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ON THE HYPOELLIPTICITY AND THE GLOBAL ANALYTIC-HYPOELLIPTICITY OF PSEUDO-DIFFERENTIAL OPERATORS

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Introduction

In the recent paper [13] Kumano-go and Taniguchi have studied by using oscillatory integrals when pseudo-differential operators in $\mathbb{R}^n$ are Fredholm type and examined whether or not the operators $L_k(x, D_x, D_y) = D_x - ix^k D_y$ in Mizohata [15] and $L_+(x, D_x, D_y) = D_x \pm ix D_y$ in Kannai [6] are hypoelliptic by a unified method. In the present paper we shall give the detailed description for results obtained in [13] and study the hypoellipticity for the operator of the form

$$L = \sum_{|\alpha| + |\beta| + |\gamma| \leq 1} a_{\alpha\beta\gamma} x^{\alpha} y^{\gamma} D_x^{\alpha} D_y^{\beta}$$

with semi-homogeneity in $(x, y, D_x, D_y)$ by deriving the similar inequality to that of Grushin [4] for the elliptic case. Then we can treat the semi-elliptic case as well as the elliptic case. We shall also give a theorem on the global analytic-hypoellipticity of a non-elliptic operator, and applying it give a necessary and sufficient condition for the operator $L(x, D_x, D_y)$ to be hypoelliptic, when the coefficients of $L$ are independent of $y^\gamma$ (see Theorem 3.1).

In Section 1 we shall describe pseudo-differential operators of class $S_{\lambda, \rho, \delta}$ which is defined by using a basic weight function $\lambda - \lambda(x, \xi)$ varying in $x$ and $\xi$ (cf. [13] and also [1]). In Section 2 we shall study the global analytic-hypoellipticity of a non-elliptic pseudo-differential operator and give an example which indicates that the condition (2.3) is necessary in general. In Section 3 we shall consider the local hypoellipticity for the operator $L$ and give some examples.

The author wishes to thank Prof. H. Kumano-go for suggesting this problem and his helpful advice.

1. Algebras and $L^1$-boundedness

Definition 1.1. For $-\infty < m < \infty$, $0 \leq \delta < 1$ and a sequence $\tau_1, \tau_2, \ldots$, we define a Fréchet space $\mathcal{A}_{m, \tau}^\infty$ by the set of $C^\infty$-functions $f(x, \xi)$ in $\mathbb{R}_x^m \xi$ for which each semi-norm
\[ |\mathcal{H}|_{\mathcal{H}_2}^m = \sup_{\xi, \xi'} \{ |\mathcal{H}(\xi, x)| \langle \xi^* \rangle^{-m-\delta} \} \]

is finite, where \( \mathcal{H}(\xi, x) = \partial_x^\alpha D_x^\beta \mathcal{H}, D_x = -i\zeta/\partial_x, \partial_{\xi_j} = \partial/\partial_{\xi_j}, j = 1, \ldots, n, \)

\[ \langle \zeta \rangle = \sqrt{1 + |\zeta|^2}, \quad \langle \xi \rangle = \sqrt{1 + |\xi|^2}. \]

We define the oscillatory integral \( O_s[p] \) for \( p(\xi, x) \in \mathcal{H}_2^m \) by

\[ O_s[p] = \lim_{\epsilon \to 0} \iint e^{-ix \cdot \zeta} p(\xi, x) dx d\xi, \]

where \( d\xi = (2\pi)^{-n} d\xi, x \cdot \xi = x_1 \xi_1 + \cdots + x_n \xi_n \) and \( \chi_\epsilon(\xi, x) = \chi(\epsilon \xi, \epsilon x) \) \((0 < \epsilon \leq 1)\) for a \( \chi(\xi, x) \in \mathcal{S} \) (the class of rapidly decreasing functions of Schwartz) in \( R^m \)

such that \( \chi(0, 0) = 1 \) (cf. ([11], [13]).

**Remark.** We can easily obtain the following statements (cf. [11]).

1°) For \( p \in \mathcal{H}_2^m \) we have

\[ O_s[p] = \iint e^{-ix \cdot \zeta} p(\xi, x) dx d\xi \]

by taking integers \( l, l' \) such that \( -2l(1-\delta)+m < -n \) and \( -2l' + \tau s < -n. \)

2°) Let \( \{p_\epsilon\}_{0 < \epsilon < 1} \) be a bounded set in \( \mathcal{H}_2^m \) and converges to a \( p_0(\xi, x) \in \mathcal{H}_2^m \) as \( \epsilon \to 0 \) uniformly on any compact set of \( R^m \). Then we have

\[ \lim_{\epsilon \to 0} O_s[p_\epsilon] = O_s[p_0]. \]

3°) For \( p \in \mathcal{H}_2^m \) we have

\[ O_s[p_\epsilon] = O_s[D_x^\beta p] \quad \text{and} \quad O_s[p_\epsilon^\alpha] = O_s[D_x^\beta p]. \]

**Definition 1.2.** We say that a \( C^\infty \)-function \( \lambda(x, \xi) \) in \( R^m_{x, \xi} \) is a basic weight function when \( \lambda(x, \xi) \) satisfies conditions:

1. \( A_0^{-1} |\xi|^\alpha \lambda(x, \xi) \leq A_1 (1 + |x|^\tau_0 + |\xi|) \quad (\tau_0 \geq 0, \alpha > 0) \),

2. \( |\lambda(\xi^\alpha)(x, \xi)| \leq A_{1\delta} \lambda(x, \xi)^{1-|\alpha|+\delta} \quad (0 \leq \delta < 1) \),

3. \( \lambda(x+y, \xi) \leq \lambda(x, \xi) \lambda(y, \xi) \quad (\tau_1 \geq 0) \)

for positive constants \( A_0, A_{1\delta}, A_1. \)

**Definition 1.3.** We say that a \( C^\infty \)-function \( p(x, \xi) \) in \( R^m_{x, \xi} \) belongs to \( S^m_{\alpha, \beta, \delta} \), \( -\infty < m < \infty, 0 \leq \delta \leq \rho < 1, \delta < 1 \), when for any multi-index \( \alpha, \beta \)

\[ 1° \) For a basic weight function \( \lambda(x, \xi) \) satisfying (1.1)–(1.3) we can always find an equivalent basic weight function \( \lambda'(x, \xi) \) with \( \delta = 0 \) in (1.2) to \( \lambda(x, \xi) \), i.e., \( C^{-1} \lambda(x, \xi) \leq \lambda'(x, \xi) \leq C \lambda(x, \xi). \)
For \( p(x, \xi) \in S_{\lambda, \delta, \beta} \) we define pseudo-differential operator \( P = p(X, D_x) \) with the symbol

\[
\sigma(P)(x, \xi) = \langle \xi \rangle^\delta p(x, \xi)
\]

by

\[
\sum_{\gamma \leq N} \frac{1}{\gamma!} \langle \xi \rangle^\gamma r_{\gamma, \delta}(x, \xi) + \int_0^1 (1 - \theta)^{N-1} r_{\gamma, \delta}(x, \xi) d\theta
\]

where \( \{ r_{\gamma, \delta}(x, \xi) \} \) is bounded in \( S^{n_1 + n_2 - \rho |\xi|} \).

The set \( \{ r_{\gamma, \delta}(x, \xi) \} \) is bounded in \( S^{n_1 + n_2 - \rho |\xi|} \).

Proof. By the same method of the Theorem 2.5 and 2.6 in [11] we can prove the formula (1.6) if we have only to prove \( \{ r_{\gamma, \delta} \} \) is a bounded set in \( S^{n_1 + n_2 - \rho |\xi|} \). Since \( \partial_x^\alpha D_{\xi}^\beta r_{\gamma, \delta} \) is represented as the linear combination of

\[
\sum_{\alpha + \beta = \gamma} \frac{\langle \xi \rangle^\beta}{\gamma!} p(x, \xi)\frac{\langle \xi \rangle^\alpha}{\gamma!} p(x, \xi)
\]

we have only to prove that each term of the form (1.7) is estimated by \( C \langle \xi \rangle^{n_1 + n_2 - \rho |\xi| - \rho |\xi|} \). Here and in what follows we omit the notation \( O_x \).

We have

\[
\left| \left| \int e^{-i\xi \cdot \eta} e^{i\xi \cdot \eta} p(x, \xi + \theta \eta) p(x, \xi) \right| dy d\eta \right|
\]

\[
= \left| \left| \int e^{-i\xi \cdot \eta} \langle \xi \rangle^{n_1 + n_2 - \rho |\xi|} p(x, \xi + \theta \eta) p(x, \xi) \right| dy d\eta \right|
\]
where $w_0 = 2([n/2] + 1)$, $m_1 = \text{Max}(m, 0)$, $l_1, l_2$ are integers such that

$$-2l_1 + \tau_1 |m_1 - \rho| \alpha_1 | \leq -n, \quad -2l_2 + m_1 + n + 1 \leq \text{Min}(0, m_1 - \rho |\gamma - \rho| |\alpha_1|),$$

and $C_0$ is a constant such that

$$\text{(1.8)} \quad \frac{1}{2} \lambda(\xi, \xi) \leq \lambda(\xi, \xi + \eta) \leq \frac{3}{2} \lambda(\xi, \xi) \quad \text{if } |\eta| \leq C_0 \lambda(\xi, \xi).$$

We can prove the following two theorems by the same method.

**Theorem 1.5.** Let $S_{\Lambda, \rho, 0}^{m, m'}$ denote a set of double symbols $p(\xi, \xi', \xi')$, which satisfy

$$|p_{(\alpha, \beta)}^{(\sigma, \omega)}(\xi, \xi', \xi')| \leq C_{m, m'} \lambda(\xi', \xi')^{m - p |\alpha|} \lambda(\xi', \xi')^{m' - p |\omega'|},$$

and define operators $P = p(D_x, X', D_x')$ by

$$\hat{P}u(\xi) = O_s - \int e^{-ix'\cdot(x'-\xi')} p(\xi, \xi', \xi') \hat{u}(\xi') d\xi' dx' \quad \text{for } u \in S.$$

Then $P$ belongs to $S_{\Lambda, \rho, 0}^{m, m'}$ and we can write $\sigma(P)(x, \xi)$ in the form (1.6) for any $N > 0$, where

$$\left\{ \begin{array}{l}
\rho_{\sigma, \sigma}(x, \xi) = p_{(\alpha, \beta)}^{(\sigma, \omega)}(\xi, x, \xi) \quad (\in S_{\Lambda, \rho, 0}^{m, m' - p |\alpha|}) \\
\tau_{\sigma, \sigma}(x, \xi) = O_s - \int e^{-iy \cdot (x+y, \xi)} p_{(\gamma, \gamma)}(\xi + \theta \eta, x+y, \xi) dy d\eta.
\end{array} \right.$$
Theorem 1.6. For $P = p(X, D_x) \in S_{m, 0}^{\alpha, \beta, 0}$, the operator $P^{(*)}$ defined by

$$(Pu, v) = (u, P^{(*)}v) \quad \text{for } u, v \in S$$

belongs to $S_{m, 0}^{\alpha, \beta, 0}$ and we have for any $N > 0$

$$\sigma(P^{(*)})(x, \xi) = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \widehat{P_{\alpha}^{(*)}}(x, \xi) + N \sum_{|\gamma| = \infty} \frac{(1 - \theta)^{N-1}}{\gamma!} \widehat{r^{(*)}_{\gamma}}(x, \xi) d\theta,$$

where

$$\begin{align*}
\widehat{p_{\alpha}^{(*)}}(x, \xi) &= (-1)^{|\alpha|} \widehat{p_{\alpha}}(x, \xi) \quad (\in S_{m, 0}^{\alpha, \beta, 0}) \\
\widehat{r^{(*)}_{\gamma}}(x, \xi) &= O_2 - \int \epsilon^{-i\gamma \cdot \eta} (-1)^{|\gamma|} \widehat{p^{(*)}_{\gamma}}(x + y, \xi + \theta \eta) dy d\eta.
\end{align*}$$

The set $\{\widehat{r^{(*)}_{\gamma}}(x, \xi)\}$ is bounded in $S_{m, 0}^{\alpha, \beta, 0}$.

Remark. The maps

$$S_{m, 0}^{\alpha, \beta, 0} \times S_{m, 0}^{\alpha, \beta, 0} \ni (p_1, p_2) \mapsto p_1 \circ p_2 \in S_{m, 0}^{\alpha, \beta, 0}$$

and

$$S_{m, 0}^{\alpha, \beta, 0} \ni p \mapsto p^{(*)} \in S_{m, 0}^{\alpha, \beta, 0}$$

are continuous.

Let $q(\sigma)$ be a $C^\infty$- and even-function such that $q(\sigma) \geq 0$, $\int q(\sigma)' d\sigma = 1$ and $\text{supp } q \subset \{\sigma \in R^n; |\sigma| \leq 1\}$, and set

$$F(x, \xi; \zeta) = \lambda(x, \xi)^{-\eta} q((\xi - \eta)/\lambda(x, \xi)^{i\tau}).$$

Theorem 1.7. For $P = p(X, D_x) \in S_{m, 0}^{\alpha, \beta, 0}$, we define the Friedrichs part $P_F = p_F(D_x, X', D_x)$ by

$$p_F(x, \xi; \zeta) = \int F(x', \xi; \eta) p(x', \xi) F(x', \xi; \zeta) d\xi.$$

Then we have

(i) $p_F(x, \xi; \zeta)$ belongs to $S_{m, 0}^{\alpha, \beta, 0}$,

(ii) The operator $P_F$ belongs to $S_{m, 0}^{\alpha, \beta, 0}$ and $P - P_F \in S_{m, 0}^{\alpha, \beta, 0}$, and $\sigma(P_F)$ has the form

$$\sigma(P_F)(x, \xi) \sim p(x, \xi) + \sum_{|\alpha| + |\beta| \geq 2} \psi_{\alpha \beta}(x, \xi) \widehat{p^{(*)}_{\beta}}(x, \xi)$$

where $\psi_{\alpha \beta} \in S_{m, 0}^{\alpha, \beta, 0}$,

(iii) If $p(x, \xi)$ is real-valued and non-negative, we have

$$(p_F(D_x, X', D_x)u, v) = (u, p_F(D_x, X', D_x)v) \quad \text{for } u, v \in S, \quad (p_F(D_x, X', D_x)u, u) \geq 0 \quad \text{for } u \in S.$$
Proof is carried out by the similar way to that in [9].

**Theorem 1.8.** We can extend $P=p(X, D_x)\in S^0_{\lambda,0,0}$ to a bounded operator on $L^2$ and we get

\[(1.9) \quad \|Pu\|_{L^2} \leq C |p|_{l_0} \|u\|_{L^2},\]

where $C$ and $l_0$ are independent of $P$ and $u$.

Since $S^0_{\lambda,0,0} \subseteq S^0_{\xi>0,0}$, this theorem is a corollary of Calderón-Vaillancourt's theorem in [2].

2. **Global analytic-hypoellipticity**

**Definition 2.1.** We say that $L\in S^m_{\lambda,1,0}$ is globally analytic-hypoelliptic if the following statement holds for $L$:

If $u \in L^2(R^n)$ is a solution of the equation

\[L(X, D_x)u = f \quad \text{for} \quad f \in C^\infty(R^n)\]

and $f$ satisfies for some $M>0$

\[|D_x^\alpha f|_{L^2} \leq M^{1+|\alpha|} \lambda!, \]

then $u$ is analytic and we have

\[|D_x^\alpha u|_{L^2} \leq M_1^{1+|\alpha|} \lambda! \]

for another constant $M_1>0$.

**Theorem 2.2.** Let $L \in S^m_{\lambda,1,0} (m>0)$ satisfy the following conditions:

\[|L(x, \xi)| \geq C\lambda(x, \xi)^m \quad \text{for} \quad |\xi| \geq R\]

for some $C>0$ and $R \geq 0$, and for any multi-index $\alpha$ there exists $M_\alpha$ such that

\[|L^{(\alpha)}(x, \xi)| \leq M^{1+|\alpha|} \lambda(x, \xi)^{|\alpha|}.\]

Then the operator $L(X, D_x)$ is globally analytic-hypoelliptic.

**Example 2.3.** Let $L(x_1, x_2, D_{x_1}, D_{x_2})=D^2_{x_1} + D^2_{x_2} + x_1^2 + x_2^4 - 15x_1^4 + 45x_2^2 - 16$. Then we can prove that $L$ satisfies the conditions (2.3) and (2.4) by taking $\lambda(x_1, x_2, \xi_1, \xi_2) = (1 + |L(x_1, x_2, \xi_1, \xi_2)|^2)^{1/2}$ as a basic weight function. The equation $L(X_1, X_2, D_{x_1}, D_{x_2})u = 0$ has a non-trivial solution $e^{-(x_1^2 + x_2^2)^{1/2}}$.

As a generalization of the above example we have

**Example 2.4 (cf. [5]).** Let $L(x, D_x) = \sum_{|\alpha| \leq m_1} a_\alpha(x) D_\alpha^2$ be a hypoelliptic differential operator of order $m_1$ with analytic coefficients. Suppose that $L$ satisfies following conditions for constants $\tau_0 \geq 0$, $0 < \rho \leq 1$, $C_1 > 0$, $C_2 > 0$, $M > 0$, $\lambda > 0$.

\[\lambda(x, \xi) \geq \lambda(x, \xi)^m \quad \text{for} \quad |\xi| \geq R\]

for some $C>0$ and $R \geq 0$, and for any multi-index $\alpha$ there exists $M_\alpha$ such that

\[|L^{(\alpha)}(x, \xi)| \leq M^{1+|\alpha|} \lambda(x, \xi)^{|\alpha|}.\]

Then the operator $L(X, D_x)$ is globally analytic-hypoelliptic.
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(0) \[ |\partial^\beta a(x)| \leq M^{1+|\beta|/\beta!} \] \quad if \quad |\beta| \geq m_1 \tau_0 \quad and \quad |\alpha| \leq m_1,

\( (i) \quad \xi^{m_1} L(0, \xi) \leq C_1 |L(x, \xi)| \) \quad for \quad large \quad |\xi|,

\( (ii) \quad |L(\beta)(x, \xi)/L(x, \xi)| \leq M^{1+|\beta|/\beta!} \xi^{|\xi| + |x|^{\tau_0}} \quad for \quad large \quad |\xi| + |x|^{\tau_0}, \)

\( (iii) \quad |L(\beta)(x, \xi)| \leq C_3 (1 + |L(0, \xi)|) \quad if \quad |\beta| \geq m_1 \tau_0. \)

Then we can see that \( L \) satisfies the conditions of Theorem 2.2 by taking \( \lambda(x, \xi) = (1 + |L(x, \xi)|^{3/2})^{1/m} \) for a large \( m \) as a basic weight function.

Proof. From (0) we can choose a positive constant \( m' \) such that

\[ |L(x, \xi)| \leq C (|\xi| + |x|^{\tau_0})^{m'} \quad for \quad |\xi| + |x|^{\tau_0} \geq 1. \]

We put \( m = m'/\rho \) and \( \lambda(x, \xi) = (1 + |L(x, \xi)|^{3/2})^{1/m} \). Then we have (2.4) from (0) and (ii). By usual calculus we have (1.2) for \( \delta = 0 \). From (i) we have (1.1) for \( a = \rho m/m \) and (2.3). Finally we can get (1.3) by (i) and (iii).

Example 2.5. Let \( L(x_1, x_2, D_{x_1}, D_{x_2}) = iD_{x_1} + 2ix_2D_{x_2} - 2ix^2D_{x_2} + x_1 - x_2 \).

Then \( L \) is a semi-elliptic operator and \( Lu = 0 \) has a non-analytic solution \( u = e^{-\xi x_1/2 + x_2^2} \sum_{n=0}^{\infty} \frac{f^{(n)}(x_1)}{(2n)!} x_2^m \) where \( f(x_1) \in C_0(R^2) \) and belongs to the Gevrey class \( \rho < (3/2) \). This fact means the conditions are necessary in general.

In fact let \( L \) satisfy (2.3) and (2.4). Then we have the following contrary:

\[ 1 = |\partial_{x_1} L(-t^2, 0, 0, t)| \leq C \lambda(-t^2, 0, 0, t)^m \leq |L(-t^2, 0, 0, t)| = 0 \]

for large \( t \).

Proof of Theorem 2.2. Define \( \{E_j(x, \xi)\}_{j=0,1,...} \) for \( |\xi| \geq R \) inductively by

\[ E_0(x, \xi) = L(x, \xi)^{-1}, \]

\[ (2.5) \quad E_j(x, \xi) = -\sum_{\gamma=0}^{j-1} \sum_{\gamma=j-1}^{j} \frac{1}{\gamma!} E_{\gamma}(x, \xi) L(\gamma, \xi) E_0(x, \xi) \quad (j \geq 1), \]

then we have \( |E_j(\beta)| \leq C_{j, \rho \beta} \lambda(x, \xi)^{-m-j-|\alpha|} \) if \( |\xi| \geq R \). Taking \( \varphi_R(\xi) \in C^\infty \) such that \( \varphi_R = 1 \) if \( |\xi| \geq 2R \) and \( \varphi_R = 0 \) if \( |\xi| \leq R \), and an integer \( N \) such that \( aN \geq 1 \), we define

\[ (2.6) \quad E(x, \xi) = \varphi_R(\xi) \sum_{j=0}^{N-1} E_j(x, \xi) \in S^{1-m}_{a, \rho, 0}. \]

Then we have

\[ (2.7) \quad EL = I - K, \quad K \in S^{1-m}_{a, 0, 0}. \]

In fact by the same method of Theorem 1.4 we have

\[ (2.8) \quad \sigma(EL)(x, \xi) - 1 \]

\[ = \sum_{j=0}^{N-1} \sum_{|\gamma| < |\beta| - j} \frac{1}{\gamma!} \varphi_R(\xi) E_{\gamma}(x, \xi) L(\gamma, \xi) - 1 \]
\[ + \sum_{j=0}^{N-1} \sum_{\gamma_1+\gamma_2=N-j, \gamma_1\neq 0} \frac{1}{\gamma_1! \gamma_2!} \partial_t^j \varphi_R(\xi) E_j^{(\gamma_2)}(x, \xi) L_{\gamma_1+\gamma_2}(x, \xi) \]
\[ + \sum_{j=0}^{N-1} \sum_{\gamma_1+\gamma_2=N-j} (N-j) \int_0^1 (1-\theta)^{N-j-1} \frac{1}{\gamma_1! \gamma_2!} r_{\gamma_1\gamma_2}(x, \xi) d\theta \]
\[ \equiv I_1 + I_2 + I_3, \]

where

\[ r_{\gamma_1\gamma_2}(x, \xi) = \iint e^{-iy\cdot\eta} \partial_t^j \varphi_R(\xi + \theta \eta) E_j^{(\gamma_2)}(x, \xi + \theta \eta) L_{\gamma_1+\gamma_2}(x+y, \xi) dy d\eta. \]

From (2.5) we have

\[ I_1 = \varphi_R(\xi) - \iint S_{\xi,0,0}. \]

From the fact that \( \partial_t^j \varphi_R(\xi) \) has compact support if \( \gamma_1 \neq 0 \), we get

\[ I_2 \equiv S_{\xi,0,0}. \]

Next we prove that \( \{ r_{\gamma_1\gamma_2} \} \) is bounded in \( S_{\xi,0,0} \). Since \( \partial_t^j \varphi_R(\xi) \) is a linear combination of

\[ r_\delta(x, \xi) = \iint e^{-iy\cdot\eta} \partial_t^j \varphi_R(\xi + \theta \eta) E_j^{(\gamma_2)}(x, \xi + \theta \eta) L_{\gamma_1+\gamma_2}(x+y, \xi) dy d\eta \]

such that \( \alpha_1 + \alpha_2 + \alpha_3 = \alpha, \beta_1 + \beta_2 = \beta \). Hence we have only to prove for a constant \( C \)

\[ |r_\delta| \leq C \langle \xi \rangle^{-1}. \]

We take a constant \( C_0 \) such that (1.8) is satisfied and integers \( l_1, l_2, l_3 \) such that

\[-2l_1 + ml_2 \leq -n, \quad -2l_2 + m \leq -n, \quad -2l_3 + n + 1 \leq -m - 1/a.\]

Then we have

\[ |r_\delta(x, \xi)| \]

\[ = \iint e^{-iy\cdot\eta} \langle y \rangle^{-2l_1} \langle D_\eta \rangle^{2l_1} \left\{ \partial_{x+\eta}^{l_1+\gamma_2} \varphi_R(x+\theta \eta) E_j^{(\gamma_2)}(x, \xi + \theta \eta) \right\} \]

\[ \cdot L_{\gamma_1+\gamma_2+y_2}(x+y, \xi) dy d\eta \]

\[ \leq \int_{|y| \geq C_0^2} \langle \eta \rangle^{-2l_2} d\eta \int |\langle D_\eta \rangle^{2l_2} \left\{ \langle y \rangle^{-2l_1} \langle D_\eta \rangle^{2l_1} \left\{ \partial_{x+\eta}^{l_1+\gamma_2} \varphi_R(x+\theta \eta) \right\} \right\} | d\eta \]

\[ + \int_{|y| \leq C_0^2} |\eta|^{-2l_2} d\eta \int \left| (-\Delta_y)^{l_2} \left\{ \langle y \rangle^{-2l_1} \langle D_\eta \rangle^{2l_1} \left\{ \partial_{x+\eta}^{l_1+\gamma_2} \varphi_R(x+\theta \eta) \right\} \right\} \right| dy \]

\[ \equiv J_1 + J_2. \]

To estimate \( J_1 \) we divide into two cases.
(i) When $\alpha_1 + \gamma_1 = 0$ we have, noting that $|\gamma_2| = N - j$
\[
J_1 \leq C \int_{|\eta| \leq C \phi} \langle \eta \rangle^{-2l_1} d\eta \int \langle y \rangle^{-2l_1} \lambda(x, \xi + \theta \eta)^{-m-N} \lambda(x+y, \xi)^m dy
\leq C \lambda(x, \xi)^{-N} \int \langle \eta \rangle^{-2l_1} d\eta \int \langle y \rangle^{-2l_1+m+1} dy \leq C \langle \xi \rangle^{-1}.
\]

(ii) When $\alpha_1 + \gamma_1 = 0$ we have, noting that $\partial_{\xi}^{\alpha_1+\gamma_1} \varphi_R$ has compact support
\[
J_1 \leq C \int_{|\eta| \leq C \phi} \langle \eta \rangle^{-2l_1} d\eta \int \langle y \rangle^{-2l_1} \langle \xi + \theta \eta \rangle^{-1} \lambda(x, \xi + \theta \eta)^{-m} \lambda(x+y, \xi)^m dy
\leq C \langle \xi \rangle^{-1} \int \langle \eta \rangle^{-2l_1+1} d\eta \int \langle y \rangle^{-2l_1+m+1} dy \leq C \langle \xi \rangle^{-1}.
\]

Next for $J_2$ we have
\[
J_2 \leq C \int_{|\eta| \geq C \phi} |\eta|^{-2l_1} d\eta \int \langle y \rangle^{-2l_1} \lambda(x+y, \xi)^m dy
\leq C \lambda(x, \xi)^{-2l_1+m+n} \int \langle y \rangle^{-2l_1+m+1} dy \leq C \lambda(x, \xi)^{-1/n} \leq C \langle \xi \rangle^{-1}.
\]

Hence we get $I_3 \in S_{<\frac{1}{2}, 0, 0}$ and combining (2.8)-(2.10) we get (2.7). From (2.4) and (2.6) we see also that there exists $M_x$ independent of $\gamma$ such that

\begin{equation}
|\sigma(EL\gamma)|_{t_0, t_0} \leq M_x^{\frac{1}{1+|\gamma|}} \text{ for } t_0 \text{ in Theorem 1.8.}
\end{equation}

Moreover from (2.7) there exists constant $C_1$ such that

\begin{equation}
|K(x, \xi_j)|_{t_0, t_0} \leq C_1 \text{ for any } j = 1, \ldots, n.
\end{equation}

Suppose that for $u \in L^2$ $Lu = f$ satisfies (2.1). We have $u = ELu + Ku = Ef + Ku$ from (2.7) and so it is clear that $u$ is a $C^m$-function. Therefore we have only to prove that $u$ satisfies (2.2), since (2.2) implies the analyticity of $u$ by Sobolev's lemma. Take $M_x$ sufficiently large such that

\begin{align}
|\sigma(EL\gamma)|_{t_0, t_0} &\leq M_x^{\frac{1}{1+|\gamma|}} \text{ for } t_0 \text{ in Theorem 1.8.} \\
|K(x, \xi_j)|_{t_0, t_0} &\leq C_1 \text{ for any } j = 1, \ldots, n.
\end{align}

(2.13) \quad 3C_2 C_1 \leq M_1, \\
(2.14) \quad 3C_2 M |E|_{t_0, t_0} \leq M_1, \quad M \leq M_1, \\
(2.15) \quad 3 \cdot 2^n C_2 M_x^2 \leq M_1, \quad 2M_x \leq M_1, \\
(2.16) \quad ||u||_{L^2} \leq M_1,
\]

where $C_2$ is a constant satisfying (1.9).

From (2.16), (2.2) is trivial when $\alpha = 0$, so we show (2.2) by induction on $|\alpha|$. From (2.7), $D_\alpha^* u = ELD_\alpha^* u + KD_\alpha^* u$ ($\alpha \neq 0$). Then we have

\begin{equation}
||D_\alpha^* u|| \leq ||ELD_\alpha^* u|| + ||KD_\alpha^* u||.
\end{equation}
Since \( \alpha \not= 0 \) there exists multi-index \( \alpha_2 \) such that \( |\alpha_2| = 1, \alpha = \alpha_1 + \alpha_2 \). By (2.12), (2.13) and Theorem 1.8 we get

\[
(2.18) \quad \|KD_\alpha^a u\| = \|(KD_\alpha^a)D_\alpha^a u\| \leq C_2 C_1 \|D_\alpha^a u\| \leq C_2 C_1 M_1^{1+|\alpha_1|} |\alpha_1| \leq M_1^{1+|\alpha_1|}/3.
\]

By Leibniz' formula, we have

\[
LD_\alpha^a = D_\alpha^a L - \sum_{\alpha_1 < \alpha} \frac{\alpha!}{\alpha_1!(\alpha - \alpha_1)!} I_{\alpha_1 < \alpha} D_\alpha^a.
\]

Then

\[
(2.19) \quad \|ELD_\alpha^a u\| \leq \|ED_\alpha^a f\| + \sum_{\alpha_1 < \alpha} \frac{\alpha!}{\alpha_1!(\alpha - \alpha_1)!} \|EL_{\alpha_1 < \alpha} D_\alpha^a u\|.
\]

From (2.1), (2.6) and (2.14) we have

\[
(2.20) \quad \|ED_\alpha^a f\| \leq C_2 E_1^{(\alpha)} \|D_\alpha^a f\| \leq C_2 E_1^{(\alpha)} M_1^{1+|\alpha|} |\alpha| \leq M_1^{1+|\alpha|}/3.
\]

Finally we have from (2.11), (2.15) and the assumption of induction

\[
(2.21) \quad \sum_{\alpha_1 < \alpha} \frac{\alpha!}{\alpha_1!(\alpha - \alpha_1)!} \|EL_{\alpha_1 < \alpha} D_\alpha^a u\| \leq \sum_{\alpha_1 < \alpha} \frac{\alpha!}{\alpha_1!(\alpha - \alpha_1)!} M_2^{1+|\alpha-\alpha_1|} |\alpha-\alpha_1|! \leq M_1^{1+|\alpha|}/3.
\]

Therefore from (2.17)–(2.21) we get (2.2).

**Corollary 2.6.** Let \( L \) satisfy the same conditions as Theorem 2.2. If a bounded and continuous function \( u \) is a solution of \( Lu = f \) and \( f \in C^\infty(R^n) \) satisfies for some \( M_s \)

\[
(2.22) \quad |D_\alpha^a f| \leq M_s^{1+|\alpha|} |\alpha|,
\]

then we have for another constant \( M_4 \)

\[
(2.23) \quad |D_\alpha^a u| \leq M_4^{1+|\alpha|} |\alpha| <^n 0 \quad \text{for an even number } n_0 > n.
\]

**Proof.** We write \( Lu = f \) in the form

\[
<\!X\!>^{-n_0} L(X, D_x) <\!X\!>^{n_0} u_i = f_i,
\]

where \( u_i(x) = <\!x\!>^{-n_0} u(x), f_i(x) = <\!x\!>^{-n_0} f(x) \).

We write simplified symbol of \( <\!X\!>^{-n_0} L(X, D_x) <\!X\!>^{n_0} \) by \( L_i(X, D_x) \). Then the pair (\( L_i, u_i, f_i \)) satisfies the conditions of the theorem and we get \( \|D_\alpha^a u_i\| \leq M_5^{1+|\alpha|} |\alpha| \) for some \( M_5 > 0 \). Hence from Sobolev's lemma we can get (2.23).

**Remark.** In Theorem 2.2 we may assume (2.4) only for \( |\alpha| \leq l_0 \) with \( l_0 \) in Theorem 1.8, and in Corollary 2.6 for \( |\alpha| \leq 2l_0 \).
3. Local hypoellipticity

In this section we shall study a differential operator $L(x, y, D_x, D_y)$ in $R^n \times R^k$ with polynomial coefficients of the form

$$L(x, y, \xi, \eta) = \sum_{|\alpha| + |\alpha'| \leq 1} a_{\alpha\alpha'} x^\alpha y^\alpha \xi^\alpha \eta^\beta,$$

where $y = (y_1, \ldots, y_s)$, $\overline{y} = (y_{s+1}, \ldots, y_k)$ for $s \leq k$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha' = (\alpha_1', \ldots, \alpha_n')$, $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\gamma' = (\gamma_1', \ldots, \gamma_n')$, $0, \ldots, 0)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for multi-indices $m = (m_1, \ldots, m_n)$, $m' = (m_1', \ldots, m_n')$ of positive integers $m_j$ and $m'_j$. We say that $L$ is hypoelliptic if $u^S(f) \in C(\Omega)$ when $Lu$ belongs to $C(\Omega)$ for any open set $\Omega$ of $R^n$. Now setting $m = \max \{m_1, m_2\}$, we assume that there exist four real vectors $\rho, \rho', \sigma, \sigma'$ of the form $\rho = (\rho_1, \ldots, \rho_n)$, $\rho' = (\rho_1', \ldots, \rho_n')$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma' = (\sigma_1', \ldots, \sigma_n', 0, \ldots, 0)$ such that

$$\begin{align*}
\begin{cases}
(i) & \rho_j = \sigma_j = m \text{ for } j = 1, \ldots, n \\
(ii) & \rho'_j > \sigma'_j \geq 0, \quad m_j \rho'_j \geq m_j \text{ for } j = 1, \ldots, k
\end{cases}
\end{align*},$$

and

$$L(t^{-\sigma} x, t^{-\sigma'} y, t^{\rho} \xi, t^{\rho'} \eta) = t^m L(x, y, \xi, \eta) \quad \text{for } t > 0,$$

where $t^{-\sigma} x = (t^{-\sigma_1} x_1, \ldots, t^{-\sigma_n} x_n)$, $t^{-\sigma'} y = (t^{-\sigma'_1} y_1, \ldots, t^{-\sigma'_n} y_n)$,

$$t^{\rho} \xi = (t^{\rho_1} \xi_1, \ldots, t^{\rho_n} \xi_n), \quad t^{\rho'} \eta = (t^{\rho'_1} \eta_1, \ldots, t^{\rho'_n} \eta_n).$$

Condition 1. If we put

$$L_o(x, y, \xi, \eta) = \sum_{|\alpha| + |\alpha'| = 1} a_{\alpha\alpha'} x^\alpha y^\alpha \xi^\alpha \eta^\beta,$$

then we have

$$L_o(x, y, \xi, \eta) \neq 0 \quad \text{for } |x| + |y| \neq 0 \text{ and } (\xi, \eta) \neq 0,$$

which means that $L(x, y, \xi, \eta)$ is semi-elliptic for $|x| + |y| = 0$.

Condition 2. The equation $L(X, y, D_x, D_y)v(x) = 0$ in $R^n$ has no non-trivial solution in $\mathcal{S}(R^n)$ for $|\eta| = 1$.

**Theorem 3.1.** We consider the operator $L(x, y, D_x, D_y)$ under Condition 1 and the assumption

$$\max \{\sigma_j'\} < \min \{m_j \rho'_j / m'_j\}.$$

Then we have

(S) If Condition 2 holds, then $L(x, y, D_x, D_y)$ is hypoelliptic.

(N) If the coefficients of $L$ are independent of $y$, i.e., $s = 0$, then Condition 2 is necessary for the hypoellipticity of the operator $L$. 
EXAMPLES 3.2.

i) \( L = (-\Delta_x)'^{m} + |x|^\gamma'(-\Delta_y)'^m \) in \( R^n \times R^k \) (cf. [3], [7], [14]).

We set \( \rho = \cdots = \rho_m = \sigma_i = \cdots = \sigma_n = l_0/l \), \( \rho' = \cdots = \rho'_m = (\nu/l + 1)b/l' \), \( \sigma' = \cdots = \sigma'_n = 0 \), where \( b = \text{Max}(l, l') \). Then we can see that \( L \) is always hypoelliptic.

ii) \( L(x, D_x, D_y) = D_x \pm i\alpha' D_y^\alpha \) in \( R^n \times R^k \) (cf. [6], [8], [15]).

We set \( \gamma_1 = \gamma_1 = m \), \( \rho = l + 1 \), \( \sigma' = 0 \). Then we see the following three cases:

a) If \( l \) is even, \( L(x, D_x, \pm 1)v = 0 \) and \( L_-(X, D_x, \pm 1)v = 0 \) have no non-trivial solution in \( S \).

b) If \( l \) is odd and \( m \) is even, \( L_+(X, D_x, \pm 1)v = 0 \) has no non-trivial solution in \( S \) and \( L_-(X, D_x, \pm 1)v = 0 \) has non-trivial solution \( e^{-x_1^2 + x_2^2/2} \in S \).

c) If \( l \) and \( m \) are odd, \( L_+(X, D_x, \pm 1)v = 0 \) has no non-trivial solution \( e^{-x_1^2 + x_2^2/2} \in S \) and \( L_-(X, D_x, 1)v = 0 \) has non-trivial solution \( e^{-x_1^2 + x_2^2/2} \in S \). Consequently we see from (N) and (S) that \( L_+ \) is hypoelliptic if and only if \( "l \) is even", or \( "l \) is odd and \( m \) is even", and \( L_- \) is hypoelliptic if and only if \( "l \) is even".

iii) \( L = D_x^2 + D_y^2 + (x_1^2 + x_2^2)D_y^0 - 15x_1^2D_y^6 + 45x_1x_2D_y^6 - 16D_y^6 \) in \( R^n \times R^k \).

We set \( \rho = \sigma = 3 \), \( \rho = \sigma = 1 \), \( \rho' = 2 \), \( \sigma' = 0 \). We can see that \( L \) does not satisfy Condition 2. In fact for \( \eta = 1 \) \( L(X_1, X_2, D_x, D_y, 1)v(x_1, x_2) = 0 \) is an equation given in Example 2.3 and has non-trivial solution \( v = e^{-x_1^2 + x_2^2/2} \).

Therefore applying (N) we can see that \( L \) is not hypoelliptic.

For the proof of the theorem we need several lemmas. We introduce notations:

- \( |x, y|_{(\sigma, \gamma)} = \sum_{j=1}^n |x_j|^{1/\sigma_j} + \sum_{j=1}^n |y_j|^{1/\gamma_j} \)
- \( |\eta|_{\sigma', \gamma'} = \sum_{j=1}^n |\eta_j|^{1/\gamma_j} \)
- \( \mu(x, y, \eta) = |x|^{(m' \rho_j - m)} |y|^{(m' \rho_j') - m} |\eta_j|^{m' \gamma_j} \)

First we estimate the monomials of the form \( x^\gamma y^\sigma \gamma'^\gamma \).

**Lemma 3.3.** Let \( \alpha, \alpha', \gamma \) and \( \gamma' \) be multi-indices of dimension \( n, k, n, k \), respectively, such that \( |\alpha|: m + |\alpha'|: m' \leq 1 \) and \( \gamma' = 0 \) for \( j \geq s + 1 \). We put

\[
\theta = (\sigma, \gamma) + (\sigma', \gamma') + m - (\rho, \alpha) - (\rho', \alpha') \tag{3.6}
\]

If we denote \( \rho_0 = \min_{i \leq s, j \leq k} (m' \rho_j') \), then we have

i) If there exists \( \theta' \geq 0 \) such that \( m|\alpha|: m + |\alpha'|: m'| + (\theta + \theta')|\rho_0| \leq m \), we have

\[
|x, y|_{(\sigma, \gamma)}^{\theta'} \leq C(|\eta|_{\sigma', \gamma'}^{\theta'}) \tag{3.7}
\]

ii) If \( m|\alpha|: m + |\alpha'|: m'| + \theta|\rho_0| > m \), we have

\[
|x, y|_{(\sigma, \gamma)}^{\theta'} \leq C(|\eta|_{\sigma', \gamma'}^{\theta'}) \tag{3.8}
\]
for \(|x| \leq \delta, |\mathcal{Y}| \leq \delta\) and \(|\eta| \geq 1\), where \(\delta\) is some positive constant.

We can prove this by the same method as Lemma 3.1 and 3.2 in [4].

**Lemma 3.4.** Under condition 1 we have for a constant \(C > 0\)

\[
C^{-1}|L_{\alpha}(x, \mathcal{Y}, \xi, \eta)| \leq \{\sum_{j=1}^{n} |\xi_j|^{m_j} + \mu(x, \mathcal{Y}, \eta)\} \leq C|L_{\alpha}(x, \mathcal{Y}, \xi, \eta)|.
\]

**Proof.** In case \(|x| + |\mathcal{Y}| = 0\), it is sufficient for the sake of semi-homogeneity to prove when \(|x| + |\mathcal{Y}| = 1\), and this is true because of Condition 1. In case \(|x| + |\mathcal{Y}| = 0\), (3.9) is clear by letting \(|x| + |\mathcal{Y}| \to 0\).

Define \(\lambda_h(x, \xi)\) with parameter \(h = (\mathcal{Y}, \eta)(|\eta| = 1)\) by \(\lambda_h(x, \xi) = \{1 + |L(x, \mathcal{Y}, \xi, \eta)|^\delta\}^{1/\mu}\) and set \(p_h(x, \xi) = L(x, \mathcal{Y}, \xi, \eta)\). Then we have

**Proposition 3.5.**

(i) \(\lambda_h(x, \xi)\) satisfies (1.1)–(1.3).

(ii) \(|p_h(x, \xi)|\) is bounded in \(\{S_{h, \alpha, \beta}\}\) in the sense that for any \(\alpha, \beta\) there exists a bounded function \(C_{\alpha\beta}(x, \mathcal{Y})\) which is independent of \(\eta(|\eta| = 1)\) and tends to zero as \(|x| + |\mathcal{Y}| \to \infty\) when \(\beta = 0\), such that

\[
|p_h(\alpha\beta)(x, \xi)| \leq C_{\alpha\beta}(x, \mathcal{Y})\lambda_h(x, \xi)^{m - |\alpha|}.
\]

(iii) There exists a constant \(C\) independent of \(h\) such that

\[
|p_h(x, \xi)| \geq C\lambda_h(x, \xi)^m \quad \text{for large} \quad |x| + |\mathcal{Y}| + |\xi|.
\]

**Proof.** Set \(\lambda_h(x, \xi) = \{1 + |L(x, \mathcal{Y}, \xi, \eta)|^\delta\}^{1/\mu}\). Then from Lemma 3.3 (i) and Lemma 3.4 we can prove

\[
|L(x, \mathcal{Y}, \xi, \eta)| \geq C\lambda_h(x, \xi)^m \quad \text{for large} \quad |x| + |\mathcal{Y}| + |\xi|,
\]

which induces

\[
C^{-1}\lambda_h(x, \xi) \leq \lambda_h(x, \xi) \leq C\lambda_h'(x, \xi).
\]

For each term \(a_{\alpha\beta\gamma\eta}(x, \mathcal{Y}, \xi, \eta)\) in \(L\), we have from Lemma 3.3

\[
|\partial_{\xi_\delta}^\gamma \partial_{\xi_\delta}^\eta (a_{\alpha\beta\gamma\eta}(x, \mathcal{Y}, \xi, \eta))| \leq C \min(1, |x, \mathcal{Y}| - (\frac{\alpha}{\alpha_0}, \frac{\beta}{\beta_0}) + (1 + \mu(x, \mathcal{Y}, \eta)))^{1 - \frac{m}{|\alpha|}} \min(1, \sum_{j=1}^{n} |\xi_j|^{m_j}) \min(1, \sum_{j=1}^{n} |\xi_j|^{m_j})^{1 - \frac{m}{|\alpha|}} \leq C \min(1, |x, \mathcal{Y}| - (\frac{\alpha}{\alpha_0}, \frac{\beta}{\beta_0}) \lambda_h(x, \xi)^{m - |\alpha|}) \text{ for } \alpha_i \geq \alpha_0.
\]

Here we use the fact that \(|\eta| = 1\). Therefore we have

\[
|p_h(x, \xi)| \leq C \min(1, |x, \mathcal{Y}| - (\frac{\alpha}{\alpha_0}, \frac{\beta}{\beta_0}) \lambda_h(x, \xi)^{m - |\alpha|}).
\]

First we check (i). From (3.12) \(\lambda_h\) satisfies (1.1) for \(a = \min\{m_j/m\}\). By usual
calculus (1.2) follows by (3.13). Since \( p_h \) is a polynomial in \( x \), we have using Taylor series
\[
|p_h(x+z, \xi)| \leq \sum_{|\alpha| \leq \sigma} |z^\alpha| p_{\alpha}(x, \xi) / |\alpha| \leq \sum_{|\alpha| \leq \sigma} |x^\alpha| \lambda_\alpha(x, \xi)^m \leq C \langle x \rangle^{m+1} \lambda_\alpha(x, \xi)^m
\]
for some \( \tau_1 \). So (1.3) holds for \( \lambda_h \). Consequently we get (i), (ii) and (iii) follow at once by (3.11)-(3.13).

**Lemma 3.6.** Let a basic weight function \( \lambda(x, \xi) \) satisfy
\[
A^{-1}(1+|x|+|\xi|)^{\sigma'} \leq \lambda(x, \xi) \leq A_0(1+|x|^\sigma+|\xi|)
\]
\((\alpha', A_0 > 0, \tau_0 > 0)\) instead of (1.1). Suppose that \( p(x, \xi) \in S_{\lambda, \sigma, \tau_0}^m (m > 0) \) satisfies
\[
|p(x, \xi)| \geq C \lambda(x, \xi)^m \quad \text{for large } |x| + |\xi|.
\]
Then for any \( u \in L^1(R^n) \), \( Pu = p(X, D_x)u(x) = 0 \) implies \( u \in S(R^n) \).

**Proof.** Let \( Q \in S_{\lambda, \sigma, \tau_0}^m \) be a parametrix such that \( QP = I - K, K \in S_{\lambda, \sigma, \tau_0}^m \) \((= \cap_{m < \infty} S_{\lambda, \sigma, \tau_0}^m)\). Then we have \( u = Ku \). For any positive number \( r \) and \( t \), \( \langle X \rangle \langle D_x \rangle K(X, D_x) \) belongs to \( S_{\lambda, \sigma, \tau_0}^m \) and we get \( \langle X \rangle \langle D_x \rangle u \in L^2 \). Therefore we get \( u \in S \).

**Proposition 3.7.** If Condition 1 and 2 hold, then for any \( v \in C_0^\infty(R^n) \) we have
\[
||v||_L^2 \leq C \int |p_h(X, D_x)v(x)|^2 dx,
\]
where \( C \) is independent of \( v \) and \( h \) with \( |\eta| = 1 \).

**Proof.** From (3.10) there exists a parametrix \( \{Q_h\} \) which is bounded in \( S_{\lambda, \sigma, \tau_0}^m \) such that
\[
Q_h P_h = I - K_h,
\]
where \( \{K_h\} \) is bounded in \( S_{\lambda, \sigma, \tau_0}^m \), \( \lim_{|x| + |\xi| \to \infty} |K_h(x, \xi)| = 0 \) and for any multi-index \( \alpha, \beta \)
\[
\sup_{x, \xi} |K_{\beta}(x, \xi) - K_{\alpha}(x, \xi)| \to 0 \quad \text{as } h \to h_0.
\]
Therefore we have
\[
||v|| \leq ||Q_h P_h v|| + ||K_h v|| \leq C (||P_h v|| + ||K_h v||).
\]
Since \( \{K_h\} \) is bounded in \( S_{\lambda, \sigma, \tau_0}^m \) and \( \lim_{|\xi| \to \infty} \sup_{(x, \xi) \in R^{2n} \text{large}} |K_h(x, \xi)| = 0 \), we have for a constant \( k_0 \) in Theorem 1.8
\[
|K_h|_{\lambda, \sigma, \tau_0}^{0} \to 0 \quad \text{as } |\gamma| \to \infty.
\]
Then for a sufficiently large constant $M > 0$

$$||K_h v|| \leq \frac{1}{2} ||v|| \quad \text{for } |\mathcal{F}| \geq M,$$

and we get (3.15) for $|\mathcal{F}| \geq M$.

Now assume that for $|\mathcal{F}| \leq M$ (3.15) does not hold. Then we can choose sequences $\{h_n\}, \{v_n\}$ such that

(3.18) $||v_n|| = 1,$

(3.19) $||P_{h_n} v_n|| \to 0$ as $n \to \infty,$

(3.20) $h_n = (\mathcal{F}_n, \eta_n), \quad \text{where } |\mathcal{F}_n| \leq M, \quad |\eta_n| = 1.$

From (3.20) we may assume that

(3.21) $h_n \to h_0$

for some $h_0 = (\mathcal{F}_0, \eta_0)$. Applying $v_n$ to (3.16) we get

(3.22) $Q_{h_0} P_{h_0} v_n = v_n - K_{h_0} v_n.$

From (3.19) and (3.21) we have $Q_{h_0} P_{h_0} v_n \to 0$ in $L^2$ as $n \to \infty$, and from the fact that $\{K_{h_n}\}$ is bounded in $\{S_{\Omega, 1, 0}\}$, $\lim \sup |K_{h_0}(x, \xi)| = 0$ and (3.17) we get $K_h$ is uniformly continuous and $K_{h_0}$ is a compact operator in $L^2$ (cf. [10], [12]). So writing $K_{h_n} v_n = (K_{h_n} - K_{h_0}) v_n + K_{h_0} v_n$ we can choose a convergent subsequence $\{K_{h_n} v_{n'}\}$ in account of (3.18). Therefore from (3.22) we can choose an element $v_0 \in L^2$ such that

(3.23) $v_n \to v_0 \quad \text{in } L^2.$

Then from (3.19) and (3.21) $P_{h_n} v_n = 0$. When $\gamma_j = 0$ for all $j$ such that $m_j \rho_j = m$, we have $v_\omega = 0$ since $p_h(x, \xi) = \sum a_{\alpha \beta \gamma \delta}(\gamma^a \xi^\beta)$. Otherwise (3.12) implies (3.14) and we get $v_\omega = 0$ from Lemma 3.6 and Condition 2. This is the contrary to (3.18) and (3.23). Then Proposition 3.7 is proved.

**Theorem 3.8.** If Condition 1 and 2 hold, we can get the following formulas for $|\mathcal{F}| < \delta$, $|\eta| \geq 1$ and $v \in C_0^\infty(\{x; |x| < \delta\})$, where $\delta$ is a number which was taken in Lemma 3.3.

(3.24) $\sum_{|\alpha| : |m| \leq 1} \int |(\mu(x, \mathcal{F}, \eta) + |\eta|^m)^{1 - |\alpha| : |m|} D^\alpha_x v(x)|^2 dx$

$$\leq C \int |L(X, \mathcal{F}, D_x, \eta)v(x)|^2 dx.$$

For any $k$-dimensional multi-index $\alpha$, $\beta$, we have
(3.25) \[ ||\partial^\alpha_x \partial^\beta_y L(X, \beta, D_x, \gamma)v||_{L^2} \leq C ||\eta||_{\rho_0 (m_1 + \sigma_0 ||\beta||)} ||L(X, \beta, D_x, \gamma)v||_{L^2} \]

where \( \rho_0 = \min_{1 \leq j \leq \lambda} (m_j^\alpha \rho_j^\lambda m_j^\lambda), \sigma_0 = \max (\sigma_j^\lambda) \).

Proof. Let \( r(x, \beta) \) be a positive root of the equation

\[ \sum_{j=1}^{\lambda} \frac{x_j^2}{r_j} + \sum_{j=1}^{\lambda} \frac{y_j^2}{r_j} = 1. \]

Then \( r(x, \beta) \) is a \( C^\infty \)-function in \( R^\lambda_+ \setminus \{0, 0\} \) and

(3.26) \[ r(x, \beta) \sim |x, \beta|_{(\sigma, \omega)}. \]

Let \( \chi(x, \beta) \) be a \( C^\infty \)-function such that \( \chi = 1 \) if \( |x| + |\beta| \geq 1 \) and \( \chi = 0 \) if \( |x| + |\beta| \leq (1/2) \). For any multi-index \( \alpha (|\alpha| = 1) \) and \( h = (\beta, \gamma) (|\gamma| = 1) \) we define \( R_{\alpha h} \) by

\[ R_{\alpha h}(x, \beta) = (\sum_{j=1}^{\lambda} \chi(x, \beta)r(x, \beta)^{m_j^\alpha \beta_j^\lambda m_j^\lambda}) \eta_j |m_j^\lambda + 1|^{-|\lambda|} x^\alpha. \]

Then \( \{R_{\alpha h}\} \) is bounded in \( \{S^m_{\omega, \alpha, \beta}\} \). From (3.16) we can write for any \( v \in C^\infty (R^\lambda_+) \)

\[ R_{\alpha h}(X, D_x)\psi (X', D_x)\psi = R_{\alpha h}(X, D_x)v - R_{\alpha h}(X, D_x)K_h(X', D_x)v \]

Noting that \( \{R_{\alpha h}(X, D_x)\psi (X', D_x)\} \), \( \{R_{\alpha h}(X, D_x)K_h(X', D_x)\} \) are bounded in \( \{S^m_{\omega, \alpha, \beta}\} \), we get from Proposition 3.7

\[ \|\left( \sum_{j=1}^{\lambda} \chi(x, \beta)r(x, \beta)^{m_j^\alpha \beta_j^\lambda m_j^\lambda} \eta_j |m_j^\lambda + 1|^{-|\lambda|} x^\alpha \| = \|R_{\alpha h}(X, D_x)v\| \]

\[ \leq \|R_{\alpha h}P_h v\| + \|R_{\alpha h}K_h v\| \leq C(\|P_h v\| + \|v\|) \leq C(\|P_h v\|). \]

Considering (3.26) we have for \( |\gamma| = 1 \)

\[ \sum_{|\lambda| \leq 1} \int (|\mu(x, \beta, \gamma)| + |\gamma| \rho_0^{\alpha} \|D^\sigma \nu\|^2) dx \leq \|L(X, \beta, D_x, \gamma)v\|^2 \leq C \int |L(X, \beta, D_x, \gamma)v| dx. \]

From the semi-homogeneity we get (3.24). Using Lemma 3.3 and (3.24) we can get (3.25) by the same method as Lemma 3.6 in [4].

Proof of (S) in Theorem 3.1. By the same method as [4] we can prove (S) by using Theorem 3.8.

Proof of (N) of Theorem 3.1 (cf. [3]). Let there exist non-trivial solution \( v(x) \in S \) of \( p_h(X, D_x)v(x) = L(X, D_x, \gamma)v(x) = 0 \) for some \( h = \gamma \) with \( |\gamma| = 1 \). From Proposition 3.5 we can apply Theorem 2.2 and we get that \( v(x) \) is analytic, and therefore there exists multi-index \( \alpha_0 \) such that

(3.27) \[ \partial^\alpha_0 v(0) \neq 0. \]

We may assume \( \gamma_1 \neq 0 \). We set \( m_0 = \max (m, |\alpha_0|) \) and take even number \( l_1 \) and
positive number $b$ such that $\{(p, \alpha_0) - (p'_0 - 1) + b, p'_0\}$ is an even number (we denote it by $l_2$) and $2l_2, p'_0 \geq \max(p, p'_0) + 2 + b$. We define

$$u(x, y) = \int_0^\infty e^{i\theta t} t^m \sum_{\nu} \frac{\partial^\nu y(t^{p_0} x_1, \ldots, t^{p_0} x_n) t^{b} dt}{(1 + t^{2 p_1})^{l_1}}.$$ 

Then $u \in C^{m_0}$ and $L(X, D_x, D_y) u = 0$. But $u \in C^{\infty}$. In fact operating $\partial^\nu$ and substituting $x = 0, y = \ldots = y_k = 0$, we get

$$\partial^\nu u(0, y_1, 0, \ldots, 0) = \int_0^\infty e^{i\theta t} t^m \sum_{\nu} \frac{\partial^\nu y(t^{p_0} x_1, \ldots, t^{p_0} x_n) t^{b} dt}{(1 + t^{2 p_1})^{l_1}}.$$ 

By changing the variable $t$ by $\theta = t^\nu$, we get

$$\partial^\nu u(0, y_1, 0, \ldots, 0) = \frac{\partial^\nu y}{p'_0} \int_0^\infty e^{i\theta t} \frac{\theta^{l_2}}{(1 + \theta^2)^{l_1}} d\theta.$$ 

Noting $l_2$ is an even number we can write

$$\text{Re} \int_0^\infty e^{i\theta t} \frac{\theta^{l_2}}{(1 + \theta^2)^{l_1}} d\theta = P(|y_1|) + e^{-|y_1|^{l_1}}$$

for some polynomial $P$ of order $l_2 - 1$. Therefore we get from (3.27) $\partial^\nu u(0, y_1, 0, \ldots, 0) \in C^{\infty}$. Consequently (N) holds.

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References


