdot\eta}\langle y\rangle^{-2l_1}\langle D_\eta\rangle^{2l_1}p^{(\alpha_1+\gamma)}_{1(\beta_1)}(x, \xi+\theta\eta)p^{(\alpha_2)}_{2(\beta_2+\gamma)}(x+y, \xi)dyd\eta\right|$$

$$\leqq \left| \int_{|\eta|\leqq C_0\lambda} \langle\eta\rangle^{-n_0} \bar{d}\eta \int e^{-iy\cdot\eta} \langle D_y\rangle^{n_0} \{\langle y\rangle^{-2l_1} \langle D_\eta\rangle^{2l_1} p_{1(\beta_1)}^{(\alpha_1+\gamma)}(x, \xi+\theta\eta) \right.$$
$$\left. \cdot p_{2(\beta_2+\gamma)}^{(\alpha_2)}(x+y, \xi)\} dy \right|$$
$$+ \left| \int_{|\eta|\geqq C_0\lambda} |\eta|^{-2l_2} \bar{d}\eta \int e^{-iy\cdot\eta} (-\Delta_y)^{l_2} \{\langle y\rangle^{-2l_1} \langle D_\eta\rangle^{2l_1} p_{1(\beta_1)}^{(\alpha_1+\gamma)}(x, \xi+\theta\eta) \right.$$
$$\left. \cdot p_{2(\beta_2+\gamma)}^{(\alpha_2)}(x+y, \xi)\} dy \right|$$
$$\leqq C \left\{ \int_{|\eta|\leqq C_0\lambda} \langle\eta\rangle^{-n_0} \bar{d}\eta \int \langle y\rangle^{-2l_1} \lambda(x, \xi+\theta\eta)^{m_1-\rho|\gamma|-\rho|\alpha_1|} \lambda(x+y, \xi)^{m_2-\rho|\alpha_2|} dy \right.$$
$$\left. + \int_{|\eta|\geqq C_0\lambda} |\eta|^{-2l_2} \bar{d}\eta \int \langle y\rangle^{-2l_1} \lambda(x, \xi+\theta\eta)^{m_1-\rho|\gamma|-\rho|\alpha_1|} \lambda(x+y, \xi)^{m_2-\rho|\alpha_2|} dy \right\}$$
$$\leqq C \left\{ \lambda(x, \xi)^{m_1+m_2-\rho|\gamma|-\rho|\alpha|} \int \langle\eta\rangle^{-n_0} \bar{d}\eta \int \langle y\rangle^{-2l_1+\tau_1|m_2-\rho|\alpha_2||} dy \right.$$
$$\left. + \lambda(x, \xi)^{m_2-\rho|\alpha_2|} \int_{|\eta|\geqq C_0\lambda} |\eta|^{-2l_2+m_{1+}} \bar{d}\eta \int \langle y\rangle^{-2l_1+\tau_1|m_2-\rho|\alpha_2||} dy \right\}$$
$$\leqq C\lambda(x, \xi)^{m_1+m_2-\rho|\gamma|-\rho|\alpha|},$$

where $n_0=2([n/2]+1)$, $m_{1+}=\mathrm{Max}(m_1, 0)$, l_1, l_2 are integers such that

$$-2l_1+\tau_1|m_2-\rho|\alpha_2||<-n, \quad -2l_2+m_{1+}+n+1\leqq \mathrm{Min}(0, m_1-\rho|\gamma|-\rho|\alpha_1|),$$

and C_0 is a constant such that

$$\frac{1}{2}\lambda(x, \xi)\leqq\lambda(x, \xi+\eta)\leqq\frac{3}{2}\lambda(x, \xi) \qquad \text{if } |\eta|\leqq C_0\lambda(x, \xi). \tag{1.8}$$

We can prove the following two theorems by the same method.

Theorem 1.5. *Let $S_{\lambda,\rho,0}^{m,m'}$ denote a set of double symbols $p(\xi, x', \xi')$, which satisfy*

$$|p_{(\beta)}^{(\alpha,\alpha')}(\xi, x', \xi')|\leqq C_{\alpha\alpha'\beta}\lambda(x', \xi)^{m-\rho|\alpha|}\lambda(x', \xi')^{m'-\rho|\alpha'|},$$

and define operators $P=p(D_x, X', D_{x'})$ by

$$\widehat{Pu}(\xi) = O_s\text{-}\iint e^{-ix'\cdot(\xi-\xi')} p(\xi, x', \xi')\hat{u}(\xi') \bar{d}\xi' dx' \qquad \text{for } u\in\mathcal{S}.$$

Then P belongs to $S_{\lambda,\rho,0}^{m+m'}$ and we can write $\sigma(P)(x, \xi)$ in the form (1.6) *for any $N>0$, where*

$$\begin{cases} p_\alpha(x, \xi) = p_{(\alpha)}^{(\alpha,0)}(\xi, x, \xi) \qquad (\in S_{\lambda,\rho,0}^{m+m'-\rho|\alpha|}) \\ r_{\gamma,\theta}(x, \xi) = O_s\text{-}\iint e^{-iy\cdot\eta} p_{(\gamma)}^{(\gamma,0)}(\xi+\theta\eta, x+y, \xi) dy \bar{d}\eta\,. \end{cases}$$

The set $\{r_{\gamma,\theta}(x, \xi)\}_{|\theta|\leqq1}$ is bounded in $S_{\lambda,\rho,0}^{m+m'-\rho|\gamma|}$.

Theorem 1.6. *For $P=p(X, D_x)\in S^m_{\lambda,\rho,0}$, the operator $P^{(*)}$ defined by*

$$(Pu, v) = (u, P^{(*)}v) \qquad \text{for } u, v\in \mathcal{S}$$

belongs to $S^m_{\lambda,\rho,0}$ and we have for any $N>0$

$$\sigma(P^{(*)})(x, \xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} p^{(*)}_{\alpha}(x, \xi) + N \sum_{|\gamma|=N} \int_0^1 \frac{(1-\theta)^{N-1}}{\gamma!} r^{(*)}_{\gamma,\theta}(x, \xi) d\theta ,$$

where

$$\begin{cases} p^{(*)}_{\alpha}(x, \xi) = (-1)^{|\alpha|} \overline{p^{(\alpha)}_{(\alpha)}(x, \xi)} \qquad (\in S^{m-\rho|\alpha|}_{\lambda,\rho,0}) \\ r^{(*)}_{\gamma,\theta}(x, \xi) = O_s-\iint e^{-iy\cdot\eta}(-1)^{|\gamma|} \overline{p^{(\gamma)}_{(\gamma)}(x+y, \xi+\theta\eta)} dy d\eta . \end{cases}$$

The set $\{r^{(*)}_{\gamma,\theta}(x, \xi)\}_{|\theta|\leq 1}$ is bounded in $S^{m-\rho|\gamma|}_{\lambda,\rho,0}$.

REMARK. The maps

$$S^{m_1}_{\lambda,\rho,0}\times S^{m_2}_{\lambda,\rho,0} \ni (p_1, p_2) \to p_1\circ p_2 \in S^{m_1+m_2}_{\lambda,\rho,0}$$

and

$$S^m_{\lambda,\rho,0} \ni p \to p^{(*)} \in S^m_{\lambda,\rho,0}$$

are continuous.

Let $q(\sigma)$ be a C^∞- and even-function such that $q(\sigma)\geq 0$, $\int q(\sigma)^2 d\sigma=1$ and $\operatorname{supp} q\subset \{\sigma\in R^n;\ |\sigma|\leqq 1\}$, and set

$$F(x, \xi; \zeta) = \lambda(x, \xi)^{-n/4} q((\zeta-\xi)/\lambda(x, \xi)^{1/2}) .$$

Theorem 1.7. *For $P=p(X, D_x)\in S^m_{\lambda,1,0}$, we define the Friedrichs part $P_F=p_F(D_x, X', D_{x'})$ by*

$$p_F(\xi, x', \xi') = \int F(x', \xi; \zeta) p(x', \zeta) F(x', \xi'; \zeta) d\zeta .$$

Then we have

(i) *$p_F(\xi, x', \xi')$ belongs to $S^{2m,0}_{\sqrt{\lambda},1,0}$,*

(ii) *The operator P_F belongs to $S^m_{\lambda,1,0}$ and $P-P_F\in S^{m-1}_{\lambda,1,0}$, and $\sigma(P_F)$ has the form*

$$\sigma(P_F)(x, \xi) \sim p(x, \xi) + \sum_{|\alpha+\beta+\gamma|\geq 2} \psi_{\alpha\beta\gamma}(x, \xi) p^{(\alpha)}_{(\beta)}(x, \xi)$$

where $\psi_{\alpha\beta\gamma}\in S^{(|\alpha|-|\beta|-|\gamma|)/2}_{\lambda,1,0}$,

(iii) *If $p(x, \xi)$ is real-valued and non-negative, we have*

$$(p_F(D_x, X', D_{x'})u, v) = (u, p_F(D_x, X', D_{x'})v) \qquad \text{for } u, v\in \mathcal{S},$$

$$(p_F(D_x, X', D_{x'})u, u) \geqq 0 \qquad \text{for } u\in \mathcal{S} .$$

Proof is carried out by the similar way to that in [9].

Theorem 1.8. *We can extend $P=p(X, D_x)\in S^0_{\lambda,0,0}$ to a bounded operator on L^2 and we get*

$$\|Pu\|_{L^2}\leqq C|p|^{(0)}_{l_0,l_0}\|u\|_{L^2}, \tag{1.9}$$

where C and l_0 are independent of P and u.

Since $S^0_{\lambda,0,0}\subset S^0_{<\xi>,0,0}$, this theorem is a corollary of Calderón-Vaillancourt's theorem in [2].

2. Global analytic-hypoellipticity

DEFINITION 2.1. We say that $L\in S^m_{\lambda,1,0}$ is globally analytic-hypoelliptic if the following statement holds for L:
If $u\in L^2(R^n)$ is a solution of the equation

$$L(X, D_x)u=f \qquad \text{for} \quad f\in C^\infty(R^n)$$

and f satisfies for some $M>0$

$$\|D_x^\alpha f\|_{L^2}\leqq M^{1+|\alpha|}\alpha!, \tag{2.1}$$

then u is analytic and we have

$$\|D_x^\alpha u\|_{L^2}\leqq M_1^{1+|\alpha|}\alpha! \tag{2.2}$$

for another constant $M_1>0$.

Theorem 2.2. *Let $L\in S^m_{\lambda,1,0}$ $(m>0)$ satisfy the following conditions:*

$$|L(x, \xi)|\geqq C\lambda(x, \xi)^m \qquad \text{for} \quad |\xi|\geqq R \tag{2.3}$$

for some $C>0$ and $R\geqq 0$, and for any multi-index α there exists M_α such that

$$|L^{(\alpha)}_{(\beta)}(x, \xi)|\leqq M_\alpha^{1+|\beta|}\beta!\,\lambda(x, \xi)^{m-|\alpha|}. \tag{2.4}$$

Then the operator $L(X, D_x)$ is globally analytic-hypoelliptic.

EXAMPLE 2.3. Let $L(x_1, x_2, D_{x_1}, D_{x_2})=D^2_{x_1}+D^6_{x_2}+x_1^2+x_2^6-15x_2^4+45x_2^2-16$. Then we can prove that L satisfies the conditions (2.3) and (2.4) by taking $\lambda(x_1, x_2, \xi_1, \xi_2)=(1+|L(x_1, x_2, \xi_1, \xi_2)|^2)^{1/12}$ as a basic weight function. The equation $L(X_1, X_2, D_{x_1}, D_{x_2})u=0$ has a non-trivial solution $e^{-(x_1^2+x_2^2)/2}$.

As a generalization of the above example we have

EXAMPLE 2.4 (cf. [5]). Let $L(x, D_x)=\sum_{|\alpha|\leqq m_1} a_\alpha(x)D_x^\alpha$ be a hypoelliptic differential operator of order m_1 with analytic coefficients. Suppose that L satisfies following conditions for constants $\tau_0\geqq 0$, $0<\rho\leqq 1$, $C_1>0$, $C_2>0$, $M>0$,

(0) $|\partial_x^\beta a_\alpha(x)| \leqq M^{1+|\beta|}\beta!$ if $|\beta| \geqq m_1\tau_0$ and $|\alpha| \leqq m_1$,
(i) $C_1^{-1}\langle\xi\rangle^{\rho m_1} \leq |L(0, \xi)| \leqq C_1 |L(x, \xi)|$ for large $|\xi|$,
(ii) $|L_{(\beta)}^{(\alpha)}(x, \xi)/L(x, \xi)| \leqq M^{1+|\beta|}\beta!(|\xi|+|x|^{\tau_0})^{-\rho|\alpha|}$ for large $|\xi|+|x|^{\tau_0}$,
(iii) $|L_{(\beta)}(x, \xi)| \leqq C_2(1+|L(0, \xi)|)$ if $|\beta| \geqq m_1\tau_0$.

Then we can see that L satisfies the conditions of Theorem 2.2 by taking $\lambda(x, \xi)=(1+|L(x, \xi)|^2)^{1/2m}$ for a large m as a basic weight function.

Proof. From (0) we can choose a positive constant m' such that

$$|L(x, \xi)| \leqq C(|\xi|+|x|^{\tau_0})^{m'} \quad \text{for} \quad |\xi|+|x|^{\tau_0} \geqq 1 .$$

We put $m=m'/\rho$ and $\lambda(x, \xi)=(1+|L(x, \xi)|^2)^{1/2m}$. Then we have (2.4) from (0) and (ii). By usual calculus we have (1.2) for $\delta=0$. From (i) we have (1.1) for $a=\rho m_1/m$ and (2.3). Finally we can get (1.3) by (i) and (iii).

EXAMPLE 2.5. Let $L(x_1, x_2, D_{x_1}, D_{x_2})=iD_{x_1}+D_{x_2}^2-2ix_2{}^3D_{x_2}+x_1-x_2{}^6-3x_2{}^2$. Then L is a semi-elliptic operator and $Lu=0$ has a non-analytic solution $u=e^{-(x_1{}^2/2+x_2{}^4/4)}\sum_{m=0}^\infty \frac{f^{(m)}(x_1)}{(2m)!}x_2{}^{2m}(\in \mathcal{S})$ where $f(x_1)\in C_0^\infty(R^1)$ and belongs to the Gevrey class $\rho(<(3/2))$. This fact means the conditions are necessary in general. In fact let L satisfy (2.3) and (2.4). Then we have the following contrary:

$$1 = |\partial_{x_1}L(-t^2, 0, 0, t)| \leqq C\lambda(-t^2, 0, 0, t)^m \leqq |L(-t^2, 0, 0, t)| = 0 \quad \text{for large } t .$$

Proof of Theorem 2.2. Define $\{E_j(x, \xi)\}_{j=0,1\dots}$ for $|\xi| \geqq R$ inductively by

$$\begin{aligned} &E_0(x, \xi) = L(x, \xi)^{-1} , \\ &E_j(x, \xi) = -\sum_{l=0}^{j-1}\sum_{|\gamma|=j-l}\frac{1}{\gamma!}E_l^{(\gamma)}(x, \xi)L_{(\gamma)}(x, \xi)E_0(x, \xi) \qquad (j\geqq 1) , \end{aligned} \tag{2.5}$$

then we have $|E_{j(\beta)}^{(\alpha)}| \leqq C_{j\alpha\beta}\lambda(x, \xi)^{-m-j-|\alpha|}$ if $|\xi| \geqq R$. Taking $\varphi_R(\xi)\in C^\infty$ such that $\varphi_R=1$ if $|\xi| \geqq 2R$ and $\varphi_R=0$ if $|\xi| \leqq R$, and an integer N such that $aN \geqq 1$, we define

$$E(x, \xi) = \varphi_R(\xi)\sum_{j=0}^{N-1}E_j(x, \xi)\in S_{\lambda,0,0}^{-m} . \tag{2.6}$$

Then we have

$$EL = I-K, \quad K\in S_{<\xi>,0,0}^{-1} . \tag{2.7}$$

In fact by the same method of Theorem 1.4 we have

$$\begin{aligned} &\sigma(EL)(x, \xi)-1 \\ &= \sum_{j=0}^{N-1}\sum_{|\gamma|<N-j}\frac{1}{\gamma!}\varphi_R(\xi)E_j^{(\gamma)}(x, \xi)L_{(\gamma)}(x, \xi)-1 \end{aligned} \tag{2.8}$$

$$+\sum_{j=0}^{N-1}\sum_{|\gamma_1+\gamma_2|<N-j,\gamma_1\neq 0}\frac{1}{\gamma_1!\gamma_2!}\partial_\xi^{\gamma_1}\varphi_R(\xi)E_j^{(\gamma_2)}(x,\xi)L_{(\gamma_1+\gamma_2)}(x,\xi)$$
$$+\sum_{j=0}^{N-1}\sum_{|\gamma_1+\gamma_2|=N-j}(N-j)\int_0^1\frac{(1-\theta)^{N-j-1}}{\gamma_1!\gamma_2!}r_{j\gamma_1\gamma_2\theta}(x,\xi)d\theta$$
$$\equiv I_1+I_2+I_3,$$

where

$$r_{j\gamma_1\gamma_2\theta}(x,\xi)=\iint e^{-iy\cdot\eta}\partial_\xi^{\gamma_1}\varphi_R(\xi+\theta\eta)E_j^{(\gamma_2)}(x,\xi+\theta\eta)L_{(\gamma_1+\gamma_2)}(x+y,\xi)dyd\eta\,.$$

From (2.5) we have

$$I_1=\varphi_R(\xi)-1\in S^{-1}_{<\xi>,0,0}\,. \tag{2.9}$$

From the fact that $\partial_\xi^{\gamma_1}\varphi_R(\xi)$ has compact support if $\gamma_1\neq 0$, we get

$$I_2\in S^{-1}_{<\xi>,0,0}\,. \tag{2.10}$$

Next we prove that $\{r_{j\gamma_1\gamma_2\theta}\}_{|\theta|\leqq 1}$ is bounded in $S^{-1}_{<\xi>,0,0}$. Since $\partial_\xi^\alpha D_x^\beta r_{j\gamma_1\gamma_2\theta}$ is a linear combination of

$$r'_\theta(x,\xi)=\iint e^{-iy\cdot\eta}\partial_\xi^{\alpha_1+\gamma_1}\varphi_R(\xi+\theta\eta)E_{j(\beta_1)}^{(\alpha_2+\gamma_2)}(x,\xi+\theta\eta)L^{(\alpha_3)}_{(\beta_2+\gamma_1+\gamma_2)}(x+y,\xi)dyd\eta$$

such that $\alpha_1+\alpha_2+\alpha_3=\alpha$, $\beta_1+\beta_2=\beta$. Hence we have only to prove for a constant C

$$|r'_\theta|\leqq C\langle\xi\rangle^{-1}\,.$$

We take a constant C_0 such that (1.8) is satisfied and integers l_1, l_2, l_3 such that $-2l_1+m\tau_1<-n$, $-2l_2+1<-n$, $-2l_3+n+1\leqq -m-1/a$. Then we have

$$\begin{aligned}
&|r'_\theta(x,\xi)|\\
&=\Big|\iint e^{-iy\cdot\eta}\langle y\rangle^{-2l_1}\langle D_\eta\rangle^{2l_1}\{\partial_\xi^{\alpha_1+\gamma_1}\varphi_R(\xi+\theta\eta)E_{j(\beta_1)}^{(\alpha_2+\gamma_2)}(x,\xi+\theta\eta)\\
&\qquad\qquad\cdot L^{(\alpha_3)}_{(\beta_2+\gamma_1+\gamma_2)}(x+y,\xi)\}dyd\eta\Big|\\
&\leqq\int_{|\eta|\leqq C_0\lambda}\langle\eta\rangle^{-2l_2}d\eta\int|\langle D_y\rangle^{2l_2}[\langle y\rangle^{-2l_1}\langle D_\eta\rangle^{2l_1}\{\partial_\xi^{\alpha_1+\gamma_1}\varphi_R(\xi+\theta\eta)\\
&\qquad\qquad\cdot E_{j(\beta_1)}^{(\alpha_2+\gamma_2)}(x,\xi+\theta\eta)L^{(\alpha_3)}_{(\beta_2+\gamma_1+\gamma_2)}(x+y,\xi)\}]|dy\\
&+\int_{|\eta|\geqq C_0\lambda}|\eta|^{-2l_3}d\eta\int|(-\Delta_y)^{l_3}[\langle y\rangle^{-2l_1}\langle D_\eta\rangle^{2l_1}\{\partial_\xi^{\alpha_1+\gamma_1}\varphi_R(\xi+\theta\eta)\\
&\qquad\qquad\cdot E_{j(\beta_1)}^{(\alpha_2+\gamma_2)}(x,\xi+\theta\eta)L^{(\alpha_3)}_{(\beta_2+\gamma_1+\gamma_2)}(x+y,\xi)\}]|dy\\
&\equiv J_1+J_2\,.
\end{aligned}$$

To estimate J_1 we devide into two cases.

(i) When $\alpha_1+\gamma_1=0$ we have, noting that $|\gamma_2|=N-j$

$$J_1\leqq C\int_{|\eta|\leqq C_0\lambda}\langle\eta\rangle^{-2l_2}d\eta\int\langle y\rangle^{-2l_1}\lambda(x,\ \xi+\theta\eta)^{-m-N}\lambda(x+y,\ \xi)^m dy$$
$$\leqq C\lambda(x,\ \xi)^{-N}\int\langle\eta\rangle^{-2l_2}d\eta\int\langle y\rangle^{-2l_1+m\tau_1}dy\leqq C\langle\xi\rangle^{-1}\,.$$

(ii) When $\alpha_1+\gamma_1\neq 0$ we have, noting that $\partial_\xi^{\alpha_1+\gamma_1}\varphi_R$ has compact support

$$J_1\leqq C\int_{|\eta|\leqq C_0\lambda}\langle\eta\rangle^{-2l_2}d\eta\int\langle y\rangle^{-2l_1}\langle\xi+\theta\eta\rangle^{-1}\lambda(x,\ \xi+\theta\eta)^{-m}\lambda(x+y,\ \xi)^m dy$$
$$\leqq C\langle\xi\rangle^{-1}\int\langle\eta\rangle^{-2l_2+1}d\eta\int\langle y\rangle^{-2l_1+m\tau_1}dy\leqq C\langle\xi\rangle^{-1}\,.$$

Next for J_2 we have

$$J_2\leqq C\int_{|\eta|\geqq C_0\lambda}|\eta|^{-2l_3}d\eta\int\langle y\rangle^{-2l_1}\lambda(x+y,\ \xi)^m dy$$
$$\leqq C\lambda(x,\ \xi)^{-2l_3+m+n}\int\langle y\rangle^{-2l_1+m\tau_1}dy\leqq C\lambda(x,\ \xi)^{-1/a}\leqq C\langle\xi\rangle^{-1}\,.$$

Hence we get $I_3\in S^{-1}_{\langle\xi\rangle,0,0}$ and combining (2.8)–(2.10) we get (2.7). From (2.4) and (2.6) we see also that there exists M_2 independent of γ such that

$$|\sigma(EL_{(\gamma)})|^{(0)}_{l_0,l_0}\leqq M_2^{1+|\gamma|}\gamma! \qquad \text{for } l_0 \text{ in Theorem 1.8.} \tag{2.11}$$

Moreover from (2.7) there exists constant C_1 such that

$$|K(x,\ \xi)\xi_j|^{(0)}_{l_0,l_0}\leqq C_1 \qquad \text{for any } j=1,\cdots,n\,. \tag{2.12}$$

Suppose that for $u\in L^2$ $Lu=f$ satisfies (2.1). We have $u=ELu+Ku$ $=Ef+Ku$ from (2.7) and so it is clear that u is a C^∞-function. Therefore we have only to prove that u satisfies (2.2), since (2.2) implies the analyticity of u by Sobolev's lemma. Take M_1 sufficiently large such that

$$3C_2C_1\leqq M_1\,, \tag{2.13}$$

$$3C_2M|E|^{(0)}_{l_0,l_0}\leqq M_1\,, \qquad M\leqq M_1\,, \tag{2.14}$$

$$3\cdot 2^n C_2M_2^2\leqq M_1\,, \qquad 2M_2\leqq M_1\,, \tag{2.15}$$

$$\|u\|_{L^2}\leqq M_1\,, \tag{2.16}$$

where C_2 is a constant satisfying (1.9).
From (2.16), (2.2) is trivial when $\alpha=0$, so we show (2.2) by induction on $|\alpha|$. From (2.7), $D_x^\alpha u=ELD_x^\alpha u+KD_x^\alpha u$ $(\alpha\neq 0)$. Then we have

$$\|D_x^\alpha u\|\leqq\|ELD_x^\alpha u\|+\|KD_x^\alpha u\|\,. \tag{2.17}$$

Since $\alpha\neq 0$ there exists multi-index α_2 such that $|\alpha_2|=1$, $\alpha=\alpha_1+\alpha_2$. By (2.12), (2.13) and Theorem 1.8 we get

$$(2.18)\quad \|KD_x^\alpha u\| = \|(KD_x^{\alpha_2})D_x^{\alpha_1}u\| \leqq C_2C_1\|D_x^{\alpha_1}u\| \leqq C_2C_1M_1^{1+|\alpha_1|}\alpha_1! \leqq M_1^{1+|\alpha|}\alpha!/3.$$

By Leibniz' formula, we have

$$LD_x^\alpha = D_x^\alpha L - \sum_{\alpha_1<\alpha}\frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!}L_{(\alpha-\alpha_1)}D_x^{\alpha_1}.$$

Then

$$(2.19)\qquad \|ELD_x^\alpha u\| \leqq \|ED_x^\alpha f\| + \sum_{\alpha_1<\alpha}\frac{\alpha!}{\alpha_1!(\alpha-\alpha_1!)}\|EL_{(\alpha-\alpha_1)}D_x^{\alpha_1}u\|.$$

From (2.1), (2.6) and (2.14) we have

$$(2.20)\qquad \|ED_x^\alpha f\| \leqq C_2|E|_{l_0,l_0}^{(0)}\|D_x^\alpha f\| \leqq C_2|E|_{l_0,l_0}^{(0)}M^{1+|\alpha|}\alpha! \leqq M_1^{1+|\alpha|}\alpha!/3.$$

Finally we have from (2.11), (2.15) and the assumption of induction

$$\begin{aligned}(2.21)\qquad &\sum_{\alpha_1<\alpha}\frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!}\|EL_{(\alpha-\alpha_1)}D_x^{\alpha_1}u\| \\ &\leqq \sum_{\alpha_1<\alpha}C_2\frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!}M_2^{1+|\alpha-\alpha_1|}(\alpha-\alpha_1)!M_1^{1+|\alpha_1|}\alpha_1! \\ &= M_1^{1+|\alpha|}\alpha!(C_2M_2^2/M_1)\sum_{\alpha_1<\alpha}(M_2/M_1)^{|\alpha-\alpha_1|-1} \leqq M_1^{1+|\alpha|}\alpha!/3.\end{aligned}$$

Therefore from (2.17)–(2.21) we get (2.2).

Corollary 2.6. *Let L satisfy the same conditions as Theorem 2.2. If a bounded and continuous function u is a solution of $Lu=f$ and $f\in C^\infty(R^n)$ satisfies for some M_3*

$$(2.22)\qquad |D_x^\alpha f| \leqq M_3^{1+|\alpha|}\alpha!,$$

then we have for another constant M_4

$$(2.23)\qquad |D_x^\alpha u| \leqq M_4^{1+|\alpha|}\alpha!\langle x\rangle^{n_0} \quad \textit{for an even number } n_0>n.$$

Proof. We write $Lu=f$ in the form

$$\langle X\rangle^{-n_0}L(X, D_x)\langle X'\rangle^{n_0}u_1 = f_1,$$

where $u_1(x)=\langle x\rangle^{-n_0}u(x)$, $f_1(x)=\langle x\rangle^{-n_0}f(x)$.
We write simplified symbol of $\langle X\rangle^{-n_0}L(X, D_x)\langle X'\rangle^{n_0}$ by $L_1(X, D_x)$. Then the pair (L_1, u_1, f_1) satisfies the conditions of the theorem and we get $\|D_x^\alpha u_1\| \leqq M_5^{1+|\alpha|}\alpha!$ for some $M_5>0$. Hence from Sobolev's lemma we can get (2.23).

REMARK. In Theorem 2.2 we may assume (2.4) only for $|\alpha|\leqq l_0$ with l_0 in Theorem 1.8, and in Corollary 2.6 for $|\alpha|\leqq 2l_0$.

3. Local hypoellipticity

In this section we shall study a differential operator $L(x, \mathfrak{y}, D_x, D_y)$ in $R^n_x \times R^k_y$ with polynomial coefficients of the form

$$L(x, \mathfrak{y}, \xi, \eta) = \sum_{|\alpha:\mathfrak{m}|+|\alpha':\mathfrak{m}'|\leqq 1} a_{\alpha\alpha'\gamma\gamma'} x^\gamma \mathfrak{y}^{\gamma'} \xi^\alpha \eta^{\alpha'}, \tag{3.1}$$

where $y=(\mathfrak{y}, \tilde{\mathfrak{y}})$, $\mathfrak{y}=(y_1, \cdots, y_s)$, $\tilde{\mathfrak{y}}=(y_{s+1}, \cdots, y_k)$ for $s\leqq k$, $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\alpha'=(\alpha'_1, \cdots, \alpha'_k)$, $\gamma=(\gamma_1, \cdots, \gamma_n)$, $\gamma'=(\gamma'_1, \cdots, \gamma'_s, 0, \cdots, 0)$ and $|\alpha:\mathfrak{m}|=\alpha_1/m_1+\cdots+\alpha_n/m_n$, $|\alpha':\mathfrak{m}'|=\alpha'_1/m'_1+\cdots+\alpha'_k/m'_k$ for multi-indices $\mathfrak{m}=(m_1, \cdots, m_n)$, $\mathfrak{m}'=(m'_1, \cdots, m'_k)$ of positive integers m_j and m'_l. We say that L is hypoelliptic if $u\in\mathcal{D}'(R^{n+k}_{x,y})$ belongs to $C^\infty(\Omega)$ when Lu belongs to $C^\infty(\Omega)$ for any open set Ω of $R^{n+k}_{x,y}$. Now setting $m=\mathrm{Max}\,\{m_j, m'_l\}$, we assume that there exist four real vectors ρ, ρ', σ, σ' of the form $\rho=(\rho_1, \cdots, \rho_n)$, $\rho'=(\rho'_1, \cdots, \rho'_k)$, $\sigma=(\sigma_1, \cdots, \sigma_n)$, $\sigma'=(\sigma'_1, \cdots, \sigma'_s, 0, \cdots, 0)$ such that

$$\begin{cases} \text{(i)} & \rho_j=\sigma_j=m/m_j \quad \text{for } j=1, \cdots, n \\ \text{(ii)} & \rho'_j>\sigma'_j\geqq 0\,, \quad m'_j\rho'_j\geqq m \quad \text{for } j=1, \cdots, k \end{cases} \tag{3.2}$$

and

$$L(t^{-\sigma}x, t^{-\sigma'}\mathfrak{y}, t^\rho\xi, t^{\rho'}\eta)=t^m L(x, \mathfrak{y}, \xi, \eta) \quad \text{for} \quad t>0\,, \tag{3.3}$$

where $t^{-\sigma}x=(t^{-\sigma_1}x_1, \cdots, t^{-\sigma_n}x_n)$, $t^{-\sigma'}\mathfrak{y}=(t^{-\sigma_1'}y_1, \cdots, t^{-\sigma_s'}y_s)$,

$$t^\rho\xi=(t^{\rho_1}\xi_1, \cdots, t^{\rho_n}\xi_n), \quad t^{\rho'}\eta=(t^{\rho_1'}\eta_1, \cdots, t^{\rho_k'}\eta_k)\,.$$

Condition 1. If we put

$$L_0(x, \mathfrak{y}, \xi, \eta) = \sum_{|\alpha:\mathfrak{m}|+|\alpha':\mathfrak{m}'|=1} a_{\alpha\alpha'\gamma\gamma'} x^\gamma \mathfrak{y}^{\gamma'} \xi^\alpha \eta^{\alpha}\,, \tag{3.4}$$

then we have

$$L_0(x, \mathfrak{y}, \xi, \eta)\neq 0 \qquad \text{for} \quad |x|+|\mathfrak{y}|\neq 0 \text{ and } (\xi, \eta)\neq 0\,, \tag{3.5}$$

which means that $L(x, \mathfrak{y}, \xi, \eta)$ is semi-elliptic for $|x|+|\mathfrak{y}|\neq 0$.

Condition 2. The equation $L(X, \mathfrak{y}, D_x, \eta)v(x)=0$ in R^n_x has no non-trivial solution in $\mathcal{S}(R^n_x)$ for $|\eta|=1$.

Theorem 3.1. *We consider the operator $L(x, \mathfrak{y}, D_x, D_y)$ under Condition* 1 *and the assumption*

$$\underset{1\leqq j\leqq k}{\mathrm{Max}}\,\{\sigma'_j\}<\underset{1\leqq j,l\leqq k}{\mathrm{Min}}\,\{m'_j\rho'_j/m'_l\}\,.$$

Then we have

(S) *If Condition 2 holds, then $L(x, \mathfrak{y}, D_x, D_y)$ is hypoelliptic.*

(N) *If the coefficients of L are independent of $\mathfrak{y}$, i.e., $s=0$, then Condition 2 is necessary for the hypoellipticity of the operator L.*

EXAMPLES 3.2.

i) $L=(-\Delta_x)^l+|x|^{2\nu}(-\Delta_y)^{l'}$ in $R^n_x\times R^k_y$ (cf. [3], [7], [14]).
We set $\rho_1=\cdots=\rho_n=\sigma_1=\cdots=\sigma_n=l_0/l$, $\rho'_1=\cdots=\rho'_k=(\nu/l+1)l_0/l'$, $\sigma'_1=\cdots=\sigma'_k=0$, where $l_0=\mathrm{Max}(l, l')$. Then we can see that L is always hypoelliptic.

ii) $L_\pm(x, D_x, D_y)=D_x\pm ix^lD^m_y$ in $R^1_x\times R^1_y$ (cf. [6], [8], [15]).
We set $\rho_1=\sigma_1=m$, $\rho'_1=l+1$, $\sigma'_1=0$. Then we see the following three cases:

a) If l is even, $L_+(X, D_x, \pm 1)v=0$ and $L_-(X, D_x, \pm 1)v=0$ have no non-trivial solution in $\mathcal{S}$.

b) If l is odd and m is even, $L_+(X, D_x, \pm 1)v=0$ has no non-trivial solution in $\mathcal{S}$ and $L_-(X, D_x, \pm 1)v=0$ has non-trivial solution $e^{-x^{l+1}/(l+1)}\in\mathcal{S}$.

c) If l and m are odd, $L_+(X, D_x, -1)v=0$ has non-trivial solution $e^{-x^{l+1}/(l+1)}\in\mathcal{S}$ and $L_-(X, D_x, 1)v=0$ has non-trivial solution $e^{-x^{l+1}/(l+1)}\in\mathcal{S}$.

Consequently we see from (N) and (S) that L_+ is hypoelliptic if and only if "l is even", or "l is odd and m is even", and L_- is hypoelliptic if and only if "l is even".

iii) $L=D^2_{x_1}+D^6_{x_2}+(x_1{}^2+x_2{}^6)D^6_y-15x_2{}^4D^5_y+45x_2{}^2D^4_y-16D^3_y$ in $R^2_x\times R^1_y$.
We set $\rho_1=\sigma_1=3$, $\rho_2=\sigma_2=1$, $\rho'_1=2$, $\sigma'_1=0$. We can see that L does not satisfy Condition 2. In fact for $\eta=1$ $L(X_1, X_2, D_{x_1}, D_{x_2}, 1)v(x_1, x_2)=0$ is an equation given in Example 2.3 and has non-trivial solution $v=e^{(-x_1{}^2+x_2{}^2)/2}$. Therefore applying (N) we can see that L is not hypoelliptic.

For the proof of the theorem we need several lemmas. We introduce notations: $|x, y|_{(\sigma,\sigma')}=\sum_{j=1}^n|x_j|^{1/\sigma_j}+\sum_{j=1}^s|y_j|^{1/\sigma'_j}$,

$$|\eta|_{\rho'}=\sum_{j=1}^k|\eta_j|^{1/\rho'_j},\quad \mu(x, y, \eta)=\sum_{j=1}^k|x, y|^{(m_j'\rho_j'-m)}_{(\sigma,\sigma')}|\eta_j|^{m_j'}.$$

First we estimate the monomials of the form $x^\gamma y^{\gamma'}\eta^{\alpha'}$.

Lemma 3.3. *Let α, α', γ and γ' be multi-indices of dimension n, k, n, k, respectively, such that $|\alpha:\mathrm{m}|+|\alpha':\mathrm{m}'|\leqq 1$ and $\gamma'_j=0$ for $j\geqq s+1$. We put*

$$\theta=(\sigma, \gamma)+(\sigma', \gamma')+m-(\rho, \alpha)-(\rho', \alpha').\tag{3.6}$$

If we denote $\rho'_0=\mathrm{Min}_{1\leqq j\leqq k}(m'_j\rho'_j/m)$, then we have

(i) *If there exists $\theta'\geqq 0$ such that $m(|\alpha:\mathrm{m}|+|\alpha':\mathrm{m}'|)+(\theta+\theta')/\rho'_0\leqq m$, we have*

$$|x, y|^{\theta'}_{(\sigma,\sigma')}|x^\gamma y^{\gamma'}\eta^{\alpha'}|\;|\eta|^{\theta+\theta'}_{\rho'}\leqq C(|\eta|^m_{\rho'}+\mu(x, y, \eta))^{1-|\alpha:\mathrm{m}|}.\tag{3.7}$$

(ii) *If $m(|\alpha:\mathrm{m}|+|\alpha':\mathrm{m}'|)+\theta/\rho'_0>m$, we have*

$$|x^\gamma y^{\gamma'}\eta^{\alpha'}|\,|\eta|^{(1-|\alpha:\mathrm{m}|-|\alpha':\mathrm{m}'|)m\rho'_0}_{\rho'}\leqq C(|\eta|^m_{\rho'}+\mu(x, y, \eta))^{1-|\alpha:\mathrm{m}|}\tag{3.8}$$

for $|x|\leqq\delta$, $|y|\leqq\delta$ *and* $|\eta|\geqq 1$, *where* δ *is some positive constant.*

We can prove this by the same method as Lemma 3.1 and 3.2 in [4].

Lemma 3.4. *Under condition* 1 *we have for a constant* $C>0$

$$(3.9)\qquad C^{-1}|L_0(x, y, \xi, \eta)|\leqq\{\sum_{j=1}^{n}|\xi_j|^{m_j}+\mu(x, y, \eta)\}\leqq C|L_0(x, y, \xi, \eta)|.$$

Proof. In case $|x|+|y|\neq 0$, it is sufficient for the sake of semi-homogeneity to prove when $|x|+|y|=1$, and this is true because of Condition 1. In case $|x|+|y|=0$, (3.9) is clear by letting $|x|+|y|\to 0$.

Define $\lambda_h(x, \xi)$ with parameter $h=(y, \eta)$ ($|\eta|=1$) by $\lambda_h(x, \xi)=\{1+|L(x, y, \xi, \eta)|^2\}^{1/2m}$ and set $p_h(x, \xi)=L(x, y, \xi, \eta)$. Then we have

Proposition 3.5.
(i) $\lambda_h(x, \xi)$ *satisfies* (1.1)–(1.3).
(ii) $\{p_h(x, \xi)\}$ *is bounded in* $\{S^m_{\lambda_h,1,0}\}$ *in the sense that for any* α, β *there exists a bounded function* $C_{\alpha\beta}(x, y)$ *which is independent of* η ($|\eta|=1$) *and tends to zero as* $|x|+|y|\to\infty$ *when* $\beta\neq 0$, *such that*

$$|p_{h(\beta)}^{(\alpha)}(x, \xi)|\leqq C_{\alpha\beta}(x, y)\lambda_h(x, \xi)^{m-|\alpha|}.$$

(iii) *There exists a constant C independent of h such that*

$$(3.10)\qquad |p_h(x, \xi)|\geqq C\lambda_h(x, \xi)^m\qquad \text{for large}\quad |x|+|y|+|\xi|.$$

Proof. Set $\lambda_h'(x, \xi)=\{1+\sum_{j=1}^{n}|\xi_j|^{m_j}+\mu(x, y, \eta)\}^{1/m}$. Then from Lemma 3.3 (i) and Lemma 3.4 we can prove

$$(3.11)\qquad |L(x, y, \xi, \eta)|\geqq C\lambda_h'(x, \xi)^m\qquad \text{for large } |x|+|y|+|\xi|,$$

which induces

$$(3.12)\qquad C^{-1}\lambda_h'(x, \xi)\leqq\lambda_h(x, \xi)\leqq C\lambda_h'(x, \xi).$$

For each term $a_{\alpha\alpha'\gamma\gamma'}x^\gamma y^{\gamma'}\xi^\alpha\eta^{\alpha'}$ in L, we have from Lemma 3.3

$$\begin{aligned}&|\partial_x^{\beta_1}\partial_\xi^{\alpha_1}(a_{\alpha\alpha'\gamma\gamma'}x^\gamma y^{\gamma'}\xi^\alpha\eta^{\alpha'})|\\ &\leqq C\,\mathrm{Min}(1, |x, y|_{(\sigma,\sigma')}^{-(\sigma,\beta_1)})(1+\mu(x, y, \eta))^{1-|\alpha:\mathfrak{m}|}(1+\sum_{j=1}^{n}|\xi_j|^{m_j})^{|\alpha:\mathfrak{m}|-|\alpha_1:\mathfrak{m}|}\\ &\leqq C\,\mathrm{Min}(1, |x, y|_{(\sigma,\sigma')}^{-(\sigma,\beta_1)})\lambda_h'(x, \xi)^{m-|\alpha_1|}\qquad(\alpha_1\leqq\alpha).\end{aligned}$$

Here we use the fact that $|\eta|=1$. Therefore we have

$$(3.13)\qquad |p_{h(\beta)}^{(\alpha)}(x, \xi)|\leqq C\,\mathrm{Min}(1, |x, y|_{(\sigma,\sigma')}^{-(\sigma,\beta)})\lambda_h'(x, \xi)^{m-|\alpha|}.$$

First we check (i). From (3.12) λ_h satisfies (1.1) for $a=\underset{1\leqq j\leqq n}{\mathrm{Min}}\{m_j/m\}$. By usual calculus (1.2) follows by (3.13). Since p_h is a polynomial in x, we have using Taylor series

$$|p_h(x+z,\ \xi)|\leqq\sum_{|\alpha|\leqq N}|z^\alpha p_{h(\alpha)}(x,\ \xi)|/\alpha!\leqq C\langle z\rangle^{m\tau_1}\lambda_h'(x,\ \xi)^m\leqq C\langle z\rangle^{m\tau_1}\lambda_h(x,\ \xi)^m$$

for some τ_1. So (1.3) holds for λ_h. Consequently we get (i). (ii) and (iii) follow at once by (3.11)-(3.13).

Lemma 3.6. *Let a basic weight function $\lambda(x,\ \xi)$ satisfy*

$$A_0^{-1}(1+|x|+|\xi|)^{a'}\leqq\lambda(x,\ \xi)\leqq A_0(1+|x|^{\tau_0}+|\xi|) \qquad (a'>0,\ A_0>0,\ \tau_0>0) \tag{3.14}$$

instead of (1.1). *Suppose that $p(x,\ \xi)\in S^m_{\lambda,1,0}$ $(m>0)$ satisfies*

$$|p(x,\ \xi)|\geqq C\lambda(x,\ \xi)^m \qquad \textit{for large } |x|+|\xi|.$$

Then for any $u\in L^2(R^n_x)$, $Pu=p(X,\ D_x)u(x)=0$ implies $u\in\mathcal{S}(R^n_x)$.

Proof. Let $Q\in S^{-m}_{\lambda,1,0}$ be a parametrix such that $QP=I-K$, $K\in S^{-\infty}_{\lambda,1,0}$ $(=\bigcap_{-\infty<m<\infty}S^m_{\lambda,1,0})$. Then we have $u=Ku$. For any positive number r and t, $\langle X\rangle^r\langle D_x\rangle^t K(X',\ D_{x'})$ belongs to $S^{-\infty}_{\lambda,1,0}$ and we get $\langle X\rangle^r\langle D_x\rangle^t u\in L^2$. Therefore we get $u\in\mathcal{S}$.

Proposition 3.7. *If Condition* 1 *and* 2 *hold, then for any $v\in C^\infty_0(R^n_x)$ we have*

$$||v||^2_{L^2}\leqq C\int|p_h(X,\ D_x)v(x)|^2dx\ , \tag{3.15}$$

where C is independent of v and h with $|\eta|=1$.

Proof. From (3.10) there exists a parametrix $\{Q_h\}$ which is bounded in $\{S^{-m}_{\lambda_h,1,0}\}$ such that

$$Q_hP_h=I-K_h\ , \tag{3.16}$$

where $\{K_h\}$ is bounded in $\{S^{-m}_{\lambda_h,1,0}\}$, $\lim_{|x|+|\mathfrak{y}|\to\infty}\ \sup_{\xi\in R^n,|\eta|=1}|K_h(x,\ \xi)|=0$ and for any multi-index α, β

$$\sup_{x,\xi}|K_{h(\beta)}^{(\alpha)}(x,\ \xi)-K_{h_0(\beta)}^{(\alpha)}(x,\ \xi)|\to 0 \quad \text{as} \quad h\to h_0\ . \tag{3.17}$$

Therefore we have

$$||v||\leqq||Q_hP_hv||+||K_hv||\leqq C||P_hv||+||K_hv||\ .$$

Since $\{K_h\}$ is bounded in $\{S^{-m}_{\lambda_h,1,0}\}$ and $\lim_{|\mathfrak{y}|\to\infty}\ \sup_{(x,\xi)\in R^{2n},|\eta|=1}|K_h(x,\ \xi)|=0$, we have for a constant l_0 in Theorem 1.8

$$|K_h|^{(0)}_{l_0,l_0}\to 0 \quad \text{as} \quad |\mathfrak{y}|\to\infty\ .$$

Then for a sufficiently large constant $M>0$

$$||K_h v|| \leqq \frac{1}{2}||v|| \qquad \text{for} \quad |\mathfrak{y}| \geqq M,$$

and we get (3.15) for $|\mathfrak{y}| \geqq M$.

Now assume that for $|\mathfrak{y}| \leqq M$ (3.15) does not hold. Then we can choose sequences $\{h_\nu\}$, $\{v_\nu\}$ such that

$$(3.18) \qquad ||v_\nu|| = 1,$$

$$(3.19) \qquad ||P_{h_\nu} v_\nu|| \to 0 \quad \text{as} \quad \nu \to \infty,$$

$$(3.20) \qquad h_\nu = (\mathfrak{y}^\nu, \eta^\nu), \quad \text{where} \quad |\mathfrak{y}^\nu| \leqq M, \ |\eta^\nu| = 1.$$

From (3.20) we may assume that

$$(3.21) \qquad h_\nu \to h_0$$

for some $h_0=(\mathfrak{y}^0, \eta^0)$. Applying v_ν to (3.16) we get

$$(3.22) \qquad Q_{h_\nu} P_{h_\nu} v_\nu = v_\nu - K_{h_\nu} v_\nu.$$

From (3.19) and (3.21) we have $Q_{h_\nu} P_{h_\nu} v_\nu \to 0$ in L^2 as $\nu \to \infty$, and from the fact that $\{K_h\}$ is bounded in $\{S^{-m}_{\lambda_h,1,0}\}$, $\lim_{|x|\to\infty} \sup_{\xi} |K_{h_0}(x, \xi)| = 0$ and (3.17) we get K_h is uniformly continuous and K_{h_0} is a compact operator in L^2 (cf. [10], [12]). So writing $K_{h_\nu} v_\nu = (K_{h_\nu} - K_{h_0}) v_\nu + K_{h_0} v_\nu$ we can choose a convergent subsequence $\{K_{h_{\nu'}} v_{\nu'}\}$ in account of (3.18). Therefore from (3.22) we can choose an element $v_0 \in L^2$ such that

$$(3.23) \qquad v_{\nu'} \to v_0 \qquad \text{in} \quad L^2.$$

Then from (3.19) and (3.21) $P_{h_0} v_0 = 0$. When $\eta^0_j = 0$ for all j such that $m'_j \rho'_j \neq m$, we have $v_0 = 0$ since $p_{h_0}(x, \xi) = \sum a_{\alpha\alpha'00} (\eta^0)^{\alpha'} \xi^\alpha$. Otherwise (3.12) implies (3.14) and we get $v_0 = 0$ from Lemma 3.6 and Condition 2. This is the contrary to (3.18) and (3.23). Then Proposition 3.7 is proved.

Theorem 3.8. *If Condition* 1 *and* 2 *hold, we can get the following formulas for* $|\mathfrak{y}| < \delta$, $|\eta| \geqq 1$ *and* $v \in C_0^\infty(\{x;\ |x| < \delta\})$, *where* δ *is a number which was taken in Lemma* 3.3.

$$(3.24) \qquad \sum_{|\alpha:\mathfrak{m}| \leqq 1} \int |(\mu(x, \mathfrak{y}, \eta) + |\eta|^{\frac{m}{\rho'}})^{1-|\alpha:\mathfrak{m}|} D_x^\alpha v(x)|^2 dx$$
$$\leqq C \int |L(X, \mathfrak{y}, D_x, \eta) v(x)|^2 dx.$$

For any k-dimensional multi-index α_1, β_1 *we have*

(3.25) $$\|\partial_\eta^{\alpha_1}\partial_y^{\beta_1}L(X, y, D_x, \eta)v\|_{L^2}\leqq C|\eta|_{\rho'}^{-\rho_0|\alpha_1|+\sigma_0|\beta_1|}\|L(X, y, D_x, \eta)v\|_{L^2}$$

where $\rho_0=\underset{1\leqq j,\, l\leqq k}{\mathrm{Min}}(m'_j\rho'_j/m'_l)$, $\sigma_0=\underset{1\leqq j\leqq k}{\mathrm{Max}}(\sigma'_j)$.

Proof. Let $r(x, y)$ be a positive root of the equation

$$\sum_{j=1}^{n}\frac{x_j^2}{r^{2\sigma_j}}+\sum_{j=1}^{s}\frac{y_j^2}{r^{2\sigma'_j}}=1\,.$$

Then $r(x, y)$ is a C^∞-function in $R_x^n\times R_y^s\setminus\{0, 0\}$ and

(3.26) $$r(x, y)\sim|x, y|_{(\sigma,\sigma')}\,.$$

Let $\chi(x, y)$ be a C^∞-function such that $\chi=1$ if $|x|+|y|\geqq 1$ and $\chi=0$ if $|x|+|y|\leqq(1/2)$. For any multi-index α ($|\alpha\colon \mathfrak{m}|\leqq 1$) and $h=(y, \eta)$ ($|\eta|=1$) we define $R_{\alpha h}$ by

$$R_{\alpha h}(x, \xi)=(\sum_{j=1}^{k}\chi(x, y)r(x, y)^{(m_j'\rho_j'-m)}|\eta_j|^{m_j'}+1)^{1-|\alpha\colon\mathfrak{m}|}\xi^\alpha\,.$$

Then $\{R_{\alpha h}\}$ is bounded in $\{S^m_{\lambda_h,1,0}\}$. From (3.16) we can write for any $v\in C_0^\infty(R_x^n)$

$$R_{\alpha h}(X, D_x)Q_h(X', D_{x'})p_h(X'', D_{x''})v=R_{\alpha h}(X, D_x)v-R_{\alpha h}(X, D_x)K_h(X', D_{x'})v$$

Noting that $\{R_{\alpha h}(X, D_x)Q_h(X', D_{x'})\}$, $\{R_{\alpha h}(X, D_x)K_h(X', D_{x'})\}$ are bounded in $\{S^0_{\lambda_h,1,0}\}$, we get from Proposition 3.7

$$\|(\sum_{j=1}^{k}\chi(x, y)r(x, y)^{m_j'\rho_j'-m}|\eta_j|^{m_j'}+1)^{1-|\alpha\colon\mathfrak{m}|}D_x^\alpha v\|=\|R_{\alpha h}(X, D_x)v\|$$

$$\leqq\|R_{\alpha h}Q_hP_hv\|+\|R_{\alpha h}K_hv\|\leqq C(\|P_hv\|+\|v\|)\leqq C\|P_hv\|\,.$$

Considering (3.26) we have for $|\eta|=1$

$$\sum_{|\alpha\colon\mathfrak{m}|\leqq 1}\int|(\mu(x, y, \eta)+|\eta|_{\rho'}^{m})^{1-|\alpha\colon\mathfrak{m}|}D_x^\alpha v|^2dx\leqq C\int|L(X, y, D_x, \eta)v|^2dx\,.$$

From the semi-homogeneity we get (3.24). Using Lemma 3.3 and (3.24) we can get (3.25) by the same method as Lemma 3.6 in [4].

Proof of (S) in Theorem 3.1. By the same method as [4] we can prove (S) by using Theorem 3.8.

Proof of (N) of Theorem 3.1 (cf. [3]). Let there exist non-trivial solution $v(x)\in\mathcal{S}$ of $p_h(X, D_x)v(x)=L(X, D_x, \eta)v(x)=0$ for some $h=\eta$ with $|\eta|=1$. From Proposition 3.5 we can apply Theorem 2.2 and we get that $v(x)$ is analytic, and therefore there exists multi-index α_0 such that

(3.27) $$\partial_x^{\alpha_0}v(0)\neq 0\,.$$

We may assume $\eta_1\neq 0$. We set $m_0=\mathrm{Max}(m, |\alpha_0|)$ and take even number l_1 and positive number b such that $\{(\rho, \alpha_0)-(\rho_1'-1)+b\}/\rho_1'$ is an even number (we denote it by l_2) and $2l_1\rho_1'\geqq m_0\cdot\mathrm{Max}(\rho_j, \rho_j')+2+b$. We define

$$u(x, y)=\int_0^\infty e^{iy\cdot t^{\rho'}\eta}\frac{v(t^{\rho_1}x_1, \cdots, t^{\rho_n}x_n)t^b}{(1+t^{2\rho_1'})^{l_1}}dt\,.$$

Then $u\in C^{m_0}$ and $L(X, D_x, D_y)u=0$. But $u\notin C^\infty$. In fact operating $\partial_x^{\alpha_0}$ and substituting $x=0$, $y_2=\cdots=y_k=0$, we get

$$\partial_x^{\alpha_0}u(0, y_1, 0, \cdots, 0)=\int_0^\infty e^{iy_1t^{\rho_1'}\eta_1}\frac{\partial^{\alpha_0}v(0)t^{(\rho,\alpha_0)+b}}{(1+t^{2\rho_1'})^{l_1}}dt\,.$$

By changing the variable t by $\theta=t^{\rho_1'}$, we get

$$\partial_x^{\alpha_0}u(0, y_1, 0, \cdots, 0)=\frac{\partial_x^{\alpha_0}v(0)}{\rho_1'}\int_0^\infty e^{iy_1\theta\eta_1}\frac{\theta^{l_2}}{(1+\theta^2)^{l_1}}d\theta\,.$$

Noting l_2 is an even number we can write

$$Re\int_0^\infty e^{iy_1\theta\eta_1}\frac{\theta^{l_2}}{(1+\theta^2)^{l_1}}d\theta=P(|y_1|)e^{-|y_1||\eta_1|}$$

for some polynomial P of order l_1-1. Therefore we get from (3.27) $\partial_x^{\alpha_0}u(0, y_1, 0, \cdots, 0)\notin C^\infty$. Consequently (N) holds.

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