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Comprehensive study of Higgs bundles, Harmonic bundles, and Non-Abelian Hodge Correspondence
（ヒッグス束, 調和束, 非可換ホッジ対応の総合的研究）

大野　高志

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Introduction

Non-Abelian Hodge Correspondence

Let E be a complex vector bundle over a compact Kähler manifold X. We denote by $\overline{\partial}_E$ a holomorphic structure on E. A Higgs bundle over X is a pair consisting of a holomorphic vector bundle $(E, \overline{\partial}_E)$ and an End$E$-valued holomorphic 1 form θ satisfying $\theta \wedge \theta = 0$. The form θ is called the *Higgs field*. Let h be a hermitian metric of E, ∂_h be the $(1,0)$-part of the Chern connection with respect to $\overline{\partial}_E$ and h, and $\theta_h^\dagger$ be the formal adjoint of θ with respect to h. We say h is a *harmonic metric* if the connection $D := \partial_h + \overline{\partial}_E + \theta + \theta_h^\dagger$ is a flat connection (i.e. $D^2 = 0$). We say that $(E, \overline{\partial}_E, \theta, h)$ is a harmonic bundle if h is a harmonic metric for $(E, \overline{\partial}_E, \theta)$.

The existence of a harmonic metric for a Higgs bundle $(E, \overline{\partial}_E, \theta)$ is equivalent to the stability of the Higgs bundle. This equivalence is called *the non-Abelian Hodge Correspondence* and stated as follows:

Theorem 0.0.1 ([H, S1]). *Suppose X is a compact Kähler manifold. $(E, \overline{\partial}_E, \theta)$ admits a harmonic metric if and only $(E, \overline{\partial}_E, \theta)$ is a polystable Higgs bundle and $c_1(E) = c_2(E) = 0$. If h_1 and h_2 are harmonic metrics, then there exists a decomposition $(E, \overline{\partial}_E, \theta) = \oplus_i (E_i, \overline{\partial}_{E_i}, \theta_i)$ such that (i) the decomposition is orthonormal with respect to both h_1 and h_2 (ii) there exist an $a_i > 0$ such that $h_1|_{E_i} = a_i h_2|_{E_i}$ for each i.*

Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle. We have a flat bundle $(E, D = \partial_h + \overline{\partial}_E + \theta + \theta_h^\dagger)$. From a flat bundle, we obtain a representation $\rho : \pi_1(X) \to GL(r)$ by the monodromy of the flat bundle. We say that a flat bundle is *reductive* if the corresponding representation is semisimple. It was shown in [Co, S4] that a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ induces a semisimple representation $\rho_D : \pi_1(X) \to GL(r)$. Equivalently, (E, D) is reductive. The converse was also proved: from a semisimple representation, $\rho : \pi_1(X) \to GL(r)$, we obtain a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$.

Hence, combining [Co, H, S1, S3, S4], the following three objects are equivalent on a compact Kähler manifold.

- Polystable Higgs bundle with vanishing Chern classes.
- Reductive flat bundle.
- Semisimple representation of fundamental group.

This is really surprising and fascinating because the first object is a holomorphic object (or an algebro geometric object), the second object is a differential geometric object, and the third one is a topological object. Sometimes, the Non-Abelian Hodge Correspondence also means this equivalence among them.

In this thesis, we focus on the first two objects and discuss some problems related to them.

Hitchin Equation

Since a compact Riemann Surface M is a compact Kähler manifold, the non-Abelian Hodge Correspondence holds. To connect a polystable Higgs bundle to a reductive flat bundle, we need an intermediate object which is *the Hitchin equation.* Let E be a complex vector bundle, h a hermitian metric, and D be a connection of E. Then D has the decomposition

$$D = \nabla_h + \Phi,$$

where ∇_h is a metric connection and Φ is the self-adjoint 1-form with respect to h. Let F_{∇_h} be the curvature of ∇_h and $*$ be the Hodge star of M. Then the Hitchin equation is defined as

$$\begin{aligned} F_{\nabla_h} - \Phi \wedge \Phi &= 0, \\ \nabla_h \Phi &= 0, \\ \nabla_h * \Phi &= 0. \end{aligned}$$

If we have a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$, then $\nabla_h := \partial_h + \overline{\partial}_E, \Phi := \theta + \theta_h^\dagger$ solve the Hitchin equation. Conversely, if a metric connection ∇_h and self-adjoint 1-form Φ solve the Hitchin equation, $(E, \overline{\partial}_E := \nabla_n^{0,1}, \theta := \Phi^{1,0})$ is a Higgs bundle which is polystable with degree 0.

If we have a reductive flat bundle (E, D), then there exists a hermitian metric h such that the metric connection and self-adjoint 1-form which we obtain from the decomposition of D, satisfy the Hitchin equation. Conversely, if ∇_h and Φ satisfy the Hitchin equation, $D := \nabla_h + \Phi$ is a reductive flat bundle.

As a consequence, the moduli of the Hitchin equation $\mathcal{M}_{Hit}$ is equivalent to the moduli of polystable Higgs bundles, the moduli space of reductive flat bundles, and the moduli space of semisimple representations of $\pi_1(M)$. This moduli space is often called Hitchin moduli. Since Higgs bundles, flat bundles, and representations of $\pi_1(M)$ appear in a wide range of mathematical fields, many areas of mathematics intersect in this space, making its geometry especially rich and intricate. In particular, the study of the geometry of the Hitchin moduli space $\mathcal{M}_{\mathrm{Hit}}$ has become a central and active topic in modern mathematics.

Harmonic Bundles on Non-Compact Manifolds

We now introduce harmonic bundles and Higgs bundles defined over $X - H$, where X is a smooth projective variety over $\mathbb{C}$ and $H \subset X$ is a simple normal-crossing divisor. We also have an analog of the non-Abelian Hodge correspondence for this context. Since the objects are defined on a noncompact space, we need to assume some conditions on the asymptotic behavior along the divisor H.

Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle on $X - H$. We say that $(E, \overline{\partial}_E, \theta, h)$ is a *tame* (resp. *wild*) harmonic bundle if the Higgs field has logarithmic (resp. meromorphic) eigenvalue along H. The study of tame harmonic bundles on a non-compact curve was initiated by Simpson in [S2]. He proved that a tame harmonic bundle is equivalent to a polystable filtered regular Higgs bundle with degree 0. Here tame harmonic bundle is a harmonic bundle with a tame Higgs field. A filtered bundle is a locally free $\mathcal{O}_X(*H)$-module with a filtration. We review this notion in Chapter 4. Biquard and Boalch expanded this correspondence to the *wild* harmonic bundle on curves. Later, Mochizuki generalized all of this correspondence to the higher-dimensional case [M2, M3]. As a consequence, we have the following

Theorem 0.0.2 ([BB, M2, M3, S1, S2]). *Let X be a smooth projective variety, H be a normal crossing divisor of X, and L be an ample line bundle of X. Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle on $X - H$. Then $(\mathcal{P}_*^h E, \theta)$ is a μ_L-polystable good filtered Higgs bundle with $\mu_L(\mathcal{P}_*^h E) = 0$ and $\int_X \mathrm{ch}_2(\mathcal{P}_*^h E) c_1(L)^{\dim X - 2} = 0$.*

Conversely, let $(\mathcal{P}_* \mathcal{V}, \theta)$ be a μ_L-polystable good filtered Higgs bundle satisfying the following vanishing condition:

$$\mu_L(\mathcal{P}_* \mathcal{V}) = 0, \int_X \mathrm{ch}_2(\mathcal{P}_* \mathcal{V}) c_1(L)^{\dim X - 2} = 0. \tag{1}$$

Let $(E, \overline{\partial}_E, \theta)$ be the Higgs bundle which we obtain from the restriction of $(\mathcal{P}_* \mathcal{V}, \theta)$ to $X - H$. Then there exists a pluri-harmonic metric h for $(E, \overline{\partial}_E, \theta)$ such that $(\mathcal{V}, \theta)|_{X \setminus H} \simeq (E, \theta)$ extends to $(\mathcal{P}_* \mathcal{V}, \theta) \simeq (\mathcal{P}_*^h E, \theta)$.

Overview of the Thesis

We now explain the content of this thesis. At the beginning of each Chapter, we give a detailed background, so we give a brief introduction here.

In Chapter 1, we study the geometry of the Kuranishi space of a pair of a compact Kähler manifold X and a polystable Higgs bundle with vanishing Chern classes $(E, \overline{\partial}_E, \theta)$. We proved that under this assumption, the

Kuranishi space of the pair is isomorphic to the direct product of the Kuranishi space of X and the Kuranishi space of $(E, \overline{\partial}_E, \theta)$. We prove this by studying the Differential Graded Lie Algebra (DGLA) which controls the deformation of the pair. This Chapter is based on [Ono1, Ono2].

In Chapter 2, we study a deformation problem of a certain triple. In Chapter 2, we study the deformation problem of the triple of two Higgs bundles $(E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F)$ and a Higgs bundle morphism $(\overline{\partial}_E, \theta) \to (F, \overline{\partial}_F, \theta_F)$. We call the triple $(f, (E, \overline{\partial}_E, \theta_e), (F, \overline{\partial}_F, \theta_F))$ Higgs triple. Let L be the DGLA that controls the deformation of $(f, (E, \overline{\partial}_E, \theta), (F, \overline{\partial}_F, \theta_F))$. We show that L is formal if both $(E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F)$ has harmonic metrics. This Chapter is based on [Ono3].

In Chapter 3, we introduce *the basic Hitchin equation* on a Sasakian three-fold. Sasakian manifolds are odd-dimensional manifolds, and they are the odd-dimensional counterpart of Kähler manifolds. Therefore Sasakian three-folds are a three-dimensional analog of Riemann surfaces. The non-Abelian Hodge Correspondence on compact Sasakian manifolds was established in [BH1]. Motivated by this work, the author defined the basic Hitchin equation on Sasakian three-folds, which is a three-dimensional analog of the Hitchin equation. In Chapter 3, we construct the moduli space of the solutions of the basic Hitchin equation and prove that the moduli space is a hyper-Kähler manifold. This Chapter is based on [Ono4].

In Chapter 4, we study a good wild harmonic bundle with a symplectic structure. Symplectic structure is a symmetry of a wild harmonic bundle. We show that a wild harmonic bundle with a symplectic structure is equivalent to a polystable good filtered Higgs bundle with a perfect skew-symmetric pairing. This Chapter is based on [Ono5].

Chapter 1

Structure of the Kuranishi Spaces of pairs of Kähler manifolds and Polystable Higgs bundles

1.1 Abstract of Chapter 1

Let X be a compact Kähler manifold and $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle over it. We study the structure of the Kuranishi space for the pair (X, E, θ) when the Higgs bundle admits a harmonic metric or equivalently when the Higgs bundle is polystable and the Chern classes are 0. Under such assumptions, we show that the Kuranishi space of the pair (X, E, θ) is isomorphic to the direct product of the Kuranishi space of (E, θ) and the Kuranishi space of X. Moreover, when X is a Riemann surface and $(E, \overline{\partial}_E, \theta)$ is stable and the degree is 0, we show that the deformation of the pair (X, E, θ) is unobstructed and calculate the dimension of the Kuranishi space.

1.2 Introduction of Chapter 1

Let X be a complex manifold and $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle over X. We call a pair X and $(E, \overline{\partial}_E, \theta)$ a *holomoprhic-Higgs triple*. In the paper [Ono1], we studied the deformation problem of holomorphic-Higgs triple differential geometrically: we studied the deformation problem when X and $(E, \overline{\partial}_E, \theta)$ deform simultaneously. We constructed the DGLA $(L, [,]_L, d_L)$ which governs the deformation differential geometrically, and constructed the Kuranishi space $Kur_{(X,E,\theta)}$. The Kuranishi space is an analytic space such that it contains all information of small deformations of the given holomorphic-Higgs pair. See section 2.4.1 for the details of the DGLA $(L, [,]_L, d_L)$.

In this Chapter, we study the structure of the Kuranishi space. We study the structure of the Kuranishi space when X is a compact Kähler manifold and the Higgs bundle $(E, \overline{\partial}_E, \theta)$ admits a harmonic metric h. Hence, in other words, we study the Kuranishi space $Kur_{(X,E,\theta)}$ when the Higgs bundle is polystable and its Chern classes are 0. From now on, we assume X to be a compact Kähler manifold.

Let Kur_X be the Kuranishi space of X, $Kur_{(E,\theta)}$ be the Kuranishi space of the Higgs bundle (E, θ), and $Kur_{(E,D)}$ be the Kuranishi space of the flat bundle (E, D). The flat bundle D is obtained from the Higgs bundle and the harmonic metric.

Theorem 1.2.1 (Theorem 1.5.2)**.** *Let (X, ω) be a compact Kähler manifold, $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle over X and, K be a harmonic metric. Then*

$$(Kur_{(X,E,\theta)}, 0) \simeq (Kur_{(E,\theta)} \times Kur_X, 0),$$
$$(Kur_{(X,E,\theta)}, 0) \simeq (Kur_{(E,D)} \times Kur_X, 0)$$

holds as germs of analytic spaces.

We prove this theorem by showing the DGLA $(L, [,]_L, d_L)$ is quasi-isomorphic to certain DGLA.

Theorem 2.4.1 predicts that once we construct a moduli space of a pair of a compact Kähler manifold and a polystable Higgs bundle with vanishing Chern classes, such moduli space should locally decompose to the direct product of the Kuranishi space of the manifold and the Kuranishi space of the Higgs bundle which we cannot expect globally. The moduli space of pairs of Kähler manifolds and stable bundles was considered in [H, ST]. However, the author couldn't find a work that deals with pairs of Kähler manifolds and stable Higgs bundles.

We have some consequences from Theorem 2.4.1 for specific cases. Let M be a Riemann surface with genus $g \geq 2$ and $(E, \overline{\partial}_E, \theta)$ be a stable Higgs bundle of degree 0. Under these assumptions, each deformations of M and $(E, \overline{\partial}_E, \theta)$ are unobstructed and the dimensions of Kur_X is $3g-3$ and $Kur_{(E,\theta)}$ is $2+r^2(2g-2)$ [MK, N]. Here r is the rank of E. The following is straightforward from Theorem 2.4.1.

Corollary 1.2.1 (Corollary 2.5.1)**.** *Let M be a Riemann surface with genus $g \geq 2$ and $(E, \overline{\partial}_E, \theta)$ be a stable Higgs bundle of degree 0. Then the deformation of pair (M, E, θ) is unobstructed. Moreover, $Kur_{(M,E,\theta)}$ is a complex manifold and its dimension is $g(2r^2+3) - 2r^2 - 1$.*

The Corollary predicts that the moduli space of a pair of Riemann surfaces and stable Higgs bundles of degree 0 is smooth in a stable locus and its dimension is $g(2r^2+3) - 2r^2 - 1$.

1.2.1 Differential graded Lie algebras

In this section, we review the notion of the Differential graded Lie algebra (DGLA for short). We work over a field $\mathbb{K}$ of characteristic 0. This section is based on [Ma].

Definition 1.2.1. *A Differential-Graded vector space (DG vector space) is a pair (L, d_L) such that $L = \oplus_i L^i$ is a $\mathbb{Z}$-graded vector space and $d : L \to L$ is a linear map such that $d(L^i) \subset L^{i+1}$ and $d \circ d = 0$.*

Let (L, d_L) be a DG vector space. A sub DG-vector space $(W = \oplus_{i\in\mathbb{Z}} W^i, d_W)$ of (L, d_L) is a DG vector space such that for each i, $W^i \subset L^i$ is a sub vector space and d_W is the restriction of d_L to W: $d_L(W) \subset W$ holds and $d_W = d_L|_W$.

A morphism $f : (L_1, d_{L_1}) \to (L_2, d_{L_2})$ of DG vector spaces is a morphism of vector spaces $f : L_1 \to L_2$ such that it commutes with the differentials. We note that f induces a morphism $H^i(f) : H^i(L_1) \to H^i(L_2)$. Here $H^i(L_j)(j = 1, 2)$ is the i-th cohomology of (L_j, d_{L_j}). Let (L, d_L) be a DG vector space and (W, d_W) be a sub DG vector space of it. Then the inclusion of W^i to L^i induces a morphism of DG vector spaces $i : (W, d_W) \to (L, d_L)$.

Definition 1.2.2. *A* Differential graded Lie algebra *(DGLA) $(L, [,], d)$ is the data of a $\mathbb{Z}$-graded vector space $L = \oplus_{i\in Z} L^i$ with a bilinear bracket $[,] \colon L \times L \to L$ and a linear map $d \colon L \to L$ satisfying the following condition:*

1. *$[,]$ is homogeneous skewsymmetric: $[L^i, L^j] \subset L^{i+j}$ and $[a, b] + (-1)^{\overline{a}\overline{b}}[b, a] = 0$ for every a, b homogeneous.*
2. *Every triple of homogeneous elements a, b, c satisfy the Jacobi identity*
$$[a, [b, c]] = [[a, b], c] + (-1)^{\overline{a}\overline{b}}[b, [a, c]].$$
3. *$d(L^i) \subset L^{i+1}$, $d \circ d = 0$ and $d[a, b] = [da, b] + (-1)^{\overline{a}}[a, db]$ holds. The map d is called the differential of L.*

Definition 1.2.3. *The Maurer-Cartan equation of a DGLA $(L, [,], d)$ is*
$$da + \frac{1}{2}[a, a] = 0, a \in L^1.$$
The solutions of the Maurer-Cartan equation are called Maurer-Cartan elements of the DGLA $(L, [,], d)$.

Let $(L, [,], d_L)$ be a DGLA. We can consider (L, d_L) as a DG vector space. We say a DGLA $(W, [,]_W, d_W)$ is a sub-DGLA of $(L, [,], d_L)$ if (W, d_W) is a sub-DG vector space of (L, d_L) and the bracket $[,]_W$ is the restriction of $[,]$ to W: $[W, W] \subset W$ holds.

Definition 1.2.4. *Let $(L_1, [,]_1, d_{L_1})$ and $(L_2, [,]_2, d_{L_2})$ be DGLAs. A morphism $f : (L_1, [,]_1, d_{L_1}) \to (L_2, [,]_2, d_{L_2})$ of DGLAs is a morphism of DG vector spaces $f : (L_1, d_{L_1}) \to (L_2, d_{L_2})$ such that it commutes with brackets.*

Let (L_1, d_{L_1}) and (L_2, d_{L_2}) be DG vector spaces. We say that (L_1, d_{L_1}) and (L_2, d_{L_2}) are *quasi-ismorphic* if there exists a morphism of DG vector spaces $f : (L_1, d_{L_1}) \to (L_2, d_{L_2})$ such that $H^i(f)$ is an isomorphism for each i.

Let $(L_1, [,]_1, d_{L_1})$ and $(L_2, [,]_2, d_{L_2})$ be DGLAs. We say that $(L_1, [,]_1, d_{L_1})$ and $(L_2, [,]_2, d_{L_2})$ are *quasi-ismorphic* if there exists a family of DGLA $\{(W_i, [,]_{W_i}, d_{W_i})\}_{i=1}^n$ and a family of morphism of DGLA $\{f_i\}_{i=1}^{n+1}$ such that

$$L_1 \xleftarrow{f_1} W_1 \xrightarrow{f_2} W_2 \xleftarrow{f_3} \cdots \xrightarrow{f_{n-1}} W_{n-1} \xleftarrow{f_n} W_n \xrightarrow{f_{n+1}} L_2$$

and each f_i is a quasi-isomorphism of DG vector spaces.

Let $(L, [,], d_L)$ be a DGLA. Since d_L satisfies the Leibniz rule, $(H^\bullet(L) := \oplus_i H^i(L), [,], 0)$ has the structure of DGLA.

Definition 1.2.5. *Let $(L, [,], d_L)$ be a DGLA. $(L, [,], d_L)$ is called formal if it is quasi-isomorphic to $(H^\bullet(L), [,], 0)$.*

Remark 1.2.1. *DG vector spaces can be regarded as DGLA with trivial brackets. In this case, the definition of quasi-isomorphic coincides: when two DG vector spaces are quasi-isomorphic as DGLA then we can show that it is quasi-isomorphic as DG vector spaces. This is because any DG vector space decomposes to two DG vector spaces which have the property called minimal and acyclic. See [Ma] for example.*

1.2.2 Homotopy invariance of the Kuranishi Space

In this section, we review the homotopy invariance of the Kuranishi spaces based on [GM1, GM2].

Let X be an analytic space and $x \in X$. We denote the germ of X at x as (X, x). We denote by $\mathcal{O}_{(X,x)}$ the corresponding analytic local ring consisting of germs of functions on X which are analytic at x. Let A be a local ring. We denote the completion of A with respect to its maximal ideal as $\widehat{A}$: the complete local ring of (X, x) is $\widehat{\mathcal{O}}_{(X,x)}$.

Let $\mathbb{K}$ be a field. Let R be a local $\mathbb{K}$-algebra, $Art_{\mathbb{K}}$ be the category of Artin local $\mathbb{K}$-algebras with residue field $\mathbb{K}$ and Set be the category of sets. We have a naturally defined functor

$$\mathrm{Hom}(R, \cdot) : Art_{\mathbb{K}} \to Set$$

which we denote F_R. Let $F : Art_{\mathbb{K}} \to Set$ be a functor. We say that an analytic germ (X, x) *pro-represents* F if F and $F_{\widehat{\mathcal{O}}_{(X,x)}}$ are naturally isomorphic. In this chapter, $\mathbb{K}$ is often $\mathbb{C}$. Using the results of [A, G], it was proved in [GM1] that the following four conditions are equivalent:

(1) The analytic germs of (X, x) and (Y, y) are analytic isomorphic.

(2) The analytic local rings $\mathcal{O}_{(X,x)}$ and $\mathcal{O}_{(Y,y)}$ are isomorphic.

(3) The complete local rings $\widehat{\mathcal{O}}_{(X,x)}$ and $\widehat{\mathcal{O}}_{(Y,y)}$ are isomorphic.

(4) The functor $F_{\widehat{\mathcal{O}}_{(X,x)}}$ and $F_{\widehat{\mathcal{O}}_{(Y,y)}}$ are naturally isomorphic.

Let $(L, [,], d_L)$ be a DGLA. Let $C^1(L)$ be a complement of the 1-coboundary $B^1(L) \subset L^1$. We define a functor $Y_L : Art_{\mathbb{K}} \to Set$ such that for $A \in Art_{\mathbb{K}}$

$$Y_L(A) = \left\{ \eta \in C^1(L) \otimes m_A : d\eta + \frac{1}{2}[\eta, \eta] = 0 \right\}.$$

Here, m_A is the maximal ideal of the Artin local $\mathbb{K}$-algebra A. It was proved in [GM2] that Y_L is *pro-representable*: that is, there exists a complete local $\mathbb{K}$-algebra R_L such that Y_L and F_{R_L} are naturally isomorphic.

Let $(L_i, [,]_i, d_{L_i})$ $(i = 1, 2)$ be DGLAs and $C^1(L_i)$ $(i = 1, 2)$ be choices for the complement of the coboundaries $B^1(L_i)$. Let $f : (L_1, [,]_1, d_{L_1}) \to (L_2, [,]_2, d_{L_2})$ be a morphism of DGLA. We assume that

(i) $H^1(f)$ is an isomorphism.

(ii) $H^2(f)$ is an injection.

Then it was proved in [GM2] that, if a morphism $f : (L_1, [,]_1, d_{L_1}) \to (L_2, [,]_2, d_{L_2})$ satisfies (i) and (ii), then R_{L_1} and R_{L_2} are isomorphic.

We next introduce the notion of *analytic DGLA*. A *normed DGLA* $(L, [,], d_L)$ is a DGLA such that each L^i is a normed vector space and with respect to the norms

(1) $d_L : L^i \to L^{i+1}$ is continuous.

(2) $[,] : L^1 \otimes L^1 \to L^2$ is contionous.

We let $\widehat{L}^i$ to be the completion of L^i with respect to the norm.

An analytic DGLA is a normed DGLA $(L, [,], d_L)$ such that it has finite-dimensional cohomology in degrees 0 and 1 and each $\widehat{L}^i$ has continuous splitting:

$$0 \to Z^j(\widehat{L}) \to \widehat{L}^j \to B^{j+1}(\widehat{L}) \to 0$$

and

$$0 \to B^j(\widehat{L}) \to Z^j(\widehat{L}) \to H^i(\widehat{L}) \to 0.$$

It was proved in [GM2, Section 2] that for an analytic DGLA $(L, [,], d_L)$, there exists a germ of analytic space $(Kur_L, 0)$ such that $F_{\mathcal{O}_{(Kur_L,0)}}$ is naturally isomorphic to Y_L. Therefore, R_L is isomorphic to $\widehat{\mathcal{O}}_{(Kur_L,0)}$. See [Ma, Chapter 3] for more details for the functors of Artin rings.

Based on the above discussions, the following Theorem was proved in [GM2].

Theorem 1.2.2 ([GM2, Theorem 4.8.]). *Suppose $(L_1, [,]_1, d_{L_1})$ and $(L_2, [,]_2, d_{L_2})$ are analytic DGLAs which are quasi-isomorphic as DGLAs. Then the analytic germs $(Kur_{L_1}, 0)$ and $(Kur_{L_2}, 0)$ are analytically isomorphic.*

Remark 1.2.2. *The DGLAs that appear in this chapter are analytic DGLAs by the standard Sobolev norms.*

Remark 1.2.3. *The construction of $(Kur_L, 0)$ is based on [Ku]. When a DGLA $(L, [,], d_L)$ comes from a differential geometric object, the complement $C^1(L)$ can be chosen by the Hodge decomposition of the differential d_L. In this case, Kur_L is the standard versal deformation space. For example, when $(A^*(TX), [,]_{SN}, \overline{\partial}_{TX})$ is the Kodaira-Spencer algebra of a complex manifold X, then $Kur_{A^*(TX)}$ is exactly the Kuranishi space of X.*

1.3 Deformation of holomorphic-Higgs pairs

In this section, we review our previous work [Ono1].

Let X be a complex manifold and $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle over X. We called the pair (X, E, θ) a *holomorphic-Higgs pair*. In our paper [Ono1], we considered the deformation problem of holomorphic-Higgs pairs and constructed the DGLA that governs the deformation and constructed the Kuranishi space. We give the definition of the deformation of the holomorphic-Higgs pair (X, E, θ).

Definition 1.3.1. *Let (X, E, θ) be a holomorphic-Higgs pair. A family of deformation of holomorphic-Higgs pair over a small ball Δ centered at the origin of $\mathbb{C}^d$, a complex manifold $\mathcal{X}$, a proper holomorphic submersion*

$$\pi : \mathcal{X} \to \Delta$$

and a Higgs bundle $(\mathcal{E}, \Theta)$ such that, $\pi^{-1}(0) = X, \mathcal{E}|_{\pi^{-1}(0)}$, and $\Theta|_{\pi^{-1}(0)} = \theta$.

1.3.1 DGLA

In this section, we introduce the DGLA which governs the deformation of holomorphic-Higgs pair based on [Ono1].

Let (X, E, θ) be a holomorphic-Higgs pair and TX be the holomorphic tangent bundle of X. We fix a hermitian metric h on E. Let ∂_h be the (1,0)-part of the Chern connection with respect to $\overline{\partial}_E$ and h. For $\phi \in A^{(0,i)}(TX)$ and ∂_h, we define

$$\{\partial_h, \phi\lrcorner\} := \partial_h(\phi\lrcorner) + (-1)^i \phi\lrcorner\partial_h.$$

Here, $\phi\lrcorner$ is the contraction with respect to ϕ.

Let $L^i := \bigoplus_{p+q=i} A^{p,q}(\mathrm{End}E) \bigoplus A^{0,i}(TX)$ and $L := \bigoplus_i L^i$. Let $(A,\phi) \in L^i$ and $(B,\psi) \in L^j$. We define,

$$[(A,\phi),(B,\psi)]_L := ((-1)^i\{\partial_h,\psi\lrcorner\}A - (-1)^{(i+1)j}\{\partial_h,\phi\lrcorner\}B - [A,B]_{\mathrm{End}E}, [\phi,\psi]_{SN}).$$

Here, $[,]_{SN}$ is the standard Schouten-Nijenhuys bracket defined on $\bigoplus_i A^{0,i}(TX)$.

We define $B \in A^{0,1}(\mathrm{Hom}(TX,\mathrm{End}E))$ and a $\mathbb{C}$-linear map $C_i : A^{0,i}(TX) \to A^{1,i}(\mathrm{End}E)$ such that they act on $\phi \in A^{0,i}(TX)$ as

$$B(\phi) := (-1)^i \phi\lrcorner F_{d_h}, C_i(\phi) := \{\partial_h,\phi\lrcorner\}\theta.$$

We define a $\mathbb{C}$-linear map $d_L : L^i \to L^{i+1}$ as

$$d_L := \begin{pmatrix} \overline{\partial}_{\mathrm{End}E} & B \\ 0 & \overline{\partial}_{TX} \end{pmatrix} + \begin{pmatrix} \theta & C_i \\ 0 & 0 \end{pmatrix}.$$

After some calculations, we obtain the following theorem.

Theorem 1.3.1 ([Ono1, Theorem 3.1.]). *$(L,[,]_L,d_L)$ is a DGLA.*

This DGLA governs the deformation of (X,E,θ). Actually, let $\eta = (A,\phi) \in L^1$. Then η defines a holomorphic-Higgs pair if and only if η is a Maurer-Cartan element. This was proved in [Ono1, Theorem 3.6.].

We assume $\eta = (A,\phi) \in L^1$ to be a Maurer-Cartan element and let $(X_\eta, E_\eta, \theta_\eta)$ be the holomorphic-Higgs pair which η determines. We briefly recall its construction. Let $A^{1,0}$ (resp. $A^{0,1}$) be the (1,0) (resp. (0,1))-part of A. We define $\overline{\partial}_{E,\eta} := \overline{\partial}_E + \{\partial_h,\phi\lrcorner\} + A^{0,1}$ and $\theta_\eta := \theta + A^{1,0} + \phi\lrcorner(\theta + A^{1,0})$. In [Ono1], we showed that the following holds:

- $\overline{\partial}_{TX}\phi + \frac{1}{2}[\phi,\phi]_{SN} = 0,$
- $(\overline{\partial}_{E,\eta} + \theta_\eta)^2 = 0.$

From the first equation, we obtain a new complex structure I_η on X. We denote this complex manifold as X_η. We also showed that $\overline{\partial}_{E,\eta}$ is a (0,1)-type operator and θ_η is a End$E$ valued (1,0)-form w.r.t X_η. From the second equation, we obtain a holomorphic bundle $E_\eta := (E, \overline{\partial}_\eta)$ and a Higgs field θ_η.

1.3.2 Kuranishi Space

We use the same notation as the previous section. In this section, we introduce the Kuranishi Space and the Kuranishi family for a given holomorphic-Higgs pair (X,E,θ). Briefly, Kuranishi Space is an analytic space and the Kuranishi family is a family of holomorphic-Higgs pairs parametrized by Kuranishi Space such that every holomorphic-Higgs pair which comes from a small deformation of (X,E,θ) is isomorphic to a holomorphic-Higgs pair which is in the Kuranishi family.

Since $(L,[,]_L,d_L)$ is constructed differential geometrically, we can apply the Kuranishi's work [Ku] to construct the Kuranishi family. Let d_L^* be the formal adjoint of d_L with respect to L^2-inner product, $\Delta_L := d_L d_L^* + d_L^* d_L$ be the Laplacian and G_L be the Green operator associated to Δ_L. Let $\mathbb{H}^i := \ker(\Delta_L : L^i \to L^i)$ and $H : L^i \to \mathbb{H}^i$ be the projection. By the classical Hodge theory, we know that $\dim\mathbb{H}^i$ has a finite dimension for each i. Let $\{\eta_i\}_{i=1}^n$ be an orthogonal bases of $\mathbb{H}^1$ with respect to L^2-inner product. For each $t = (t_1,\dots,t_n) \in \mathbb{C}^n$, we set $\epsilon_1(t) := \sum_i t_i\eta_i$.

Lemma 1.3.1 ([Ono1, Proposition 4.1, Proposition 4.2.]). *For any $t \in \mathbb{C}^n$ and $|t| << 1$, there is a $\epsilon(t) \in L^1$ such that $\epsilon(t)$ satisfies the equation*

$$\epsilon(t) = \epsilon_1(t) + \frac{1}{2} d_L^* G_L[\epsilon(t),\epsilon(t)]_L$$

and $\epsilon(t)$ depends holomorphically with respect to the variable t.

Moreover, $\epsilon(t)$ satisfies the Maurer-Cartan equation if and only if

$$H[\epsilon(t),\epsilon(t)] = 0.$$

Let $\Delta \subset \mathbb{C}^n$ be a small ball such that $\epsilon(t)$ is holomorphic on Δ. We set,

$$Kur_{(X,E,\theta)} := \{t \in \Delta : H[\epsilon(t), \epsilon(t)] = 0\}.$$

Since the dimension of $\mathbb{H}^2$ is finite, $Kur_{(X,E,\theta)}$ is an analytic space. We call $Kur_{(X,E,\theta)}$ the Kuranishi space of (X, E, θ). Since a Maurer-Cartan element defines a holomorphic-Higgs pair, we obtain a family of holomorphic-Higgs pair $\{(X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)})\}_{t \in Kur_{(X,E,\theta)}}$. We call this family the Kuranishi family of (X, E, θ). The Kuranishi space and the Kuranishi family contain all small deformation of (X, E, θ). Actually, let $|\cdot|_k$ be the k-th Sobolev norm of L^1 and let $\eta \in L^1$ be a Maurer-Cartan element. If $|\eta|_k << 1$, then there exists a $t \in Kur_{(X,E,\theta)}$ such that

$$(X_\eta, E_\eta, \theta_\eta) \simeq (X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)}).$$

Here $(X_\eta, E_\eta, \theta_\eta)$ is the holomorphic-Higgs pair which η determines. This was proved in [Ono1, Theorem 4.2.].

1.4 Harmonic bunldes

1.4.1 Kähler Identities

Let X be a compact Kähler manifold and $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle.

Let ∂_h be the (1,0)-part of the Chern connection associated with $\overline{\partial}_E$ and K and $\theta_h^\dagger$ be the adjoint of θ with respect to h. We set $D'_h := \partial_h + \theta_h^\dagger$ and $D'' := \overline{\partial}_E + \theta$. The connection $D := D'_h + D''_E$ is flat.

We define a L^2-metric on $A^p(E)$ by using the Riemannian metric g on X and the Hermitian metric h on E. Let D_E^*, $(D'_h)^*$ and $(D''_E)^*$ be the formal adjoint of D_E, D'_h and D''_E with respect to the L^2 inner product. Let Λ_ω be the contraction with respect to the Kahler form ω. The following Kähler identities were proved in [S1, Lemma 3.1.].

Lemma 1.4.1. *Let (X, ω) be a compact Kähler manifold, $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle over X and h be a hermitian metric. Let D'_h, D''_E, $(D'_h)^*$ and $(D''_E)^*$ be as above. Then the following equalities hold.*

$$(D'_h)^* = \sqrt{-1}[\,\Lambda_\omega, D''_E]\ , (D''_E)^* = -\sqrt{-1}[\,\Lambda_\omega, D'_h]\ .$$

We define the laplacians as follows,

$$\begin{aligned}
\Delta_E &:= D_E D_E^* + D_E^* D_E, \\
\Delta''_E &:= D''_E (D''_E)^* + (D''_E)^* D''_E, \\
\Delta'_h &:= D'_h (D'_h)^* + (D'_h)^* D'_h.
\end{aligned}$$

We assume h to be a harmonic metric. Under this assumption, we have $D''_E D'_h + D'_h D''_E = 0$ and by the Kähler identities, we obtain the following equalities.

$$\Delta_E = 2\Delta''_E = 2\Delta'_h.$$

Let G_E, G''_E and G'_h be the Green operators asscoiated to Δ_E, Δ''_E and Δ'_h. By the above relations of Laplacians, we have $2G_E = G''_E = G'_h$. We set $\mathbb{H}^i := \ker \Delta_E$. By the classical Hodge theory, we have the following orthogonal decompositions with respect to the L^2-inner product.

$$\begin{aligned}
A^i(E) &= \mathbb{H}^i \oplus \mathrm{im} D_E \oplus \mathrm{im} D_E^*, \\
A^i(E) &= \mathbb{H}^i \oplus \mathrm{im} D''_E \oplus \mathrm{im}(D''_E)^*, \\
A^i(E) &= \mathbb{H}^i \oplus \mathrm{im} D'_h \oplus \mathrm{im}(D'_h)^*.
\end{aligned}$$

The next lemma was proved in [S2]. This is an analog of the Kähler case which was proved in [DGMS].

Lemma 1.4.2 ([S2, Lemma 2.1.])**.** *Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle on X. Then*

$$\ker D'_h \cap \ker D''_E \cap (\mathrm{im} D'_h + \mathrm{im} D''_E) = \mathrm{im} D'_h D''_E.$$

Proof. We give the proof for convenience. This lemma was originally proved in [S2].

Let $\gamma \in A^i(E)$. Suppose $\gamma = D'_h\alpha + D''_E\beta$, $D'_h\gamma = 0$ and $D''_E\gamma = 0$. Let $\beta = \beta_0 + D'_h\beta_1 + (D'_h)^*\beta_2$ be the Hodge decomposition with respect to D'_K with β_0 harmonic. Since $\Delta'_h = \Delta''_E$, $\Delta''_E\beta_0 = 0$. Thus we have

$$D''_E\beta = D''_E D'_h\beta_1 + D''_E(D'_h)^*\beta_2.$$

From the Kähler identities, we have $D''_E(D'_h)^* = \sqrt{-1}D''_E[\,\Lambda_\omega, D''_E]\ = \sqrt{-1}D''_E\Lambda_\omega D''_E = -\sqrt{-1}'[\,\Lambda_\omega, D''_E]\ D'' = -(D'_h)^*D''_E$. Hence we have

$$\gamma = D'_h\alpha + D''_E D'_h\beta_1 - (D'_h)^*D''_E\beta_2.$$

Since γ is D'_h-closed, $(D'_h)^*D''_E\beta_2$ is also. From the equation

$$\big((D'_h)^*D''_E\beta_2, (D'_h)^*D''_E\beta_2\big)_{L^2} = \big(D''_E\beta_2, D'_h(D'_h)^*D''_E\beta_2\big)_{L^2} = 0,$$

we obtain $(D'_h)^*D''_E\beta_2 = 0$ and therefore, $D''_E\beta = D''_E D'_h\beta_1$. Here $(,)_{L^2}$ is the L^2-norm. We can show $D'_h\alpha = D''_E D'_h\alpha_1$ by using exactly the same argument as β. Hence the claim is proved. $\square$

1.4.2 Formality

Let (X,ω) be a compact Kähler manifold, $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle on X. We obtain three DG vector spaces $(A^*(E) := \oplus_i A^i(E), D_E)$, $(A^*(E), D''_E)$, and $(A^*(E), D'_h)$. We define $\mathbb{H}^i_{DR}$, $\mathbb{H}^i_{Dol}$, and $\mathbb{H}^i_{D'_h}$ to be the i-th cohomology of $(A^*(E), D_E)$, $(A^*(E), D''_E)$, and $(A^*(E), D'_h)$. These DG vector spaces satisfy formality conditions.

Lemma 1.4.3 ([GM1, P.83.],[S2, Lemma 2.2.])**.** *The natural morphisms induce quasi-isomorphisms of the following DG vector spaces.*

$$\begin{aligned}
(\ker D'_h, D''_E) &\to (A^*(E), D_E),\\
(\ker D'_h, D''_E) &\to (A^*(E), D''_E),\\
(\ker D'_h, D''_E) &\to (\mathbb{H}^*_{DR}, 0),\\
(\ker D'_h, D''_E) &\to (\mathbb{H}^*_{Dol}, 0)\\
(\ker D'_h, D''_E) &\to (\mathbb{H}^*_{D'_h}, 0).
\end{aligned}$$

Proof. We only prove the quasi-isomorphism of $i : (\ker D'_h, D''_E) \to (A^*(E), D_E)$.

$H^*(i)$ is surjective: Let $\alpha \in \mathrm{Ker} D_E$. We now consider $D'_h\alpha$ and show it is D''_E-closed. Since h is a harmonic metric, $D'_h D''_E + D''_E D'_h = 0$. Therefore, $D''_E D'_h\alpha = -D'_h D''_E\alpha = D'_h D'_h\alpha = 0$. The second equation follows from the assumption of α. As $D'_h\alpha$ is D'_h-closed, we can apply the $D'_h D''_E$-lemma. Hence there exists a β such that

$$D'_h\alpha = D'_h D''_E\beta.$$

Moreover

$$D''_E\alpha = -D'_h D''_E\beta.$$

Now we consider $\alpha - D_E\beta$. From the equations

$$\begin{aligned}
D'_h(\alpha - D_E\beta) &= D'_h\alpha - D'_h D''_E\beta = 0,\\
D''_E(\alpha - D_E\beta) &= D''_E\alpha - D''_E D'_h\beta = 0,
\end{aligned}$$

We have $\alpha - D_E\beta \in \mathrm{Ker} D'_h \cap \mathrm{Ker} D''_E$. Hence $\alpha - D_E\beta$ defines a cohomology class of $(\ker D'_h, D''_E)$. $H^*(i)$ maps the cohomology class of $\alpha - D_E\beta$ to the cohomology class of α. Hence $H^*(i)$ is surjective.

$H^*(i)$ is injective: Let $\alpha \in \mathrm{Ker} D'_h \cap \mathrm{Ker} D''_E$ and we assume that there exists a β such that $\alpha = D_E\beta$. Under this assumption, We can apply the $D'_h D''_E$-lemma to $D'_h\beta$. Then there exists a γ such that

$$D'_h\beta = D'_h D''_E\gamma.$$

Now we consider $\beta - D_E\gamma$. From the equations

$$D'_h(\beta - D_E\gamma) = D'_h\beta - D'_h D''_E\gamma = 0,$$
$$D''_E(\beta - D_E\gamma) = D''_E\beta - D''_E D'_h\gamma = D''_E\beta + D'_h\beta = D_E\beta = \alpha,$$

we obtain $\alpha \in D''_E(\mathrm{Ker}D'_h)$. Hence the cohomology class which α defines in $(\ker D'_h, D''_E)$ is 0. Therefore $H^*(i)$ is injective. □

Let E^* be the dual of E. For any $p, q \in \mathbb{Z}_{\geq 0}$, $E^{*\otimes p} \otimes E^{\otimes q}$ has a induced harmonic metric from E. Hence, $(A^*(E^{*\otimes p} \otimes E^{\otimes q}), D_E)$ and $(A^*(E^{*\otimes p} \otimes E^{\otimes q}), D''_E)$ satisfy formality condition. We now focus on $p = q = 1$ case. In this case, $E^* \otimes E = \mathrm{End}E$ and $A^*(\mathrm{End}E)$ has a naturally defined bracket $[,]_{\mathrm{End}E}$ such that for $A \in A^i(\mathrm{End}E)$ and $B \in A^j(\mathrm{End}E)$,

$$[A, B]_{\mathrm{End}E} = A \wedge B - (-1)^{ij} B \wedge A.$$

We give a local description for convenience. Assume that $A = A_i dz_i$ and $B = B_j dz_j$ locally. Here A_i and B_j are matrix-valued functions. Then the bracket $[A, B]_{\mathrm{End}E}$ is calculated as

$$[A, B]_{\mathrm{End}E} = A_i B_j dz_i \wedge dz_j - (-1)^{ij} B_j A_i dz_j \wedge dz_i..$$

By some calculation, we can show that $(A^*(\mathrm{End}E), [,]_{\mathrm{End}E}, D_E)$ and $(A^*(\mathrm{End}E), [,]_{\mathrm{End}E}, D''_E)$ are DGLA. Therefore, by Lemma 1.4.3, we have the following lemma.

Lemma 1.4.4. $(A^*(\mathrm{End}E), [,]_{\mathrm{End}E}, D_E)$ *and* $(A^*(\mathrm{End}E), [,]_{\mathrm{End}E}, D''_E)$ *are formal as DGLA.*

1.5 Structure of Kuranishi space

In this section, we study the structure of the analytic germ $(Kur_{(X,E,\theta)}, 0)$ when (X, ω) is a compact Kähler manifold and the Higgs bundle $(E, \overline{\partial}_E, \theta)$ on X has a harmonic metric K. We prove that $(Kur_{(X,E,\theta)}, 0) \simeq (Kur_{(E,\theta)} \times Kur_X, 0)$ as analytic germs. We prove that by showing certain DGLAs are quasi-isomorphic and apply Theorem 1.2.2.

Throughout this section, the DGLA $(L, [,]_L, d_L)$ is the DGLA in the Theroem 1.3.1.

1.5.1 DGLA

In this section, we study the differential of $(L, [,]_L, d_L)$ when (X, ω) is a compact Kähler manifold and $(E, \overline{\partial}_E, \theta, h)$ is a harmonic bundle.

Proposition 1.5.1. *When X is a compact Kähler manifold and $(E, \overline{\partial}_E, \theta, h)$ is a harmonic bundle over X, then the differential d_L of the DGLA $(L, [,]_L, d_L)$ acts on $(A, \phi) \in L^i$ as*

$$d_L \begin{pmatrix} A \\ \phi \end{pmatrix} = \begin{pmatrix} D''_E A + D'_h(\phi \lrcorner \theta) \\ \overline{\partial}_{TX}\phi \end{pmatrix}.$$

Proof. The second row is from the definition of d_L. From the definition of d_L, the first row of

$$d_L \begin{pmatrix} A \\ \phi \end{pmatrix}$$

is

$$\overline{\partial}_{\mathrm{End}E} A + (-1)^i \phi \lrcorner F_{d_h} + [\theta, A] + \{\partial_h, \phi\lrcorner\}\theta.$$

Since h is a harmonic metric, $D = D'_h + D''$ is flat. Therefore, the $(2,0)$-part and the $(1,1)$-part of D^2 is 0. The $(2,0)$-part is $\partial_h\theta$ and the $(1,1)$-part is $F_{d_h} + [\theta, \theta^\dagger_h]$. Hence we have the equality

$$\begin{aligned}\overline{\partial}_{\mathrm{End}E} A + (-1)^i \phi \lrcorner F_{d_h} + [\theta, A] + \{\partial_h, \phi\lrcorner\}\theta &= D''_E A - (-1)^i \phi\lrcorner[\theta, \theta^\dagger_h] + \partial_K(\phi\lrcorner\theta) \\ &= D''_E A + [\theta^\dagger_h, \phi\lrcorner\theta] + \partial_K(\phi\lrcorner\theta) \\ &= D''_E A + D'_h(\phi\lrcorner\theta).\end{aligned}$$

Hence the claim is proved. □

1.5.2 Quasi-isomorphisms of DGLAs

In this section, we prove quasi-isomorphisms of certain DGLAs. Let $(A^*(TX), [,]_{SN}, \overline{\partial}_{TX})$ be the Kodaira-Spencer algebra. We first state the main result of this section.

Theorem 1.5.1. *$(L, [,]_L, d_L)$ is quasi-isomorphic to $(A^*(\mathrm{End}E), [,]_{\mathrm{End}E}, D) \oplus (A^*(TX), [,]_{SN}, \overline{\partial}_{TX})$.*

We first prove the following Proposition.

Proposition 1.5.2. *$(\mathrm{Ker}D'_h \oplus A^*(TX), [,]_L, d_L)$ is a sub DGLA of $(L, [,], d_L)$ and the canonical morphism $i : (\mathrm{Ker}D'_h \oplus A^*(TX), [,]_L, d_L) \to (L, [,]_L, d_L)$ is a quasi-isomorphism.*

Proof. By the definition of d_L, it is easy to see that $(\mathrm{Ker}D'_h \oplus A^*(TX), d_L)$ is a sub DG vector space of (L, d_L). We show that $[,]_L$ preserves $\mathrm{Ker}D'_h \oplus A^*(TX)$. Let $\alpha := (A, \phi), \beta := (B, \psi) \in \mathrm{Ker}D'_h \oplus A^*(TX)$. We have

$$\begin{aligned}
[\alpha, \beta]_L &= \begin{pmatrix} (-1)^i\{\partial_h, \psi\lrcorner\}A - (-1)^{(i+1)j}\{\partial_h, \phi\lrcorner\}B - [A, B] \\ [\phi, \psi]_{SN} \end{pmatrix} \\
&= \begin{pmatrix} (-1)^i\{\partial_h, \psi\lrcorner\}A - (-1)^{(i+1)j}\{\partial_h, \phi\lrcorner\}B - [A, B] \\ [\phi, \psi]_{SN} \end{pmatrix}
\end{aligned}$$

Since $A \in \mathrm{Ker}D'_h$, we have

$$\begin{aligned}
\{\partial_h, \psi\lrcorner\}A &= \partial_h(\psi\lrcorner A) + (-1)^j \psi\lrcorner\partial_h A \\
&= \partial_h(\psi\lrcorner A) - (-1)^j \psi\lrcorner\theta_h^\dagger A \\
&= \partial_h(\psi\lrcorner A) + \theta_h^\dagger \psi\lrcorner A \\
&= D'_h(\psi\lrcorner A).
\end{aligned}$$

Hence we have

$$[\alpha, \beta]_L = \begin{pmatrix} (-1)^i D'_h(\psi\lrcorner A) - (-1)^{(i+1)j} D'_h(\phi\lrcorner B) - [A, B] \\ [\phi, \psi]_{SN} \end{pmatrix}. \tag{1.1}$$

Hence $[\alpha, \beta]_L \in \mathrm{Ker}D'_h \oplus A^*(TX)$. Therefore, $(\mathrm{Ker}D'_h \oplus A^*(TX), [,]_L, d_L)$ is a sub DGLA of $(L, [,]_L, d_L)$.

We show that the natural morphism $i : (\mathrm{Ker}D'_K \oplus A^*(TX), [,]_L, d_L) \to (L, [,], d_L)$ is a quasi-isomorphism.

$H^*(i)$ is surjective: Let $\eta := (A, \phi) \in \mathrm{Ker}d_L$. We want to show that there exist

$$\eta' = (A', \phi') \in \Big(\mathrm{Ker}D'_h \oplus A^*(TX)\Big) \cap \ker d_L$$

and $\gamma \in L$ such that

$$\eta - \eta' = d_L\gamma.$$

By Proposition 1.5.1, we have

$$d_L\eta = \begin{pmatrix} D''_E A + D'_h(\phi\lrcorner\theta) \\ \overline{\partial}_{TX}\phi \end{pmatrix} = 0.$$

Let $\mathcal{A}$ be the harmonic projection of A with respect to D''_E. The Hodge decomposition of A with respect to D''_E is

$$\begin{aligned}
A = \mathcal{A} + G''_E\Delta'' A &= \mathcal{A} + G''_E D''_E (D''_E)^* A + G''_E (D''_E)^* D''_E A \\
&= \mathcal{A} + D''_E (D''_E)^* G''_E A - \sqrt{-1} G''_E [\,\Lambda_\omega, D'_h]\; D''_E A \\
&= \mathcal{A} + D''_E (D''_E)^* G''_E A - \sqrt{-1} G''_E \Lambda_\omega D'_h D''_E A + \sqrt{-1} G''_E D'_h \Lambda_\omega D''_E A \\
&= \mathcal{A} + D''_E (D''_E)^* G''_E A + \sqrt{-1} G''_E \Lambda_\omega D'_h D'_h(\phi\lrcorner\theta) + \sqrt{-1} D'_h G''_E \Lambda_\omega D''_E A \\
&= \mathcal{A} + D''_E (D''_E)^* G''_E A + \sqrt{-1} D'_h G''_E \Lambda_\omega D''_E A.
\end{aligned}$$

The compatibility of D'_h and G''_E follows from the fact that $G''_E = G'_h$.

We set

$$\eta' := \begin{pmatrix} \mathcal{A} + \sqrt{-1}D'_h G''_E \Lambda_\omega D''_E A \\ \phi \end{pmatrix},$$
$$\gamma := \begin{pmatrix} (D''_E)^* G''_E A \\ 0 \end{pmatrix}.$$

Since $D''_E(\mathcal{A} + \sqrt{-1}D'_h G''_E \Lambda_\omega D''_E A) = \sqrt{-1}D''_E D'_h G''_E \Lambda_\omega D''_E A = D''_E A$ and $\mathcal{A}$ is the harmonic projection of A, $\eta' \in \Big(\mathrm{Ker}D'_h \oplus A^*(TX)\Big) \cap \mathrm{ker}d_L$. By direct calculation, we can check $\eta - \eta' = d_L\gamma$. Hence, $H^*(i)$ is surjective.

$H^*(i)$ is injective: Let $\eta := (A, \phi) \in \Big(\mathrm{Ker}D'_h \oplus A^*(TX)\Big) \cap \mathrm{ker}d_L$. We assume that there exsits a $\beta := (B, \psi) \in L$ such that $\alpha = d_L\beta$. Under this assumption, we have

$$\begin{pmatrix} A \\ \phi \end{pmatrix} = \alpha = d_L\beta = \begin{pmatrix} D''_E B + D'_h(\psi \lrcorner \theta) \\ \overline{\partial}_{TX}\psi \end{pmatrix}.$$

Since $A \in \mathrm{ker}D'_h$, $D''_E B \in \mathrm{ker}D'_h \cap \mathrm{ker}D''_E \cap (\mathrm{im}D''_E + \mathrm{im}D'_h)$. Hence we can apply $D'_h D''_E$-lemma to $D''_E B$. Let $C \in A^*(\mathrm{End}E)$ such that $D''_E B = D''_E D'_h C$. We set,

$$\gamma := \begin{pmatrix} D'_h C \\ \psi \end{pmatrix}.$$

Then $\gamma \in \mathrm{Ker}D'_h \oplus A^*(TX)$ and $\alpha = d_L\gamma$ stands. Hence we showed that $H^*(i)$ is injective. □

For $A \in A^*(E)$, let $[A]_{D'_h}$ be the cohomology class in $\mathbb{H}^*_{D'_h}$. Let $Q : \mathrm{Ker}D'_h \to \mathbb{H}^*_{D'_h}$ be the $\mathbb{C}$-linear map such that $Q(A) = [A]_{D'_h}$.

Proposition 1.5.3. *The morphism*

$$\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix} : (\mathrm{Ker}D'_h \oplus A^*(TX), [,]_L, d_L) \to \left(\mathbb{H}^*_{D'_h} \oplus A^*(TX), [,]_{\mathrm{End}E} \oplus [,]_{SN}, \begin{pmatrix} 0 & 0 \\ 0 & \overline{\partial}_{TX} \end{pmatrix}\right)$$

is a morphism of DGLA and it is a quasi-isomorphism.

Proof. We first show that $\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix}$ is a morphism of DG vector spaces. Let $\alpha := (A, \phi) \in \big(\mathrm{Ker}D'_h \oplus A^*(TX)\big)$. We have,

$$\begin{pmatrix} 0 & 0 \\ 0 & \overline{\partial}_{TX} \end{pmatrix}\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix}\begin{pmatrix} A \\ \phi \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \overline{\partial}_{TX} \end{pmatrix}\begin{pmatrix} [A]_{D'_h} \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{\partial}_{TX}\phi \end{pmatrix},$$
$$\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix} \circ d_L \begin{pmatrix} A \\ \phi \end{pmatrix} = \begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix}\begin{pmatrix} D''_E A + D'_h(\phi \lrcorner \theta) \\ \overline{\partial}_{TX}\phi \end{pmatrix}$$
$$= \begin{pmatrix} [D''_E A + D'_h(\phi \lrcorner \theta)]_{D'_h} \\ \overline{\partial}_{TX}\phi \end{pmatrix}.$$

Since $A \in \mathrm{Ker}D'_h$, we can apply $D'_h D''_E$-lemma to $D''_E A$. Hence there is a $B \in A^*(\mathrm{End}E)$ such that $D''_E A = D'_h D''_E B$. Therefore

$$[D''_E A + D'_h(\phi \lrcorner \theta)]_{D'_h} = [D'_h D''_E B + D'_h(\phi \lrcorner \theta)]_{D'_h} = [D'_h(D''_E B + \phi \lrcorner \theta)]_{D'_h} = 0.$$

Hence $\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix}$ is a morphism of DG vector spaces. We next show that it is compatible with the brackets. Let $\alpha := (A, \phi), \beta := (B, \psi) \in \mathrm{Ker}D'_h \oplus A^*(TX)$. By (1.1), we have

$$[\alpha, \beta]_L = \begin{pmatrix} (-1)^i D'_h(\psi \lrcorner A) - (-1)^{(i+1)j} D'_h(\phi \lrcorner B) - [A, B] \\ [\phi, \psi]_{SN} \end{pmatrix}.$$

Hence we have

$$\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix} [\alpha, \beta]_L = \begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix} \begin{pmatrix} (-1)^i D'_h(\psi \lrcorner A) - (-1)^{(i+1)j} D'_h(\phi \lrcorner B) - [A, B] \\ [\phi, \psi]_{SN} \end{pmatrix}$$
$$= \begin{pmatrix} -[(-1)^i D'_h(\psi \lrcorner A) - (-1)^{(i+1)j} D'_h(\phi \lrcorner B) - [A, B]_{\mathrm{End}E}]_{D'_h} \\ [\phi, \psi]_{SN} \end{pmatrix}$$
$$= \begin{pmatrix} [[A, B]_{\mathrm{End}E}]_{D'_h} \\ [\phi, \psi]_{SN} \end{pmatrix} = \begin{pmatrix} [[A]_{D'_h}, [B]_{D'_h}]_{\mathrm{End}E} \\ [\phi, \psi]_{SN} \end{pmatrix}.$$

Hence $\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix}$ is a morphism of DGLA.

We next show that $\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix}$ is a quasi-isomorphism.

$H^*\left(\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix}\right)$ is surjective: Let $([A]_{D'_h}, \phi) \in \left(\mathbb{H}^*_{D'_h} \oplus A^*(TX)\right) \cap \mathrm{Ker} \begin{pmatrix} 0 & 0 \\ 0 & \overline{\partial}_{TX} \end{pmatrix}$. We first show that

$$-D''_E A + D'_h(\phi \lrcorner \theta) \in \mathrm{Ker} D''_E \cap \mathrm{Ker} D'_h \cap \mathrm{im} D_E.$$

Since $A \in \mathrm{Ker} D'_h$, $-D''_E A + D'_h(\phi \lrcorner \theta) \in \mathrm{Ker} D'_h$.

Since $\overline{\partial}_{TX} \phi = 0$, we have

$$D''_E(\phi \lrcorner \theta) = \overline{\partial}_{\mathrm{End}E}(\phi \lrcorner \theta) + [\theta, \phi \lrcorner \theta]_{\mathrm{End}E}$$
$$= \overline{\partial}_{TX} \phi \lrcorner \theta - \phi \lrcorner \overline{\partial}_{\mathrm{End}E} \theta + \frac{1}{2} \phi \lrcorner [\theta, \theta]_{\mathrm{End}E}$$
$$= 0.$$

Hence

$$D''_E(-D''_E A + D'_h(\phi \lrcorner \theta)) = -D'_h D''_E(\phi \lrcorner \theta) = 0.$$

Moreover, we have

$$D_E(-A + \phi \lrcorner \theta) = D''_E A + D'_h(\phi \lrcorner \theta).$$

Hence we proved $-D''_E A + D'_h(\phi \lrcorner \theta) \in \mathrm{Ker} D''_E \cap \mathrm{Ker} D'_h \cap \mathrm{im} D_E$. Therefore, we can apply $D'_h D''_E$-lemma to $-D''_E A + D'_h(\phi \lrcorner \theta)$. Hence there is a $B \in A^*(\mathrm{End}E)$ such that

$$-D''_E A + D'_h(\phi \lrcorner \theta) = D''_E D'_h B.$$

Equivalently, we have

$$-D''_E(A + D'_h B) + D'_h(\phi \lrcorner \theta) = 0.$$

Therefore

$$\begin{pmatrix} -A - D'_h B \\ \phi \end{pmatrix} \in \mathrm{Ker} D'_h \oplus A^*(TX) \cap \mathrm{Ker} d_L,$$

and

$$\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix} \begin{pmatrix} -A - D'_h B \\ \phi \end{pmatrix} = \begin{pmatrix} [A + D'_h B]_{D'_h} \\ \phi \end{pmatrix} = \begin{pmatrix} [A]_{D'_h} \\ \phi \end{pmatrix}.$$

Hence $H^*\left(\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix}\right)$ is surjective.

$H^*\left(\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix}\right)$ is injective: Let $(A, \phi) \in \left(\mathrm{Ker} D'_h \oplus A^*(TX)\right) \cap \mathrm{Ker} d_L$. We assume that the cohomology class of $([A]_{D'_h}, \phi)$ in $\left(\mathbb{H}^*_{D'_h} \oplus A^*(TX), \begin{pmatrix} 0 & 0 \\ 0 & \overline{\partial}_{TX} \end{pmatrix}\right)$ is 0. Hence there exsit a $B \in A^*(\mathrm{End}E)$ and a $\psi \in A^*(TX)$ such that

$$A = D'_h B,$$
$$\phi = \overline{\partial}_{TX} \psi.$$

We show that
$$A - D'_h(\psi \lrcorner \theta) \in \mathrm{Ker} D''_E \cap \mathrm{Ker} D'_h \cap \mathrm{im} D'_h.$$
Since $A = D'_h B$ and $A \in \mathrm{Ker} D'_h$, $A - D'_h(\psi \lrcorner \theta) \in \mathrm{Ker} D'_h \cap \mathrm{im} D'_h$. We also have
$$\begin{aligned} D''_E(A - D'_h(\psi \lrcorner \theta)) &= D''_E A + D'_h D''_E(\psi \lrcorner \theta) \\ &= D''_E A + D'_h(\overline{\partial}_{TX}\psi \lrcorner \theta + \frac{1}{2}\psi \lrcorner [\theta, \theta]_{\mathrm{End}E}) \\ &= D''_E A + D'_h(\phi \lrcorner \theta). \end{aligned}$$
Since $(A, \phi) \in \mathrm{Ker} d_L$, we have
$$d_L \begin{pmatrix} A \\ \phi \end{pmatrix} = \begin{pmatrix} D''_E A + D'_h(\phi \lrcorner \theta) \\ \overline{\partial}_{TX}\phi \end{pmatrix} = 0.$$
Therefore we have
$$D''_E(A - D'_h(\psi \lrcorner \theta) = D''_E A + D'_h(\phi \lrcorner \theta) = 0.$$
Hence we showed that $A - D'_h(\psi \lrcorner \theta) \in \mathrm{Ker} D''_E \cap \mathrm{Ker} D'_h \cap \mathrm{im} D'_h$. Therefore we can apply $D'_h D''_E$-lemma to $A - D'_h(\psi \lrcorner \theta)$. Hence there exists a $C \in A^*(\mathrm{End}E)$ such that
$$A - D'_h(\psi \lrcorner \theta) = D''_E D'_h C.$$
We note that $(D'_h C, \psi) \in \mathrm{Ker} D'_h \oplus A^*(TX)$ and
$$d_L \begin{pmatrix} D'_h C \\ \psi \end{pmatrix} = \begin{pmatrix} D''_E D'_h C + D'_h(\psi \lrcorner \theta) \\ \overline{\partial}_{TX}\psi \end{pmatrix} = \begin{pmatrix} A \\ \phi \end{pmatrix}.$$
Therefore the cohomology class of (A, ϕ) in $(\mathrm{Ker} D'_h \oplus A^*(TX), d_L)$ is 0. Hence $H^*\left(\begin{pmatrix} -Q & 0 \\ 0 & Id_{TX} \end{pmatrix}\right)$ is injective. □

Proof of Theorem 1.5.1. Combining Lemma 1.4.3, Proposition 1.5.2, and Proposition 1.5.3 we have the following chain of DGLAs
$$\begin{aligned} (L, [,]_L, d_L) &\leftarrow (\mathrm{Ker} D'_h \oplus A^*(TX), [,]_L, d_L) \\ &\rightarrow \left(\mathbb{H}^*_{D'_h} \oplus A^*(TX), [,]_{\mathrm{End}E} \oplus [,]_{SN}, \begin{pmatrix} 0 & 0 \\ 0 & \overline{\partial}_{TX} \end{pmatrix}\right) \\ &\leftarrow \left(\mathrm{Ker} D'_h \oplus A^*(TX), [,]_{\mathrm{End}E} \oplus [,]_{SN}, \begin{pmatrix} D_E & 0 \\ 0 & \overline{\partial}_{TX} \end{pmatrix}\right) \\ &\rightarrow \left(A^*(\mathrm{End}E) \oplus A^*(TX), [,]_{\mathrm{End}E} \oplus [,]_{SN}, \begin{pmatrix} D_E & 0 \\ 0 & \overline{\partial}_{TX} \end{pmatrix}\right) \end{aligned}$$
such that each morphism is quasi-isomorphism. Hence the claim is proved. □

Corollary 1.5.1. *$(L, [,]_L, d_L)$ is quasi-isomorphic to $(A^*(\mathrm{End}E), [,]_{\mathrm{End}E}, D''_E) \oplus (A^*(TX), [,]_{SN}, \overline{\partial}_{TX})$.*

Let Kur_X be the Kuranishi space of X, $Kur_{(E,\theta)}$ be the Kuranishi space of the Higgs bundle (E, θ), and $Kur_{(E,D)}$ be the Kuranishi space of the flat bundle (E, D).

Combining Theorem 1.2.2 and Theorem 1.5.1, we have the following theorem.

Theorem 1.5.2. *Let (X, ω) be a compact Kähler manifold, $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle over X and, K be a harmonic metric. Then*
$$\begin{aligned} (Kur_{(X,E,\theta)}, 0) \simeq (Kur_{(E,\theta)} \times Kur_X, 0), \\ (Kur_{(X,E,\theta)}, 0) \simeq (Kur_{(E,D)} \times Kur_X, 0) \end{aligned}$$
holds as germs of analytic spaces.

We have some consequences from Theorem 1.5.2 for specific cases. Let M be a Riemann surface with genus $g \geq 2$ and $(E, \overline{\partial}_E, \theta)$ be a stable Higgs bundle of degree 0. Under these assumptions, each deformations of M and $(E, \overline{\partial}_E, \theta)$ are unobstruced. Hence Kur_M and $Kur_{(E,\theta)}$ are complex manifolds. Moreover, the dimensions of Kur_X is $3g - 3$ and $Kur_{(E,\theta)}$ is $2 + r^2(2g - 2)$ [MK, N]. Here r is the rank of E. The following is straightforward from Theorem 1.5.2.

Corollary 1.5.2. *Let M be a Riemann surface with genus $g \geq 2$ and $(E, \overline{\partial}_E, \theta)$ be a stable Higgs bundle of degree 0. Then the deformation of pair (M, E, θ) is unobstructed. Moreover, $Kur_{(M,E,\theta)}$ is a complex manifold and its dimension is $g(2r^2 + 3) - 2r^2 - 1$.*

Chapter 2

Deformation of Higgs Triples

2.1 Abstract of Chapter 2

Let $(E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F)$ be Higgs bundles and $f : (E, \overline{\partial}_E, \theta) \to (F, \overline{\partial}_F, \theta)$ be a morphism between them. In this chapter, we study the deformation of the triple $(f, (E, \overline{\partial}_E, \theta), (F, \overline{\partial}_F, \theta))$. We call this triple Higgs triples. We construct the DGLA L which governs the deformation of Higgs triples and study the property of L when $(E, \overline{\partial}_E, \theta_E)$ and $(F, \overline{\partial}_F, \theta_F)$ admit harmonic metrics. In particular, we show that L is formal.

2.2 Introduction of Chapter 2

Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle. We have a Higgs bundle $(E, \overline{\partial}_E, \theta)$ and a flat bundle $(E, D = \partial_h + \overline{\partial}_E + \theta + \theta_h^\dagger)$. Let $L_{Dol,E}$ (resp. $L_{DR,E}$) be the Differential Graded Lie algebra (DGLA) that controls the deformation of the Higgs bundle (resp. the flat bundle). It was proved in [GM1, S2] that $(i) L_{Dol,E}$ and $L_{DR,E}$ are *formal*, (ii) $L_{Dol,E}$ and $L_{DR,E}$ are *quasi-isomorphic* (See Section 1.2.1 for details). Formality of $L_{DR,E}$ was used in [GM1] to prove that the deformation space of $\pi_1(X)$ is quadratic at D. The quasi-isomorphism between $L_{Dol,E}$ and $L_{DR,E}$ was used in [S4] to prove the *isosingularity principle*: the singularity of the Moduli space of Higgs bundle M_{Dol} and the Moduli space of flat bundle M_{DR} are formally isomorphic at corresponding points. However, the global homeomorphism between M_{Dol} and M_{DR} is not a complex isomorphism, so the local isomorphism is not directly related to the global map. See the introduction and Section 10 of [S4] for details.

We say a triple $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ is a *Higgs triple* over X if each $(E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ is a Higgs bundle over X and f is a morphism of Higgs bundles (i.e. f is a vector bundle morphism and $(\overline{\partial}_F + \theta_F) \circ f = f \circ (\overline{\partial}_E + \theta_E)$ holds). Note that if $\theta_E = \theta_F = 0$, then the Higgs triple $(f, (E, \overline{\partial}_E, 0), (F, \overline{\partial}_F, 0))$ is the *holomorphic triples* which is deeply studied in [BO, O].

In this chapter, we are interested in the deformation problem of Higgs triples $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$. We first construct the DGLA which controls the deformation of the Higgs triple.

Let

$$L^i := A^i(\mathrm{End}F) \oplus A^{i-1}(\mathrm{Hom}(E, F) \oplus A^i(\mathrm{End}E) \ \ (i \in \mathbb{Z}),$$
$$L := \oplus_i L^i.$$

We define a linear map $\overline{\partial}_f : L^i \to L^{i+1}$ such that for $(A, C, B) \in L^i$

$$\overline{\partial}_f \begin{pmatrix} A \\ C \\ B \end{pmatrix} := \begin{pmatrix} (\overline{\partial}_F + \theta_F)A \\ (\overline{\partial}_{\mathrm{Hom}(E,F)} + \theta_{\mathrm{Hom}(E,F)})C + (-1)^{i-1}Af - (-1)^{i-1}fB \\ (\overline{\partial}_E + \theta_E)B \end{pmatrix}.$$

We next a bilinear map $[,]_L : L^i \times L^j \to L^{i+j}$ as follows: Let $(A_i, C_i, B_i) \in L^i, (A_j, C_j, B_j) \in L^j$ then

$$[(A_i, C_i, B_i), (A_j, C_j, B_j)]_L := \begin{pmatrix} [A_i, A_j]_{\mathrm{End}F} \\ A_i \wedge C_j - (-1)^{i(j-1)} C_j \wedge B_i - (-1)^{ij} A_j \wedge C_i + (-1)^j C_i \wedge B_j \\ [B_i, B_j]_{\mathrm{End}E} \end{pmatrix}.$$

Then

Theorem 2.2.1 (Theorem 2.4.1, 2.4.2)**.** *$(L, [,]_L, \overline{\partial}_f)$ is a DGLA. Moreover $(L, [,]_L, \overline{\partial}_f)$ governs the deformation of the Higgs triple $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$: $x = (A, C, B) \in L^1$ defines a Higgs triple $(f + C, (E, \overline{\partial}_E + A^{0,1}, \theta_E + A^{1,0}), (F, \overline{\partial}_F + C^{0,1}, \theta_F + C^{1,0}))$ if and only if x satisfies the Maurer-Cartan equation. Here $A^{1,0}$, $B^{1,0}$ is the (1,0)-part of A, B and $A^{0,1}$, $B^{0,1}$ is the (0,1)-part of A, B.*

We are interesed in the property of $(L, [,]_L, \overline{\partial}_f)$ when the Higgs bundles $(E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F)$ has harmonic metrics h_E and h_F or equivalently, both Higgs bundles are polystable and $c_1(E) = c_1(F) = c_2(E) = c_2(F) = 0$. Under this assumption, morphisms of Higgs bundles between E and F are parameterized by a suitable matrix space: the space of Higgs bundle morphisms between E and F is isomorphic to $n \times m$ complex-valued matrix $M(n, m, \mathbb{C})$ as vector space. This is an application of the result that the morphism between stable Higgs bundles is $a\mathrm{Id}(a \in \mathbb{C})$ or 0. See [Ko] for details. Hence, once we fix $(E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F)$ with both polystable and $c_1(E) = c_1(F) = c_2(E) = c_2(F) = 0$, the deformation of Higgs bundles morphism $f : E \to F$ is not complicated. This observation and the fact the DGLAs $L_{Dol,E}$ and $L_{Dol,F}$ are formal when E and F has harmonic metrics, gives us a hope that $(L, [,]_L, \overline{\partial}_f)$ has some additional property. Actually, we prove the following

Theorem 2.2.2 (Theorem 2.5.2)**.** *Let $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ be a Higgs triple over a compact Kähler manifold X. We assume that each E and F has a harmonic metric h_E and h_F. Then $(L, [,]_L, \overline{\partial}_f)$ is formal.*

We say a triple $(f, (E, D_E), (F, D_F))$ is a *flat triple* over X if each $(E, D_E), (F, D_F))$ is a flat bundle over X and f is a morphism of flat bundles (i.e. f is a vector bundle morphism and $D_F \circ f = f \circ D_E$ holds). In this chapter, we also construct the DGLA $(L, [,]_L, d_f)$ which controls the deformation of $(f, (E, D_E), (F, D_F))$.

We study the property of $(L, [,]_L, d_f)$ when (E, D_E) and (F, D_F) comes from harmonic bundles $(E, \overline{\partial}_E, \theta_E, h_E)$, $(F, \overline{\partial}_F, \theta_F, h_F)$ (i.e. $D_E = \partial_{h_E} + \overline{\partial}_E + \theta_E + \theta^{\dagger}_{h_E}$, $D_F = \partial_{h_F} + \overline{\partial}_F + \theta_E + \theta^{\dagger}_{h_F}$). By [S2, Corollary 1.3], a Higgs triples $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ with harmonic metrics h_E, h_F, gives a flat triple $(f, (E, D_E), (F, D_F))$. Conversly, a flat triples $(f, (E, D_E), (F, D_F))$ which both flat bundles comes from harmonic bundles $(E, \overline{\partial}_E, \theta_E, h_E)$, $(F, \overline{\partial}_F, \theta_F, h_F)$, gives a Higgs triple $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$.

Hence it is natrual to compare $(L, [,]_L, d_f)$ and $(L, [,]_L, \overline{\partial}_f)$ under the assumption of exsitence of harmonic metrics.

Theorem 2.2.3 (Theorem 2.5.3)**.** *Let $(f, (E, D_E), (F, D_F))$ be a flat triple over a compact Kähelr manifold X. Assume that (E, D_E) and (F, D_F) comes form harmonic bundles $(E, \overline{\partial}_E, \theta_E, h_E), (F, \overline{\partial}_F, \theta_F, h_F)$. Let $(L, [,]_L, d_f)$ be the DGLA which controls the deformation of $(f, (E, D_E), (F, D_F))$ and $(L, [,]_L, \overline{\partial}_f)$ be the DGLA which controls the deformation of $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$.*

Then

- *$(L, [,]_L, d_f)$ is formal.*
- *$(L, [,]_L, d_f)$ is quasi-isomorphic to $(L, [,]_L, \overline{\partial}_f)$.*

This is an analog of $L_{Dol,E}$ and $L_{DR,E}$ being quasi-isomorohic for a harmonic bundle E.

Notation

We use the notation of 1.2.1 and 1.4.

For a vector bundle E, we denote the space of smooth sections of E as $A(E)$. We denote the space of E-valued smooth p-forms as $A^p(E)$.

Let F be a vector bundle. We freely identify $E^{\vee} \otimes F$ and $\mathrm{Hom}(E, F)$.

2.3 Preliminary

2.3.1 Leibniz Rule

Let $(E, \overline{\partial}_E, \theta_E)$ be Higgs bundles. Then for the dual bundle $E^\vee$, we have a dual holomorphic structure $\overline{\partial}_{E^\vee}$ and the dual Higgs field $\theta^\vee$ and hence we have a dual Higgs bundle $(E^\vee, \overline{\partial}_{E^\vee}, \theta_E^\vee)$. The dual Higgs field $\theta^\vee$ is defined as $\theta^\vee = -\theta^t$. Here θ^t is the transpose of θ. Let h_E be a harmonic metric. Then the dual metric $h_E^\vee$ is a harmonic metric for $(E^\vee, \overline{\partial}_{E^\vee}, \theta_E^\vee)$.

Let $(F, \overline{\partial}_F, \theta_F, h_F)$ be a harmonic bundle. Then we have a harmonic bundle $(E^\vee \otimes F, \overline{\partial}_{E^\vee \otimes F}, \theta_{E^\vee \otimes F}, h_E^\vee \otimes h_F)$. We also have associated operators $D''_{E^\vee \otimes F}, D'_{h_E^\vee \otimes h_F}, D_{E^\vee \otimes F}$.

Let $A \in A^i(\mathrm{End}F), B \in A^i(\mathrm{End}E), C \in A^j(E^\vee \otimes F),$. Then since $E^\vee \otimes F = \mathrm{Hom}(E, F)$, $A \wedge C, C \wedge B \in A^{i+j}(\mathrm{Hom}(E, F)) = A^{i+j}(E^\vee \otimes F)$.

The following result is an application of Leibniz rule and will be used throughout the chapter. We sometimes use it without mention.

Lemma 2.3.1. *Let $A \in A^i(\mathrm{End}F), C \in A^j(E^\vee \otimes F), B \in A^k(\mathrm{End}E)$. Then*

$$\begin{aligned} D''_{E^\vee \otimes F}(A \wedge C) &= (D''_F A) \wedge C + (-1)^i A \wedge D''_{E^\vee \otimes F}(C), \\ D''_{E^\vee \otimes F}(C \wedge B) &= D''_{E^\vee \otimes F}(C) \wedge B + (-1)^j C \wedge (D''_E B). \end{aligned}$$

This also holds for $D'_{h_E^\vee \otimes h_F}, D_{E^\vee \otimes F}$.

2.4 Deformation of Higgs Triples

Let X be a compact complex manifold and $(E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F)$ be Higgs bundles over X. We say that $f : E \to F$ is a morphism of Higgs bundles $(E, \overline{\partial}_E, \theta_E)$ and $(F, \overline{\partial}_F, \theta_F)$ if f is a morphism of vector bundles and $(D''_F) \circ f = f \circ (\overline{\partial}_E + \theta_E)$ holds.

We say $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ is a *Higgs tripe* over X if $(E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F)$ are Higgs bundles and $f : E \to F$ is a morphism of Higgs bundles over X and a pair if X is clear.

In this section, we study the simultaneous deformation problem of a given Higgs tripe $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$. The goal of this section is to construct the DGLA which governs the deformation: We construct a DGLA L such that $A \in L^1$ defines another Higgs triple $(f_A, (E_A, \overline{\partial}_{E_A}, \theta_{E_A}), (F_A, \overline{\partial}_{F_A}, \theta_{F_A}))$ if and only if A is a Maurer-Cartan element.

We now fix a Higgs triper $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ and let $(f_1, (E_1, \overline{\partial}_{E_1}, \theta_{E_1}), (F_1, \overline{\partial}_{F_1}, \theta_{F_1}))$ be another Higgs triple. Let

$$\begin{aligned} A &:= D''_{F_1} - D''_F, \\ C &:= f_1 - f, \\ B &:= D''_{E_1} - D''_E. \end{aligned}$$

Note that $A \in A^1(\mathrm{End}F), C \in A(\mathrm{Hom}(E, F)), B \in A^1(\mathrm{End}E)$. Since $(D''_F + A)^2 = (D''_{F_1})^2 = 0$ and $(D''_E + B)^2 = (D''_{E_1})^2 = 0$ holds, we have

$$\begin{aligned} D''_F A + \frac{1}{2}[A, A]_{\mathrm{End}F} = 0, \\ D''_E B + \frac{1}{2}[B, B]_{\mathrm{End}E} = 0. \end{aligned}$$

The brackets $[,]_{\mathrm{End}F}, [,]_{\mathrm{End}E}$ are defined in Section 1.4.2. From now on, we denote $[,]_{\mathrm{End}F}, [,]_{\mathrm{End}E}$ as $[,]$ if there is no risk of confusion.

Moreover since $(D''_F + A) \circ (f + C) - (f + C) \circ (D''_E + B) = D''_{F_1} \circ f_1 - f_1 \circ D''_{E_1} = 0$ holds, we have

$$\begin{aligned} &(D''_F + A) \circ (f + C) - (f + C) \circ (D''_E + B) \\ =&D''_F \circ f + D''_F C + Af + AC - f \circ D''_E - fB - CD''_E + CB \\ =&D''_{E^\vee \otimes F} C + Af - fB + AC - CB = 0 \end{aligned}$$

Here $E^\vee$ is the dual bundle of E.

Conversely, let $A \in A^1(\mathrm{End}F), C \in A(\mathrm{Hom}(E,F)), B \in A^1(\mathrm{End}E)$ and assume they satisifes

$$\begin{aligned} D''_F A + \frac{1}{2}[A,A] = 0, \\ D''_E B + \frac{1}{2}[B,B] = 0, \\ D''_{E^\vee \otimes F} C + Af - fB + AC - CB = 0. \end{aligned}$$

Then, it is clear that $(f + C, (E, \overline{\partial}_E + A^{0,1}, \theta_E + A^{1,0}), (F, \overline{\partial}_F + B^{0,1}, \theta_F + B^{1,0}))$ is a Higgs triple. Here $A^{1,0}$, $B^{1,0}$ is the (1,0)-part of A, B and $A^{0,1}$, $B^{0,1}$ is the (0,1)-part of A, B.

From this observation, we will now construct the DGLA which governs the deformation of a given Higgs triple $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$.

Let

$$\begin{aligned} &L^i := A^i(\mathrm{End}F) \oplus A^{i-1}(\mathrm{Hom}(E,F)) \oplus A^i(\mathrm{End}E) \ \ (i \in \mathbb{Z}), \\ &L := \oplus_i L^i. \end{aligned}$$

We define a linear map $\overline{\partial}_f : L^i \to L^{i+1}$ such that for $(A, C, B) \in L^i$

$$\overline{\partial}_f \begin{pmatrix} A \\ C \\ B \end{pmatrix} := \begin{pmatrix} D''_F A \\ D''_{E^\vee \otimes F} C + (-1)^{i-1} Af - (-1)^{i-1} fB \\ D''_E B \end{pmatrix}.$$

Since $A \in A^i(\mathrm{End}F)$, $B \in A^i(\mathrm{End}E)$, $Af \in A^i(\mathrm{Hom}(E,F))$ and $fB \in A^i(\mathrm{Hom}(E,F))$. Then since $L^{i+1} = A^{i+1}(\mathrm{End}F) \oplus A^i(\mathrm{Hom}(E,F)) \oplus A^{i+1}(\mathrm{End}E)$, $\overline{\partial}_f(A,C,B)$ is indeed an element of L^{i+1}.

We next define the bracket $[,]_L : L^i \times L^j \to L^{i+j}$ as follows: Let $(A_i, C_i, B_i) \in L^i, (A_j, C_j, B_j) \in L^j$ then

$$[(A_i, C_i, B_i), (A_j, C_j, B_j)]_L := \begin{pmatrix} [A_i, A_j] \\ A_i \wedge C_j - (-1)^{i(j-1)} C_j \wedge B_i - (-1)^{ij} A_j \wedge C_i + (-1)^j C_i \wedge B_j \\ [B_i, B_j] \end{pmatrix}.$$

The first result of this chapter is as follows:

Theorem 2.4.1. *$(L, [,]_L, \overline{\partial}_f)$ is a DGLA.*

Since the proof consists of a lengthy computation, we give it in the next section.

Theorem 2.4.2. *Let $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ be a Higgs triple. Then the DGLA $(L, [,]_L, \overline{\partial}_f)$ controls the deformation of the Higgs triple: $x = (A, C, B) \in L^1$ defines a Higgs triple $(f + C, (E, \overline{\partial}_E + A^{0,1}, \theta_E + A^{1,0}), (F, \overline{\partial}_F + B^{0,1}, \theta_F + B^{1,0}))$ if and only if x is a Maurer-Cartan element.*

Proof. Let $x = (A, C, B) \in L^1$ be a Maurer-Cartan element. Then by definition

$$\overline{\partial}_f x + \frac{1}{2}[x,x]_L = 0.$$

Then, computing each row, we have

$$\begin{aligned} D''_F A + \frac{1}{2}[A,A] = 0, \\ D''_E B + \frac{1}{2}[B,B] = 0, \\ D''_{E^\vee \otimes F} C + Af - fB + AC - CB = 0. \end{aligned}$$

Then x defines a Higgs triple $(f + C, (E, \overline{\partial}_E + A^{0,1}, \theta_E + A^{1,0}), (F, \overline{\partial}_F + B^{0,1}, \theta_F + B^{1,0}))$ by the above observation. The converse is also true by the observation above. □

2.4.1 Proof of Theorem 2.4.1

Let $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ be a Higgs triple and let $(L, [,]_L, \overline{\partial}_f)$ be the pair of the graded vector space, the bracket, and the linear map we defined in the previous section. In this section, we prove that it is actually a DGLA. We first prove that the bracket is graded skew-symmetry.

Proposition 2.4.1. *Let $x = (A_i, C_i, B_i) \in L^i, y = (A_j, C_j, B_j) \in L^j$. Then*

$$[x, y]_L + (-1)^{ij}[y, x]_L = 0$$

holds.

Proof. By the definition of the bracket, we have

$$[x, y]_L = \begin{pmatrix} [A_i, A_j] \\ A_i \wedge C_j - (-1)^{i(j-1)} C_j \wedge B_i - (-1)^{ij} A_j \wedge C_i + (-1)^j C_i \wedge B_j \\ [B_i, B_j] \end{pmatrix}$$

and

$$\begin{aligned} (-1)^{ij}[y, x]_L &= (-1)^{ij} \begin{pmatrix} [A_j, A_i] \\ A_j \wedge C_i - (-1)^{j(i-1)} C_i \wedge B_j - (-1)^{ij} A_i \wedge C_j + (-1)^i C_j \wedge B_i \\ [B_j, B_i] \end{pmatrix} \\ &= \begin{pmatrix} (-1)^{ij}[A_j, A_i] \\ (-1)^{ij} A_j \wedge C_i - (-1)^{-j} C_i \wedge B_j - A_i \wedge C_j + (-1)^{ij+i} C_j \wedge B_i \\ (-1)^{ij}[B_j, B_i] \end{pmatrix}. \end{aligned}$$

Since the bracket of End$F$ valued forms is graded skew symmetry

$$[x, y]_L + (-1)^{ij}[y, x]_L = 0.$$

□

We next prove that the bracket satisfies the graded Jacobi identity.

Proposition 2.4.2. *Let $x = (A_i, C_i, B_i) \in L^i, y = (A_j, C_j, B_j) \in L^j, z = (A_k, C_k, B_k) \in L^k$. Then*

$$[x, [y, z]_L]_L = [[x, y]_L, z]_L + (-1)^{ij}[y, [x, z]_L]_L$$

holds.

Proof. We first calculate each element.

$$\begin{aligned} [x, [y, z]_L]_L &= \begin{bmatrix} A_i & [A_j, A_k] \\ C_i & A_j \wedge C_k - (-1)^{j(k-1)} C_k \wedge B_j - (-1)^{jk} A_k \wedge C_j + (-1)^k C_j \wedge B_k \\ B_i & [B_j, B_k] \end{bmatrix}_L \\ &= \begin{pmatrix} [A_i, [A_j, A_k]] \\ C_{[x,[y,z]_L]_L} \\ [B_i, [B_j, B_k]] \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} C_{[x,[y,z]_L]_L} =& A_i \wedge A_j \wedge C_k - (-1)^{j(k-1)} A_i \wedge C_k \wedge B_j - (-1)^{jk} A_i \wedge A_k \wedge C_j + (-1)^k A_i \wedge C_j \wedge B_k \\ & - (-1)^{i(j+k-1)} \left\{ A_j \wedge C_k \wedge B_i - (-1)^{j(k-1)} C_k \wedge B_j \wedge B_i - (-1)^{jk} A_k \wedge C_j \wedge B_i + (-1)^k C_j \wedge B_k \wedge B_i \right\} \\ & - (-1)^{i(j+k)} [A_j, A_k] \wedge C_i + (-1)^{j+k} C_i \wedge [B_j, B_k] \\ =& A_i \wedge A_j \wedge C_k - (-1)^{j(k-1)} A_i \wedge C_k \wedge B_j - (-1)^{jk} A_i \wedge A_k \wedge C_j + (-1)^k A_i \wedge C_j \wedge B_k \\ & - (-1)^{i(j+k-1)} A_j \wedge C_k \wedge B_i - (-1)^{i(j+k-1)+j(k-1)} C_k \wedge B_j \wedge B_i - (-1)^{i(j+k-1)+jk} A_k \wedge C_j \wedge B_i \\ & + (-1)^{i(j+k-1)+k} C_j \wedge B_k \wedge A_i - (-1)^{i(j+k)} [A_j, A_k] \wedge C_i + (-1)^{j+k} C_i \wedge [B_j, B_k]. \end{aligned}$$

Similarly, we have

$$[[x,y]_L,z]_L = \begin{bmatrix} [A_i,A_j] & A_k \\ A_i\wedge C_j-(-1)^{i(j-1)}C_j\wedge B_i-(-1)^{ij}A_j\wedge C_i+(-1)^jC_i\wedge B_j & C_k \\ [B_i,B_j] & B_k \end{bmatrix}_L = \begin{pmatrix} [[A_i,A_j],A_k] \\ C_{[[x,y]_L,z]_L} \\ [[B_i,B_j],B_k] \end{pmatrix}$$

where

$$\begin{aligned} C_{[[x,y]_L,z]_L} =& [A_i,A_j]\wedge C_k-(-1)^{(i+j)(k-1)}C_k\wedge [B_i,B_j] \\ &-(-1)^{(i+j)k}\Big\{A_k\wedge A_i\wedge C_j-(-1)^{i(j-1)}A_k\wedge C_j\wedge B_i-(-1)^{ij}A_k\wedge A_j\wedge C_i+(-1)^jA_k\wedge C_i\wedge B_j\Big\} \\ &+(-1)^k\Big\{A_i\wedge C_j\wedge B_k-(-1)^{i(j-1)}C_j\wedge B_i\wedge B_j-(-1)^{ij}A_j\wedge C_i\wedge B_k+(-1)^jC_i\wedge B_j\wedge B_k\Big\} \end{aligned}$$

and

$$(-1)^{ij}[y,[x,z]_L]_L = (-1)^{ij}\begin{bmatrix} A_j & [A_i,A_k] \\ C_j & A_i\wedge C_k-(-1)^{i(k-1)}C_k\wedge B_i-(-1)^{ik}A_k\wedge C_i+(-1)^kC_i\wedge B_k \\ B_j & [B_i,B_k] \end{bmatrix}_L = (-1)^{ij}\begin{pmatrix} [A_j,[A_i,A_k]] \\ C_{[y,[x,z]_L]_L} \\ [B_j,[B_i,B_k]] \end{pmatrix}$$

where

$$\begin{aligned} C_{[y,[x,z]_L]_L} =& A_j\wedge A_i\wedge C_k-(-1)^{i(k-1)}A_j\wedge C_k\wedge B_i-(-1)^{ik}A_j\wedge A_k\wedge C_i+(-1)^kA_j\wedge C_i\wedge B_k \\ &-(-1)^{j(i+k-1))}\Big\{A_i\wedge C_k\wedge B_j-(-1)^{i(k-1)}C_k\wedge B_i\wedge B_j-(-1)^{ik}A_k\wedge C_i\wedge B_j+(-1)^kC_i\wedge B_k\wedge B_j\Big\} \\ &-(-1)^{j(i+k)}[A_i,A_k]\wedge C_j+(-1)^{j+k}C_j\wedge [B_i,B_k] \end{aligned}$$

Since $(A^\bullet(\text{End}F),[,],D''_F),(A^\bullet(\text{End}F),[,],D''_F)$ are DGLA, we only need to prove

$$C_{[x,[y,z]_L]_L} = C_{[[x,y]_L,z]_L} + (-1)^{ij}C_{[y,[x,z]_L]_L}$$

$$
\begin{aligned}
&C_{[[x,y]_L,z]_L} + (-1)^{ij} C_{[y,[x,z]_L]_L}\\
=&[A_i, A_j] \wedge C_k - (-1)^{(i+j)(k-1)} C_k \wedge [B_i, B_j]\\
&- (-1)^{(i+j)k} \Big\{ A_k \wedge A_i \wedge C_j - (-1)^{i(j-1)} A_k \wedge C_j \wedge B_i - (-1)^{ij} A_k \wedge A_j \wedge C_i + (-1)^j A_k \wedge C_i \wedge B_j \Big\}\\
&+ (-1)^k \Big\{ A_i \wedge C_j \wedge B_k - (-1)^{i(j-1)} C_j \wedge B_i \wedge B_j - (-1)^{ij} A_j \wedge C_i \wedge B_k + (-1)^j C_i \wedge B_j \wedge B_k \Big\}\\
&+ (-1)^{ij} A_j \wedge A_i \wedge C_k - (-1)^{i(j+k-1)} A_j \wedge C_k \wedge B_i - (-1)^{i(j+k)} A_j \wedge A_k \wedge C_i + (-1)^{ij+k} A_j \wedge C_i \wedge B_k\\
&- (-1)^{j(k-1)} \Big\{ A_i \wedge C_k \wedge B_j - (-1)^{i(k-1)} C_k \wedge B_i \wedge B_j - (-1)^{ik} A_k \wedge C_i \wedge B_j + (-1)^k C_i \wedge B_k \wedge B_j \Big\}\\
&- (-1)^{jk} [A_i, A_k] \wedge C_j + (-1)^{ij+j+k} C_j \wedge [B_i, B_k]\\
=&[A_i, A_j] \wedge C_k + (-1)^{ij} A_j \wedge A_i \wedge C_k - (-1)^{j(k-1))} A_i \wedge C_k \wedge B_j\\
&- (-1)^{(i+j)k} A_k \wedge A_i \wedge C_j - (-1)^{jk} [A_i, A_k] \wedge C_j + (-1)^k A_i \wedge C_j \wedge B_k - (-1)^{i(j+k-1)} A_j \wedge C_k \wedge B_i\\
&- (-1)^{(i+j)(k-1)} C_k \wedge [B_i, B_j] + (-1)^{(i+j)(k-1)} C_k \wedge B_i \wedge B_j + (-1)^{(i+j)k+i(j-1)} A_k \wedge C_j \wedge B_i\\
&- (-1)^{i(j-1)+k} C_j \wedge B_i \wedge B_j + (-1)^{ij+j+k} C_j \wedge [B_i, B_k]\\
&+ (-1)^{(i+j)k+ij} A_k \wedge A_j \wedge C_i - (-1)^{i(j+k)} A_j \wedge A_k \wedge C_i\\
&+ (-1)^{j+k} C_i \wedge B_j \wedge B_k - (-1)^{j(k-1)+k} C_i \wedge B_k \wedge B_j\\
&- (-1)^{(i+j)k+j} A_k \wedge C_i \wedge B_j + (-1)^{ik+j(k-1)} A_k \wedge C_i \wedge B_j\\
&- (-1)^{ij+k} A_j \wedge C_i \wedge B_k + (-1)^{ij+k} A_j \wedge C_i \wedge B_k\\
=&A_i \wedge A_j \wedge C_k - (-1)^{j(k-1)} A_i \wedge C_k \wedge B_j\\
&- (-1)^{jk} A_i \wedge A_k \wedge C_j + (-1)^k A_i \wedge C_j \wedge B_k - (-1)^{i(j+k-1)} A_j \wedge C_k \wedge B_i\\
&- (-1)^{i(j+k-1)+j(k-1)} C_k \wedge B_j \wedge B_i - (-1)^{i(j+k-1)+jk} A_k \wedge C_j \wedge B_i\\
&+ (-1)^{i(j+k-1)+k} C_j \wedge B_k \wedge A_i\\
&- (-1)^{i(j+k)} [A_j, A_k] \wedge C_i\\
&+ (-1)^{j+k} C_i \wedge [B_j, B_k]\\
=&C_{[x,[y,z]_L]_L}.
\end{aligned}
$$

□

We next prove that the square of $\overline{\partial}_f$ is zero.

Proposition 2.4.3. *Let $x = (A, C, B) \in L^i$. Then*

$$\overline{\partial}_f \circ \overline{\partial}_f(x) = 0$$

holds.

Proof. By the definition of d_f, we have

$$
\begin{aligned}
\overline{\partial}_f \circ \overline{\partial}_f(x) &= \overline{\partial}_f \circ \overline{\partial}_f \begin{pmatrix} A \\ B \\ C \end{pmatrix}\\
&= \overline{\partial}_f \begin{pmatrix} D''_F A \\ D''_{E^\vee \otimes F} C + (-1)^{i-1} A f - (-1)^{i-1} f B \\ D''_E B \end{pmatrix}\\
&= \begin{pmatrix} D''_F (D''_F A) \\ C_{\overline{\partial}_f \circ \overline{\partial}_f(x)} \\ D''_E (D''_E B) \end{pmatrix}
\end{aligned}
$$

where

$$\begin{aligned}
C_{\overline{\partial}_f\circ\overline{\partial}_f(x)} &= D''_{E^\vee\otimes F}(D''_{E^\vee\otimes F}C + (-1)^{i-1}Af - (-1)^{i-1}fB) + (-1)^i(D''_FA)f - (-1)^i f(D''_EB) \\
&= D''_{E^\vee\otimes F}(D''_{E^\vee\otimes F}C) + (-1)^{i-1}(D''_FA)f - (-1)^{i-1}f(D''_EB) + (-1)^i(D''_FA)f - (-1)^i f(D''_EB) \\
&= 0.
\end{aligned}$$

Moreover, $D''_F \circ D''_F = D''_E \circ D''_E = 0$ and thus the claim follows. □

We next prove that the $\overline{\partial}_f$ satisfies the Leibniz rule for the brackets.

Proposition 2.4.4. *Let $x = (A_i, C_i, B_i) \in L^i, y = (A_j, C_j, B_j) \in L^j$. Then*

$$\overline{\partial}_f[x,y]_L = [\overline{\partial}_f x, y]_L + (-1)^i[x, \overline{\partial}_f y]_L$$

holds.

Proof. We first calculate each element. We have

$$\begin{aligned}
\overline{\partial}_f[x,y]_L =& \overline{\partial}_f \begin{pmatrix} [A_i,A_j] \\ A_i\wedge C_j - (-1)^{i(j-1)}C_j\wedge B_i - (-1)^{ij}A_j\wedge C_i + (-1)^jC_i\wedge B_j \\ [B_i,B_j] \end{pmatrix} \\
=& \begin{pmatrix} D''_F[A_i,A_j] \\ C_{\overline{\partial}_f[x,y]_L} \\ D''_E[B_i,B_j] \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
C_{\overline{\partial}_f[x,y]_L} =& D''_{E^\vee\otimes F}\Big(A_i\wedge C_j - (-1)^{i(j-1)}C_j\wedge B_i - (-1)^{ij}A_j\wedge C_i + (-1)^jC_i\wedge B_j\Big) \\
&+ (-1)^{i+j-1}[A_i,A_j]f - (-1)^{i+j-1}f[B_i,B_j],
\end{aligned}$$

$$\begin{aligned}
[\overline{\partial}_f x, y]_L &= \begin{bmatrix} D''_FA_i & A_j \\ D''_{E^\vee\otimes F}C_i + (-1)^{i-1}A_if - (-1)^{i-1}fB_i & C_j \\ D''_EB_i & B_j \end{bmatrix}_L \\
&= \begin{pmatrix} [D''_FA_i, A_j] \\ C_{[\overline{\partial}_f x,y]_L} \\ [D''_EB_i, B_j] \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
C_{[\overline{\partial}_f x,y]_L} =& D''_FA_i\wedge C_j - (-1)^{(i+1)(j-1)}C_j\wedge D''_EB_i \\
&- (-1)^{(i+1)j}A_j\wedge D''_{E^\vee\otimes F}C_i - (-1)^{i(j+1)+j-1}A_j\wedge A_i\wedge f + (-1)^{i(j+1)+j-1}A_j\wedge f\wedge B_i \\
&+ (-1)^jD''_{E^\vee\otimes F}C_i\wedge B_j + (-1)^{i+j-1}A_if\wedge B_j - (-1)^{i+j-1}fB_i\wedge B_j,
\end{aligned}$$

and

$$\begin{aligned}
[x, \overline{\partial}_f y]_L &= \begin{bmatrix} A_i & D''_FA_j \\ C_i & D''_{E^\vee\otimes F}C_j + (-1)^{j-1}A_jf - (-1)^{j-1}fB_j \\ B_i & D''_EB_j \end{bmatrix}_L \\
&= \begin{pmatrix} [A_i, D''_FA_j] \\ C_{[x,\overline{\partial}_f y]_L} \\ [B_i, D''_EB_j] \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}C_{[x,\overline{\partial}_f y]_L} =& A_i \wedge D''_{E^\vee\otimes F} C_j + (-1)^{j-1} A_i \wedge A_j f - (-1)^{j-1} A_i f \wedge B_j \\ & - (-1)^{ij} D''_{E^\vee\otimes F} C_j \wedge B_i - (-1)^{ij+j-1} A_j f \wedge B_i + (-1)^{ij+j-1} f B_j \wedge B_i \\ & - (-1)^{i(j+1)} D''_F A_j \wedge C_i + (-1)^{j+1} C_i \wedge D''_E B_j.\end{aligned}$$

Since $(A^\bullet(\mathrm{End}F), [,], D''_F), (A^\bullet(\mathrm{End}F), [,], D''_F)$ are DGLA, we only need to prove

$$C_{\overline{\partial}_f [x,y]_L} = C_{[\overline{\partial}_f x,y]_L} + (-1)^i C_{[x,\overline{\partial}_f y]_L}.$$

For $\alpha \in \{\overline{\partial}_f [x,y]_L, [\overline{\partial}_f x, y]_L, [x, \overline{\partial}_f y]_L\}$, we denote as $C_{\alpha,0}$ as the sum of the elements where f does not appear and $C_{\alpha,1}$ as where f appears. For example, for $\alpha = \overline{\partial}_f [x,y]_L$

$$\begin{aligned}C_{\overline{\partial}_f [x,y]_L,0} &= D''_{E^\vee\otimes F}\left(A_i \wedge C_j - (-1)^{i(j-1)} C_j \wedge B_i - (-1)^{ij} A_j \wedge C_i + (-1)^j C_i \wedge B_j \right), \\ C_{\overline{\partial}_f [x,y]_L,1} &= (-1)^{i+j-1} [A_i, A_j] f - (-1)^{i+j-1} f [B_i, B_j].\end{aligned}$$

It is clear that $C_\alpha = C_{\alpha,0} + C_{\alpha,1}$ and hence to prove

$$C_{\overline{\partial}_f [x,y]_L} = C_{[\overline{\partial}_f x,y]_L} + (-1)^i C_{[x,\overline{\partial}_f y]_L},$$

We only have to prove

$$\begin{aligned}C_{\overline{\partial}_f [x,y]_L,0} &= C_{[\overline{\partial}_f x,y]_L,0} + (-1)^i C_{[x,\overline{\partial}_f y]_L,0}, \\ C_{\overline{\partial}_f [x,y]_L,1} &= C_{[\overline{\partial}_f x,y]_L,1} + (-1)^i C_{[x,\overline{\partial}_f y]_L,1}.\end{aligned}$$

We now prove these two equations. Recall that C_i is a $E^\vee \otimes F$-valued $(i-1)$-form and C_j is a $(j-1)$-form.

$$\begin{aligned}C_{[\overline{\partial}_f x,y]_L,0} + (-1)^i C_{[x,\overline{\partial}_f y]_L,0} =& D''_F A_i \wedge C_j - (-1)^{(i+1)(j-1)} C_j \wedge D''_E B_i \\ & - (-1)^{(i+1)j} A_j \wedge D''_{E^\vee\otimes F} C_i + (-1)^j D''_{E^\vee\otimes F} C_i \wedge B_j \\ & + (-1)^i A_i \wedge D''_{E^\vee\otimes F} C_j - (-1)^{ij+i} D''_{E^\vee\otimes F} C_j \wedge B_i \\ & - (-1)^{ij} D''_F A_j \wedge C_i + (-1)^{i+j+1} C_i \wedge D''_E B_j \\ =& D''_F A_i \wedge C_j + (-1)^i A_i \wedge D''_{E^\vee\otimes F} C_j \\ & - (-1)^{ij+i} \left(D''_{E^\vee\otimes F} C_j \wedge B_i + (-1)^{j-1} C_j \wedge D''_E B_i \right) \\ & - (-1)^{ij} \left(D''_F A_j \wedge C_i + (-1)^j A_j \wedge D''_{E^\vee\otimes F} C_i \right) \\ & + (-1)^j \left(D''_{E^\vee\otimes F} C_i \wedge B_j + (-1)^{i+1} C_i \wedge D''_E B_j \right) \\ =& D''_{E^\vee\otimes F}\left(A_i \wedge C_j - (-1)^{i(j-1)} C_j \wedge B_i - (-1)^{ij} A_j \wedge C_i + (-1)^j C_i \wedge B_j \right) \\ =& C_{\overline{\partial}_f [x,y]_L,0}.\end{aligned}$$

$$
\begin{aligned}
C_{[\overline{\partial}_f x,y]_L,1}+(-1)^i C_{[x,\overline{\partial}_f y]_L,1} =& -(-1)^{i(j+1)+j-1}A_j\wedge A_i\wedge f+(-1)^{i(j+1)+j-1}A_j\wedge f\wedge B_i\\
&+(-1)^{i+j-1}A_i f\wedge B_j-(-1)^{i+j-1}fB_i\wedge B_j\\
&(-1)^{i+j-1}A_i\wedge A_j f-(-1)^{i+j-1}A_i f\wedge B_j\\
&-(-1)^{ij+i+j-1}A_j f\wedge B_i+(-1)^{ij+i+j-1}fB_j\wedge B_i\\
=&(-1)^{i+j-1}\Big(A_i\wedge A_j f-(-1)^{ij}A_j\wedge A_i f\Big)\\
&-(-1)^{i+j-1}\Big(fB_i\wedge B_j-(-1)^{ij}fB_j\wedge B_i\Big)\\
&+(-1)^{i+j-1}A_i f\wedge B_j-(-1)^{i+j-1}A_i f\wedge B_j\\
&+(-1)^{i(j+1)+j-1}A_j\wedge f\wedge B_i-(-1)^{i(j+1)+j-1}A_j\wedge f\wedge B_i\\
=&(-1)^{i+j-1}[A_i,A_j]f-(-1)^{i+j-1}f[B_i,B_j]\\
=&C_{\overline{\partial}_f[x,y]_L,1}.
\end{aligned}
$$

Hence, the claim is proved. □

It is clear that $(L,[,]_L,\overline{\partial}_f)$ is a DGLA from Proposition 2.4.1, 2.4.2, 2.4.3, and 2.4.4. Hence, Theorem 2.4.1 is proved.

2.5 Formality

We freely use the notation of Section 1.4.

Let X be a compact complex manifold and $(f,(E,\overline{\partial}_E,\theta_E),(F,\overline{\partial}_F,\theta_F))$ be a Higgs triple over X. In the previous section, we constructed the DGLA $(L,[,]_L,\overline{\partial}_f)$ which governs the deformation of the Higgs triple.

Recall that a DGLA L is called formal if it is quasi-isomorphic to $H^\bullet(L)$ as a DGLA. Note that L is always quasi-isomorphic to $H^\bullet(L)$ as a DG vector space. However, they are not always as DGLAs.

From now on, we assume X is a compact Kähler manifold and *each* $(E,\overline{\partial}_E,\theta_E)$ and $(F,\overline{\partial}_F,\theta_F)$ has a harmonic metric h_E and h_F.

We show that the DGLA $(L,[,]_L,\overline{\partial}_f)$ is formal under this assumption in the coming sections. Before we proceed, we prepare some results and notions.

Lemma 2.5.1 ([S2, Lemma 1.2])**.** *Let $(E,\overline{\partial}_E,\theta)$ be a Higgs bundle over a compact Kähler manifold X. Let h be a harmonic metric of $(E,\overline{\partial}_E,\theta)$. Then*

$$\mathrm{Ker}D''_E\cap A(E)=\mathrm{Ker}D'_h\cap A(E)=\mathrm{Ker}D_E\cap A(E).$$

Assume that each E and F has a harmonic metric h_E and h_F. Then $h_E^\vee\otimes h_F$ is also a harmonic metric for the Higgs bundlle $(E^\vee\otimes F,\overline{\partial}_{E^\vee\otimes F},\theta_{E^\vee\otimes F})$.

The next Lemma is straightforward from Lemma 2.5.1. However, it plays a core role in Section 2.5.2.

Lemma 2.5.2 ([S2, Corollary 1.3])**.** *Let $(f,(E,\overline{\partial}_E,\theta_E),(F,\overline{\partial}_F,\theta_F))$ be a Higgs triple over a compact Kähler manifold X. We assume that each E and F has a harmonic metric h_E and h_F. Then*

$$D'_{h_E^\vee\otimes h_F}f=D_{E^\vee\otimes F}f=0.$$

Proof. We regard f as a section of $E^\vee\otimes F$, and since it is a morphism between Higgs bundles, we have

$$D''_{E^\vee\otimes F}f=D''_F\circ f-f\circ D''_E=0.$$

We can now apply Lemma 2.5.1 to f. □

In order to prove formality, we need a suitable sub-DGLA of $(L, [,]_L, \overline{\partial}_f)$. This idea goes back to [DGMS, GM1].

For each $i \in \mathbb{Z}$, we define the subspace $(\mathrm{Ker} D'_{h_E,h_F})^i \subset L^i$ as

$$(\mathrm{Ker} D'_{h_E,h_F})^i := \left(\mathrm{Ker} D'_{h_F} \cap A^i(\mathrm{End} F)\right) \oplus \left(\mathrm{Ker} D'_{h_E^\vee \otimes h_F} \cap A^{i-1}(\mathrm{Hom}(E,F)\right) \oplus \left(\mathrm{Ker} D'_{h_F} \cap A^i(\mathrm{End} E)\right).$$

We define the graded subspace $(\mathrm{Ker} D'_{h_E,h_F})^\bullet := \oplus_{i\in\mathbb{Z}} (\mathrm{Ker} D'_{h_E,h_F})^i$. We denote $(\mathrm{Ker} D'_{h_E,h_F})^\bullet$ as $\mathrm{Ker} D'_{h_E,h_F}$ for simplicity.

Proposition 2.5.1. *Let $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ be a Higgs tripleover a compact Kähler manifold X. We assume that each E and F has a harmonic metric h_E and h_F. Then $(\mathrm{Ker} D'_{h_E,h_F}, [,]_L, \overline{\partial}_f)$ is a sub DGLA of $(L, [,]_L, \overline{\partial}_f)$.*

Proof. Since $\mathrm{Ker} D'_{h_E,h_F}$ is a graded sub-vector space of L, we only need to check that $\mathrm{Ker} D'_{E,F}$ is closed under the bracket $[,]_L$ and $\overline{\partial}_f$.

Let $x = (A_i, C_i, B_i) \in (\mathrm{Ker} D'_{h_E,h_F})^i, y = (A_j, C_j, B_j) \in (\mathrm{Ker} D'_{h_E,h_F})^j$. Then since

$$[x,y]_L = \begin{pmatrix} [A_i, A_j] \\ A_i \wedge C_j - (-1)^{i(j-1)} C_j \wedge B_i - (-1)^{ij} A_j \wedge C_i + (-1)^j C_i \wedge B_j \\ [B_i, B_j] \end{pmatrix},$$

it is clear that $[x,y]_L \in (\mathrm{Ker} D'_{h_E,h_F})^{i+j}$ by Lemma 2.3.1. Recall

$$\overline{\partial}_f(x) = \overline{\partial}_f \begin{pmatrix} A_i \\ C_i \\ B_i \end{pmatrix} = \begin{pmatrix} D''_F A_i \\ D''_{E^\vee \otimes F} C_i + (-1)^{i-1} A_i f - (-1)^{i-1} f B_i \\ D''_E B_i \end{pmatrix}.$$

Then $\overline{\partial}_f(x) \in (\mathrm{Ker} D'_{h_E,h_F})^{i+1}$ follows from the compatiblity of D'_{h_α} and D''_α ($\alpha \in \{E, E^\vee \otimes F, F\}$), Lemma 2.3.1, and Lemma 2.5.1. □

We note that this Proposition holds because of harmonic metrics.

2.5.1 Formality for f=0

In this section, we fix a compact Kähler manifold X. Let $(E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F)$ be Higgs bundles with harmonic metrics h_E and h_F. Let $0 : E \to F$ be a trivial morphism (i.e, the zero section of $E^\vee \otimes F$). It is clear that $(0, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ is a Higgs triple. Let $(L, [,]_L, \overline{\partial}_0)$ be the DGLA which controls the deformation of $(0, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$.

In this section, we show that $(L, [,]_L, \overline{\partial}_0)$ is formal as a DGLA. Although this follows from the result of the next section, we give a proof since this case immediately follows from Section 1.4.2.

Theorem 2.5.1. *$(L, [,]_L, \overline{\partial}_0)$ is formal as a DGLA.*

Proof. For $x = (A_i, C_i, B_i) \in L^i$

$$\overline{\partial}_0 \begin{pmatrix} A_i \\ C_i \\ B_i \end{pmatrix} = \begin{pmatrix} D''_F A_i \\ D''_{E^\vee \otimes F} C_i \\ D''_E B_i \end{pmatrix}.$$

Recall that $(\mathrm{Ker} D'_{h_E,h_F}, [,]_L, \overline{\partial}_0)$ is sub-DGLA of $(L, [,]_L, \overline{\partial}_0)$ by Propostion 2.5.1. Let $i : \mathrm{Ker} D'_{h_E,h_F} \to L$ be the inclusion map. This map is also a morphism of DGLA. Moreover, by Section 1.4.2, i is a quasi-isomorphism.

Let By the definition of $\overline{\partial}_0$, for the i-the cohomology $H^i(L)$ of $(L, \overline{\partial}_0)$, we have

$$H^i(L) = H^i_{Dol,F} \oplus H^{i-1}_{Dol,E^\vee \otimes F} \oplus H^i_{Dol,E}.$$

Let $q_\alpha : (\mathrm{Ker} D'_{h_\alpha})^i \to H^i_{Dol,\alpha}$ ($\alpha \in \{E, E^\vee \otimes F, F\}$) be the natural projection. Then by Section 1.4.2, q_α is a quasi-isomorphism and also a morphism of DGLA.

Hence $(L, [,]_L, \overline{\partial}_0)$ is quasi-isomorphic to $(H^\bullet(L), [,], 0)$ as a DGLA and the claim is proved. □

We note that this proof only works for $f=0$. This is because the cohomology of the DGLA $(L,[,]_L,\overline{\partial}_0)$ is rather simple.

2.5.2 Formality for arbitrarily f

In this section, we fix a compact Kähler manifold X and a Higgs triple $(f,(E,\overline{\partial}_E,\theta_E),(F,\overline{\partial}_F,\theta_F))$ over it. We assume that each $(E,\overline{\partial}_E,\theta_E),(F,\overline{\partial}_F,\theta_F)$ has a harmonic metric h_E and h_F. Let $(L,[,]_L,\overline{\partial}_f)$ be the DGLA which controls the deformation of $(f,(E,\overline{\partial}_E,\theta_E),(F,\overline{\partial}_F,\theta_F))$. We show that $(L,[,]_L,\overline{\partial}_f)$ is formal. The following will be used in the proof of formality.

Lemma 2.5.3. *Let $(f,(E,\overline{\partial}_E,\theta_E),(F,\overline{\partial}_F,\theta_F))$ be a Higgs triple over a compact Kähler manifold X. We assume that each E and F has a harmonic metric h_E and h_F. Then for any $A\in \mathrm{Ker}D'_{h_F}\cap A^i(\mathrm{End}F), B\in \mathrm{Ker}D'_{h_E}\cap A^i(\mathrm{End}E)$*

$$G'_{h_E^\vee\otimes h_F}(Af)=(G'_{h_F}A)f,$$
$$G'_{h_E^\vee\otimes h_F}(fB)=f(G'_{h_E}B)$$

holds.

Proof. We only prove the first equation. The second one can be proved by the same argument. Recall that for $C\in A^i(\mathrm{Hom}(E,F))$, $G'_{h_E^\vee\otimes h_F}C$ is the unique element in $(\mathbb{H}^i)^\perp$ that satisfies

$$\Delta'_{h_E^\vee\otimes h_F}G'_{h_E^\vee\otimes h_F}C=C-H^i(C).$$

Here $(\mathbb{H}^i)^\perp$ is the L^2-orthogonal space of $\mathbb{H}^i$ and H^i is the harmonic projection $H^i: A^i(\mathrm{Hom}(E,F)\to \mathbb{H}^i$.

Hence to prove the equation, we need to prove $(i)(G'_{h_F}A)f\in(\mathbb{H}^i)^\perp$, $(ii)\Delta'_{h_E^\vee\otimes h_F}((G'_{h_F}A)f)=Af-H^i(Af)$. We first prove (i). By the Hodge decomposition for $G'_{h_F}A$, we have

$$\begin{aligned}G'_{h_F}A&=D_{h_F}(D_{h_F})^*(G'_{h_F})^2A+(D_{h_F})^*D_{h_F}(G'_{h_F})^2A\\&=D_{h_F}(D_{h_F})^*(G'_{h_F})^2A+\sqrt{-1}[\ \Lambda_\omega,D''_E]D_{h_F}(G'_{h_F})^2A\\&=D_{h_F}(D_{h_F})^*(G'_{h_F})^2A+\sqrt{-1}(\Lambda_\omega D''_E-D''_E\Lambda_\omega)D_{h_F}(G'_{h_F})^2A.\end{aligned}$$

We used the Käher identity in the second equation. Then we have

$$\begin{aligned}(G'_{h_F}A)f&=\Big(D_{h_F}(D_{h_F})^*(G'_{h_F})^2A+\sqrt{-1}(\Lambda_\omega D''_E-D''_E\Lambda_\omega)D_{h_F}(G'_{h_F})^2A\Big)f\\&=D'_{h_E^\vee\otimes h_F}\Big(((D_{h_F})^*(G'_{h_F})^2A)f\Big)+\sqrt{-1}(\Lambda_\omega D''_{E^\vee\otimes F}-D''_{E^\vee\otimes F}\Lambda_\omega)\Big((D_{h_F}(G'_{h_F})^2A)f\Big)\\&=D'_{h_E^\vee\otimes h_F}\Big(((D_{h_F})^*(G'_{h_F})^2A)f\Big)+(D'_{h_E^\vee\otimes h_F})^*\Big((D_{h_F}(G'_{h_F})^2A)f\Big).\end{aligned}$$

The second equation follows from Lemma 2.3.1, Lemma 2.5.1, and f is a section (i.e. f is a 0-form). Hence $(G'_{h_F}A)f\in(\mathbb{H}^i)^\perp$.

We next prove (ii). We first prove $H^i(Af)=0$. Recall that we assumed $A\in\mathrm{Ker}D''_E$, the Hodge decomposition of A is

$$A=D''_E(D''_E)^*G''_EA.$$

Hence

$$\begin{aligned}Af&=(D''_E(D''_E)^*G''_EA)f\\&=D''_{E^\vee\otimes F}\Big(((D''_E)^*G''_EA)f\Big).\end{aligned}$$

Hence $H^i(Af) = 0$.

To finish the proof, we prove $\Delta'_{h^\vee_E \otimes h_F}((G'_{h_F}A)f) = Af$. By Kähler identity and Lemma 2.3.1, we have

$$\begin{aligned}\Delta'_{h^\vee_E \otimes h_F}((G'_{h_F}A)f) &= D'_{h^\vee_E \otimes h_F}(D'_{h^\vee_E \otimes h_F})^*((G'_{h_F}A)f) + (D'_{h^\vee_E \otimes h_F})^*(D'_{h^\vee_E \otimes h_F})((G'_{h_F}A)f)\\ &= \Big(D'_{h_F}(D'_{h_F})^* G'_{h_F}A\Big)f + \Big((D'_{h_F})^*(D'_{h_F})G'_{h_F}A\Big)f\\ &= (\Delta'_{h_F}G'_{h_F}A)f\\ &= Af.\end{aligned}$$

The last equation follows from $A \in \mathrm{Ker}D'_{h_F}$. The claim is proved. □

Proposition 2.5.2. *Let $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ be a Higgs triple over a compact Kähler manifold X. We assume that each E and F has a harmonic metric h_E and h_F.*

Let $\iota : \mathrm{Ker}D'_{h_E,h_F} \to L$ be the inclusion. Then $\iota : (\mathrm{Ker}D'_{h_E,h_F}, [,]_L, \overline{\partial}_f) \to (L, [,]_L, \overline{\partial}_f)$ is a quasi-isomorphism.

Proof. By Proposition 2.5.1, ι is a morphism of DGLA. Hence, it induces a map between cohomologies $H^i(\iota) : H^i(\mathrm{Ker}D'_{h_E,h_F}) \to H^i(L) (i \in \mathbb{Z})$. We show that $H^i(\iota)$ is an isomorphism.

$(i) H^i(\iota)$ is surjective: Let $x = (A, C, B) \in \mathrm{Ker}d_f \cap L^i$. We set as

$$\begin{aligned}\alpha &:= -\sqrt{-1}\Lambda_\omega D'_{h_F}G'_{h_F}A,\\ \gamma &:= \sqrt{-1}D''_{E^\vee \otimes F}\Lambda_\omega G'_{h^\vee_E \otimes h_F}D'_{h^\vee_E \otimes h_F}C,\\ \beta &:= -\sqrt{-1}\Lambda_\omega D'_{h_E}G'_{h_E}B.\end{aligned}$$

Note that $y = (\alpha, \gamma, \beta) \in L^{i-1}$. We show that $x - \overline{\partial}_f y \in (\mathrm{Ker}D'_{h_E,h_F})^i$. This proves the surjectivity of $H^i(\iota)$. We first show that $A - D''_F\alpha \in \mathrm{Ker}D'_{h_F}, B - D''_E\beta \in \mathrm{Ker}D'_{h_E}$. By the Hodge decomposition, Kähler idenity, $D'_h D''_F + D''_F D'_h = 0$, and $A \in \mathrm{Ker}D''_F$, we have

$$\begin{aligned}A =& D'_{h_F}(D'_{h_F})^* G'_{h_F}A + (D'_{h_F})^* D'_{h_F}G'_{h_F}A\\ =& D'_{h_F}(D'_{h_F})^* G'_{h_F}A - \sqrt{-1}D''_F\Lambda_\omega D'_{h_F}G'_{h_F}A.\end{aligned}$$

Then

$$\begin{aligned}A - D''_F\alpha =& A - \sqrt{-1}D''_F\Lambda_\omega D'_{h_F}G'_{h_F}A\\ =& D'_{h_F}(D'_{h_F})^* G'_{h_F}A.\end{aligned}$$

This proves $A - D''_F\alpha \in \mathrm{Ker}D'_{h_F}$. $B - D''_E\beta \in \mathrm{Ker}D'_{h_E}$ follows from the same argument.

Since $x = (A, C, B) \in \mathrm{Ker}\overline{\partial}_f \cap L^i$, we have

$$D''_{E^\vee \otimes F}C + (-1)^{i-1}Af - (-1)^{i-1}fB = 0. \tag{2.1}$$

Recall that we have to prove

$$C - D''_{E^\vee \otimes F}\gamma - (-1)^{i-2}\alpha f + (-1)^{i-2}f\beta \in \mathrm{Ker}D'_{h^\vee_E \otimes h_F}.$$

The Hodge decomposition of $D'_{h_E^\vee\otimes h_F}C$ is

$$
\begin{aligned}
D'_{h_E^\vee\otimes h_F}C =&(D''_{E^\vee\otimes F})^*D''_{E^\vee\otimes F}G''_{E^\vee\otimes F}D'_{h_E^\vee\otimes h_F}C + D''_{E^\vee\otimes F}(D''_{E^\vee\otimes F})^*G''_{E^\vee\otimes F}D'_{h_E^\vee\otimes h_F}C\\
=&-\sqrt{-1}[\Lambda_\omega, D'_{h_E^\vee\otimes h_F}]D''_{E^\vee\otimes F}G''_{E^\vee\otimes F}D'_{h_E^\vee\otimes h_F}C\\
&-\sqrt{-1}D''_{E^\vee\otimes F}[\Lambda_\omega, D'_{h_E^\vee\otimes h_F}]G''_{E^\vee\otimes F}D'_{h_E^\vee\otimes h_F}C\\
=&\sqrt{-1}D'_{h_E^\vee\otimes h_F}\Lambda_\omega D''_{E^\vee\otimes F}G''_{E^\vee\otimes F}D'_{h_E^\vee\otimes h_F}C\\
&+\sqrt{-1}D''_{E^\vee\otimes F}D'_{h_E^\vee\otimes h_F}\Lambda_\omega G''_{E^\vee\otimes F}D'_{h_E^\vee\otimes h_F}C\\
=&-\sqrt{-1}D'_{h_E^\vee\otimes h_F}\Lambda_\omega D'_{h_E^\vee\otimes h_F}G''_{E^\vee\otimes F}D''_{E^\vee\otimes F}C\\
&+\sqrt{-1}D''_{E^\vee\otimes F}D'_{h_E^\vee\otimes h_F}\Lambda_\omega G''_{E^\vee\otimes F}D'_{h_E^\vee\otimes h_F}C\\
=&-\sqrt{-1}D'_{h_E^\vee\otimes h_F}\Lambda_\omega D'_{h_E^\vee\otimes h_F}G''_{E^\vee\otimes F}D''_{E^\vee\otimes F}C + D''_{E^\vee\otimes F}\gamma.
\end{aligned}
\tag{2.2}
$$

We calculate $-\sqrt{-1}D'_{h_E^\vee\otimes h_F}\Lambda_\omega D'_{h_E^\vee\otimes h_F}G''_{E^\vee\otimes F}D''_{E^\vee\otimes F}C$. By Proposition 2.5.3 and (2.1), we have

$$
\begin{aligned}
&-\sqrt{-1}D'_{h_E^\vee\otimes h_F}\Lambda_\omega D'_{h_E^\vee\otimes h_F}G''_{E^\vee\otimes F}D''_{E^\vee\otimes F}C\\
=&(-1)^{i-1}\sqrt{-1}D'_{h_E^\vee\otimes h_F}\Lambda_\omega D'_{h_E^\vee\otimes h_F}G''_{E^\vee\otimes F}(Af) - (-1)^{i-1}\sqrt{-1}D'_{h_E^\vee\otimes h_F}\Lambda_\omega D'_{h_E^\vee\otimes h_F}G''_{E^\vee\otimes F}(fB)\\
=&(-1)^{i-1}D'_{h_E^\vee\otimes h_F}\left((\sqrt{-1}\Lambda_\omega D'_{h_F}G''_FA)f\right) - (-1)^{i-1}D'_{h_E^\vee\otimes h_F}\left(f(\sqrt{-1}\Lambda_\omega D'_{h_E}G''_EB)\right)\\
=&(-1)^{i-2}D'_{h_E^\vee\otimes h_F}(\alpha f) - (-1)^{i-2}D'_{h_E^\vee\otimes h_F}(f\beta).
\end{aligned}
$$

Then by (2.2), we have

$$
\begin{aligned}
D'_{h_E^\vee\otimes h_F}C =&-\sqrt{-1}D'_{h_E^\vee\otimes h_F}\Lambda_\omega D'_{h_E^\vee\otimes h_F}G''_{E^\vee\otimes F}D''_{E^\vee\otimes F}C + D''_{E^\vee\otimes F}\gamma\\
=&(-1)^{i-2}D'_{h_E^\vee\otimes h_F}(\alpha f) - (-1)^{i-2}D'_{h_E^\vee\otimes h_F}(f\beta) + D''_{E^\vee\otimes F}\gamma.
\end{aligned}
$$

Hence

$$
D'_{h_E^\vee\otimes h_F}C - D''_{E^\vee\otimes F}\gamma - (-1)^{i-2}D'_{h_E^\vee\otimes h_F}(\alpha f) + (-1)^{i-2}D'_{h_E^\vee\otimes h_F}(f\beta) = 0.
$$

Hence $C - D''_{E^\vee\otimes F}\gamma - (-1)^{i-2}\alpha f + (-1)^{i-2}f\beta \in \mathrm{Ker}D'_{h_E^\vee\otimes h_F}$. We proved $H^i(\iota)$ is surjective.

$(ii)H^i(\iota)$ is injective: Let $x = (A, B, C) \in \mathrm{Ker}D'_{h_E,h_F} \cap \mathrm{Ker}\overline{\partial}_f$. We assume that there exists a $y = (\alpha, \gamma, \beta) \in L^{i-1}$ such that $x = \overline{\partial}_f y$. We prove that there exists a $z \in (\mathrm{Ker}D'_{h_E,h_F})^i$ such that $x = \overline{\partial}_f z$. This proves the injectivity of $H^i(\iota)$. By the assumption, we have

$$
\begin{aligned}
A &= D''_F\alpha,\\
B &= D''_E\beta.
\end{aligned}
$$

Then since $A \in \mathrm{Ker}D'_{h_F}$, $D''_F D'_{h_F}\alpha = D'_{h_F}D'_{h_F}\alpha = 0$ holds. Then by Lemma 1.4.2, we have a α' such that $D'_{h_F}\alpha = D'_{h_F}D''_F\alpha'$. Define $\alpha'' := \alpha - D''_F\alpha'$. Then $\alpha'' \in \mathrm{Ker}D'_{h_F} \cap A^{i-2}(\mathrm{End}F)$ and $D''_F\alpha'' = A$. By the same argument, we can construct $\beta' \in A^{i-2}(\mathrm{End}E)$ such that $\beta'' := \beta - D''_E\beta' \in \mathrm{Ker}D'_{h_E} \cap A^{i-1}(\mathrm{End}E)$ and $D''_F\beta'' = B$.

We set as $z' := (\alpha'', \gamma + (-1)^{i-2}\alpha' f - (-1)^{i-2}f\beta', \beta'')$. Since $x = \overline{\partial}_f y$, we have

$$
\begin{aligned}
&D''_{E^\vee\otimes F}\left(\gamma + (-1)^{i-2}\alpha' f - (-1)^{i-2}f\beta'\right) + (-1)^{i-2}\alpha'' f - (-1)^{i-2}f\beta''\\
=&D''_{E^\vee\otimes F}\gamma + (-1)^{i-2}(D''_E\alpha' + \alpha'')f - (-1)^{i-2}f(D''_F\beta' + \beta'')\\
=&D''_{E^\vee\otimes F}\gamma + (-1)^{i-2}\alpha f - (-1)^{i-2}f\beta\\
=&C.
\end{aligned}
$$

Hence

$$D''_{E^\vee \otimes F}\Big(\gamma + (-1)^{i-2}\alpha' f - (-1)^{i-2} f\beta'\Big) = C - (-1)^{i-2}\alpha'' f + (-1)^{i-2} f\beta''$$

Since $C \in \mathrm{Ker} D'_{h_E^\vee \otimes h_F}, \alpha'' \in \mathrm{Ker} D'_{h_F}, \beta \in \mathrm{Ker} D'_{h_E}$, we have

$$D''_{E^\vee \otimes F}\Big(\gamma + (-1)^{i-2}\alpha' f - (-1)^{i-2} f\beta'\Big) \in \mathrm{Ker} D'_{h_E^\vee \otimes h_F} \cap \mathrm{Ker} D''_{E^\vee \otimes F}.$$

Then we can apply Lemma 1.4.2 to $D''_{E^\vee \otimes F}\Big(\gamma + (-1)^{i-2}\alpha' f - (-1)^{i-2} f\beta'\Big)$ and show that there exsits a γ' such that

$$D''_{E^\vee \otimes F}\Big(\gamma + (-1)^{i-2}\alpha' f - (-1)^{i-2} f\beta'\Big) = D''_{E^\vee \otimes F} D'_{h_E^\vee \otimes h_F}\gamma'.$$

We set $z := (\alpha'', D'_{h_E^\vee \otimes h_F}\gamma', \beta'')$. Then it is clear that $z \in (\mathrm{Ker} D'_{h_E, h_F})^i$ and $x = \overline{\partial}_f z$. This proves the injectivity of $H^i(\iota)$. Hence ι is quasi-isomorphic. □

We next prove $(\mathrm{Ker} D'_{h_E, h_F}, [,]_L, \overline{\partial}_f)$ is quasi-isomorphic to $(H^\bullet(L), [,]_L, 0)$.

Before, we recall standard results from elliptic operator theory. See [Wells, Chapter 4] for details. For each $i \in \mathbb{Z}$ we define the L^2-metric for L^i. The complex $(L, \overline{\partial}_f)$ is obviously an elliptic complex. Let $(\overline{\partial}_f)^*$ be the L^2-adjoint of $\overline{\partial}_f$. Then for each $\overline{\partial}_f : L^i \to L^{i+1}$ we set

$$\Delta_i := \overline{\partial}_f(\overline{\partial}_f)^* + (\overline{\partial}_f)^*(\overline{\partial}_f),$$
$$\mathbb{H}^i := \mathrm{Ker}(\Delta_i).$$

Then

$$H^i(L) \simeq \mathbb{H}^i.$$

Let $H^i : L^i \to \mathbb{H}^i$ be the projection and $q : L^i \to H^i(L)$ be the map such that $q(x) = [H^i(x)]$. Here $[H^i(x)]$ is the cohomology class of $H^i(x)$ in $H^i(L)$.

Proposition 2.5.3. *Let $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ be a Higgs triple over a compact Kähler manifold X. We assume that each E and F has a harmonic metric h_E and h_F.*

Then the map $q : L \to H^\bullet(L)$ induces a quasi-isomorphism $q : (\mathrm{Ker} D'_{h_E, h_F}, [,]_L, \overline{\partial}_f) \to (H^\bullet(L), [,]_L, 0)$.

Proof. This is clear from Proposition 2.5.2. □

Theorem 2.5.2. *Let $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ be a Higgs triple over a compact Kähler manifold X. We assume that each E and F has a harmonic metric h_E and h_F. Let $(L, [,]_L, \overline{\partial}_f)$ be the DGLA which controls the deformation of $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$. Then $(L, [,]_L, \overline{\partial}_f)$ is formal.*

Proof. Combine Proposition 2.5.2 and 2.5.3. □

2.5.3 Relation to Deformation of Flat Triples

Let X be a compact smooth manifold. We say a pair $(f, (E, D_E), (F, D_F))$ is a *flat triple over* X if $(E, D_E), (F, D_F)$ are flat bundles over X and $f : E \to F$ is a vector bundle morphism such that $D_F \circ f = f \circ D_E$. Then by the same argument as in Section 2.4, we can show that the DGLA which controls the deformation of the flat triples $(f, (E, D_E), (F, D_F))$ is $(L, [,]_L, d_f)$ where the graded vector space L and the bracket $[,]_L$ is same as in Section 2.4 and d_f is defined as

$$d_f(x) = d_f \begin{pmatrix} A \\ C \\ B \end{pmatrix} = \begin{pmatrix} D_F A \\ D_{E^\vee \otimes F} C + (-1)^{i-1} A f - (-1)^{i-1} f B \\ D''_E B \end{pmatrix} (x \in L^i).$$

We can show that $(L, [,]_L, d_f)$ is actually a DGLA by a little modification of the proof of Theorem 2.4.1.

From now on, we assume X is a compact Kähler manifold and (E, D_E) and (F, D_F) comes form harmonic bundles $(E, \overline{\partial}_E, \theta_E, h_E), (F, \overline{\partial}_F, \theta_F, h_F)$ (i.e. $D_E = D'_{h_E} + D''_E, D_F = D'_{h_F} + D''_F$).

Then by Lemma 2.5.1 and 4.4.2, the flat triple $(f, (E, D_E), (F, D_F))$ induces a Higgs triples $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$. If we have a Higgs triple $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ and each Higgs bundle has harmonic metric, then it induces a flat triple $(f, (E, D_E = D'_{h_E} + D''_E), (F, D_F = D'_{h_F} + D''_F))$.

By a similar proof of Proposition 2.5.1, we have

Proposition 2.5.4. *Let $(f, (E, D_E), (F, D_F))$ be a flat triple over a compact Kähelr manifold X. Assume that (E, D_E) and (F, D_F) comes form harmonic bundles $(E, \overline{\partial}_E, \theta_E, h_E), (F, \overline{\partial}_F, \theta_F, h_F)$. Let $(L, [,]_L, d_f)$ be the DGLA which controls the deformation of $(f, (E, D_E), (F, D_F))$.*

Then $(\mathrm{Ker} D'_{h_E, h_F}, [,]_L, \overline{\partial}_f)$ is a sub DGLA of $(L, [,]_L, d_f)$. Moreover, the inclusion $\iota : \mathrm{Ker} D'_{h_E, h_F} \to L$ is a quasi-isomorphism.

Moreover, by a similar proof of Theorem 2.5.2, we have

Theorem 2.5.3. *Let $(f, (E, D_E), (F, D_F))$ be a flat triple over a compact Kähelr manifold X. Assume that (E, D_E) and (F, D_F) comes form harmonic bundles $(E, \overline{\partial}_E, \theta_E, h_E), (F, \overline{\partial}_F, \theta_F, h_F)$. Let $(L, [,]_L, d_f)$ be the DGLA which controls the deformation of $(f, (E, D_E), (F, D_F))$ and $(L, [,]_L, \overline{\partial}_f)$ be the DGLA which controls the deformation of $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$.*

Then

- *$(L, [,]_L, d_f)$ is formal.*
- *$(L, [,]_L, d_f)$ is quasi-isomorphic to $(L, [,]_L, \overline{\partial}_f)$.*

Proof. $(\mathrm{Ker} D'_{h_E, h_F}, [,]_L, \overline{\partial}_f)$ is quasi-isomoporhic to $(L, [,]_L, \overline{\partial}_f)$ by Proposition 2.5.2 and also quasi-isomorphic to $(L, [,]_L, d_f)$ by the last Proposition. Hence $(L, [,]_L, d_f)$ is quasi-isomorphic to $(L, [,]_L, \overline{\partial}_f)$. □

Hence the deformation problems of $(f, (E, D_E), (F, D_F))$ and $(f, (E, \overline{\partial}_E, \theta_E), (F, \overline{\partial}_F, \theta_F))$ are same.

Chapter 3

Moduli Spaces of the Basic Hitchin equations on Sasakian three-folds

3.1 Abstract of Chapter 3

In this Chapter, we introduce an equation which we call the Basic Hitchin equation. This is an equation defined on Sasakian three-folds and is a three-dimensional analog of the Hitchin equation which is defined on Riemann Surfaces. We construct the moduli space of the basic Hitchin equation and show such space admits a hyperKähler metric. This also shows that the moduli space of flat bundles over Sasakian three-folds admits a hyperKähelr metric. We also calculate the dimension of the moduli space under certain assumptions.

3.2 Introduction of Chapter 3

Let X be a compact Riemann surface of a genus bigger than two. Let E be a complex vector bundle over X and h be a Hermitian metric. Let (∇_h, Φ) be a pair of a h-unitary connection and a skew-symmetric 1-form w.r.t. h. As we introduced in the Introduction the Hitchin equation is

$$\begin{aligned} F_{\nabla_h} - \Phi \wedge \Phi &= 0, \\ \nabla_h \Phi &= 0, \\ \nabla_h * \Phi &= 0. \end{aligned}$$

Here F_{∇_h} is the curvature of ∇_h and $*$ is the hodge star. We say (∇_h, Φ) is a Hitchin pair if it satisfies the Hitchin equation and irreducible if the connection $D := \nabla_h + \sqrt{-1}\Phi$ is irreducible. In [H], he also constructed the moduli space $\mathcal{M}_{\mathrm{Hit}}$ of irreducible Hitchin pair by infinite-dimensional hyperKähler reduction.

Let M be a compact Sasakian manifold. Sasakian manifolds are odd-dimension analogs of Kähler manifolds. See [BG] for more details about Sasakian manifolds. In this Chapter, we focus on the case of $\dim M = 3$. We call such M a Sasakian three-fold. In this case, M is a three-dimensional analog of the Riemann surface.

We introduce the Sasakian analog of the Hitchin equation which we call the *basic Hitchin equation.* Let E be a *basic* complex vector bundle and h be a *basic* hermitian metric (See Section 3.4.1 for definitions about basic vector bundles and metrics). Let (∇_h, Φ) is a pair of basic h-unitary connection and Φ be a basic skew-symmetric 1-form w.r.t. h. Then the basic Hitchin equation is the following equations:

$$\begin{aligned} F_{\nabla_h} - \Phi \wedge \Phi &= 0, \\ \nabla_h \Phi &= 0, \\ \nabla_h \star_\xi \Phi &= 0. \end{aligned}$$

Here $\star_\xi$ is the basic Hodge star (See Section 3.3.2). We call a pair (∇_h, Φ) a basic Hitchin pair if the pair satisfies the basic Hitchin equation. The main result of this chapter is the construction of the moduli space $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ of irreducible basic Hitchin pairs. Moreover, we have

Theorem 3.2.1 (Theorem 3.5.1)**.** *$\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ is an empty set or a smooth hyperKähler manifold.*

As like the Riemann surface case, the basic Hitchin equation is related to flat bundles and Higgs bundles. Since Higgs bundles are holomorphic objects, we need basic Higgs bundles. Hence we can regard $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ as a moduli space of simple flat bundles with a fixed basic structure and stable basic Higgs bundles of degree 0.

We also calculate the dimension of $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ under the some assumptions.

Theorem 3.2.2 (Theorem 3.5.2)**.** *Let $(M,(T^{1,0},S,I),(\eta,\xi))$ be a regular Sasakian threefold (See Section 3.3.1 for the definition of regular). Let E be a regular basic bundle (See Section 3.5.4) and h be a basic Hermitian metric. Let g be the genus of M/S^1. We assume $g \geq 2$. Then*

$$\dim_{\mathbb{R}} \mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} = 4(\mathrm{rk}E)^2(g-1)+4.$$

Relation to other works

For the higher dimensional case, there is a work by Kasuya [K]. He studied the moduli of the flat bundle over general Sasakian manifolds and showed that the moduli have stratification by the basic structure.

3.3 Sasakian manifolds

3.3.1 Sasakian manifolds

Let M be a $(2n+1)$-dimensional real smooth manifold. Let $TM \otimes \mathbb{C}$ be the complexified tangent bundle of TM. A *CR-structure* on M is a rank n complex sub-bundle $T^{1,0}$ of $TM \otimes \mathbb{C}$ such that $T^{1,0} \cap \overline{T^{1,0}} = 0$ and $T^{1,0}$ is integrable. We denote $\overline{T^{1,0}}$ as $T^{0,1}$. For a CR-structure $T^{1,0}$ on M, there is an unique sub-bundle of rank $2n$ of real tangent bundle TM with a vector bundle homomorphism $I : S \to S$ such that the following properties holds:

- $I^2 = -\mathrm{Id}_S$,
- $T^{1,0}$ is the $\sqrt{-1}$-eigen bundle of I.

A $(2n+1)$-dimensional manifold M is equipped with a triple $(T^{1,0},S,I)$ is called a *CR-manifold*. A *contact 1-form* η of M is a non-degenerate 1-form of M (i.e. $\eta \wedge (d\eta)^n$ is everywhere non-zero). By the non-degeneracy of η, there exists a vector field ξ called *Reeb vector field* such that it satisfies

$$\eta(\xi) = 1, \xi \lrcorner (d\eta)^n = 0.$$

A *contact CR manifold* is a CR-manifold M with a contact 1-form η such that $\mathrm{Ker}(\eta) = S$. For a contact CR-manifold, the above $I : S \to S$ extends to the entire TM by setting $I(\xi) = 0$. Here ξ is the Reeb vector field of η.

Definition 3.3.1. *A contact CR-manifold $(M,(T^{1,0},S,I),(\eta,\xi))$ is a strongly pseudo-convex CR-manifold if the Hermitian form L_η on S_x defined by $L_\eta(X,Y) = d\eta(X,IY), X,Y \in S_x$ is positive definite for every point $x \in M$.*

For a strongly pseudo-convex CR-manifold $(M,(T^{1,0},S,I),(\eta,\xi))$, we have a canonical Riemann metric g_η on M which is defined by

$$g_\eta(X,Y) := L_\eta(X,Y) + \eta(X)\eta(Y), X,Y \in T_xM.$$

Definition 3.3.2. *A Sasakian manifold is a strongly pseudo-convex CR-manifold*

$$(M,(T^{1,0},S,I),(\eta,\xi))$$

such that for any section ζ of $T^{1,0}$, $[\xi,\zeta]$ is also a section of $T^{1,0}$. For a Sasakian manifold, we call g_η as Sasaki metric.

For a Sasakian manifold $(M,(T^{1,0},S,I),(\eta,\xi))$, the metric cone of (M,g_η) is a Kähler manifold. We can also define a Sasakian manifold as a contact metric manifold whose metric cone is Kähler.

Let M be a Sasakian manifold. If the orbits of the Reeb vector field ξ are all closed, and hence it is a circle, then ξ induces a S^1-action on M. Since ξ is nowhere zero, then the action is locally free. We say that M is *regular* if the S^1-action is free and *quasi-regular* if it is locally free. When the orbit of ξ is not all closed, then we say M is *irregular*.

3.3.2 Basic Differential forms

Thoughrout this section, let $(M,(T^{1,0},S,I),(\eta,\xi))$ be a $2n+1$-dimensional compact Sasakian manifold.

The Reeb vector field ξ defines a 1-dimensional foliation $\mathcal{F}_\xi$ on M. It is known the map $I:TM\to TM$ associated with the CR-structure $T^{1,0}$ defines a transversely complex structure on the foliated manifold $(M,\mathcal{F}_\xi)$. Furthermore, the closed 2-form $d\eta$ is a transversely Kähler structure with respect to this transversely complex structure.

A differential form ω of M is called a *basic diffrential form* if

$$i_\xi\omega=0,\mathcal{L}_\xi\omega=0.$$

For simplicity, we call a differential form *basic* if it is a basic differential form. We note that η is not basic but $d\eta$ is basic. We denote $A^*_B(M)$ as the space of real basic differential forms. We note that $A^*_B(M)$ forms a sub-complex of deRham complex $A^*(M)$. We denote as $H^i_B(M)$ to be the i-th cohomology of $(A^*_B(M),d)$.

Corresponding to the decomposition $S_{\mathbb{C}}=T^{1,0}\oplus T^{0,1}$, we have the bigrading

$$A^r_B(M)_{\mathbb{C}}=\bigoplus_{p+q=r}A^{p,q}_B(M)$$

as well as the decomposition of the exterior differential

$$d|_{A^r_B(M)_{\mathbb{C}}}=\partial_\xi+\overline{\partial}_\xi$$

on $A^r_B(M)_{\mathbb{C}}$, so that

$$\partial_\xi:A^{p,q}_B(M)\to A^{p+1,q}_B(M),$$
$$\overline{\partial}_\xi:A^{p,q}_B(M)\to A^{p,q+1}_B(M).$$

We also have the transverse Hodge theory ([EKA, KT]). Let

$$*:A^r(M)\to A^{2n+1-r}(M)$$

be the usual Hodge star operator associated with the Sasaki metric g_η and let

$$\delta:=-*d*:A^r(M)\to A^{r-1}(M)$$

be the formal adjoint of the exterior derivative with respect to the L^2-norm.

We define the linear operator

$$\star_\xi:A^r_B(M)\to A^{2n-r}_B(M)$$

such that $\star_\xi$ acts on $\omega\in A^r_B(M)$ as

$$\star_\xi\omega=*(\eta\wedge\omega).$$

We also define a few more operators:

$$\delta_\xi:=-\star_\xi d\star_\xi:A^r_B(M)\to A^{r-1}_B(M),$$
$$\partial^*_\xi:=-\star_\xi\overline{\partial}_\xi\star_\xi:A^{p,q}_B(M)\to A^{p-1,q}_B(M),$$
$$\overline{\partial}^*_\xi:=-\star_\xi\partial_\xi\star_\xi:A^{p,q}_B(M)\to A^{p,q-1}_B(M),$$
$$\Lambda:=-\star_\xi\partial_\xi\star_\xi\,.$$

They are the formal adjoints of $d, \partial_\xi, \overline{\partial}_\xi$ and $d\eta\wedge$ with respect to the pairing

$$A_B^r(M) \times A_B^r(M) : (\alpha, \beta)_B \to \int_M \eta \wedge \alpha \wedge \star_\xi \beta. \tag{3.1}$$

The following Proposition might be well-known for specialists, however, we give its detailed proof since it is crucial to define the hyperKähelr metric for the moduli spaces.

Proposition 3.3.1. *Assume* $\dim M = 3$. *Then*

$$\star_\xi \circ \star_\xi|_{A_B^1(M)} = -\mathrm{Id}_{A_B^1(M)}.$$

Proof. To show the equation holds, it is enough to show it holds pointwise. Let $p \in M$ and (U, x, y, z) be a local coordinate around p. We assume

$$S_p = \mathbb{R}\left(\frac{\partial}{\partial x}\right)_p \oplus \mathbb{R}\left(\frac{\partial}{\partial y}\right)_p$$

and

$$\left(\frac{\partial}{\partial x}\right)_p \perp_{g_\eta} \left(\frac{\partial}{\partial y}\right)_p \perp_{g_\eta} \xi_p.$$

Under the assumption we have

$$\begin{aligned} A^1(M)_p &= \mathbb{R}(dx)_p \oplus \mathbb{R}(dy)_p \oplus \mathbb{R}\eta_p, \\ A_B^1(M)_p &= \mathbb{R}(dx)_p \oplus \mathbb{R}(dy)_p, \\ vol_p &= \eta_p \wedge (dx)_p \wedge (dy)_p. \end{aligned}$$

Hence we have

$$\begin{aligned} \star_\xi(dx)_p &= *(\eta_p \wedge (dx)_p) = (dy)_p, \\ \star_\xi(dy)_p &= *(\eta_p \wedge (dy)_p) = -(dx)_p. \end{aligned}$$

Hence the claim is proved. □

3.4 Basic bundles

3.4.1 Basic vector bundles

Throughout this section, let $(M, (T^{1,0}, S, I), (\eta, \xi))$ be a compact Sasakian manifold.

Let E be a rank r complex vector bundle over M. We say that E is basic if there exists a local trivialization $\{U_\alpha\}_{\alpha\in A}$ of E such that the associated transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_r(\mathbb{C})$ is basic (i.e. $i_\xi dg_{\alpha\beta} = 0$).

Let E be a basic bundle. A E-valued differential form ω is called basic if for every $\alpha \in A$, $\omega|_{U_\alpha} \in A_B^p(U_\alpha) \otimes E$. This is well-defined since E is basic. We denote the space of basic E-valued p-form as $A_B^p(E)$. Let D be a connection of E. We call D basic if for all $\alpha \in A$, $D|_{U_\alpha} = d + A_\alpha, A_\alpha \in A_B^1(\mathrm{End}E)$. If D is basic, we have a homomorphism $D : A_B^*(E) \to A_B^{*+1}(E)$. If D is a flat connection, we regard it as a basic connection because of the flat frame ([Ko]).

Let h be a Hermitian metric of E. Note that $h \in A(E^\vee \otimes \overline{E}^\vee)$. Here $E^\vee$ is the dual of E. We say the h is basic if $h \in A_B(E^\vee \otimes \overline{E}^\vee)$. Although hermitian metric always exists, basic hermitian metric might not exist. The next section shows that E admits a basic hermitian metric when a flat connection D satisfies certain conditions.

We now fix a basic bundle E, a basic connection D, and a basic hermitian metric h. As it is well-known D has a decomposition

$$D = \nabla_h + \sqrt{-1}\Phi \tag{3.2}$$

such that ∇_h is a metric connection and Φ is skew-symmetric w.r.t. h. Since D and h are basic, ∇_h and Φ are also. We say the (E, D) is *irreducible* if there does not exist a basic sub-bundle F of E with $D(F) \subset A^1_B(F)$. We say (E, D) is *reductive* if (E, D) is a direct sum of irreducible ones.

We define some notations. Let

$$\begin{aligned} A(\mathfrak{u}(E)) &:= \{f \in A(\mathrm{End}E) : h(fu, v) + h(u, fv) = 0\}, \\ A_r(\mathfrak{u}(E)) &:= \{f \in A(\mathfrak{u}(E)) : \int_M \mathrm{tr}(f) = 0\}, \\ A^i(\mathfrak{u}(E)) &:= A^i \otimes A(\mathfrak{u}(E)), \\ A^i_r(\mathfrak{u}(E)) &:= A^i \otimes A_r(\mathfrak{u}(E)), \\ A^i_B(\mathfrak{u}(E)) &:= A^i_B \otimes A(\mathfrak{u}(E)), \\ A^i_{B,r}(\mathfrak{u}(E)) &:= A^i_B \otimes A_r(\mathfrak{u}(E)). \end{aligned}$$

We say $A_r(\mathfrak{u}(E))$ (resp. $A_{B,r}(\mathfrak{u}(E))$) as (basic) reduced section. We note that we have the following L^2-decomposition.

$$\begin{aligned} A(\mathfrak{u}(E)) &= A_r(\mathfrak{u}(E)) \oplus \sqrt{-1}\mathbb{R}\mathrm{Id}_E, \\ A_B(\mathfrak{u}(E)) &= A_{B,r}(\mathfrak{u}(E)) \oplus \sqrt{-1}\mathbb{R}\mathrm{Id}_E. \end{aligned}$$

The following result is used for the calculation of the dimension of the moduli space.

Proposition 3.4.1. *The following are equivalent.*

- (E, D) *is irreduicible.*
- *We define a differential operator* $D_1 : A_B(\mathfrak{u}(E)) \to A^1_B(\mathfrak{u}(E)) \oplus A^1_B(\mathfrak{u}(E))$ *as follows:*

  $$D_1(f) := (\nabla_h f, [\Phi, f]).$$

  Then $\mathrm{Ker}(D_1) = \sqrt{-1}\mathbb{R}\mathrm{Id}_E$.

Proof. Assume (E, D) is irreducible. Suppose we have a $f \in A_B(\mathfrak{u}(E))\backslash\sqrt{-1}\mathbb{R}\mathrm{Id}_E$ such that $D_1 f = 0$. By the definition of D_1, we have $\nabla_h f = 0$. From [LT, p.25, Proposition 1.1.17], we have the eigendecomposition of E with respect to f:

$$E = \bigoplus_\lambda E_\lambda.$$

Since f is basic, each E_λ is basic. The decomposition is h-othogonal and $D_1(E_\lambda) \subset A^1_B(E_\lambda)$. Since each E_λ is eigen bundle of f and $\Phi f - f\Phi = [\Phi, f] = 0$, we have $\Phi(E_\lambda) \subset A^1_B(E_\lambda)$. Hence we have $D(E_\lambda) \subset A^1_B(E_\lambda)$. This contradicts the assumption.

Assume $\mathrm{Ker}(D_1) = \sqrt{-1}\mathbb{R}\mathrm{Id}_E$. Suppose (E, D) is reducible. We have a following h-othogonal decomposition:

$$(E, D) = (E_\alpha, D_\alpha) \oplus (E_\beta, D_\beta).$$

Let pr_α and pr_β be the orthogonal projection to E_α and E_β. By definition, $\sqrt{-1}pr_\alpha, \sqrt{-1}pr_\beta \in A_B(\mathfrak{u}(E))$. It is straight forward to check $\sqrt{-1}pr_\alpha - \sqrt{-1}pr_\beta \in A_B(\mathfrak{u}(E))\backslash\sqrt{-1}\mathbb{R}\mathrm{Id}_E$ and $D(\sqrt{-1}pr_\alpha - \sqrt{-1}pr_\beta) = 0$ and hence $D_1(\sqrt{-1}pr_\alpha - \sqrt{-1}pr_\beta) = 0$. Hence contradicts. □

Remark 3.4.1. *In [BHe2], the authors defined a h-unitary basic connection ∇_h is irreducible if* $\mathrm{Ker}(\nabla_h)|_{A_B(\mathfrak{u}(E))} = \sqrt{-1}\mathbb{R}\mathrm{Id}_E$. *Proposition 3.4.1 tells us that the definition of our irreducibility and their irreducibility coincide when* $\Phi = 0$.

Let $A_B(GL(E))$ be the automorphism group of the basic bundle E. We define the gauge group

$$\mathcal{G}_B := \{f \in A_B(GL(E)) : h(fu, fv) = h(u, v)\}.$$

We moreover define the reduced gauge group as

$$\mathcal{G}_{B,r} := \mathcal{G}_B / S^1 \mathrm{Id}_E.$$

The Lie algebra of $\mathcal{G}_B$ is $A(\mathfrak{u}(E))$ and $\mathcal{G}_{B,r}$ is $A_r(\mathfrak{u}(E))$.

Let $\mathcal{A}_{h,B}$ be the space of the h-unitary basic connection. This is an affine space that is modeled on $A^1_B(\mathfrak{u}(E))$. We define

$$\mathcal{A}_B := \mathcal{A}_{h,B} \times A^1_B(\mathfrak{u}(E)).$$

Since any basic connection D has the decomposition (4.5.1), we regard $\mathcal{A}_B$ as the space of connections. $\mathcal{G}_B(E)$ acts on $\mathcal{A}^k_B$ as

$$\begin{aligned} \mathcal{G}_B \times \mathcal{A}_B &\longrightarrow \mathcal{A}_B \\ (g, \nabla_h, \Phi) &\longmapsto (g^{-1}\nabla_h g, g^{-1}\Phi g). \end{aligned} \tag{3.3}$$

Degree of basic bundles

Let E be a basic bundle and D be a basic connection. Let F_D be the curvature of D. Since E and D are basic, $F_D \in A^2_B(\mathrm{End}E)$. For any $0 \leqslant i \leqslant n$, we define $c_{i,B}(E, D) \in A^{2i}_B(M)$ by

$$\det\left(\mathrm{Id}_E - \frac{F_D}{2\pi\sqrt{-1}}\right) = 1 + \sum_{i=1}^{2n} c_{i,B}(E, D).$$

Then, as the case of the usual Chern-Weil theory, the cohomology class,

$$c_{i,B}(E) \in H^{2i}_B(M)$$

of each $c_{i,B}(E, D)$ is independent of the choice of a basic connection D.

We define the *degree* of E as

$$\deg(E) := \frac{1}{2\pi\sqrt{-1}} \int_M \mathrm{Tr}(\Lambda F_D).$$

We also have

$$\deg(E) = \int_M c_{1,B}(M) \wedge (d\eta)^{n-1} \wedge \eta.$$

Hence $\deg(E)$ only depends on E.

L^2-metric, Adjoints, and Brackets

In this section, we review some operations around $A^i_B(\mathfrak{u}(E))$. The results in this section are nothing new. However, we write this section for completeness.

Let (E, h) be a basic vector bundle with a basic Hermitian metric on a Sasakian manifold M. Let $A, B \in A^i(\mathrm{End}E)$. Recall that the L^2-inner product $(A, B)_{L^2}$ is defined as

$$(A, B)_{L^2} = \int_M \mathrm{Tr}(A \wedge *B^\dagger_h).$$

Here recall that $B^\dagger_h$ is the formal adjoint of B w.r.t. h and $*$ is the ordinary Hodge star. Hence if we assume $B \in A^i(\mathfrak{u}(E))$, we have

$$(A, B)_{L^2} = \int_M \mathrm{Tr}(A \wedge *B^\dagger_h) = -\int_M \mathrm{Tr}(A \wedge *B).$$

We study the L^2-metric restricted to $A^i_B(\mathfrak{u}(E))$. Let $\alpha \in A^i(M)$. The usual Hodge star $*$ and the basic Hodge star $\star_\xi$ have the following relation ([KT]):

$$*\alpha = \star_\xi \alpha \wedge \eta.$$

Hence if $A, B \in A^i_B(\mathfrak{u}(E))$, we have

$$(A, B)_{L^2} = -\int_M \mathrm{Tr}(A \wedge *B) = -\int_M \mathrm{Tr}(A \wedge \star_\xi B) \wedge \eta.$$

Let $\nabla_h \in \mathcal{A}_{h,B}$ and $\Phi \in A^1_B(\mathfrak{u}(E))$. Let ∇^*_h and Φ^* be the formal adjoints of ∇_h and Φ w.r.t. the L^2-inner product i.e. for $A \in A^i(\mathrm{End}(E))$ and $B \in A^{i+1}(\mathrm{End}(E))$, the following holds

$$(\nabla_h A, B)_{L^2} = (A, \nabla^*_h B)_{L^2},$$
$$([\Phi, A], B)_{L^2} = (A, [\Phi^*, B])_{L^2}.$$

We give the explicit formula of ∇^*_h and Φ^* when we restrict the L^2-inner product to $A^*_B(\mathfrak{u}(E))$. Since the Sasakian manifold has no basic $2n+1$-form, for $A \in A^i_B(\mathfrak{u}(E))$ and $B \in A^{i+1}_B(\mathfrak{u}(E))$, we have

$$(\nabla_h A, B)_{L^2} = (A, \nabla^*_h B)_{L^2} = -(A, \star_\xi \nabla_h \star_\xi B)_{L^2},$$
$$([\Phi, A], B)_{L^2} = (A, [\Phi^*, B])_{L^2} = (A, \star_\xi[\Phi^\dagger_h, \star_\xi B])_{L^2} = -(A, \star_\xi[\Phi, \star_\xi B])_{L^2}.$$

Hence we have

Lemma 3.4.1. *When we restrict the action of ∇_h and Φ to $A^*_B(\mathfrak{u}(E))$, those formal adjoints ∇^*_h, Φ^* w.r.t. the L^2-inner product has the form*

$$\nabla^*_h = -\star_\xi \nabla_h \star_\xi,$$
$$\Phi^* = -\star_\xi \Phi \star_\xi .$$

This can be shown by a standard calculation. We emphasize that this equality holds since M is Sasakian and we restricted the L^2-inner product to $A^*_B(\mathfrak{u}(E))$. We cannot expect this equality to hold for general foliated manifolds or for general sections.

We state one more result which we use later. From now on we assume $\dim M = 3$.

Lemma 3.4.2. *Let $A, B \in A^1_B(\mathrm{End}E)$. Then*

$$[\star_\xi A, B] = -[A, \star_\xi B]$$

holds.

Proof. We only have to prove it pointwisely. Let $p \in M$. We use the coordinate which we used in Proposition 3.3.1.

$$\begin{aligned}[\star_\xi A, B]_p &= [\star_\xi(A_x(dx)_p + A_y(dy)_p), B_x(dx)_p + B_y(dy)_p] \\ &= [A_x(dy)_p - A_y(dx)_p, B_x(dx)_p + B_y(dy)_p] \\ &= -\Big([A_x, B_x] + [A_y, B_y]\Big)(dx)_p \wedge (dy)_p.\end{aligned}$$

$$\begin{aligned}[A, \star_\xi B]_p &= [A_x(dx)_p + A_y(dy)_p, \star_\xi(B_x(dx)_p + B_y(dy)_p)] \\ &= [A_x(dx)_p + A_y(dy)_p, B_x(dy)_p - B_y(dx)_p] \\ &= \Big([A_x, B_x] + [A_y, B_y]\Big)(dx)_p \wedge (dy)_p.\end{aligned}$$

Hence the Lemma is proved. □

3.5 The Moduli space of Basic Hitchin equations

Throughout this section, we assume $(M, (T^{1,0}, S, I), (\eta, \xi))$ to be a compact Sasakian manifold of dimsension three. We also fix a basic bundle E and a basic metric h.

3.5.1 Basic Hitchin equation

Recall that we defined $\mathcal{A}_{h,B}$ to be the set of basic h-unitary connection and $A^1_B(\mathfrak{u}(E))$ be End$E$-valued skew-hermitian 1-form (See section 3.4.1). Note that $\mathcal{A}_{h,B}$ is an affine space modeled on $A^1_B(\mathfrak{u}(E))$.

Let $(\nabla_h, \Phi) \in \mathcal{A}_B = \mathcal{A}_{h,B} \times A^1_B(\mathfrak{u}(E))$. We say that (∇_h, Φ) satisfies the *basic Hitchin equation* if

$$\begin{cases} F_{\nabla_h} - \Phi \wedge \Phi = 0, \\ \nabla_h \Phi = 0, \\ \nabla_h \star_\xi \Phi = 0. \end{cases} \tag{3.4}$$

Here F_{∇_h} is the curvature of ∇_h. If (∇_h, Φ) satisfies the Hitchin equation we call (∇_h, Φ) a *basic Hitchin pair.* We set as

$$\mathcal{A}_{\mathrm{BaHit}} := \{(\nabla_h, \Phi) \in \mathcal{A}_{h,B} \times A^1_B(\mathfrak{u}(E)) : (\nabla_h, \Phi) \text{ is a basic Hitchin pair}\}.$$

We say that (∇_h, Φ) is *irreducible* if the connection $D = \nabla_h + \sqrt{-1}\Phi$ is irreducible (See section 3.4.1). We set as

$$\mathcal{A}^{\mathrm{irr}}_{\mathrm{BaHit}} := \{(\nabla_h, \Phi) \in \mathcal{A}_{\mathrm{BaHit}} : (\nabla_h, \Phi) \text{ is irreducible}\}.$$

Note that the action of the gauge groups $\mathcal{G}_B$ and $\mathcal{G}_{B,r}$ preserves $\mathcal{A}_{\mathrm{BaHit}}$ and $\mathcal{A}^{\mathrm{irr}}_{\mathrm{BaHit}}$. Moreover, $\mathcal{G}_{B,r}$ acts freely on $\mathcal{A}^{\mathrm{irr}}_{\mathrm{BaHit}}$.

Let $(\nabla_h, \Phi) \in \mathcal{A}_{\mathrm{BaHit}}$. Considering the linearization of the action of the gauge group $\mathcal{G}_B$ and the linearization of the Basic Hitchin equation (3.4), we obtain a complex

$$0 \longrightarrow A_B(\mathfrak{u}(E)) \xrightarrow{D_1} A^1_B(\mathfrak{u}(E))^{\oplus 2} \xrightarrow{D_2} A^2_B(\mathfrak{u}(E))^{\oplus 3} \longrightarrow 0 \tag{3.5}$$

where

$$\begin{aligned} D_1 A &:= (\nabla_h A, [\Phi, A]), \\ D_2(A, B) &:= (\nabla_h A - [\Phi, B], \nabla_h B + [A, \Phi], \nabla_h \star_\xi B + [A, \star_\xi \Phi]). \end{aligned} \tag{3.6}$$

Note that D_1 is exactly the same operator we introduced in Proposition 3.4.1. Considering the highest-order part of the differential operators D_1 and D_2, we see that the complex (3.5) is *transverse elliptic complex* (See [Wa]). We denote the i-th cohomology of the complex (3.5) as $\mathbb{H}^i$. These cohomology are finite dimensions since they are the kernel of transverse elliptic operators [EKA]. The dimension of $\mathbb{H}^1$ is expected to be the dimension of the moduli space.

We now consider the case $(\nabla_h, \Phi) \in \mathcal{A}^{\mathrm{irr}}_{\mathrm{BaHit}}$. In this case, Ker$D_1 = \sqrt{-1}\mathbb{R}\mathrm{Id}_E$ (See Proposition 3.4.1) and hence $\dim_{\mathbb{R}} \mathbb{H}^0 = 1$. We later use the following result to show the moduli space is smooth and to calculate the dimension of the moduli space.

Proposition 3.5.1. *Assume* $(\nabla_h, \Phi) \in \mathcal{A}^{\mathrm{irr}}_{\mathrm{BaHit}}$. *Then* $\dim_{\mathbb{R}} \mathbb{H}^2 = 3$. *In particular each row of* $\mathbb{H}^2$ *is spanned by the multiplication of* $\sqrt{-1}d\eta$ *and* Id_E *i.e.*

$$\mathbb{H}^2 = [\langle \sqrt{-1} d\eta \mathrm{Id}_E \rangle^{\oplus 3}_{\mathbb{R}}].$$

Here

$$\langle \sqrt{-1} d\eta \mathrm{Id}_E \rangle^{\oplus 3}_{\mathbb{R}} := \mathbb{R} \begin{pmatrix} \sqrt{-1} d\eta \mathrm{Id}_E \\ 0 \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 0 \\ \sqrt{-1} d\eta \mathrm{Id}_E \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 0 \\ 0 \\ \sqrt{-1} d\eta \mathrm{Id}_E \end{pmatrix},$$

and $[\langle \sqrt{-1} d\eta \mathrm{Id}_E \rangle^{\oplus 3}_{\mathbb{R}}]$ *is the* $\mathbb{R}$*-vector space which is spanned by the cohomology class of the basis of* $\langle \sqrt{-1} d\eta \mathrm{Id}_E \rangle^{\oplus 3}_{\mathbb{R}}$.

Proof. It is enough to show

$$\mathrm{Ker} D_2^* = \langle \sqrt{-1} d\eta \mathrm{Id}_E \rangle^{\oplus 3}_{\mathbb{R}}$$

Let $(A, B, C) \in A^2_B(\mathfrak{u}(E))^{\oplus 3}$. By direct calculation, we have

$$D_2^*(A, B, C) = (\nabla_h^* A + [(\star_\xi \Phi)^*, B] + [\Phi^*, C], -[\Phi^*, A] - \star_\xi \nabla_h^* B + \nabla_h^* C).$$

Here ∇_h^* is the formal adjoint of ∇_h w.r.t. L^2-inner product. Φ^*, $(\star_\xi \Phi)^*$ are also.

Hence $D_2^*(A,B,C)=0$ is equivalent to

$$\begin{cases} \nabla_h^* A + [(\star_\xi \Phi)^*, B] + [\Phi^*, C] = 0, \\ -[\Phi^*, A] - \star_\xi \nabla_h^* B + \nabla_h^* C = 0. \end{cases} \tag{3.7}$$

Recall that from Lemma 3.4.1, we have the explicit formula of ∇_h^*, Φ^*, and $(\Phi^{1,0})^*$:

$$\begin{aligned} \nabla_h^* &= - \star_\xi \nabla_h \star_\xi, \\ (\Phi)^* &= \star_\xi (\Phi)_h^\dagger \star_\xi = - \star_\xi \Phi \star_\xi, \\ (\star_\xi \Phi)^* &= \star_\xi (\star_\xi \Phi)_h^\dagger \star_\xi = - \star_\xi (\star_\xi \Phi) \star_\xi \,. \end{aligned}$$

The operator $\star_\xi$ induces an isomorphism

$$\star_\xi : A_B^2(\mathfrak{u}(E)) \to A_B(\mathfrak{u}(E)).$$

Hence to consider the pair $(A,B,C) \in A_B^2(\mathfrak{u}(E))^{\oplus 3}$ which satisfies the equation (3.7) is equivalent to consider the pair $(\alpha,\beta,\gamma) \in A_B(\mathfrak{u}(E))^{\oplus 3}$ which satisfies the following equations

$$\begin{cases} \nabla_h \alpha + [\star_\xi \Phi, \beta] + [\Phi, \gamma] = 0, \\ [\Phi, \alpha] + \star_\xi \nabla_h \beta - \nabla_h \gamma = 0. \end{cases} \tag{3.8}$$

Let $(,)_{L^2}$ be the L^2-inner product. Assume $(\alpha,\beta,\gamma) \in A_B(\mathfrak{u}(E))^{\oplus 3}$ satisfies the equation (3.8). Then we have

$$\begin{aligned} \|\nabla_h \alpha\|_{L^2}^2 &= (\nabla_h \alpha, \nabla_h \alpha)_{L^2} \\ &= (- \star_\xi \nabla_h \star_\xi \nabla_h \alpha, \alpha)_{L^2} \\ &= (\star_\xi \nabla_h \star_\xi [\star_\xi \Phi, \beta] + \star_\xi \nabla_h \star_\xi [\Phi, \gamma], \alpha) \\ &= (- \star_\xi \nabla_h [\Phi, \beta] + \star_\xi \nabla_h [\star_\xi \Phi, \gamma], \alpha)_{L^2} \qquad (\because \text{ Lemma 3.4.2.}) \\ &= (\star_\xi [\Phi, \nabla_h \beta] - \star_\xi [\star_\xi \Phi, \nabla_h \gamma], \alpha)_{L^2} \\ &= (\star_\xi [\Phi, \nabla_h \beta] + \star_\xi [\Phi, \star_\xi \nabla_h \gamma], \alpha)_{L^2} \\ &= (\star_\xi [\Phi, \star_\xi (- \star_\xi \nabla_h \beta + \nabla_h \gamma)], \alpha)_{L^2} \\ &= (\star_\xi [\Phi, \star_\xi [\Phi, \alpha]], \alpha)_{L^2} \\ &= -((\Phi)^* [\Phi, \alpha], \alpha)_{L^2} \\ &= -([\Phi, \alpha], [\Phi, \alpha])_{L^2} \\ &= -\|[\Phi, \alpha]\|_{L^2}^2. \end{aligned}$$

Hence we obtain $\nabla_h \alpha = [\Phi, \alpha] = 0$. This is equivalent to $\alpha \in \mathrm{Ker} D_1$. Since $(\nabla_h, \Phi) \in \mathcal{A}_{\mathrm{BaHit}}^{\mathrm{irr}}$, $\alpha = \sqrt{-1} a \mathrm{Id}_E$ for some $a \in \mathbb{R}$. Then β and γ satisfies

$$\begin{cases} [\star_\xi \Phi, \beta] + [\Phi, \gamma] = 0, \\ \star_\xi \nabla_h \beta - \nabla_h \gamma = 0. \end{cases} \tag{3.9}$$

We first calculate $\|\nabla_h\gamma\|_{L^2}^2$.

$$\begin{aligned}\|\nabla_h\gamma\|_{L^2}^2 &= (\nabla_h\gamma, \nabla_h\gamma)_{L^2}\\ &= -(\star_\xi\nabla_h\star_\xi\nabla_h\gamma, \gamma)_{L^2}\\ &= -(\star_\xi\nabla_h\star_\xi\star_\xi\nabla_h\beta, \gamma)_{L^2}\\ &= (\star_\xi\nabla_h\nabla_h\beta, \gamma)_{L^2}\\ &= (\star_\xi F_{\nabla_h}\beta, \gamma)_{L^2}\\ &= (\star_\xi[\Phi, [\Phi, \beta]], \gamma)_{L^2}\\ &= -(\star_\xi[\Phi, \star_\xi\star_\xi[\Phi, \beta]], \gamma)_{L^2}\\ &= ((\Phi)^*\star_\xi[\Phi, \beta], \gamma)_{L^2}\\ &= ([\star_\xi\Phi, \beta], [\Phi, \gamma])_{L^2}\\ &= -([\star_\xi\Phi, \beta], [\star_\xi\Phi, \beta])_{L^2}\\ &= -\|[\star_\xi\Phi, \beta]\|_{L^2}^2.\end{aligned}$$

Hence we obtain $\nabla_h\gamma = [\star_\xi\Phi, \beta] = 0$. Since β and γ satisfies the equation (3.9), we also obtain $\star_\xi\nabla_h\beta = [\Phi, \gamma] = 0$. Since $\star_\xi$ is an isomorphism, $\nabla_h\Phi = [\Phi, \beta] = 0$. Hence $\beta, \gamma \in \mathrm{Ker}D_1$, and therefore $\beta = \sqrt{-1}b\mathrm{Id}_E$ and $\gamma = \sqrt{-1}c\mathrm{Id}_E$ for some $b, c \in \mathbb{R}$.

Let $(A, B, C) \in \mathrm{Ker}D_2^*$. Then $(\alpha, \beta, \gamma) := (\star_\xi A, \star_\xi B, \star_\xi C)$ satisfies the equation (3.8). By the discussion above, $(\alpha, \beta, \gamma) = (\sqrt{-1}a\mathrm{Id}_E, \sqrt{-1}b\mathrm{Id}_E, \sqrt{-1}c\mathrm{Id}_E)$ for some $a, b, c \in \mathbb{R}$. Since we have $\star_\xi 1 = d\eta$, $A, B, C \in \langle\sqrt{-1}d\eta\mathrm{Id}_E\rangle_{\mathbb{R}}$. Hence $\ker D_2^* \subset \langle\sqrt{-1}d\eta\mathrm{Id}_E\rangle_{\mathbb{R}}^{\oplus 3}$.

Since $\star_\xi d\eta = 1$, $\langle\sqrt{-1}d\eta\mathrm{Id}_E\rangle_{\mathbb{R}}^{\oplus 3} \subset \ker D_2^*$. Hence we have

$$\ker D_2^* = \langle\sqrt{-1}d\eta\mathrm{Id}_E\rangle_{\mathbb{R}}^{\oplus 3}.$$

□

We now construct the moduli space of the irreducible basic Hitchin pair. To construct the moduli space, we introduce $\|\cdot\|_{k,2}$ the L_k^2-Sobolev norm. Let $L_k^2(A_B^1(\mathfrak{u}(E)))$ to be the completion of $A_B^1(\mathfrak{u}(E))$ with respect to the L_k^2-norm. We denote as $\mathcal{A}_{h,B}^k$ to be the space of h-unitary basic L_k^2- connection. We set

$$\mathcal{A}_B^k := \mathcal{A}_{h,B}^k \times L_k^2(A_B^1(\mathfrak{u}(E))).$$

We may regard $\mathcal{A}_B^k$ as the space of basic L_k^2-connection. Let $\mathcal{G}_B^k$ to be the L_k^2-basic gauge group and $\mathcal{G}_{r,B}^k := \mathcal{G}_B^k/S^1\mathrm{Id_E}$ to be the reduced L_k^2-basic gauge group. We take k large enough so that the basic Sobolev embedding holds [BHe2, KLW]. Then one can show as in [DK], that $\mathcal{G}_B^k$ and $\mathcal{G}_{r,B}^k$ are Hilbert Lie groups. By basic Sobolev multiplication [BHe2, KLW], $\mathcal{G}_B^{k+1}$ and $\mathcal{G}_{r,B}^{k+1}$ acts smoothly on $\mathcal{A}_B^k$ and we can show that $\mathcal{B}^k := \mathcal{A}_B^k/\mathcal{G}_B^{k+1}$ and $\mathcal{B}_r^k := \mathcal{A}_B^k/\mathcal{G}_{r,B}^{k+1}$ are Hausdorff spaces in the quotient topology. Let $\mathcal{A}_{\mathrm{BaHit}}^k \subset \mathcal{A}_B^k$ be the space of L_k^2-basic Hitchin pair. We define the moduli space of L_k^2-basic Hitchin equation $\mathcal{M}_{\mathrm{BaHit}}^k$ as

$$\mathcal{M}_{\mathrm{BaHit}}^k := \mathcal{A}_{\mathrm{BaHit}}^k/\mathcal{G}_{r,B}^{k+1}.$$

Since $\mathcal{M}_{\mathrm{BaHit}}^k \subset \mathcal{B}_r^k$, $\mathcal{M}_{\mathrm{BaHit}}^k$ is a Hausdorff space. We define $\mathcal{A}_B^{k,\mathrm{irr}} \subset \mathcal{A}_B^k$ to be the space irreducible basic L_k^2-connection and $\mathcal{A}_{\mathrm{BaHit}}^{k,\mathrm{irr}} := \mathcal{A}_{\mathrm{BaHit}}^k \cap \mathcal{A}_B^{k,\mathrm{irr}}$ to be the space of irreducible basic L_k^2-Hitchin pairs. Note that $\mathcal{G}_{r,B}^{k+1}$ acts freely on $\mathcal{A}_B^{k,\mathrm{irr}}$ and $\mathcal{A}_{\mathrm{BaHit}}^{k,\mathrm{irr}}$. We define $\mathcal{B}_r^{k,\mathrm{irr}} := \mathcal{A}_B^{k,\mathrm{irr}}/\mathcal{G}_{r,B}^{k+1}$. We finally define the moduli of irreducible L_k^2-basic Hitchin pairs as

$$\mathcal{M}_{\mathrm{BaHit}}^{k,\mathrm{irr}} := \mathcal{A}_{\mathrm{BaHit}}^{k,\mathrm{irr}}/\mathcal{G}_{r,B}^{k+1}.$$

Since $\mathcal{B}_r^{k,\mathrm{irr}} \subset \mathcal{B}_r^k$ and $\mathcal{M}_{\mathrm{BaHit}}^{k,\mathrm{irr}} \subset \mathcal{M}_{\mathrm{BaHit}}^k$, they are Hausdorff spaces. The topology of $\mathcal{M}_{\mathrm{BaHit}}^{k,\mathrm{irr}}$ do depend on k. However, we can apply the argument in [DK, LT] and show the following.

Proposition 3.5.2. *Assume that k is large enough. Then the natural map $\mathcal{M}_{\mathrm{BaHit}}^{k+1,\mathrm{irr}} \to \mathcal{M}_{\mathrm{BaHit}}^{k,\mathrm{irr}}$ is a homeomorphism.*

Since we have this Proposition, we omit the subscription k from now.

We now turn our interest to the local structure of the moduli space. Let $[(\nabla_h, \Phi)] \in \mathcal{B}_r^{\mathrm{irr}}$. We define a slice

$$S_{(\nabla_h,\Phi),\epsilon} := \{\alpha \in A_B^1(\mathfrak{u}(E))^{\oplus 2} : \|\alpha\|_{L_k^2} < \epsilon, D_1^*\alpha = 0\}. \tag{3.10}$$

We can apply the argument of [DK, LT, Pa] and show that $S_{(\nabla_h,\Phi),\epsilon}$ gives a coordinate patch for $\mathcal{B}_r^{irr}$.

From now on, we assume $[(\nabla_h, \Phi)] \in \mathcal{M}_{\mathrm{BaHit}}^{\mathrm{irr}}$. We show that $\mathcal{M}_{\mathrm{BaHit}}^{\mathrm{irr}} \cap S_{(\nabla_h,\Phi),\epsilon}$ is diffeomorphic to the neighborhood of $\mathbb{H}^1$. Before we proceed, we prepare some notations. We set $\Delta_{i,(\nabla_h,\Phi)} := D_iD_i^* + D_{i+1}^*D_{i+1}(i = 0, 1, 2)$ to be the Laplacians. We set as $D_{-1} = D_3 = 0$. Let $G_{(\nabla_h,\Phi)}$ be the Green operators and $H_{(\nabla_h,\Phi)}$ be the Harmonic projections. We denote as Δ_i, G, H if there is no confusion.

Let $\alpha = (A, B) \in S_{(\nabla_h,\Phi),\epsilon}$. Then $\alpha \in \mathcal{M}_{\mathrm{BaHit}}^{\mathrm{irr}}$ if and only if

$$D_2\alpha + \begin{pmatrix} A\wedge A - B\wedge B \\ [A,B] \\ [A, \star_\xi B] \end{pmatrix} = D_2(A,B) + \begin{pmatrix} A\wedge A - B\wedge B \\ [A,B] \\ [A, \star_\xi B] \end{pmatrix} = 0. \tag{3.11}$$

This can be checked by direct computation. To simplify the notation, we set

$$\widetilde{\alpha\wedge\alpha} := \begin{pmatrix} A\wedge A - B\wedge B \\ [A,B] \\ [A, \star_\xi B] \end{pmatrix}.$$

Note that $\widetilde{\alpha\wedge\alpha}$ is not an ordinary wedge product.

Hence we have

$$\mathcal{M}_{\mathrm{BaHit}}^{\mathrm{irr}} \cap S_{(\nabla_h,\Phi),\epsilon} = \{\alpha \in S_{(\nabla_h,\Phi),\epsilon} : D_2\alpha + \alpha\wedge\alpha = 0\}.$$

By the Hodge decomposition, the equation (3.11) is equivalent to

$$\begin{cases} D_2\alpha + D_2D_2^*G(\widetilde{\alpha\wedge\alpha}) = 0, \\ H(\widetilde{\alpha\wedge\alpha}) = 0. \end{cases} \tag{3.12}$$

We define the *Kuranishi map* $k_{(\nabla_h,\Phi)} : A_B^1(\mathfrak{u}(E))^{\oplus 2} \to A_B^1(\mathfrak{u}(E))^{\oplus 2}$ as

$$k_{(\nabla_h,\Phi)}(\alpha) = \alpha + D_2^*G(\alpha\wedge\alpha). \tag{3.13}$$

Let $\alpha \in \mathcal{M}_{\mathrm{BaHit}}^{\mathrm{irr}} \cap S_{(\nabla_h,\Phi),\epsilon}$. Then by (3.12),

$$D_1^*(k_{(\nabla_h,\Phi)}(\alpha)) = D_1^*\alpha + D_1^*D_2^*G(\alpha\wedge\alpha) = 0,$$
$$D_2(k_{(\nabla_h,\Phi)}(\alpha)) = D_2\alpha + D_2D_2^*G(\alpha\wedge\alpha) = 0.$$

Hence

$$k_{(\nabla_h,\Phi)}(\mathcal{M}_{\mathrm{BaHit}}^{\mathrm{irr}} \cap S_{(\nabla_h,\Phi),\epsilon}) \subset \mathbb{H}^1.$$

The next proposition shows that $\mathcal{M}_{\mathrm{BaHit}}^{\mathrm{irr}}$ is smooth.

Proposition 3.5.3. *Let U be a neighborhood of the origin of $\mathbb{H}^1$. If we take a U small enough, then there exists a ϵ such that $k_{(\nabla_h,\Phi)}$ induces a homeomorphism*

$$k_{(\nabla_h,\Phi)} : \mathcal{M}_{\mathrm{BaHit}}^{\mathrm{irr}} \cap S_{(\nabla_h,\Phi),\epsilon} \to U.$$

Proof. The proof is quite standard (See [Ko]). The point of this proposition is that we do not need any assumption to show $\mathcal{M}_{\mathrm{BaHit}}^{\mathrm{irr}}$ is smooth.

Let $L_k^2(A_B^1(\mathfrak{u}(E)))$ be the completion of $A_B^1(\mathfrak{u}(E))$ with respect to the L_k^2-norm. We extend the Kuranishi map to

$$k_{(\nabla_h,\Phi)} : L_k^2(A_B^1(\mathfrak{u}(E)))^{\oplus 2} \to L_k^2(A_B^1(\mathfrak{u}(E)))^{\oplus 2}.$$

Since the derivative of the Kuranishi map at the origin is the identity, we can apply the inverse function theorem of Banach spaces and show that there exist neighborhoods of origin V_1 and V_2 such that $k_{(\nabla_h,\Phi)}$ induces a homeomorphism

$$k_{(\nabla_h,\Phi)} : V_1 \to V_2.$$

Let $\beta \in V_2 \cap \mathbb{H}^1$. Let $\alpha := k^{-1}(\beta)$. We show that $\alpha \in V_1 \cap \mathrm{Ker} D_1^* \cap \mathcal{M}^{k,\mathrm{irr}}_{\mathrm{BaHit}}$. Once this is shown, shrink V_1 and we prove the proposition.

First, from the definition of α, we have

$$\beta = \alpha + D_2^* G(\widetilde{\alpha \wedge \alpha}).$$

Act the Laplacian Δ_1 and we have

$$\begin{aligned} 0 = \Delta_1 \beta &= \Delta_1 \alpha + D_2^* \Delta_2 G(\widetilde{\alpha \wedge \alpha}) \\ &= \Delta_1 \alpha + D_2^* \Delta_2 G(\widetilde{\alpha \wedge \alpha}) \\ &= \Delta_1 \alpha + D_2^* (\widetilde{\alpha \wedge \alpha}) - D_2^* H(\widetilde{\alpha \wedge \alpha}) \\ &= \Delta_1 \alpha + D_2^* (\widetilde{\alpha \wedge \alpha}). \end{aligned}$$

Hence by the transverse elliptic regularity, α is smooth. We also have

$$\begin{aligned} &0 = D_2 \beta = D_2 \alpha + D_2 D_2^* G(\widetilde{\alpha \wedge \alpha}), \\ &0 = D_1^* \beta = D_1^* \alpha. \end{aligned}$$

We now showed that $\alpha \in V_1 \cap \mathrm{Ker} D_1^*$. To show $\alpha \in \mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$, we need to show $H(\alpha \wedge \alpha) = 0$ (See (3.12)). To show this, we use Proposition 3.5.1. Recall that

$$\widetilde{\alpha \wedge \alpha} = \begin{pmatrix} A \wedge A - B \wedge B \\ [A, B] \\ [A, \star_\xi B] \end{pmatrix}.$$

From Proposition 3.5.1, there exists $a, b, c \in \mathbb{R}$ such that

$$H \begin{pmatrix} A \wedge A - B \wedge B \\ [A, B] \\ [A, \star_\xi B] \end{pmatrix} = \sqrt{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} d\eta \mathrm{Id}_E.$$

We would like to show $a = b = c = 0$. First, let

$$A_B^i(\mathfrak{su}(E)) := \{ f \in A_B^i(\mathfrak{u}(E)) : \mathrm{Tr}(f) = 0 \}.$$

Then the complex

$$0 \longrightarrow A_B(\mathfrak{su}(E)) \xrightarrow{D_1} A_B^1(\mathfrak{su}(E))^{\oplus 2} \xrightarrow{D_2} A_B^2(\mathfrak{su}(E))^{\oplus 3} \longrightarrow 0$$

forms a sub complex of (3.5). Since

$$\begin{pmatrix} A \wedge A - B \wedge B \\ [A, B] \\ [A, \star_\xi B] \end{pmatrix} \in A_B^2(\mathfrak{su}(E))^{\oplus 3},$$

we have

$$H \begin{pmatrix} A \wedge A - B \wedge B \\ [A, B] \\ [A, \star_\xi B] \end{pmatrix} \in \mathbb{H}^2 \cap A_B^2(\mathfrak{su}(E))^{\oplus 3}.$$

Hence $\mathrm{Tr}(a \cdot d\eta \mathrm{Id}_E) = \mathrm{Tr}(b \cdot d\eta \mathrm{Id}_E) = \mathrm{Tr}(c \cdot d\eta \mathrm{Id}_E)$=0. We obtain $a = b = c = 0$. □

In particular, we have the following

Corollary 3.5.1. *$\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ is an empty set or a smooth manifold. If not empty, the dimension of $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ around $[(\nabla_h,\Phi)]\in\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ is $\mathbb{H}^1$.*

We give a sufficient condition for $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ not to be empty. Recall that $T^{1,0}$ is the CR structure on M. If $c_{1,B}(T^{1,0})=-C[d\eta]$ for some positive constant C, then there exists a basic stable Higgs bundle due to [BH2, Example 3.6]. Hence if $c_{1,B}(T^{1,0})=-C[d\eta], C>0$, then $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ is not empty (See Section 3.6.1).

3.5.2 Riemannian Structure on $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$

We use the same notation of the previous section. We assume that $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ is not an empty set.

We show that the moduli space $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ of irreducible Basic Hitchin pair on a compact Sasakian three-fold M is a hyperKähler manifold. We first define a Riemannian metric g on $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$. Let $[(\nabla_h,\Phi)]\in\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ and $\alpha=(\alpha_1,\alpha_2),\beta=(\beta_1,\beta_2)\in\mathbb{H}^1\simeq T_{[(\nabla_h,\Phi)]}\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$. We define g as

$$g_{[(\nabla_h,\Phi)]}(\alpha,\beta):=-\int_M \mathrm{Tr}(\alpha_1\wedge\star_\xi\beta_1+\alpha_2\wedge\star_\xi\beta_2)\wedge\eta. \tag{3.14}$$

To show g is well-defined, we need to check that g does not depend on the gauge-equivalence class of $[(\nabla_h,\Phi)]\in\mathcal{M}^{irr}_{\mathrm{BaHit}}$. Under a gauge transformation $(\nabla_h,\Phi)\to h^{-1}(\nabla_h,\Phi)h$, the infinitesimal deformations α,β maps to $h^{-1}\alpha h, h^{-1}\beta h$ which are the corresponding harmonic repsentative (See [I] for details.). Since (3.14), the metric g is equivalent to the gauge transformation. Hence g is well-defined.

We now prove the distinguished coordinate of the moduli $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ induced by the Kuranishi map and the slice is a normal coordinate with respect to $(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}},g)$. This result will be used later to show that $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ is hyperkähler.

Let $[(\nabla_h,\Phi)]\in\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$. Then from the previous section we have the Kuranishi map $k_{(\nabla_h,\Phi)}$, Slice $S_{(\nabla_h,\Phi),\epsilon}$, and a open subset $0\in U\subset\mathbb{H}^1$ such that

$$k_{(\nabla_h,\Phi)}:\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}\cap S_{(\nabla_h,\Phi),\epsilon}\to U$$

is a homeomorphism. The derivative of the Kuranishi map at $\alpha\in A^1_B(\mathfrak{u}(E))^{\oplus 2}$ as follows

$$\begin{aligned} &d(k_{(\nabla_h,\Phi)})_\alpha: T_\alpha A^1_B(\mathfrak{u}(E))^{\oplus 2}\to T_{k_{(\nabla_h,\Phi)}(\alpha)}A^1_B(\mathfrak{u}(E))^{\oplus 2},\\ &d(k_{(\nabla_h,\Phi)})_\alpha(\beta)=\beta+D_2^*G(\widetilde{[\alpha,\beta]}). \end{aligned} \tag{3.15}$$

Here for $\alpha=(\alpha_1,\alpha_2),\beta=(\beta_1,\beta_2)\in A^1_B(\mathfrak{u}(E))^{\oplus 2}$ we defined $\widetilde{[\alpha,\beta]}$ as

$$\widetilde{[\alpha,\beta]}:=\begin{pmatrix}[\alpha_1,\beta_1]-[\alpha_2,\beta_2]\\ [\alpha_1,\beta_2]+[\beta_1,\alpha_2]\\ [\alpha_1,\star_\xi\beta_2]+[\beta_1,\star_\xi\alpha_2]\end{pmatrix}. \tag{3.16}$$

Note that $\widetilde{[\alpha,\beta]}$ is not the ordinary bracket. We call this bracket as the *modified bracket.*

Using the modified bracket, we can characterize the tangent space of $\alpha\in\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}\cap S_{(\nabla_h,\Phi),\epsilon}$ as follows

$$T_\alpha(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}\cap S_{(\nabla_h,\Phi),\epsilon})=\{\beta\in A^1_B(\mathfrak{u}(E))^{\oplus 2}: D_1^*\beta=0, D_2\beta+\widetilde{[\alpha,\beta]}=D_{2,\alpha}\beta=0\}. \tag{3.17}$$

Here $D_{2,\alpha}$ is the operator of (3.6) defined for $(\nabla_h,\Phi)+\alpha=(\nabla_h+\alpha_1,\Phi+\alpha_2)\in\mathcal{A}^{\mathrm{irr}}_{\mathrm{BaHit}}$. From (3.15) and (3.17), the restriction of $dk_{(\nabla_h,\Phi)}$ to $T_\alpha(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}\cap S_{(\nabla_h,\Phi),\epsilon})$ has the following form.

Proposition 3.5.4. *The differential of the Kuranishi map*

$$d(k_{(\nabla_h,\Phi)})_\alpha: T_\alpha(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}\cap S_{(\nabla_h,\Phi),\epsilon})\to T_{k_{(\nabla_h,\Phi)}(\alpha)}U=\mathbb{H}^1$$

has the form

$$d(k_{(\nabla_h,\Phi)})_\alpha(\beta)=H_{(\nabla_h,\Phi)}\beta.$$

Here $H_{(\nabla_h,\Phi)}: A^1_B(\mathfrak{u}(E))^{\oplus 2}\to\mathbb{H}^1$ is the harmonic projection.

Proof. Since D_2^* commutes with the Green operator, and we have (3.15) and (3.17), we have

$$\begin{aligned} d(k_{(\nabla_h,\Phi)})_\alpha(\beta) &= \beta + D_2^* G(\widetilde{[\alpha,\beta]}) \\ &= \beta - D_2^* G D_2 \beta \\ &= \beta - D_2^* D_2 G \beta \\ &= H_{(\nabla_h,\Phi)}\beta. \end{aligned}$$

□

In the previous section, we denoted $H_{(\nabla_h,\Phi)}$ just as H. We denoted as $H_{(\nabla_h,\Phi)}$ because later, we use the harmonic projection induced by different basic Hitchin pairs.

We now solve conversely an equation $d(k_{(\nabla_h,\Phi)})_\alpha(\beta) = \gamma$ for a given $\gamma \in T_{k_{(\nabla_h,\Phi)}(\alpha)}U = \mathbb{H}^1$ and $\alpha \in \mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon}$ with respect to $\beta \in T_\alpha(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon})$. We decompose β as

$$\beta = D_1\gamma_0 + \gamma_1 + D_2^*\gamma_2,$$

where $\gamma_0 \in A_B(\mathfrak{u}(E))$, $\gamma_1 \in \mathbb{H}^1$, and $\gamma_2 \in A_B^2(\mathfrak{u}(E))^{\oplus 3}$. By Proposition 3.5.4, $\gamma_1 = \gamma$. Moreover, since $D_1^*\beta = 0$, we have $D_1^* D_1 \gamma_0 = 0$ and hence $D_1\gamma_0 = 0$. Hence we obtain

$$\beta = \gamma + D_2^*\gamma_2.$$

From (3.17), γ_2 satisfies the equation

$$D_2 D_2^* \gamma_2 + [\alpha, \widetilde{\gamma + D_2^*\gamma_2}] = 0.$$

By the definition of the modified bracket, it is a bilinear map. Hence

$$D_2 D_2^* \gamma_2 + [\widetilde{\alpha, D_2^*\gamma_2}] = -[\widetilde{\alpha,\gamma}]. \tag{3.18}$$

As a consequence we have

Proposition 3.5.5. *For a given $\gamma \in \mathbb{H}^1$, the inverse image $\beta = (d(k_{(\nabla_h,\Phi)})_\alpha)^{-1}(\gamma) \in T_\alpha(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon})$ is represented by*

$$\beta = \gamma + D_2^*\gamma_2$$

where $\gamma_2 \in A_B^2(\mathfrak{u}(E))^{\oplus 3}$ is a solution of (3.18).

We note that at the origin, $T_0(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon}) = \mathbb{H}^1$ and $d(k_{(\nabla_h,\Phi)})_0 = \mathrm{Id}_{\mathbb{H}^1}$ holds.

Let $X, Y, Z \in T_0(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon}) = \mathbb{H}^1$. Since $\mathbb{H}^1$ is affine, these vectors also define vector fields on U canonically. We define a vector field $\overline{X}$ on $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon}$ as

$$\overline{X}_\alpha := d((k_{(\nabla_h,\Phi)})^{-1})_{k_{(\nabla_h,\Phi)}(\alpha)}(X),\ \alpha \in \mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon}.$$

We define $\overline{Y}, \overline{Z}$ in the same manner. From Proposition 3.5.5, $\overline{X}_\alpha$ has the form

$$\overline{X}_\alpha = X + D_2^*\gamma(\alpha, X)$$

where $\gamma(\alpha, X) \in A_B^2(\mathfrak{u}(E))^{\oplus 3}$ and it satisfies the following equation

$$D_2 D_2^* \gamma(\alpha, X) + [\widetilde{\alpha, D_2^*\gamma(\alpha, X)}] = -[\widetilde{\alpha, X}]. \tag{3.19}$$

We note that at $\alpha = 0$, $\overline{X}_0 = X$ and $D_2^*\gamma(0, X) = 0$.

Let $c(t)$ be a curve on $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon}$ defined by $c(t) := (k_{(\nabla_h,\Phi)})^{-1}(tX)$. Then we have $c(0) = 0$ and $\frac{d}{dt}c(t)|_{t=0} = X \in T_0(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon}) = \mathbb{H}^1$.

Proposition 3.5.6. *The Riemannian metric g on $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ satisfies at $\alpha = 0$ in a slice neighborhood $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon}$*

$$Xg_{[(\nabla_h,\Phi)]}(Y,Z) = 0$$

for every $X, Y, Z \in T_0(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon}) = \mathbb{H}^1$.

We remark that this Proposition shows that the coordinate obtained by the Kuranishi map is normal.

Proof. By the definition of the metric

$$\begin{aligned}
Xg_{[(\nabla_h,\Phi)]}(Y,Z) &= \frac{d}{dt} g_{[(\nabla_h,\Phi)+c(t)]}(\overline{Y}_{c(t)}, \overline{Z}_{c(t)})\Big|_{t=0} \\
&= \frac{d}{dt}\Big(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)}, H_{(\nabla_h,\Phi)+c(t)}\overline{Z}_{c(t)}\Big)_{L^2}\Big|_{t=0} \\
&= \Big(\frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})|_{t=0}, Z\Big)_{L^2} + \Big(Y, \frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Z}_{c(t)})\Big|_{t=0}\Big)_{L^2}.
\end{aligned}$$

Differentiating $H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)}$ at $t = 0$, we get

$$\frac{d}{dt}\Big(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)}\Big)\Big|_{t=0} = \Big(\frac{d}{dt}H_{(\nabla_h,\Phi)+c(t)}\Big|_{t=0}\Big)Y + H_{(\nabla_h,\Phi)}\Big(\frac{d}{dt}\overline{Y}_{c(t)}\Big|_{t=0}\Big).$$

Before we proceed, we prepare two Lemmas.

Lemma 3.5.1.

$$H_{(\nabla_h,\Phi)}\Big(\frac{d}{dt}\overline{Y}_{c(t)}\Big|_{t=0}\Big) = 0.$$

Proof. From Proposition 3.5.5, we have

$$\begin{aligned}
\frac{d}{dt}\overline{Y}_{c(t)}\Big|_{t=0} &= \frac{d}{dt}(Y + D_2^*\gamma(c(t),Y))\Big|_{t=0} \\
&= D_2^*\Big(\frac{d}{dt}\gamma(c(t),Y)\Big|_{t=0}\Big).
\end{aligned}$$

From (3.19), $\gamma(c(t),Y)$ satisfies the equation

$$D_2D_2^*\gamma(c(t),Y) + [c(t), \widetilde{D_2^*\gamma(c(t),Y)}] = -[\widetilde{c(t),Y}].$$

We differential this equation at $t = 0$ and we obtain

$$D_2D_2^*\Big(\frac{d}{dt}\gamma(c(t),Y)\Big|_{t=0}\Big) = -[\widetilde{X,Y}].$$

By Proposition 3.5.1 and the Hodge decomposition, we have $a, b, c \in \mathbb{R}$ such that

$$\begin{aligned}
\frac{d}{dt}\gamma(c(t),Y)|_{t=0} &= \sqrt{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} d\eta + GD_2D_2^*\Big(\frac{d}{dt}\gamma(c(t),Y)|_{t=0}\Big) \\
&= \sqrt{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} d\eta - [\widetilde{X,Y}].
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{d}{dt}\overline{Y}_{c(t)}|_{t=0} &= D_2^*\left(\frac{d}{dt}\gamma(c(t),Y)\Big|_{t=0}\right)\\
&= D_2^*\left(\sqrt{-1}\begin{pmatrix}a\\ b\\ c\end{pmatrix}d\eta - G\widetilde{[X,Y]}\right)\\
&= -D_2^*G\widetilde{[X,Y]}.
\end{aligned}$$

Then the Lemma is obtained by the Hodge decomposition. □

Lemma 3.5.2.

$$\begin{aligned}
\left(\frac{d}{dt}H_{(\nabla_h,\Phi)+c(t)}|_{t=0}\right)Y &= -G[X,D_1^*Y]^1 - D_1G[X,Y]^2 - D_2^*G\widetilde{[X,Y]} - G[X,D_2Y]^3\\
&= -D_1G[X,Y]^2 - D_2^*G\widetilde{[X,Y]}.
\end{aligned}$$

Here

$$\begin{aligned}
[X,D_1^*Y]^1 &:= \begin{pmatrix}[X_1,D_1^*Y]\\ [X_2,D_1^*Y]\end{pmatrix},\\
[X,Y]^2 &:= [X_1^*,Y_1]+[X_2^*,Y_2],\\
[X,D_2Y]^3 &:= \begin{pmatrix}[X_1,\nabla_hY_1-[\Phi,Y_2]]+[\star_\xi X_2^*,\nabla_hY_2+[Y_1,\Phi]]+[X_2^*,\nabla_h * Y_2+[Y_1,*\Phi]]\\ -[X_2^*,\nabla_hY_2+[Y_1,\Phi]]-[\star_\xi X_2^*,\nabla_hY_1-[\Phi,Y_2]]+[X_2^*,\nabla_h\star_\xi Y_2+[Y_1,\star_\xi\Phi]]\end{pmatrix}.
\end{aligned}$$

Proof. The second equality follows from the harmonicity of Y. We prove the first equality.

By the Hodge decomposition, we have

$$\begin{aligned}
\left(\frac{d}{dt}H_{(\nabla_h,\Phi)+c(t)}\Big|_{t=0}\right)Y &= \frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}Y)\Big|_{t=0}\\
&= -\frac{d}{dt}(G_{c(t)}\Delta_{1,(\nabla_h,\Phi)+c(t)}Y)\Big|_{t=0}\\
&= -\frac{d}{dt}\left(G_{c(t)}\Big|_{t=0}\right)\Delta_{1,(\nabla_h,\Phi)}Y + G\frac{d}{dt}(\Delta_{1,(\nabla_h,\Phi)+c(t)}Y)\Big|_{t=0}\\
&= -G\frac{d}{dt}(\Delta_{1,(\nabla_h,\Phi)+c(t)}Y)\Big|_{t=0}.
\end{aligned}$$

We now calculate $\frac{d}{dt}(\Delta_{1,(\nabla_h,\Phi)+c(t)}Y)|_{t=0}$.

$$\begin{aligned}
\frac{d}{dt}(\Delta_{1,(\nabla_h,\Phi)+c(t)}Y)\Big|_{t=0} &= \frac{d}{dt}(D_{1,(\nabla_h,\Phi)+c(t)}D^*_{1,(\nabla_h,\Phi)+c(t)}Y + D^*_{2,(\nabla_h,\Phi)+c(t)}D_{2,(\nabla_h,\Phi)+c(t)}Y)\Big|_{t=0}\\
&= [X,D_1^*Y]^1 + D_1[X,Y]^2 + D_2^*\widetilde{[X,Y]} + [X,D_2Y]^3\\
&= D_1[X,Y]^2 + D_2^*\widetilde{[X,Y]}.
\end{aligned}$$

Hence the claim is proved. □

We now prove the Proposition. From the two Lemmas above, we have

$$\begin{aligned}
Xg_{[(\nabla_h,\Phi)]}(Y,Z) &= \left(\frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})\Big|_{t=0}, Z\right)_{L^2} + \left(Y,\frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Z}_{c(t)})\Big|_{t=0}\right)_{L^2}\\
&= \left(-D_1G[X,Y]_2 - D_2^*G\widetilde{[X,Y]}, Z\right)_{L^2} + \left(Y,-D_1G[X,Z]_2 - D_2^*G\widetilde{[X,Z]}\right)_{L^2}\\
&= 0.
\end{aligned}$$

The last follows from the harmonicity of Y and Z. □

3.5.3 HyperKähler Structure on $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$

We use the same notation as the previous section. We assume that $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ is not an empty set.

We define almost complex structures $\mathcal{I}, \mathcal{J}, \mathcal{K}$ on $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$. We first fix a $(\nabla_h, \Phi) \in \mathcal{A}_{\mathrm{BaHit}}$. First, we show that $A^1_B(\mathfrak{u}(E))^{\oplus 2}$ has the structure of the quaternion vector space. Next, we show that they induce a quaternion structure to $\mathbb{H}^1$.

Let $\alpha = (\alpha_1, \alpha_2) \in A^1_B(\mathfrak{u}(E))^{\oplus 2}$. We define $I, J, K \in \mathrm{End}(A^1_B(\mathfrak{u}(E))^{\oplus 2})$ as follows

$$\begin{aligned} I \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} &:= \begin{pmatrix} \star_\xi \alpha_1 \\ - \star_\xi \alpha_2 \end{pmatrix}, \\ J \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} &:= \begin{pmatrix} -\alpha_2 \\ \alpha_1 \end{pmatrix}, \\ K \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} &:= \begin{pmatrix} - \star_\xi \alpha_2 \\ - \star_\xi \alpha_1 \end{pmatrix}. \end{aligned}$$

By Proposition 3.3.1 and definition of I, J, and, K we can check that

$$I^2 = J^2 = K^2 = -\mathrm{Id},\ K = IJ$$

and hence I, J, K defines a quaternion structure of $A^1_B(\mathfrak{u}(E))^{\oplus 2}$. To show that I, J, K induces a quaternion structure to $\mathbb{H}^1$, we only need to check that I, J, K preserves $\mathrm{Ker} D_1^* \cap \mathrm{Ker} D_2$. This can be shown by direct computation. Note that for $\alpha = (\alpha_1, \alpha_2) \in A^1_B(\mathfrak{u}(E))^{\oplus 2}$, we have

$$\begin{aligned} D_1^* \alpha &= \nabla_h^* \alpha_1 + \Phi^* \alpha_2 \\ &= - \star_\xi \nabla_h \star_\xi \alpha_1 - \star_\xi [\Phi, \star_\xi \alpha_2]. \end{aligned} \tag{3.20}$$

Hence by (3.6) and (3.20), $\alpha \in \mathrm{Ker} D_1^* \cap \mathrm{Ker} D_2$ if and only if the following equations hold

$$\begin{aligned} &\nabla_h \star_\xi \alpha_1 + [\Phi, \star_\xi \alpha_2] = 0, \\ &\nabla_h \alpha_1 - [\Phi, \alpha_2] = 0, \\ &\nabla_h \alpha_2 + [\alpha_1, \Phi] = 0, \\ &\nabla_h \star_\xi \alpha_2 + [\alpha_1, \star_\xi \Phi] = 0. \end{aligned} \tag{3.21}$$

Then it is easy to check that if $\alpha \in \mathrm{Ker} D_1^* \cap \mathrm{Ker} D_2$, then $I\alpha$, $J\alpha$, and $K\alpha$ satisfies (3.21) and hence $I\alpha, J\alpha, K\alpha \in \mathrm{Ker} D_1^* \cap \mathrm{Ker} D_2$. Hence $(\mathbb{H}^1, I, J, K)$ is a quaternion vector space. These I, J, K induce almost complex structures to $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ and we denote as $\mathcal{I}, \mathcal{J}, \mathcal{K}$ for the corresponding almost complex structures. It is clear that $\mathcal{I}, \mathcal{J}, \mathcal{K}$ satisfies the quaternion relationship.

To compatibility of g with $\mathcal{I}, \mathcal{J}, \mathcal{K}$ can be shown by using the following equality: Let $A, B \in A^1_B(\mathfrak{u}(E))$. Then we have

$$\begin{aligned} \mathrm{Tr}(A \wedge \star_\xi B) &= \mathrm{Tr}(A^{1,0} \wedge \star_\xi B^{0,1}) + \mathrm{Tr}(A^{0,1} \wedge \star_\xi B^{1,0}) \\ &= \sqrt{-1}\mathrm{Tr}(A^{1,0} \wedge B^{0,1}) - \sqrt{-1}\mathrm{Tr}(A^{0,1} \wedge B^{1,0}) \\ &= -\mathrm{Tr}(\star_\xi A^{1,0} \wedge B^{0,1}) - \mathrm{Tr}(\star_\xi A^{0,1} \wedge B^{1,0}) \\ &= -\mathrm{Tr}(\star_\xi A \wedge B). \end{aligned}$$

We now show $(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}, g, \mathcal{I}, \mathcal{J}, \mathcal{K})$ is a hyperKäher manifold. Let $\omega_\mathcal{I}, \omega_\mathcal{J}, \omega_\mathcal{K}$ be the corresponding Kähler forms. We give the explicit form of $\omega_\mathcal{I}, \omega_\mathcal{J}, \omega_\mathcal{K}$ for $[(\nabla_h, \Phi)] \in \mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ and $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{H}^1 \simeq T_{[(\nabla_h, \Phi)]} \mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ for convinience.

$$\begin{aligned} \omega_{\mathcal{I}, [(\nabla_h, \Phi)]}(\alpha, \beta) &= \int_M \mathrm{Tr}(\alpha_1 \wedge \beta_1 - \alpha_2 \wedge \beta_2) \wedge \eta, \\ \omega_{\mathcal{J}, [(\nabla_h, \Phi)]}(\alpha, \beta) &= \int_M \mathrm{Tr}(\alpha_1 \wedge \star_\xi \beta_2 - \alpha_2 \wedge \star_\xi \beta_1) \wedge \eta, \\ \omega_{\mathcal{K}, [(\nabla_h, \Phi)]}(\alpha, \beta) &= -\int_M \mathrm{Tr}(\alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1) \wedge \eta. \end{aligned}$$

Proposition 3.5.7. *The Kähler form $\omega_{\mathcal{I}}$ on $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$ satisfies at $\alpha = 0$ in a slice neighborhood $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon}$*

$$X\omega_{\mathcal{I},[(\nabla_h,\Phi)]}(Y,Z) = 0$$

for every $X, Y, Z \in T_0(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} \cap S_{(\nabla_h,\Phi),\epsilon}) = \mathbb{H}^1$.

Proof. We give the proof by direct computation.

$$\begin{aligned}
X\omega_{\mathcal{I},[(\nabla_h,\Phi)]}(Y,Z) =& \frac{d}{dt}\omega_{\mathcal{I},[(\nabla_h,\Phi)+c(t)]}(\overline{Y}_{c(t)}, \overline{Z}_{c(t)})\Big|_{t=0} \\
=& \frac{d}{dt} g_{[(\nabla_h,\Phi)+c(t)]}(\overline{Y}_{c(t)}, \mathcal{I}\overline{Z}_{c(t)})\Big|_{t=0} \\
=& \frac{d}{dt}\int_M \mathrm{Tr}\Big((H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})_1 \wedge (H_{(\nabla_h,\Phi)+c(t)}\overline{Z}_{c(t)})_1\Big) \wedge \eta\Big|_{t=0} \\
& - \frac{d}{dt}\int_M \mathrm{Tr}\Big((H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})_2 \wedge (H_{(\nabla_h,\Phi)+c(t)}\overline{Z}_{c(t)})_2\Big) \wedge \eta\Big|_{t=0} \\
=& \int_M \mathrm{Tr}\Big(\frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})_1\Big|_{t=0} \wedge Z_1\Big) \wedge \eta + \int_M \mathrm{Tr}\Big(Y_1 \wedge \frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Z}_{c(t)})_1\Big|_{t=0}\Big) \wedge \eta \\
& - \int_M \mathrm{Tr}\Big(\frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})_2\Big|_{t=0} \wedge Z_2) - \int_M \mathrm{Tr}\Big(Y_2 \wedge \frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Z}_{c(t)})_2\Big|_{t=0}\Big) \wedge \eta.
\end{aligned}$$

Here $(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})_i$ (resp. $(H_{(\nabla_h,\Phi)+c(t)}\overline{Z}_{c(t)})_i$) is the i-th componet of the $H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)}$ (resp. $H_{(\nabla_h,\Phi)+c(t)}\overline{Z}_{c(t)}$).

The following Claim will give us the proof of the Proposition.

Claim 3.5.1.

$$\int_M \mathrm{Tr}\Big(\frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})_1\Big|_{t=0} \wedge Z_1) \wedge \eta - \int_M \mathrm{Tr}\Big(\frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})_2\Big|_{t=0} \wedge Z_2) \wedge \eta = 0.$$

Proof. By Lemma 3.5.1 and 3.5.2, we have

$$\begin{aligned}
& \int_M \mathrm{Tr}\Big(\frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})_1\Big|_{t=0} \wedge Z_1\Big) \wedge \eta \\
=& \int_M \mathrm{Tr}\Big(\Big(\Big(\frac{d}{dt}H_{(\nabla_h,\Phi)+c(t)}\Big|_{t=0}\Big)Y\Big)_1 \wedge Z_1\Big) \wedge \eta \\
=& \int_M \mathrm{Tr}\Big(\Big(-D_1 G[X,Y]^2 - D_2^* G\widetilde{[X,Y]}\Big)_1 \wedge Z_1\Big) \wedge \eta \\
=& \int_M \mathrm{Tr}\Big(\big(-\nabla_h G[X,Y]^2 - \nabla_h^*\big(G\widetilde{[X,Y]}\big)_1 - [(\star_\xi\Phi)^*]\big(G\widetilde{[X,Y]}\big)_2 - [(\star_\xi\Phi)]\big(G\widetilde{[X,Y]}\big)_3\big) \wedge Z_1\Big) \wedge \eta \\
=& \Big(-\nabla_h G[X,Y]^2 - \nabla_h^*\big(G\widetilde{[X,Y]}\big)_1 - [(\star_\xi\Phi)^*, \big(G\widetilde{[X,Y]}\big)_2] - [\Phi^*, \big(G\widetilde{[X,Y]}\big)_3], \star_\xi Z_1\Big)_{L^2} \\
=& -\Big(\nabla_h G[X,Y]^2, \star_\xi Z_1\Big)_{L^2} - \Big(\nabla_h^*\big(G\widetilde{[X,Y]}\big)_1, \star_\xi Z_1\Big)_{L^2} \\
& - \Big([(\star_\xi\Phi)^*, \big(G\widetilde{[X,Y]}\big)_2], \star_\xi Z_1\Big)_{L^2} - \Big([\Phi^*, \big(G\widetilde{[X,Y]}\big)_3], \star_\xi Z_1\Big)_{L^2} \\
=& -\Big(G[X,Y]^2, \nabla_h^* \star_\xi Z_1\Big)_{L^2} - \Big(\big(G\widetilde{[X,Y]}\big)_1, \nabla_h \star_\xi Z_1\Big)_{L^2} \\
& - \Big(\big(G\widetilde{[X,Y]}\big)_2, [\star_\xi\Phi, \star_\xi Z_1]\Big)_{L^2} - \Big(\big(G\widetilde{[X,Y]}\big)_3, [\Phi, \star_\xi Z_1]\Big)_{L^2}.
\end{aligned}$$

Here $[X,Y]^2$ is the map we defined in Lemma 3.5.2. We also have

$$\begin{aligned}
&\int_M \mathrm{Tr}\Big(\frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})_2\Big|_{t=0} \wedge Z_2)\wedge\eta \\
=&\int_M \mathrm{Tr}\Big(\Big(\Big(\frac{d}{dt}H_{(\nabla_h,\Phi)+c(t)}\Big|_{t=0}\Big)Y\Big)_2 \wedge Z_2\Big)\wedge\eta \\
=&\int_M \mathrm{Tr}\Big(\Big(-D_1G[X,Y]^2 - D_2^*G\widetilde{[X,Y]}\Big)_2 \wedge Z_2\Big)\wedge\eta \\
=&\int_M \mathrm{Tr}\Big(\big(-[\Phi,G[X,Y]^2]+[\Phi^*,\big(G\widetilde{[X,Y]}\big)_1]+\star_\xi\nabla_h^*\big(G\widetilde{[X,Y]}\big)_2-\nabla_h^*\big(G\widetilde{[X,Y]}\big)_3\big)\wedge Z_2\Big)\wedge\eta \\
=&-\Big(G[X,Y]^2,[\Phi^*,\star_\xi Z_2]\Big)_{L^2}+\Big(\big(G\widetilde{[X,Y]}\big)_1,[\Phi,\star_\xi Z_2]\Big)_{L^2} \\
&-\int_M \mathrm{Tr}\Big(\nabla_h^*\big(G\widetilde{[X,Y]}\big)_2\wedge\star_\xi Z_2\Big)\wedge\eta-\Big(\big(G\widetilde{[X,Y]}\big)_3,\nabla_h\star_\xi Z_2\Big)_{L^2} \\
=&-\Big(G[X,Y]^2,[\Phi^*,\star_\xi Z_2]\Big)_{L^2}+\Big(\big(G\widetilde{[X,Y]}\big)_1,[\Phi,\star_\xi Z_2]\Big)_{L^2} \\
&-\Big(\big(G\widetilde{[X,Y]}\big)_2,\nabla_h\star_\xi\star_\xi Z_2\Big)_{L^2}-\Big(\big(G\widetilde{[X,Y]}\big)_3,\nabla_h\star_\xi Z_2\Big)_{L^2}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\int_M \mathrm{Tr}\Big(\frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})_1\Big|_{t=0}\wedge Z_1\Big)\wedge\eta-\int_M \mathrm{Tr}\Big(\frac{d}{dt}(H_{(\nabla_h,\Phi)+c(t)}\overline{Y}_{c(t)})_2\Big|_{t=0}\wedge Z_2\Big)\wedge\eta \\
=&-\Big(G[X,Y]^2,\nabla_h^*\star_\xi Z_1\Big)_{L^2}-\Big(\big(G\widetilde{[X,Y]}\big)_1,\nabla_h\star_\xi Z_1\Big)_{L^2} \\
&-\Big(\big(G\widetilde{[X,Y]}\big)_2,[\star_\xi\Phi,\star_\xi Z_1]\Big)_{L^2}-\Big(\big(G\widetilde{[X,Y]}\big)_3,[\Phi,\star_\xi Z_1]\Big)_{L^2} \\
&+\Big(G[X,Y]^2,[\Phi^*,\star_\xi Z_2]\Big)_{L^2}-\Big(\big(G\widetilde{[X,Y]}\big)_1,[\Phi,\star_\xi Z_2]\Big)_{L^2} \\
&-\Big(\big(G\widetilde{[X,Y]}\big)_2,\nabla_h\star_\xi\star_\xi Z_2\Big)_{L^2}+\Big(\big(G\widetilde{[X,Y]}\big)_3,\nabla_h\star_\xi Z_2\Big)_{L^2} \\
=&-\Big(\big(G\widetilde{[X,Y]}\big)_2,D_1^*IZ\Big)_{L^2}-\Big(\big(G\widetilde{[X,Y]}\big)_1,\big(D_2IZ\big)_1\Big)_{L^2}-\Big(\big(G\widetilde{[X,Y]}\big)_2,\big(D_2IZ\big)_3\Big)_{L^2}-\Big(\big(G\widetilde{[X,Y]}\big)_3,\big(D_2IZ\big)_2\Big)_{L^2} \\
=&0.
\end{aligned}$$

The last equation holds since I preserves $\mathbb{H}^1$. □

The Proposition follows immediately from the Claim. □

Integrability of $\mathcal{I}$ follows from Proposition 3.5.6 and 3.5.7: These two Propositions show that $\mathcal{I}$ is flat with respect to the Levi-Civita connection of g and hence $\mathcal{I}$ is integrable. Although we only proved for $\mathcal{I}$, we are able to show the integrability of $\mathcal{J}$ and $\mathcal{K}$ in the same way as $\mathcal{I}$. Hence we omit the proof. From the discussion above, we have

Theorem 3.5.1. $(\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}, g, \mathcal{I}, \mathcal{J}, \mathcal{K})$ *is a smooth hyperKähler manifold.*

3.5.4 Dimention of $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$

In this section, we calculate the dimension of $\mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}}$. We calculate it under the assumption of M being regular and E being *regular*. We recall the notion of regular for bundles later. We first prove the following proposition.

Proposition 3.5.8. *Let E be a basic bundle over M with a basic metric h. Let $(\nabla_h, \Phi) \in \mathcal{A}_{\mathrm{BaHit}}$ and let $(\overline{\partial}_E, \theta)$ be the associated basic Higgs bundle (See Section 3.6.1). Then the map*

$$\begin{array}{cccc} f\colon & A^1_B(\mathfrak{u}(E))^{\oplus 2} & \longrightarrow & A^1_B(\mathrm{End}E) \\ & \cup & & \cup \\ & (\alpha_1, \alpha_2) & \longmapsto & \alpha_1 + \sqrt{-1}\alpha_2 \end{array}$$

induces an isomorphism

$$f : \mathbb{H}^1 \to \mathbb{H}^1_{\mathrm{BaDol}}.$$

Here $\mathbb{H}^1$ is the first cohomology of the complex (3.5) and $\mathbb{H}^1_{\mathrm{BaDol}}$ is the first cohomology of the following complex:

$$0 \longrightarrow A_B(\mathrm{End}E) \overset{\overline{\partial}_E+\theta}{\longrightarrow} A^1_B(\mathrm{End}E) \overset{\overline{\partial}_E+\theta}{\longrightarrow} A^2_B(\mathrm{End}E) \longrightarrow 0.$$

Proof. It is enough to show that f induces an isomorphism

$$f : \mathrm{Ker}D_1^* \cap \mathrm{Ker}D_2 \to \mathrm{Ker}(\overline{\partial}_E + \theta)^* \cap \mathrm{Ker}(\overline{\partial}_E + \theta).$$

Here $(\overline{\partial}_E + \theta)^*$ is the L^2 adjoint of $\overline{\partial}_E + \theta$.

Let $(\alpha_1, \alpha_2) \in A^1_B(\mathfrak{u}(E))^{\oplus 2}$. We assume that $(\alpha_1, \alpha_2) \in \mathrm{Ker}D_1^* \cap \mathrm{Ker}D_2$. We first show that $f(\alpha_1, \alpha_2) = \alpha_1 + \sqrt{-1}\alpha_2 \in \mathrm{Ker}(\overline{\partial}_E + \theta)^* \cap \mathrm{Ker}(\overline{\partial}_E + \theta)$. By (3.6) and (3.20), we have

$$\nabla_h \alpha_1 - [\Phi, \alpha_2] = 0, \tag{3.22}$$
$$\nabla_h \alpha_2 + [\alpha_1, \Phi] = 0, \tag{3.23}$$
$$\nabla_h \star_\xi \alpha_2 + [\alpha_1, \star_\xi \Phi] = 0, \tag{3.24}$$
$$\nabla_h \star_\xi \alpha_1 + [\Phi, \star_\xi \alpha_2] = 0. \tag{3.25}$$

Note that from Lemma 3.4.2, (3.24) is equivalent to

$$\nabla_h \star_\xi \alpha_2 - [\star_\xi \alpha_1, \Phi] = 0. \tag{3.26}$$

Since $\star_\xi|_{A^{1,0}_B(M)} = -\sqrt{-1}\mathrm{Id}_{A^{1,0}_B(M)}$ and $\star_\xi|_{A^{0,1}_B(M)} = \sqrt{-1}\mathrm{Id}_{A^{0,1}_B(M)}$, by calculating (3.22) + $\sqrt{-1}$(3.25) and (3.23) + $\sqrt{-1}$(3.26) we have

$$\nabla^{0,1}_h \alpha_1^{1,0} - [\Phi^{1,0}, \alpha_2^{0,1}] = 0,$$
$$\nabla^{0,1}_h \alpha_2^{1,0} + [\alpha_1^{0,1}, \Phi^{1,0}] = 0.$$

Since $\overline{\partial}_E = \nabla^{0,1}_h$ and $\theta = \sqrt{-1}\Phi^{1,0}$ (See Section 3.6.1), they show $\alpha_1 + \sqrt{-1}\alpha_2 \in \mathrm{Ker}(\overline{\partial}_E + \theta)$. By using a similar argument, we can also show that $\alpha_1 + \sqrt{-1}\alpha_2 \in \mathrm{Ker}(\overline{\partial}_E + \theta)^*$. Hence $f(\mathrm{Ker}D_1^* \cap \mathrm{Ker}D_2) \subset \mathrm{Ker}(\overline{\partial}_E + \theta)^* \cap \mathrm{Ker}(\overline{\partial}_E + \theta)$ holds. We now construct the inverse of f and prove the claim. Let

$$\begin{array}{cccc} g\colon & A^1_B(\mathrm{End}E) & \longrightarrow & A^1_B(\mathfrak{u}(E))^{\oplus 2} \\ & \cup & & \cup \\ & A & \longmapsto & (\frac{A - A^\dagger_h}{2}, -\sqrt{-1}\frac{A + A^\dagger_h}{2}) \end{array}.$$

Here $A^\dagger_h$ is the formal adjoint of A with respect to h. It is straightforward to check that $f \cdot g = \mathrm{Id}$ and $g \cdot f = \mathrm{Id}$ holds. Hence, it is enough to show that $g(\mathrm{Ker}(\overline{\partial}_E + \theta)^* \cap \mathrm{Ker}(\overline{\partial}_E + \theta)) \subset \mathrm{Ker}D_1^* \cap \mathrm{Ker}D_2$ to prove the claim.

Let $A \in \mathrm{Ker}(\overline{\partial}_E + \theta)^* \cap \mathrm{Ker}(\overline{\partial}_E + \theta)$. From [BH1], we have $(\overline{\partial}_E + \theta)^* = (\nabla^{0,1}_h + \sqrt{-1}\Phi^{1,0})^* = \sqrt{-1}[\Lambda, \nabla^{1,0}_h + \sqrt{-1}\Phi^{0,1}]$. Since $A^{2,0}_B(M) = A^{0,2}_B(M) = 0$, A satisfies the following equations

$$\nabla^{0,1}_h A^{1,0} + \sqrt{-1}[\Phi^{1,0}, A^{0,1}] = 0, \tag{3.27}$$
$$\Lambda(\nabla^{1,0}_h A^{0,1} + \sqrt{-1}[\Phi^{0,1}, A^{1,0}]) = 0. \tag{3.28}$$

We wedge $d\eta \mathrm{Id}_E$ to the second equation and we obtain

$$\nabla^{1,0}_h A^{0,1} + \sqrt{-1}[\Phi^{0,1}, A^{1,0}] = 0. \tag{3.29}$$

We take the formal adjoint of (3.27) and (3.29) with respect to h and obtain

$$\nabla_h^{1,0}(A^{1,0})_h^\dagger - \sqrt{-1}[\Phi^{0,1},(A^{0,1})_h^\dagger] = 0, \tag{3.30}$$

$$\nabla_h^{0,1}(A^{0,1})_h^\dagger - \sqrt{-1}[\Phi^{1,0},(A^{1,0})_h^\dagger] = 0. \tag{3.31}$$

We now prove that $g(A) \in \mathrm{Ker}D_1^* \cap \mathrm{Ker}D_2$. We only prove that $g(A) = (\frac{A-A_h^\dagger}{2}, -\sqrt{-1}\frac{A+A_h^\dagger}{2})$ satisfies (3.22). The other can be proved by using the same argument below.

$$\begin{aligned}&\nabla_h\left(\frac{A-A_h^\dagger}{2}\right) + \sqrt{-1}\left[\Phi, \frac{A+A_h^\dagger}{2}\right]\\ =&\nabla_h^{1,0}\left(\frac{A^{0,1}-(A^{1,0})_h^\dagger}{2}\right) + \nabla_h^{0,1}\left(\frac{A^{1,0}-(A^{0,1})_h^\dagger}{2}\right) + \sqrt{-1}\left[\Phi^{1,0}, \frac{A^{0,1}+(A^{1,0})_h^\dagger}{2}\right] + \sqrt{-1}\left[\Phi^{0,1}, \frac{A^{1,0}+(A^{0,1})_h^\dagger}{2}\right]\\ =&0.\end{aligned}$$

The last equation follows from (3.27), (3.29), (3.30) and (3.31). □

From now on, we assume that $(M,(T^{1,0},S,I),(\eta,\xi))$ is regular.
Let

$$\varphi : \mathbb{R}\times M \to M, (t,x) \mapsto \varphi_t(x)$$

be the flow generated by the Reeb vector field. Let E be a basic vector bundle of rank r. Then by [BH2], we can define a natural action

$$\Phi : \mathbb{R}\times E \to E, (t,e) \mapsto \Phi_t(e)$$

such that they are compatible with the natural projection $p_E : E \to M$ (i.e. $p_E \circ \Phi_t = \varphi_t$). Since M is regular, the flow $\varphi : \mathbb{R}\times M \to M$ induces a free smooth action $\psi : S^1 \times M \to M$. This is equivalent to the existence of a positive number $r \in \mathbb{R}$ such that $\varphi_r(x) = 1$ for all $x \in M$. The minimum of all such r_{Min} is called the period of M. We assume $r_{Min} = 1$ for simplicity. We say that E is *quasiregular* if $\Phi : \mathbb{R}\times E \to E$ induces a S^1-action $\Psi : S^1 \times E \to E$. This is equivalent to the existence of a positive integer m such that $\Phi_m = \mathrm{Id}_E$. We say that E is *regular* if it is quasi-regular and $m = 1$.

Let $(E,\overline{\partial}_E,\theta)$ be a basic Higgs bundle. We say that it is a regular basic Higgs bundle if E is regular. We recall that there is a one-on-one correspondence between a regular basic Higgs bundle on M and a Higgs bundle over M/S^1 following [BH2]. Note that since M is regular, M/S^1 is a Riemann surface. From now on we assume the genus of M/S^1 is bigger than 2.

We first review the construction of a regular basic Higgs bundle over M from a Higgs bundle over M/S^1. Let $(\widetilde{E},\overline{\partial}_{\widetilde{E}},\widetilde{\theta})$ be a Higgs bundle over M/S^1. Let $\{U_\alpha\}_{\alpha\in A}$ be a open covering of M/S^1. We assume that $\widetilde{E}$ is trivialized over each U_α. Then we have a family of holomorphic transition function $\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r,\mathbb{C})$ such that it satisfies the 1-cocycle condition. Since M is the total space of a S^1-bundle over M/S^1, we can regard $\{U_\alpha \times S^1\}_{\alpha\in A}$ as an open covering of M. We define a family of maps $g_{\alpha\beta} : U_\alpha\times S^1 \cap U_\beta \times S^1 \to GL(r,\mathbb{C})$ as $g_{\alpha\beta}(x,t) := \widetilde{g}_{\alpha\beta}(x)$. This family defines a vector bundle E over M since it satisfies the 1-cocycle condition. Since E is trivialized over each $U_\alpha \times S^1$, E is regular and since the transition function is constant along the S^1-action, E is basic and finally, since $\widetilde{g}_{\alpha\beta}$ is holomorphic, E is basic holomorphic. We can also show that $\overline{\partial}_{\widetilde{E}}$ induces a basic holomorphic structure $\overline{\partial}_E$ as follows: We assume that $\overline{\partial}_{\widetilde{E}}|_{U_\alpha} = \overline{\partial} + \widetilde{A}_\alpha$ where $\widetilde{A}_\alpha \in A^{0,1}(\mathfrak{gl}(r,\mathbb{C}))$. Then $\widetilde{A}_\beta = \widetilde{g}_{\alpha\beta}^{-1}\widetilde{A}_\alpha\widetilde{g}_{\alpha\beta} + \widetilde{g}_{\alpha\beta}^{-1}\overline{\partial}\widetilde{g}_{\alpha\beta}$ holds. We define $A_\alpha \in A_B^{0,1}(\mathfrak{gl}(r,\mathbb{C}))$ as $A_\alpha(x,t) := \widetilde{A}_\alpha(x)$. This satisfies $A_\beta = g_{\alpha\beta}^{-1}A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1}\overline{\partial}_\xi g_{\alpha\beta}$. Here $\overline{\partial}_\xi$ is the $(0,1)$-part of $d|_{A_B^\bullet(M)}$. Hence $\{A_\alpha\}_{\alpha\in A}$ defines a $(0,1)$-differential operator $\overline{\partial}_E$ and hence a basic holomorphic structure on E. We can also show that $\widetilde{\theta}$ induces a basic Higgs field θ by using a similar argument.

We next review the converse construction. Let $(E,\overline{\partial}_E,\theta)$ be a regular basic Higgs bundle over M. Since M is the total space of a S^1-bundle over M/S^1 there exist an open cover of $\{U_\alpha\}_{\alpha\in A}$ of M/S^1 such that $\{U_\alpha\times S^1\}_{\alpha\in A}$ is an open cover of M. Since E is regular, we may assume E is trivialized over each $\{U_\alpha\times S^1\}_{\alpha\in A}$ after shrinking U_α appropriately. Since E is basic, the transition function $g_{\alpha\beta} : U_\alpha\times S^1 \cap U_\beta\times S^1 \to GL(r,\mathbb{C})$ of E is constant along S^1. Hence $g_{\alpha\beta}$ reduces to the function on $U_\alpha \cap U_\beta$ and defines a vector bundle $\widetilde{E}$ on M/S^1. We can use a similar argument above to show that $\overline{\partial}_E$ and θ reduces to $\widetilde{E}$ and define a holomorphic structure $\overline{\partial}_{\widetilde{E}}$ and a Higgs field $\widetilde{\theta}$ on $\widetilde{E}$.

We now assume E is regular. We show that there exists a one-on-one correspondence between the space of basic sections $A_B(E)$ over M and smooth sections $A(\widetilde{E})$ over M/S^1.

Let $s \in A_B(E)$. Then $s_\alpha := s|_{U_\alpha} = (s_{\alpha,1}, \dots, s_{\alpha,r}) : U_\alpha \times S^1 \to \mathbb{C}^r$ is a basic function. Hence s_α reduces to a function $\widetilde{s}_\alpha : U_\alpha \to \mathbb{C}^r$. We can glue $\{\widetilde{s}_\alpha\}_{\alpha \in A}$ and define a smooth section s over $\widetilde{E}$. Conversely, let $\widetilde{s} \in A(\widetilde{E})$. Then $\widetilde{s}_\alpha := \widetilde{s}|_{U_\alpha} = (\widetilde{s}_{\alpha,1}, \dots, \widetilde{s}_{\alpha,r}) : U_\alpha \to \mathbb{C}^r$ is a smooth function. We define $s_\alpha : U_\alpha \times S^1 \to \mathbb{C}^r$ as $s_\alpha(x,t) := \widetilde{s}_\alpha(x)$. We can glue $\{s_\alpha\}_{\alpha \in A}$ and define a smooth section s over E. Since s_α is constant along S^1, s is basic. We define linear maps

$$p : A_B(E) \to A(\widetilde{E}),$$
$$q : A(\widetilde{E}) \to A_B(E)$$

as $p(s) := \widetilde{s}$ and $q(\widetilde{s}) := s$. $p \circ q = q \circ p = Id$ is clear from the construction.

Proposition 3.5.9. *Let $(E, \overline{\partial}_E, \theta)$ be a regular basic Higgs bundle over M and $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta})$ be the induced Higgs bundle over M/S^1. Then p,q induces a morphism between complexes*

$$p : (A^\bullet_B(E), \overline{\partial}_E + \theta) \to (A(\widetilde{E}), \overline{\partial}_{\widetilde{E}} + \widetilde{\theta}),$$
$$q : (A(\widetilde{E}), \overline{\partial}_{\widetilde{E}} + \widetilde{\theta}) \to (A^\bullet_B(E), \overline{\partial}_E + \theta).$$

Since $p \circ q = q \circ p = Id$, p and q induce an isomorphism between the cohomologies. In particular, the dimensions of the cohomologies of the two complexes are the same.

Proof. This is clear from the construction of p, q and $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta})$. □

Let h be a basic hermitian metric and let $(\nabla_h, \Phi) \in \mathcal{A}^{\mathrm{irr}}_{\mathrm{BaHit}}$. Then $(E, \overline{\partial}_E := \nabla^{0,1}_h, \theta := \sqrt{-1}\Phi^{1,0})$ is a regular stable Higgs bundle (See Section 3.6.1). Since h is basic and E is regular, we can show that h induces a metric $\widetilde{h}$ on $\widetilde{E}$ by using the trivialization above. It is clear from the construction that $\widetilde{h}$ is a harmonic metric for $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta})$. Hence $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta})$ is polystable and degree 0. Assume that $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta})$ is not stable. Then by [S1, Proposition 3.3], there exists a sub Higgs bundle $\widetilde{V} \subset \widetilde{E}$ such that $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta}) = (\widetilde{V}, \overline{\partial}_{\widetilde{V}}, \widetilde{\theta}_V) \oplus (\widetilde{V}^\perp, \overline{\partial}_{\widetilde{V}^\perp}, \widetilde{\theta}_{\widetilde{V}^\perp})$ holds and both Higgs bundles are stable and degree 0. Here $\widetilde{V}^\perp$ is the orthogonal bundle of $\widetilde{V}$. By applying the above procedure to $\widetilde{V}$, we obtain a sub Higgs bundle $V \subset E$. The harmonic metric $\widetilde{h}|_{\widetilde{V}}$ of $\widetilde{V}$ induces a harmonic metric h_V on V. Hence $(V, \overline{\partial}_V, \theta_V)$ is degree 0. This contradicts to the stability of $(E, \overline{\partial}_E, \theta)$ and hence $(\widetilde{E}, \overline{\partial}_{\widetilde{E}}, \widetilde{\theta})$ is stable.

Let $\mathbb{H}^1_{\mathrm{Dol}}$ be the first cohomology of the folllowing complex

$$0 \longrightarrow A(\mathrm{End}\widetilde{E}) \overset{\overline{\partial}_{\mathrm{End}\widetilde{E}}+\widetilde{\theta}}{\longrightarrow} A^1(\mathrm{End}\widetilde{E}) \overset{\overline{\partial}_{\mathrm{End}\widetilde{E}}+\widetilde{\theta}}{\longrightarrow} A^2(\mathrm{End}\widetilde{E}) \longrightarrow 0.$$

Since E is regular, the dual $E^\vee$ is also regular, and so is End$E$. We can apply Proposition 3.5.9 to End$E$ and obtain $\mathbb{H}^1_{\mathrm{BaDol}} \simeq \mathbb{H}^1_{\mathrm{Dol}}$. By [N], $\dim_{\mathbb{C}} \mathbb{H}^1_{\mathrm{Dol}} = 2(\mathrm{rk}\widetilde{E})^2(g-1) + 2$. Then combining Corollary Proposition 3.5.8 and 3.5.9, we obtain

Theorem 3.5.2. *Let $(M, (T^{1,0}, S, I), (\eta, \xi))$ be a regular Sasakian threefold. Let E be a regular basic bundle and h be a basic Hermitian metric. Let g be the genus of M/S^1. Then*

$$\dim_{\mathbb{R}} \mathcal{M}^{\mathrm{irr}}_{\mathrm{BaHit}} = 4(\mathrm{rk}E)^2(g-1) + 4.$$

3.6 Appendix

3.6.1 Basic Higgs bundle

Throughout this section, let $(M, (T^{1,0}, S, I), (\eta, \xi))$ be a compact Sasakian manifold.

Let E be a basic vector bundle over M. We say that E is transverse holomorphic if there exists a local trivialization $\{U_\alpha\}_{\alpha\in A}$ of E such that the associated transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_r(\mathbb{C})$ is basic and holomorphic (i.e. $i_\xi dg_{\alpha\beta} = 0$ and $\overline{\partial}_\xi g_{\alpha\beta} = 0$). For a transversely holomorphic vector bundle E over M, we define the Dolbeult operator

$$\overline{\partial}_E : A_B(E) \to A_B^{0,1}(E)$$
$$\overline{\partial}_E|_{U_\alpha} := \overline{\partial}_\xi.$$

This is well defined since the transition function is holomorphic and satisfies $\overline{\partial}_E\overline{\partial}_E = 0$. It is canonically extended to $\overline{\partial}_E : A_B^{p,q}(E) \to A_B^{p,q+1}(E)$ and satisfies the Leibniz rule:

$$\overline{\partial}_E(\omega \wedge s) = \overline{\partial}_\xi\omega \wedge s + (-1)^{p+q}\omega \wedge \overline{\partial}_E s.$$

Conversely, if we have an operator $\overline{\partial}_E : A_B^{p,q}(E) \to A_B^{p,q+1}(E)$ such that it satisfies $\overline{\partial}_E\overline{\partial}_E = 0$ and the Leibniz rule, $\overline{\partial}_E$ defines a transverse holomorphic structure by the Frobenius theorem ([Ko]).

Definition 3.6.1. *Let $(M, (T^{1,0}, S, I), (\eta, \xi))$ be a compact Sasakian manifold. A basic Higgs bundle $(E, \overline{\partial}_E, \theta)$ over X is a pair such that*

- *E is basic and $(E, \overline{\partial}_E)$ is a transverse holomorphic bundle.*
- *$\theta \in A_B^{1,0}(\mathrm{End}E)$, $\overline{\partial}_E\theta = 0$, and $\theta \wedge \theta = 0$.*

We call θ a Higgs fields.

Let $(E, \overline{\partial}_E, \theta)$ be a basic Higgs bundle on M and h be a basic hermitian metric.

We define a connection $\nabla_h : A(E) \to A^1(E)$ as follows: Let $e_{1,\alpha}, \dots, e_{r,\alpha}$ be a local holomorphic frame of E on U_α and $H_\alpha := (h(e_{i,\alpha}, e_{j,\alpha})_{1\leqslant i,j\leqslant r})$. We define

$$\nabla_h|_{U_\alpha} := d + H_\alpha^{-1}\partial_\xi H_\alpha.$$

This is well defined and since h is basic, ∇_h is a basic connection. ∇_h is also a h-unitary connection. Note that $\nabla_h^{0,1} = \overline{\partial}_E$.

Let $\theta_h^\dagger$ be the formal adjoint of θ: For every section $u, v \in A(E)$,

$$h(\theta u, v) = h(u, \theta_h^\dagger v)$$

holds. We define a connection $D_h := \nabla_h + \theta + \theta_h^\dagger$. This is a basic connection. Let F_{D_h} be the curvature of D_h. We say that h is *Hermitie-Einstein* if

$$\Lambda F_{D_h}^\perp = 0.$$

Here $\Lambda F_{D_h}^\perp$ is the trace-free part of F_{D_h}.

The existence of Hermitie-Einstein metric is related to the stability of the Higgs bundle. We now recall the them following [BHe2, BS].

Let $(E, \overline{\partial}_E, \theta)$ be a basic Higgs bundle on M. Let $\mathcal{O}_B$ be the sheaf of basic holomorphic functions and $\mathcal{O}_B(E)$ be the sheaf of basic holomorphic sections of E. A *sub Higgs sheaf* of $(E, \overline{\partial}_E, \theta)$ is a coherent $\mathcal{O}_B$-subsheaf $\mathcal{V}$ of $\mathcal{O}_B(E)$ such that $\theta(\mathcal{V}) \subset \mathcal{V} \otimes \Omega_B^1$. Here Ω_B^1 is the sheaf of basic holomorphic 1-form. By [BHe2], if $\mathrm{rk}\mathcal{V} < \mathrm{rk}E$ and $\mathcal{O}_B(E)/\mathcal{V}$ is torsion-free, then there is a transversely analytic sub-variety $S \subset M$ of complex co-dimension at least 2 such that $\mathcal{V}|_{M\setminus S}$ is a transverse holomorphic bundle on $M\setminus S$. We define the degree of $\mathcal{V}$ as the degree of $\mathcal{V}|_{M\setminus S}$.

Definition 3.6.2. *A basic Higgs bundle $(E, \overline{\partial}_E, \theta)$ is stable if*

- *E admits a basic hermitian metric h.*
- *For every sub-Higgs sheaf $\mathcal{V} \subset \mathcal{O}_B(E)$ such that $\mathrm{rk}\mathcal{V} < \mathrm{rk}E$ and $\mathcal{O}_B(E)/\mathcal{V}$ is torsion-free,*

  $$\frac{\deg(\mathcal{V})}{\mathrm{rk}\mathcal{V}} < \frac{\deg(E)}{\mathrm{rk}E}.$$

  holds.

We say that $(E, \overline{\partial}_E, \theta)$ *is polystable if*

$$(E, \overline{\partial}_E, \theta) = \bigoplus_i (E_i, \overline{\partial}_{E_i}, \theta_i)$$

where each $(E_i, \overline{\partial}_{E_i}, \theta_i)$ *is stable and*

$$\frac{\deg(E)}{\mathrm{rk}E} = \frac{\deg(E_i)}{\mathrm{rk}E_i}.$$

Proposition 3.6.1 ([BH1, Theorem 5.2, Proposition 5.3.])**.** *For a stable basic Higgs bundle* $(E, \overline{\partial}_E, \theta)$ *over a compact Sasakian manifold* $(M, (T^{1,0}, S, I), (\eta, \xi))$*, there exsit a basic hermitian metric* h *such that* D_h *satisfies*

$$\Lambda F^{\perp}_{D_h} = 0.$$

Note that h *is a Hermite-Einstein metric.*

Moreover, if $c_{1,B}(E) = c_{2,B}(E) = 0$*, then* D_h *is flat (i.e.* $F_{D_h} = 0$*).*

If we assume some conditions for the degree of the bundle, we have the converse.

Proposition 3.6.2 ([BHe2, Theorem 4.7.],[BH1, Proposition 7.1.])**.** *Let* $(E, \overline{\partial}_E, \theta)$ *be a basic Higgs bundle over a compact Sasakian manifold* $(M, (T^{1,0}, S, I), (\eta, \xi))$ *with a* $\deg(E) = 0$*. Suppose that* h *is a basic Hermitian metric on* E *with* $\Lambda F_{D_h} = 0$*. Then* $(E, \overline{\partial}_E, \theta)$ *is a direct sum of stable basic Higgs bundles of degree zero.*

Basic Higgs bundles and Basic Hitchin equation

In this section, we clarify the relation between a stable basic Higgs bundle and an irreducible basic Hitchin pair.

Let $(\nabla_h, \Phi) \in \mathcal{A}^{\mathrm{irr}}_{\mathrm{BaHit}}$. $(E, \nabla^{0,1}_h, \sqrt{-1}\Phi^{1,0})$ is a basic Higgs bundle. We show that this Higgs bundle is stable with degree 0. Since $\Phi \in A^1_B(\mathfrak{u}(E))$, we have

$$\Phi^{0,1} = -(\Phi^{1,0})^{\dagger}_h.$$

Here $(\Phi^{1,0})^{\dagger}_h$ is the formal adjoint of $\Phi^{1,0}$. Since ∇_h is a metric connection and $\nabla^{0,1}_h \Phi^{1,0} = 0$, we have

$$\nabla^{1,0}_h \Phi^{0,1} = -\nabla^{1,0}_h (\Phi^{1,0})^{\dagger}_h = 0.$$

Hence $D = \nabla_h + \sqrt{-1}\Phi$ is a flat bundle and $\deg(E) = 0$. Stability of $(E, \nabla^{0,1}_h, \sqrt{-1}\Phi^{1,0})$ follows form Proposition 3.6.2 and irreducibilty of (∇_h, Φ).

Let $(E, \overline{\partial}_E, \theta)$ be a stable basic Higgs bundle of degree 0. Then by Proposition 3.6.1, there exists a basic hermitian metric h such that the connection $D = \nabla_h + \theta + \theta^{\dagger}_h$ is flat. Let $\Phi := -\sqrt{-1}(\theta + \theta^{\dagger}_h)$. Then (∇_h, Φ) is an irreducible Hitchin pair.

Chapter 4

Harmonic Bundles with Symplectic Structures

4.1 Abstract of Chapter 4

We study harmonic bundles with an additional structure called symplectic structure. We study them for the case of the base manifold is compact and non-compact. For the compact case, we show that a harmonic bundle with a symplectic structure is equivalent to principle $\mathrm{Sp}(2n, \mathbb{C})$-bundle with a reductive flat connection. For the non-compact case, we show that a polystable good filtered Higgs bundle with a perfect skew-symmetric pairing is equivalent to a good wild harmonic bundle with a symplectic structure.

4.2 Introduction of Chapter 4

4.2.1 Harmonic bundles on non-compact manifolds

As we explained in the introduction, the study of harmonic bundles for the non-compact case was initiated in [S1, S2]. Simpson studied them on curves and when the Higgs field has the singularity called *tame*. He established the non-Abeian Hodge Correspondence (or Kobayashi-Hitchin correspondence) in this case. In [BB], Biquard-Boalch studied the harmonic bundles on curves when the Higgs field admits a singularity called *wild* and proved the correspondence. In [M2, M3], Mochizuki fully generalized the correspondence for the higher dimensional case.

Theorem 4.2.1 ([BB, M2, M3, S1, S2])**.** *Let X be a smooth projective variety, H be a normal crossing divisor of X, and L be an ample line bundle of X. Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle on $X - H$. Then $(\mathcal{P}^h_* E, \theta)$ is a μ_L-polystable good filtered Higgs bundle with $\mu_L(\mathcal{P}^h_* E) = 0$ and $\int_X \mathrm{ch}_2(\mathcal{P}^h_* E) c_1(L)^{\dim X - 2} = 0$.*

Conversely, let $(\mathcal{P}_* \mathcal{V}, \theta)$ be a μ_L-polystable good filtered Higgs bundle satisfying the following vanishing condition:

$$\mu_L(\mathcal{P}_* \mathcal{V}) = 0, \int_X \mathrm{ch}_2(\mathcal{P}_* \mathcal{V}) c_1(L)^{\dim X - 2} = 0. \tag{4.1}$$

Let $(E, \overline{\partial}_E, \theta)$ be the Higgs bundle which we obtain from the restriction of $(\mathcal{P}_* \mathcal{V}, \theta)$ to $X - H$. Then there exists a pluri-harmonic metric h for $(E, \overline{\partial}_E, \theta)$ such that $(\mathcal{V}, \theta)|_{X \setminus H} \simeq (E, \theta)$ extends to $(\mathcal{P}_* \mathcal{V}, \theta) \simeq (\mathcal{P}^h_* E, \theta)$.

4.2.2 Harmonic Bundles with Symplectic Structures

Results

We first state the main results of this chapter. Let X be a smooth projective variety over $\mathbb{C}$ and $H \subset X$ be a normal crossing divisor.

Theorem 4.2.2 (Theorem 4.5.1)**.** *The following objects are equivalent on (X,H)*

- *Good wild harmonic bundles with a symplectic structure.*
- *Good filtered polystable Higgs bundles equipped with a perfect skew-symmetric pairing satisfying the vanishing condition (4.1).*

Symplectic structures for harmonic bundles are defined in Section 4.3. Roughly speaking, it is a skew-symmetric pairing of a vector bundle and is compatible with the Higgs field and the metric. Notions of filtered bundles and pairings of them are recalled in Section 4.4 and 4.5.

We explain the outline of the proof of Theorem 4.2.2 in the next section. We can regard this result as a Kobayashi-Hitchin correspondence with skew-symmetry.

Outline of proof

The contents here are written in Section 4.4 and 4.5.

Let X be a smooth projective variety and H be a normal crossing divisor. Let $(E,\overline{\partial}_E,\theta,h)$ be a good wild harmonic bundle on $X-H$ and $(\mathcal{P}_*\mathcal{V},\theta)$ be a good filtered Higgs bundle on (X,H). In the latter half of this chapter, we study the good wild harmonic bundles and good filtered Higgs bundles when they admit a symplectic structure and a perfect skew-symmetric pairing.

A perfect skew-symmetric pairing ω on $(\mathcal{P}_*\mathcal{V},\theta)$ is a morphism of filtered bundle

$$\omega:\mathcal{P}_*\mathcal{V}\otimes\mathcal{P}_*\mathcal{V}\to\mathcal{P}_*^{(0)}(\mathcal{O}_X(*H))$$

such that it is skew-symmetric and induces an isomorphism $\Psi_\omega:(\mathcal{P}_*\mathcal{V},\theta)\to(\mathcal{P}_*\mathcal{V}^\vee,-\theta^\vee)$. See Section 4 for more details on the pairing of filtered bundles.

In section 4.5.2, we show that when the good wild harmonic bundle admits a symplectic structure, then the good filtered Higgs bundle obtained by prolongation admits a perfect skew-symmetric pairing:

Proposition 4.2.1 (Proposition 4.5.1)**.** *Let $(E,\overline{\partial}_E,\theta,h)$ be a good wild harmonic bundle equipped with symplectic structure ω. Then $(\mathcal{P}^h_*E,\theta)$ is a μ_L-polystable good filtered Higgs bundle equipped with a perfect skew-symmetric pairing ω and satisfies the vanishing condition (4.1).*

We show that the converse also holds. In section 4.5.3, we study the structure of a good filtered Higgs bundle with a perfect skew-symmetric pairing and show that it admits a pluri-harmonic metric compatible with pairings. This completes the proof of Theorem 4.2.2.

Proposition 4.2.2 (Proposition 4.5.2 and 4.5.3)**.** *Let $(\mathcal{P}_*\mathcal{V},\theta)$ be a μ_L-polystable good filtered Higgs bundle equipped with perfect skew-symmetic pairing ω and satisfies the vanishing condition (4.1). Then there exist stable Higgs bundles $(\mathcal{P}_*\mathcal{V}_i^{(0)},\theta_i^{(0)})$ $(i=1,\dots,p(0))$, $(\mathcal{P}_*\mathcal{V}_i^{(1)},\theta_i^{(1)})$ $(i=1,\dots,p(1))$ and $(\mathcal{P}_*\mathcal{V}_i^{(2)},\theta_i^{(2)})$ $(i=1,\dots,p(2))$ of degree 0 on X such that the following holds.*

- $(\mathcal{P}_*\mathcal{V}_i^{(0)},\theta_i^{(0)})$ *is equipped with a symmetric pairing* $P_i^{(0)}$.
- $(\mathcal{P}_*\mathcal{V}_i^{(1)},\theta_i^{(1)})$ *is equipped with a skew-symmetric pairing* $P_i^{(1)}$.
- $(\mathcal{P}_*\mathcal{V}_i^{(2)},\theta_i^{(2)})\not\simeq(\mathcal{P}_*\mathcal{V}_i^{(2)},-\theta_i^{(2)})^\vee$.
- *There exists positive integers $l(a,i)$ and an isomorphism*

$$(\mathcal{P}_*\mathcal{V},\theta)\simeq\bigoplus_{i=1}^{p(0)}(\mathcal{P}_*\mathcal{V}_i^{(0)},\theta_i^{(0)})\otimes\mathbb{C}^{2l(0,i)}\oplus\bigoplus_{i=1}^{p(1)}(\mathcal{P}_*\mathcal{V}_i^{(1)},\theta_i^{(1)})\otimes\mathbb{C}^{l(1,i)}$$
$$\oplus\bigoplus_{i=1}^{p(2)}\Big(\big((\mathcal{P}_*\mathcal{V}_i^{(2)},\theta_i^{(2)})\otimes\mathbb{C}^{l(2,i)}\big)\oplus\big((\mathcal{P}_*\mathcal{V}_i^{(2)},-\theta_i^{(2)})^\vee\otimes(\mathbb{C}^{l(2,i)})^\vee\big)\Big).$$

Under this isomorphism, ω is identified with the direct sum of $P_i^{(0)}\otimes\omega_{\mathbb{C}^{2l(0,i)}}$, $P_i^{(1)}\otimes C_{\mathbb{C}^{l(1,i)}}$ and $\widetilde{\omega}_{(E_i^{(2)},\theta_i^{(2)})}\otimes C_{\mathbb{C}^{l(2,i)}}$

- $(\mathcal{P}_*\mathcal{V}_i^{(a)}, \theta_i^{(a)}) \not\simeq (\mathcal{P}_*\mathcal{V}_j^{(a)}, \theta_j^{(a)})$ $(i \neq j)$ *for a=0,1,2, and* $(\mathcal{P}_*\mathcal{V}_i^{(2)}, \theta_i^{(2)}) \not\simeq (\mathcal{P}_*\mathcal{V}_j^{(2)}, -\theta_j^{(2)})^\vee$ *for any* i, j.

Moreover, there exists a harmonic metric h *on* $(\mathcal{V}, \theta)|_{X\setminus D}$ *such that* (i) h *is adapted to* $\mathcal{P}_*\mathcal{V}$, (ii) *it is compatible with* ω.

We give more details on the harmonic metric in Proposition 4.5.3. Theorem 4.2.2 is proved by combining Proposition 4.2.1 and 4.2.2.

Relation to other works

In [LM1], Li and Mochizuki studied harmonic bundles with an additional structure called real structure. A real structure is a holomorphic non-degenerate pairing of the given bundle such that the Higgs field is symmetric with it and the harmonic metric is compatible. Although they focused on the study of generically regular semisimple Higgs bundle, they also obtained the Kobayashi-Hitchin correspondence with symmetry.

Theorem 4.2.3 ([LM1, Theorem 3.28])**.** *Let* X *be a compact Riemann surface and* $D \subset X$ *be a divisor. Then the following objects are equivalent on* (X, D).

- *Wild harmonic bundles on* (X, D) *with a real structure.*
- *Polystable good filtered Higgs bundles of degree 0 equipped with a perfect symmetric pairing.*

Although they only proved for the Riemann surface case, generalization to higher dimensions is straightforward.

In Section 4.3, we study the compact case. In [S3], Simpson established the one-on-one correspondence for reductive flat principal G-bundle and semistable G-Higgs bundle. Here, we assume G to be a complex reductive algebraic Lie group. Hence Section 4.3 is a detailed version for $G = \mathrm{Sp}(2n, \mathbb{C})$.

4.3 Harmonic bundles with symplectic structure

4.3.1 Skew-symmetric pairings of vector spaces

Let V be a complex vector space of dimension n. We fix a hermitian metric h on V. Let $V^\vee$ be the dual of V. From a hermitian metric h we have an anti-linear map:

$$\Psi_h : V \to V^\vee$$

defined as $\Psi_h(u)(v) := h(v, u)$ for $u, v \in V$.

We have an induced hermitian metric $h^\vee$ on $V^\vee$ defined as

$$h^\vee(u^\vee, v^\vee) := h(\Psi_h^{-1}(v^\vee), \Psi_h^{-1}(u^\vee)).$$

Let ω be a non-degenerate skew-symmetric bilinear form on V. We obtain a linear map,

$$\Psi_\omega : V \to V^\vee$$

defined as $\Psi_\omega(u)(v) := \omega(u, v)$.

We have an induced skew-symmetric bilinear form $\omega^\vee$ on $V^\vee$ defined as,

$$\omega^\vee(u^\vee, v^\vee) := \omega(\Psi_\omega^{-1}(u^\vee), \Psi_\omega^{-1}(v^\vee)).$$

Definition 4.3.1. *Let* (V, h) *be a vector space with hermitian metric. Let* ω *be a non-degenerate skew-symmetric bilinear form on* V. ω *is compatible with* (V, h) *if*

$$\Psi_\omega : (V, h) \to (V^\vee, h^\vee)$$

is an isometry.

The following Lemma was proved in [LM1] without proof. We give the proof for convenience.

Lemma 4.3.1. *The following conditions are equivalent*

- *h is compatible with ω.*
- *$\Psi_{h^\vee} \circ \Psi_\omega = \Psi_{\omega^\vee} \circ \Psi_h$.*
- *$\omega(u,v) = \overline{\omega^\vee(\Psi_h(u), \Psi_h(v))}$ for any $u, v \in V$.*

Proof. For a matrix A, we denote the transpose of it as A^T. Let $< e_1, \dots, e_n >$ be a basis of V and $< e_1^\vee, \dots, e_n^\vee >$ be the dual basis of $V^\vee$. Let $H := (h(e_i, e_j))_{1 \le i,j \le n}$, $\Omega := (\omega(e_i, e_j))_{1 \le i,j \le n}$. The representation matrix of Ψ_h is H, $\Psi_{h^\vee}$ is $(H^{-1})^T$, Ψ_ω is Ω^T and $\Psi_{\omega^\vee}$ is Ω^{-1}.

When h is compatible with ω, then $(H^{-1})^T = \Omega^{-1} H \overline{\Omega^{-1}}^T$ stands. $\Psi_{h^\vee} \circ \Psi_\omega = \Psi_{\omega^\vee} \circ \Psi_h$ is equivalent to $(H^{-1})^T \overline{\Omega^T} = \Omega^{-1} H$. The third condition is equivalent to the equality $\Omega^T = \overline{H^T \Omega^{-1} H}$. Hence the three conditions are equivalent. □

4.3.2 Harmonic bundles with symplectic structure

Let X be a complex manifold and $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on X.

Definition 4.3.2. *Let $(E^\vee, \overline{\partial}_{E^\vee})$ be the dual holomorphic bundle of $(E, \overline{\partial}_E)$. A skew-symmetric pairing ω of E is a global holomorphic section of $E^\vee \otimes E^\vee$ such that $\omega(u,v) = -\omega(v,u)$ holds for any section u, v of E. We say that ω is perfect if the induced morphism $\Psi_\omega : E \to E^\vee$ is an isomorphism.*

We note that when a holomorphic bundle has a perfect symplectic pairing, the rank of it is even.

Definition 4.3.3. *A skew-symmetric pairing ω of the Higgs bundle $(E, \overline{\partial}_E, \theta)$ is a skew-symmetric pairing of $(E, \overline{\partial}_E)$ such that $\omega(\theta \otimes \mathrm{Id}) = -\omega(\mathrm{Id} \otimes \theta)$ holds. We call ω perfect if it is a perfect skew-symmetric pairing of $(E, \overline{\partial}_E)$.*

A skew-symmetric pairing ω for $(E, \overline{\partial}_E, \theta)$ induces a morphism $\Psi_\omega : (E, \theta) \to (E^\vee, -\theta^\vee)$. Here $\theta^\vee$ is the Higgs field of $E^\vee$ induced from θ.

Remark 4.3.1. *A Higgs bundle with a skew-symmetric pairing is called* $\mathrm{Sp}(2n, \mathbb{C})$*-Higgs bundle in [GGM].*

Definition 4.3.4. *A symplectic structure ω of the harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ is a perfect skew-symmetric pairing of $(E, \overline{\partial}_E, \theta)$ such that $h_{|P}$ is compatible with $\omega_{|P}$ for any $P \in X$.*

4.3.3 Harmonic metrics on Principal G-bundles

Let G be a Lie group. In this section, we briefly review harmonic metrics on the principal G-bundle. Let X be a Riemannian manifold.

Definition 4.3.5. *Let $P \to X$ be a principal G-bundle and ∇ be a flat connection on it. ∇ is called reductive if the corresponding representation $\rho : \pi_1(M) \to G$ is semisimple.*

Let $K \subset G$ be a maximal compact subgroup and let P_K be a K-reduction of P. When P admits a flat connection ∇, to give a K-reduction P_K is equivalent to give a $\pi_1(X)$-equivalent smooth map

$$f : \widetilde{X} \to G/K.$$

Here $\widetilde{X}$ is the universal covering of X.

The following result was proved by Donaldson [Do] (when X is a compact Riemann surface and $G = \mathrm{SL}(2, \mathbb{C})$), Corlette [Co] (when X is compact and for semisimple Lie groups) and Simpson [S4] (when X is compact and for algebraic reductive groups).

Theorem 4.3.1 ([Co, Do, S4])**.** *Suppose X to be compact. Let $P \to X$ be a principal G-bundle with a flat connection ∇. Then there exists a $\pi_1(X)$-equivalent harmonic map $f : \widetilde{X} \to G/K$ if and only if ∇ is reductive.*

From now on, we assume X to be a compact Kähler manifold. Let $\pi : G \to \mathrm{GL}(V)$ be a linear representation. We briefly recall how to induce a Higgs bundle structure to $E := P \times^{\pi} V$ from a principle G-bundle with a reductive flat connection ∇. See [S4] for details. Let D be the induced flat connection of E. The harmonic map f induces a metric h on E. Let $D = D_h + \phi$ be the decomposition such that D_h is the metric connection and ϕ is self-adjoint w.r.t. h. Let $D_h^{0,1}$ be the (0,1)-part of D_h and θ be the (1,0)-part of ϕ. The harmonicity of f implies that $D_h^{0,1} \circ D_h^{0,1} = 0$ and $D_h^{0,1}\theta = 0$. Hence we obtain a harmonic bundle $(E, D_h^{0,1}, \theta, h)$.

4.3.4 Harmonic bundles and Principal $\mathrm{Sp}(2n, \mathbb{C})$-Bundles

Throughout this section, we assume X to be a compact Kähler manifold. In this section, we prove the following:

Proposition 4.3.1. *Let X be a compact Kähler manifold. The following objects are equivalent on X.*

- *Polystable Higgs bundle of rank 2n with vanishing Chern classes equipped with a perfect skew-symmetric pairing.*
- *Harmonic bundle of rank 2n equipped with a symplectic structure.*
- *Principal* $\mathrm{Sp}(2n, \mathbb{C})$*-bundle with a reductive flat connection.*

Proof. The equivalence of the first two objects is a consequence of Corollary 4.5.1. We give the proof of the equivalence of the last two objects in the end of the section. □

To prove Proposition 4.3.1, we prepare some Propositions.

Lemma 4.3.2. *Let $(E, \overline{\partial}_E)$ be a holomorphic bundle of rank 2n on X and ω be a perfect skew-symmetric pairing of it. Let $P_E \to X$ be the principal* $\mathrm{GL}(2n, \mathbb{C})$*-bundle associated to E. Then P_E has a reduction to $P_{E,\mathrm{Sp}(2n,\mathbb{C})}$ such that $P_{E,\mathrm{Sp}(2n,\mathbb{C})} \to X$ is a principal* $\mathrm{Sp}(2n, \mathbb{C})$*-bundle.*

Proof. To prove the claim, it is enough to prove that there exists an open covering $\{U_i\}_{i\in\Lambda}$ and a family of section $\{(e_{k,i})_{k=1}^{2n}\}_{i\in\Lambda}$ of E such that

- $(e_{k,i})_{k=1}^{2n}$ is a frame of E on U_i,
- The family of transition function $\{g_{ij}\}_{i,j\in\Lambda}$ associated to $\{(e_{k,i})_{k=1}^{2n}\}_{i\in\Lambda}$ takes value in $\mathrm{Sp}(2n, \mathbb{C})$.

To show such an open covering and frames exists, we only have to show that there exists an open covering $\{U_i\}_{i\in\Lambda}$ of X and on each U_i, we have a frame $(e_{k,i})_{k=1}^{2n}$ of E such that w.r.t $(e_{k,i})_{k=1}^{2n}$, $\omega|_{U_i}$ has the form

$$\omega|_{U_i} = \sum_{k=1}^{n} \left(e_{k,i}^{\vee} \otimes e_{k+n,i}^{\vee} - e_{k+n,i}^{\vee} \otimes e_{k,i}^{\vee}\right).$$

Here, $e_{k,i}^{\vee}$ is the dual frame of $e_{k,i}$. We note that

$$\begin{pmatrix} \omega|_{U_i}(e_{1,i}, e_{1,i}) & \dots & \omega|_{U_i}(e_{1,i}, e_{2n,i}) \\ \vdots & \ddots & \vdots \\ \omega|_{U_i}(e_{2n,i}, e_{1,i}) & \dots & \omega|_{U_i}(e_{2n,i}, e_{2n,i}) \end{pmatrix} = \mathcal{J}_n := \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}.$$

Here I_n is the $n \times n$ identity matrix. Once we showed such frames exist, then the transition functions obiously take value in $\mathrm{Sp}(2n, \mathbb{C})$.

We now prove that such frames exist around any $P \in X$. Let U_P be an open neighborhood of P and $(e_k)_{k=1}^{2n}$ be a frame of E on U_P. Since ω is perfect, there exists a $e_k (k \neq 1)$ such that $\omega(e_1, e_k)|_P \neq 0$. We may shrink U_P so that $\omega(e_1, e_k)$ does not take 0 in U_P. We may also permute $(e_k)_{k=1}^{2n}$ so we can assume $\omega(e_1, e_{n+1})$ does not take 0 in U_P. Under this assumption, we construct a new frame $(e'_k)_{k=1}^{2n}$ as

$$e'_1 := e_1,$$
$$e'_{n+1} := -\frac{e_1}{\omega(e_1, e_{n+1})},$$
$$e'_k := e_k - \omega(e_k, e'_{n+1})e'_1 + \omega(e_k, e'_1)e'_{n+1} (k : otherwise).$$

By direct calculation, we can check $\omega(e'_1, e'_{n+1}) = 1$ and $\omega(e'_k, e'_1) = \omega(e'_k, e'_{n+1}) = 0 (k \neq 1, n+1)$. It is easy to see that $(e'_k)_{k=1}^{2n}$ is actually a frame.

By the same argument as above for e'_2, we can assume that $\omega(e'_2, e'_{n+2})$ does not take 0 in U_P. We construct a new frame $(e''_k)_{k=1}^{2n}$ as

$$\begin{aligned} e''_1 &:= e'_1, \\ e''_{n+1} &:= e'_{n+1}, \\ e''_2 &:= e'_2, \\ e''_{n+2} &:= -\frac{e'_2}{\omega(e'_2, e'_{n+2})}, \\ e''_k &:= e'_k - \omega(e'_k, e''_{n+2})e''_2 + \omega(e'_k, e''_2)e''_{n+2} (k : \textit{otherwise}). \end{aligned}$$

By direct calculation, we can check $\omega(e''_i, e''_{n+i}) = 1 (i = 1, 2)$ and $\omega(e''_k, e''_i) = \omega(e''_k, e''_{n+i}) = 0 (i = 1, 2, k \neq 1, 2, n+1, n+2)$. Continuing this procedure, we finally obtain a frame $(\widetilde{e}_k)_{k=1}^{2n}$ on U_P such that

$$\begin{pmatrix} \omega|_{U_P}(\widetilde{e}_1, \widetilde{e}_1) & \dots & \omega|_{U_P}(\widetilde{e}_1, \widetilde{e}_{2n}) \\ \vdots & \ddots & \vdots \\ \omega|_{U_P}(\widetilde{e}_{2n}, \widetilde{e}_1) & \dots & \omega|_{U_P}(\widetilde{e}_{2n}, \widetilde{e}_{2n}) \end{pmatrix} = \mathcal{J}_n.$$

We can construct such a frame around for arbitrary $P \in X$. Hence we proved the claim. □

We set $\mathrm{Sp}(2n) := \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n)$. Here $\mathrm{U}(2n)$ is the set of unitary matrices. $\mathrm{Sp}(2n)$ is a maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{C})$.

Lemma 4.3.3. *Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle of rank 2n on X and ω be a symplectic structure of it. Then the associated principal $\mathrm{Sp}(2n, \mathbb{C})$-bundle $P_{E,\mathrm{Sp}(2n,\mathbb{C})}$ admits a reductive flat connection ∇.*

Proof. Since h is a pluri-harmonic metric, the connection $\nabla_h = \partial_h + \overline{\partial}_E + \theta + \theta_h^\dagger$ is a flat connection. Let $\{U_i\}_{i \in \Lambda}$ and $\{(e_{k,i})_{k=1}^{2n}\}_{i \in \Lambda}$ be the open cover and the frame which we constructed in Proposition 4.3.2. Let $\mathfrak{sp}(2n, \mathbb{C})$ be the Lie algebra of $\mathrm{Sp}(2n, \mathbb{C})$. To prove the claim, first, we show that the connection form of ∇_h w.r.t $(e_{k,i})_{k=1}^{2n}$ is a $\mathfrak{sp}(2n, \mathbb{C})$-valued 1-form on U_i. Once this is shown, since the transition functions of $\{(e_{k,i})_{k=1}^{2n}\}_{i \in \Lambda}$ take value in $\mathrm{Sp}(2n, \mathbb{C})$, we obtain a connection form on $P_{E,\mathrm{Sp}(2n,\mathbb{C})}$ and hence it induces a connection ∇. The flatness of ∇ follows from the flatness of ∇_h. Reductiveness of ∇ follows from h: From Lemma 4.3.4, we know that h defines a $\mathrm{Sp}(2n)$-reduction of $P_{E,\mathrm{Sp}(2n,\mathbb{C})}$. Since ∇ is flat, h induces a map $f_h : \widetilde{X} \to \mathrm{Sp}(2n, \mathbb{C})/\mathrm{Sp}(2n)$. f_h is harmonic since h is a pluri-harmonic metric. Reductiveness of ∇ follows immediately.

Let A_i be the connection form of ∇_h w.r.t. $(e_{k,i})_{k=1}^{2n}$. Let h_i be a $n \times n$ matrix such that

$$h_i := \begin{pmatrix} h|_{U_i}(e_{1,i}, e_{1,i}) & \dots & h|_{U_i}(e_{1,i}, e_{2n,i}) \\ \vdots & \ddots & \vdots \\ h|_{U_i}(e_{2n,i}, e_{1,i}) & \dots & h|_{U_i}(e_{2n,i}, e_{2n,i}) \end{pmatrix}.$$

From the standard argument of the connections, we have

$$A_i = h_i^{-1}\partial h_i + \theta|_{U_i} + \theta_h^\dagger|_{U_i} = h_i^{-1}\partial h_i + \theta|_{U_i} + h_i^{-1}\overline{\theta^T}|_{U_i} h_i.$$

We show that A_i takes value in $\mathfrak{sp}(2n, \mathbb{C})$. First, we show that $\theta|_{U_i}$ takes value in $\mathfrak{sp}(2n, \mathbb{C})$. Recall that the local description of ω w.r.t. $(e_{k,i})_{k=1}^{2n}$ is $\mathcal{J}_n$. Since $\omega(\theta \otimes \mathrm{Id}) = -\omega(\mathrm{Id} \otimes \theta)$ holds,

$$\theta^T|_{U_i}\mathcal{J}_n = -\mathcal{J}_n\theta|_{U_i}$$

holds. Hence we showed it.

We next prove h_i takes value in $\mathrm{Sp}(2n, \mathbb{C})$. Once this is shown, then it is obvious that $\theta_h^\dagger|_{U_i} = h_i^{-1}\overline{\theta^T}|_{U_i} h_i$ takes value in $\mathfrak{sp}(2n, \mathbb{C})$. We also can show that $h_i^{-1}\partial h_i$ takes value in it: Suppose h_i takes value in $\mathrm{Sp}(2n, \mathbb{C})$. Then we have the following

$$h_i^T \mathcal{J}_n h_i = \mathcal{J}_n.$$

Then we have

$$
\begin{aligned}
h_i^T &= -\mathcal{J}_n h_i^{-1} \mathcal{J}_n, \\
h_i &= -\mathcal{J}_n (h_i^{-1})^T \mathcal{J}_n, \\
\partial h_i^T \mathcal{J}_n h_i + h_i^T \mathcal{J}_n \partial h_i &= 0.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
0 &= \partial h_i^T \mathcal{J}_n h_i + h_i^T \mathcal{J}_n \partial h_i \\
&= \partial h_i^T \mathcal{J}_n (-\mathcal{J}_n (h_i^{-1})^T \mathcal{J}_n) + (-\mathcal{J}_n h_i^{-1} \mathcal{J}_n) \mathcal{J}_n \partial h_i \\
&= \partial h_i^T (h_i^{-1})^T \mathcal{J}_n + \mathcal{J}_n h_i^{-1} \partial h_i.
\end{aligned}
$$

Since $(h_i^{-1} \partial h_i)^T = \partial h_i^T (h_i^{-1})^T$, $h_i^{-1} \partial h_i$ takes value in $\mathfrak{sp}(2n, \mathbb{C})$. We now prove h_i takes value in $\mathrm{Sp}(2n, \mathbb{C})$. Let $(e_{k,i}^\vee)_{k=1}^{2n}$ be the dual frame of $(e_{k,i})_{k=1}^{2n}$, $h^\vee$ be the dual metric of h, and $\omega^\vee$ be the dual of ω. Then the matrix realizations of $h^\vee$ w.r.t to $(e_{k,i}^\vee)_{k=1}^{2n}$ is $(h_i^{-1})^T$. Since ω is compatible with h we can use Lemma 4.3.1 and hence we have

$$
(h_i^{-1})^T = \mathcal{J}_n h_i \mathcal{J}_n^T.
$$

Hence we have

$$
\mathcal{J}_n = h_i^T \mathcal{J}_n h_i.
$$

This shows that h_i takes value in $\mathrm{Sp}(2n, \mathbb{C})$. $\square$

Let $M(2n, \mathbb{C})$ be the set of $2n \times 2n$-matrix, $\mathfrak{p} \subset M(2n, \mathbb{C})$ be the set of hermitian matrix, and $\mathfrak{p}_+ \subset \mathfrak{p}$ be the set of positive definite ones. As it is well known the standard exponential map

$$
\exp : \mathfrak{p} \to \mathfrak{p}_+
$$

is a real analytic isomorphism. We set $\log := (\exp)^{-1}$.

Although the following Lemma might be well known to experts, we give the proof for convenience.

Lemma 4.3.4. *Let E be a complex vector bundle, h be a hermitian metric, and ω be a smooth perfect skew-symmetric structure. We assume h is compatible with ω. Under this assumption, h defines a $\mathrm{Sp}(2n)$-reduction $P_{E,\mathrm{Sp}(2n)}$ of $P_{E,\mathrm{Sp}(2n,\mathbb{C})}$.*

Proof. In Proposition 4.3.2, we constructed an open cover $\{U_i\}_{i \in \Lambda}$ and a family of frame $\{(e_{k,i})_{k=1}^{2n}\}_{i \in \Lambda}$ such that its transition functions $\{g_{ij}\}_{i,j \in \Lambda}$ takes value in $\mathrm{Sp}(2n, \mathbb{C})$. We recall that $\{g_{ij}\}_{i,j \in \Lambda}$ constructs $P_{E,\mathrm{Sp}(2n,\mathbb{C})}$. To prove h induces a $\mathrm{Sp}(2n)$-reduction, it is enough to show that on each U_i, h defines a function $s_i : U_i \to \mathrm{Sp}(2n, \mathbb{C})$ such that if $U_i \cap U_j \neq \emptyset$

$$
s_i^{-1}(x) g_{ij}(x) s_j(x) \in \mathrm{Sp}(2n), x \in U_i \cap U_j
$$

holds. Actually, if we set $g'_{ij} = s_i^{-1} g_{ij} s_j$, then it is easy to check that $\{g'_{ij}\}_{i,j \in \Lambda}$ defines a principal $\mathrm{Sp}(2n)$-bundle which is a reduction of $P_{E,\mathrm{Sp}(2n,\mathbb{C})}$.

We now construct s_i. Let h_i be the matrix realization of h w.r.t. $(e_{k,i})_{k=1}^{2n}$ as in Proposition 4.3.3. We showed that h_i takes value in $\mathrm{Sp}(2n, \mathbb{C})$. We set

$$
s_i := \exp\left(\frac{\log h_i}{2} \right).
$$

$\log h_i$ makes sense since h_i is a positive definite hermitian matrix. Since h_i takes value in $\mathrm{Sp}(2n, \mathbb{C})$, $\log h_i$ takes value in $\mathfrak{sp}(2n, \mathbb{C})$. Hence s_i is a $\mathrm{Sp}(2n, \mathbb{C})$-valued smooth function on U_i.

We next show that $s_i^{-1}g_{ij}s_j \in \mathrm{U}(n)$. We show this by direct calculation. Before going to the calculation we note that if $U_i \cap U_i \neq \emptyset$, then $h_i = g_{ij}h_j\overline{g}_{ij}^T$.

$$\begin{aligned}
\overline{s_i^{-1}g_{ij}s_j}^T s_i^{-1}g_{ij}s_j &= s_j\overline{g_{ij}}^T s_i^{-1}s_i^{-1}g_{ij}s_j \\
&= s_j\overline{g_{ij}}^T \exp\Big(-\frac{\log h_i}{2}\Big)\exp\Big(-\frac{\log h_i}{2}\Big)g_{ij}s_j \\
&= s_j\overline{g_{ij}}^T h_i^{-1}g_{ij}s_j \\
&= s_j h_j^{-1}s_j \\
&= s_j\exp\Big(-\frac{\log h_j}{2}\Big)\exp\Big(-\frac{\log h_j}{2}\Big)s_j \\
&= I_n.
\end{aligned}$$

The first equation holds since h_i is hermitian. Since s_i is $\mathrm{Sp}(2n,\mathbb{C})$-valued, $s_i^{-1}g_{ij}s_j$ takes value in $\mathrm{Sp}(2n)$. The claim is proved. □

Let $i : \mathrm{Sp}(2n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{C})$ be the standard representaion of $\mathbb{C}^{2n}$.

Lemma 4.3.5. *Let $P \to X$ be a principal $\mathrm{Sp}(2n,\mathbb{C})$-bundle. Then the associated bundle $E := P \times^i \mathbb{C}^{2n}$ admits a smooth perfect skew-symmetric pairing ω.*

Proof. By the definition of E, we have an open covering $\{U_i\}_{i\in\Lambda}$ of X and on each U_i, we have a frame $(e_{k,i})_{k=1}^{2n}$ of E such that the associated tranisition functions $\{g_{ij}\}_{i,j\in\Lambda}$ takes value in $\mathrm{Sp}(2n,\mathbb{C})$. We define a section ω_i of $E^\vee \otimes E^\vee|_{U_i}$ as

$$\omega_i := \sum_{k=1}^{n}\big(e_{k,i}^\vee \otimes e_{k+n,i}^\vee - e_{k+n,i}^\vee \otimes e_{k,i}^\vee\big).$$

Here, $e_{k,i}^\vee$ is the dual frame of $e_{k,i}$. We note that

$$\begin{pmatrix} \omega_i(e_{1,i},e_{1,i}) & \dots & \omega_i(e_{1,i},e_{2n,i}) \\ \vdots & \ddots & \vdots \\ \omega_i(e_{2n,i},e_{1,i}) & \dots & \omega_i(e_{2n,i},e_{2n,i}) \end{pmatrix} = \mathcal{J}_n.$$

Since the transition function $\{g_{ij}\}_{i,j\in\Lambda}$ takes value in $\mathrm{Sp}(2n,\mathbb{C})$, $\omega_i|_{U_i\cap U_j} = \omega_j|_{U_i\cap U_j}$ holds. Hence we can glue them and construct a global section ω of $E^\vee \otimes E^\vee$ such that $\omega|_{U_i} = \omega_i$. By the local description of ω, it is a smooth perfect skew-symmetric pairing. □

Lemma 4.3.6. *Let $P \to X$ a principle $\mathrm{Sp}(2n,\mathbb{C})$-bundle with a reductive flat connection ∇. Then we obtain a harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ and it has a symplectic structure ω.*

Proof. By the previous proposition, we have a smooth bundle E with a smooth perfect skew-symmetric pairing ω. Since ∇ is a reductive a flat bundle, we have a $\pi_1(X)$-equivalent harmonic map $f : \widetilde{X} \to \mathrm{Sp}(2n,\mathbb{C})/\mathrm{Sp}(2n)$. f induces a hermitian metric h on E and by construction, it is compatible with ω.

Let D_∇ be the flat bundle of E induced by ∇. We have a decomposition $D_\nabla = D_h + \phi$ such that D_h is a metric connection and ϕ is self-adjoint w.r.t. h. Let θ be the $(1,0)$-part of ϕ. Since ϕ is self-adjoint we have the decomposition $\phi = \theta + \theta_h^\dagger$. As we recalled in the previous section, the reductiveness of ∇ implies that $D_h^{0,1} \circ D_h^{0,1} = 0$ and $D_h^{0,1}\theta = 0$. Hence $(E, D_h^{0,1}, \theta, h)$ is a harmonic bundle.

Next, we show that θ is compatible with ω. Let $(e_{k,i})_{k=1}^{2n}$ be the frame that we used in the last proposition, and let A_i be the connection matrix of D_∇ w.r.t. $(e_{k,i})_{k=1}^{2n}$ (i.e. $D_\nabla = d + A_i$ locally). Note that A_i takes value in $\mathfrak{sp}(2n,\mathbb{C})$. We briefly recall how we obtain the decomposition $D_\nabla = D_h + \phi$. Let $D^{1,0}$ (resp. $D^{0,1}$) be the (1,0) (resp. (0,1))-part of D_∇. Let $\delta^{1,0}$ (resp. $\delta^{0,1}$) be the (1,0) (resp. (0,1))-type of the differential operator which makes $D^{1,0} + \delta^{0,1}$ and $D^{0,1} + \delta^{1,0}$ metric connections. D_h and ϕ were defined as follows

$$D_h := \frac{D^{1,0} + D^{0,1} + \delta^{1,0} + \delta^{0,1}}{2}, \phi := \frac{D^{1,0} + D^{0,1} - \delta^{1,0} - \delta^{0,1}}{2}.$$

We note that $\delta^{1,0}$ and $\delta^{0,1}$do exsits and locally they are expressed as

$$\delta^{1,0} = \partial - (A_i^{0,1})_h^\dagger + h_i^{-1}\partial h_i,$$
$$\delta^{0,1} = \overline{\partial} - (A_i^{1,0})_h^\dagger + h_i^{-1}\overline{\partial} h_i.$$

Hence θ has the form

$$\theta = \frac{A_i^{1,0} - (A_i^{0,1})_h^\dagger + h_i^{-1}\partial h_i}{2}.$$

Hence θ takes value in $\mathfrak{sp}(2n, \mathbb{C})$ and therefore it is compatible with ω.

We next prove ω is holomorphic and hence it is a symplectic structure of $(E, D_h^{0,1}, \theta, h)$. We have to show $D_h^{0,1}\omega = 0$. By the construction of D_h we have

$$D_h^{0,1} = \frac{D^{0,1} + \delta^{0,1}}{2}.$$

Let B_i be the connection matrix of $D_h^{0,1}$. From the local description of $\delta^{0,1}$, B_i is a (0,1)-form which takes value in $\mathfrak{sp}(2n, \mathbb{C})$. Hence we have

$$D_h^{0,1}\omega = \overline{\partial}\mathcal{J}_n - B_i^T\mathcal{J}_n - \mathcal{J}_n B_j = 0.$$

The first equality follows from the standard argument of connection (See [Ko], for example). Therefore we proved the claim. ☐

Proof of Proposition 4.3.1. Lemma 4.3.2 and 4.3.3 gives a path from a harmonic bundle with a symplectic structure to a principal $\mathrm{Sp}(2n, \mathbb{C})$-bundle with a reductive flat connection. The inverse path is given by Lemma 4.3.5 and 4.3.6. ☐

4.4 Good filtered Higgs bundles and Good Wild Harmonic bunldes

4.4.1 Filtered sheaves

Let X be a complex manifold and H be a simple normal crossing hypersurface of X. Let $H := \bigcup_{i\in\Lambda} H_\lambda$ be the decomposition such that each H_i is smooth.

Filtered sheaves

For any $P \in H$, a holomorphic coordinate neighborhood $(U_P, z_1, \dots, z_n)$ around P is called admissible if $H_P := H \cap U_P = \bigcup_{i=1}^{l(P)}\{z_i = 0\}$. For admissible coordinate neighborhood, we obtain a map $\rho_P : \{1, \dots, l(P)\} \to \Lambda$ such that $H_{\rho_{P(i)}} \cap U_p = \{z_i = 0\}$. We also obtain a map $\kappa_P : \mathbb{R}^\Lambda \to \mathbb{R}^{l(P)}$ by $\kappa_P(\mathbf{a}) = (a_{\rho(1)}, \dots, a_{\rho(l(P))})$.

Let $\mathcal{O}_X(*H)$ be the sheaf of meromorphic function on X which may have poles along H. Let $\mathcal{V}$ be a torsion free $\mathcal{O}_X(*H)$-module. A filtered sheaf over $\mathcal{V}$ is defined to be a tuple of coherent $\mathcal{O}_X$-submodules $\mathcal{P}_{\boldsymbol{a}}\mathcal{V} \subset \mathcal{V}$ ($\boldsymbol{a} \in \mathbb{R}^\Lambda$) such that

- $\mathcal{P}_{\boldsymbol{a}}\mathcal{E} \subset \mathcal{P}_{\boldsymbol{b}}\mathcal{E}$ if $\boldsymbol{a} \leq \boldsymbol{b}$, i.e. $a_i \leq b_i$ for any $i \in \Lambda$.
- $\mathcal{P}_{\boldsymbol{a}}\mathcal{E} \otimes \mathcal{O}_X(*H) = \mathcal{E}$ for any $\boldsymbol{a} \in \mathbb{R}^\Lambda$.
- $\mathcal{P}_{\boldsymbol{a}+\boldsymbol{n}}\mathcal{E} = \mathcal{P}_{\boldsymbol{a}}\mathcal{E} \otimes \mathcal{O}_X(\sum_{i\in\Lambda} n_i H_i)$ for any $\boldsymbol{a} \in \mathbb{R}^\Lambda$ and for any $\boldsymbol{n} \in \mathbb{Z}^\Lambda$.
- For any $\boldsymbol{a} \in \mathbb{R}^\Lambda$, there exists $\boldsymbol{\epsilon} \in \mathbb{R}^\Lambda_{>0}$ such that $\mathcal{P}_{\boldsymbol{a}+\boldsymbol{\epsilon}}\mathcal{E} = \mathcal{P}_{\boldsymbol{a}}\mathcal{E}$.
- For any $P \in H$, let $(U_P, z_1, \dots, z_n)$ be an admissible coordinate of P. Then $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}|_{U_P}$ depends only on $\kappa_P(\boldsymbol{a})$ for any $\boldsymbol{a} \in \mathbb{R}^\Lambda$.

For any coherent $\mathcal{O}_X(*H)$-submodule $\mathcal{E}' \subset \mathcal{E}$, we obtain a filtered sheaf $\mathcal{P}_*\mathcal{E}'$ over $\mathcal{E}'$ by $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}' = \mathcal{P}_{\boldsymbol{a}}\mathcal{E} \cap \mathcal{E}'$. If $\mathcal{V}'$ is saturated, i.e. $\mathcal{E}'' := \mathcal{E}/\mathcal{E}'$ is torsion-free, then we obtain a filtered sheaf $\mathcal{P}_*\mathcal{E}''$ over $\mathcal{E}''$ by $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}'' := \mathrm{Im}(\mathcal{P}_{\boldsymbol{a}}\mathcal{E} \to \mathcal{E}'')$.

A morphism of filtered sheaves $f : \mathcal{P}_*\mathcal{E}_1 \to \mathcal{P}_*\mathcal{E}_2$ is a morphism of $\mathcal{O}_X(*H)$-modules such that $f(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_1) \subset \mathcal{P}_{\boldsymbol{a}}\mathcal{E}_2$ for any $\boldsymbol{a} \in \mathbb{R}^\Lambda$.

Let $\mathcal{P}_*\mathcal{E}$ be a filtered sheaf on X. For every open subset $U \subset X$, we can induce a filtered sheaf over $\mathcal{E}|_U$ from $\mathcal{P}_*\mathcal{E}$. We denote this filtered sheaf $\mathcal{P}_*\mathcal{E}|_U$. Conversely, let $X = \bigcup_{i\in\Lambda} U_i$ be an open covering. Let $\mathcal{P}_*\mathcal{V}_i$ be a filtered sheaf on U_i. If $\mathcal{P}_*\mathcal{E}_i|_{U_i\cap U_j} = \mathcal{P}_*\mathcal{E}_i|_{U_i\cap U_j}$, we have a unique filtered sheaf $\mathcal{P}_*\mathcal{E}$ on X such that $\mathcal{P}_*\mathcal{E}|_{U_i} = \mathcal{P}_*\mathcal{E}_i$. See [M3, Section 2.1.2] for details of this paragraph.

Filtered Higgs sheaves

Let $\mathcal{E}$ be a torsion-free coherent $\mathcal{O}_X(*H)$-module. A Higgs field $\theta : \mathcal{V} \to \Omega^1_X \otimes \mathcal{V}$ is a $\mathcal{O}_X$-linear morphism of sheaves such that $\theta \wedge \theta = 0$. When $\mathcal{V}$ is equipped with a Higgs field, a sub-Higgs sheaf of $\mathcal{V}'$ is a coherent $\mathcal{O}_X(*H)$-submodule $\mathcal{V}' \subset \mathcal{V}$ such that $\theta(\mathcal{V}') \subset \Omega^1_X \otimes \mathcal{V}'$. A pair of a filtered sheaf $\mathcal{P}_*\mathcal{V}$ over $\mathcal{V}$ and a Higgs field θ of $\mathcal{V}$ is called a filtered Higgs bundle.

4.4.2 μ_L- stability condition for filtered Higgs sheaves

Throughout this section, we assume X to be a smooth projective variety, $H = \bigcup_{i\in\Lambda} H_i$ to be a normal crossing divisor of it, and L to be an ample line bundle.

Slope of filtered sheaves

Let $\mathcal{P}_*\mathcal{E}$ be a filtered sheaf on (X, H). We recall the definition of the first Chern class $c_1(\mathcal{P}_*\mathcal{E})$. Let $\boldsymbol{a} \in \mathbb{R}^\Lambda$. Let η_i be a generic point on H_i. The $\mathcal{O}_{X,\eta_i}$-module $(\mathcal{P}_{\boldsymbol{a}}\mathcal{E})_{\eta_i}$ only depends on a_i which we denote as $\mathcal{P}_{a_i}(\mathcal{E}_{\eta_i})$. We obtain a $\mathcal{O}_{H_i,\eta_i}$-module $\mathrm{Gr}^{\mathcal{P}}_{a_i}(\mathcal{E}_{\eta_i}) := \mathcal{P}_{a_i}(\mathcal{E}_{\eta_i})/\sum_{b_i<a_i}\mathcal{P}_{b_i}(\mathcal{E}_{\eta_i})$. $c_1(\mathcal{P}_*\mathcal{E})$ is defined as

$$c_1(\mathcal{P}_*\mathcal{E}) := c_1(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}) - \sum_{i\in\Lambda}\sum_{a_i-1<a\le a_i} a \cdot \mathrm{rank}\mathrm{Gr}^{\mathcal{P}}_a(\mathcal{E}_{\eta_i}) \cdot [H_i].$$

Here, $[H_i] \in H^2(X, \mathbb{R})$ is the cohomology class induced by H_i.

The slope $\mu_L(\mathcal{P}_*\mathcal{E})$ of a filtered sheaf $\mathcal{P}_*\mathcal{E}$ with respect to L is defined as

$$\mu_L(\mathcal{P}_*\mathcal{E}) = \frac{1}{\mathrm{rank}\mathcal{E}}\int_X c_1(\mathcal{P}_*\mathcal{E}) \cdot c_1(L)^{\dim X - 1}.$$

μ_L-stablity condition

Let $(\mathcal{P}_*\mathcal{E}, \theta)$ be a filtered Higgs bundle over (X, H). We say that $(\mathcal{P}_*\mathcal{E}, \theta)$ is μ_L-stable (resp. μ_L-semistable) if for every sub Higgs sheaf $\mathcal{E}' \subset \mathcal{E}$ such that $0 < \mathrm{rank}\mathcal{E}' < \mathrm{rank}\mathcal{E}$, $\mu_L(\mathcal{P}_*\mathcal{E}') < \mu_L(\mathcal{P}_*\mathcal{E})$ (resp. $\mu_L(\mathcal{P}_*\mathcal{E}') \le \mu_L(\mathcal{P}_*\mathcal{E})$) holds.

We say that $(\mathcal{P}_*\mathcal{E}, \theta)$ is μ_L-polystable if the following two conditions are satisfied

- $(\mathcal{P}_*\mathcal{E}, \theta)$ is μ_L-semistable.
- We have a decomposition $(\mathcal{P}_*\mathcal{E}, \theta) = \bigoplus_i(\mathcal{P}_*\mathcal{E}_i, \theta_i)$ such that each $(\mathcal{P}_*\mathcal{E}_i, \theta_i)$ is μ_L-stable and $\mu_L(\mathcal{P}_*\mathcal{E}) = \mu_L(\mathcal{P}_*\mathcal{E}_i)$ holds.

Canonical decomposition

Let $(\mathcal{P}_*\mathcal{E}_1, \theta_1)$ and $(\mathcal{P}_*\mathcal{E}_2, \theta_2)$ be filtered Higgs bundle on (X, H). We use the following result frequently without mention.

Proposition 4.4.1 ([M2, Lemma 3.10]). *Let $(\mathcal{P}_*\mathcal{E}_i, \theta_i)(i = 1, 2)$ be μ_L-semistable reflexive satrated Higgs sheaves such that $\mu_L(\mathcal{E}_1) = \mu_L(\mathcal{E}_2)$. Assume either one of the following:*

- *One of* $(\mathcal{P}_*\mathcal{E}_i, \theta_i)$ *is* μ_L*-stable and* $\mathrm{rank}\mathcal{E}_1 = \mathrm{rank}\mathcal{E}_2$ *holds.*
- *Both* $(\mathcal{P}_*\mathcal{E}_i, \theta_i)$ *are* μ_L*-stable.*

If there is a non-trivial map $f : (\mathcal{P}_*\mathcal{E}_1, \theta_1) \to (\mathcal{P}_*\mathcal{E}_2, \theta_2)$*, then* f *is an isomorphism.*

The following is straightforward from the above result.

Corollary 4.4.1. *Let* $(\mathcal{P}_*\mathcal{E}, \theta)$ *be a* μ_L*-polystable reflexive satrated Higgs sheaves. Then there exists an unique decomposition* $(\mathcal{P}_*\mathcal{E}, \theta) = \bigoplus_i (\mathcal{P}_*\mathcal{E}_i, \theta_i) \otimes \mathbb{C}^{m(i)}$ *such that (i)* $(\mathcal{P}_*\mathcal{E}_i, \theta_i)$ *are* μ_L*-stable, (ii)* $\mu_L(\mathcal{P}_*\mathcal{E}) = \mu_L(\mathcal{P}_*\mathcal{E}_i)$*, (iii)* $(\mathcal{P}_*\mathcal{E}_i, \theta_i) \not\simeq (\mathcal{P}_*\mathcal{E}_j, \theta_j)$ $(i \neq j)$*. We call the decomposition* $(\mathcal{P}_*\mathcal{E}, \theta) = \bigoplus_i (\mathcal{P}_*\mathcal{E}_i, \theta_i) \otimes \mathbb{C}^{m(i)}$ *the canonical decomposition.*

4.4.3 Filtered bundles

Local case

Let U be an open neighborhood of $0 \in \mathbb{C}^n$. Let $H_{U_i} := U \cap \{z_i = 0\}$ and $H_U := \bigcup_{i=1}^{l} H_{U_i}$. Let $\mathcal{V}$ be a locally free $\mathcal{O}_U(*H)$-module. A filtered bundle $\mathcal{P}_*\mathcal{V}$ is a family of locally free $\mathcal{O}_U$-modules $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}$ indexed by $\boldsymbol{a} \in \mathbb{R}^l$ such that

- $\mathcal{P}_{\boldsymbol{a}}\mathcal{V} \subset \mathcal{P}_{\boldsymbol{b}}\mathcal{V}$ for $\boldsymbol{a} \leq \boldsymbol{b}$.
- There exists a frame $(v_1, \dots, v_r)$ of $\mathcal{V}$ and tuples $a(v_j) \in \mathbb{R}$ $(j = 1, \dots, l)$ such that

$$\mathcal{P}_{\boldsymbol{b}}\mathcal{V} = \bigoplus_{j=1}^{r} \mathcal{O}_U \left(\sum_{i=1}^{l} [b_i - a(v_j)] H_{U_i} \right) v_j.$$

  Here for $c \in \mathbb{R}$, $[c] := \max\{a \leq c | a \in \mathbb{Z}\}$.

Hence locally, a filtered bundle is a filtered sheaf that is locally free and has a frame compatible with filtration.

Pullback of filtered bundles

We use the same notation as in the previous section. Let $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ be a map given by $\varphi(\xi_1, \dots, \xi_n) = (\xi_1^{m_1}, \dots, \xi_l^{m_l}, \xi_{l+1}, \dots, \xi_n)$. We set $U' := \varphi^{-1}(U)$ and $H_{U',i} := \varphi^{-1}(H_{U,i})$. We denote the induced ramified covering $U' \to U$ as φ.

For any $\boldsymbol{b} \in \mathbb{R}^l$, we set $\varphi^*(\boldsymbol{b}) = (m_i b_i) \in \mathbb{R}^l$. Let $\mathcal{P}_*\mathcal{V}$ be a filtered bundle on (U, H_U).

Global case

In this section, we assume X to be a complex manifold and $H = \bigcup_{i \in \Lambda} H_i$ to be a normal crossing divisor of it.

Let $\mathcal{V}$ be a locally free $\mathcal{O}_X(*H)$-module. A filtered bundle $\mathcal{P}_*\mathcal{V}$ over $\mathcal{V}$ is a filtered sheaf over $\mathcal{V}$ such that it is locally written as in Section 4.4.3. We give some examples of filtered bundles.

Let $\mathcal{P}_*\mathcal{V}_1$ and $\mathcal{P}_*\mathcal{V}_2$ be filtered bundles. For $P \in H$, we take an admiisble coordinate neighborhood $(U_P, z_1, \dots, z_n)$ such that each and any $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_i|_{U_P}$ only depends on $\kappa_P(\boldsymbol{a})$. We define filtered bundles $\mathcal{P}_*(\mathcal{V}_1|_{U_P} \oplus \mathcal{V}_2|_{U_P})$, $\mathcal{P}_*\mathcal{V}_1|_{U_P} \otimes \mathcal{P}_*\mathcal{V}_2|_{U_P}$ and $\mathcal{P}_*(\mathcal{H}om(\mathcal{V}_1|_{U_P}, \mathcal{V}_2|_{U_P}))$ on U_P as,

$$\mathcal{P}_{\boldsymbol{a}}(\mathcal{V}_1|_{U_P} \oplus \mathcal{V}_2|_{U_P}) := \mathcal{P}_{\boldsymbol{a}}\mathcal{V}_1|_{U_P} \oplus \mathcal{P}_{\boldsymbol{a}}\mathcal{V}_2|_{U_P},$$

$$\mathcal{P}_{\boldsymbol{a}}(\mathcal{V}_1|_{U_P} \otimes \mathcal{V}_2|_{U_P}) := \sum_{\boldsymbol{c}_1 + \boldsymbol{c}_2 \leq \boldsymbol{a}} \mathcal{P}_{c_1}\mathcal{V}_1|_{U_P} \otimes \mathcal{P}_{c_2}\mathcal{V}_2|_{U_P},$$

$$\mathcal{P}_{\boldsymbol{a}}(\mathcal{H}om(\mathcal{V}_1|_{U_P}, \mathcal{V}_2|_{U_P})) := \left\{ f \in \mathcal{H}om(\mathcal{V}_1|_{U_P}, \mathcal{V}_2|_{U_P}) \mid f(\mathcal{P}_{\boldsymbol{b}}\mathcal{V}_1|_{U_P}) \subset f(\mathcal{P}_{\boldsymbol{a}+\boldsymbol{b}}\mathcal{V}_2|_{U_P})(\forall \boldsymbol{b} \in \mathbb{R}^{l(P)}) \right\}.$$

Here $\boldsymbol{a} \in \mathbb{R}^{l(P)}$. We construct filtered bundles as above around for each $P \in H$. After taking a suitable covering of X, we can glue the filtered bundles and obtain unique filtered bundles $\mathcal{P}_*(\mathcal{V}_1 \oplus \mathcal{V}_2)$, $\mathcal{P}_*(\mathcal{V}_1 \otimes \mathcal{V}_2)$

and $\mathcal{P}_*(\mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2))$ such that $\mathcal{P}_*(\mathcal{V}_1 \oplus \mathcal{V}_2)|_{U_P} = \mathcal{P}_*(\mathcal{V}_1|_{U_P} \oplus \mathcal{V}_2|_{U_P})$, $\mathcal{P}_*(\mathcal{V}_1 \otimes \mathcal{V}_2)|_{U_P} = \mathcal{P}_*(\mathcal{V}_1|_{U_P} \otimes \mathcal{V}_2|_{U_P})$ and $\mathcal{P}_*(\mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2)|_{U_P}) = \mathcal{P}_*(\mathcal{H}om(\mathcal{V}_1|_{U_P}, \mathcal{V}_2|_{U_P}))$ holds for any $P \in H$. We denote these filtered bundles $\mathcal{P}_*\mathcal{V}_1 \oplus \mathcal{P}_*\mathcal{V}_2$, $\mathcal{P}_*\mathcal{V}_1 \otimes \mathcal{P}_*\mathcal{V}_2$ and $\mathcal{H}om(\mathcal{P}_*\mathcal{V}_1, \mathcal{P}_*\mathcal{V}_2)$.

Let $\mathcal{P}_*\mathcal{V}$ be a filtered bundle and let $\mathcal{V}' \subset \mathcal{V}$ be a locally free sub $\mathcal{O}_X(*H)$-module of $\mathrm{rank}\mathcal{V}' < \mathrm{rank}\mathcal{V}$. We obtain a filtered bundle $\mathcal{P}_*\mathcal{V}'$ (See section 4.4.1).

Remark 4.4.1. *Let $\mathcal{V}', \mathcal{V}'' \subset \mathcal{V}$ be locally free subsheaves. We note that even if $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''$ holds, $\mathcal{P}_*\mathcal{V} = \mathcal{P}_*\mathcal{V}' \oplus \mathcal{P}_*\mathcal{V}''$ does not always hold. Here $\mathcal{P}_*\mathcal{V}', \mathcal{P}_*\mathcal{V}''$ is the induced filtration from $\mathcal{P}_*\mathcal{V}$. We say that the $\mathcal{P}_*\mathcal{V}$ is compatible with decomposition if $\mathcal{P}_*\mathcal{V} = \mathcal{P}_*\mathcal{V}' \oplus \mathcal{P}_*\mathcal{V}''$holds.*

We give a very easy example of a filtered bundle that is not compatible with decomposition. Let U be an open neighborhood of $0 \in \mathbb{C}$. Let $\mathcal{V} := \mathcal{O}_U(*0)e_1 \oplus \mathcal{O}_U(*0)e_2$. For every $a \in \mathbb{R}$, we set

$$\mathcal{P}_a\mathcal{V} := \mathcal{O}_U\big([a]0\big)e_1 \oplus \mathcal{O}_U\left(\left[a + \frac{1}{2}\right]0\right)e_2.$$

We set $\mathcal{V}_1 := \mathcal{O}_U(*0)(e_1 + e_2)$ and $\mathcal{V}_2 := \mathcal{O}_U(*0)(e_1 - e_2)$. It is easy to see that $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ holds. Let $\mathcal{P}_*\mathcal{V}_1$ and $\mathcal{P}_*\mathcal{V}_2$ be the induced filtered bundle. The decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ is not compatible with filtration. For example, take $a = \frac{1}{2}$. Then

$$\begin{aligned} \mathcal{P}_{\frac{1}{2}}\mathcal{V} &= \mathcal{O}_U e_1 \oplus \mathcal{O}_U \frac{e_2}{z}, \\ \mathcal{P}_{\frac{1}{2}}\mathcal{V}_1 &= \mathcal{O}_U(*0)(e_1 + e_2) \cap \mathcal{P}_{\frac{1}{2}}\mathcal{V} = \mathcal{O}_U(e_1 + e_2), \\ \mathcal{P}_{\frac{1}{2}}\mathcal{V}_2 &= \mathcal{O}_U(*0)(e_1 - e_2) \cap \mathcal{P}_{\frac{1}{2}}\mathcal{V} = \mathcal{O}_U(e_1 - e_2). \end{aligned}$$

Hence the decomposition is not compatible with the filtration. Obviously, if we set $\mathcal{V}_1' := \mathcal{O}_U(*0)(e_1)$ and $\mathcal{V}_2' := \mathcal{O}_U(*0)(e_2)$, then the decomposition is compatible with the filtration.

Let $\mathrm{rank}\mathcal{V} = r$ and $\mathcal{P}_*\mathcal{V}$ be a filtered bundle over it. We obtain a filtered bundle $\mathcal{P}_*\mathcal{V}^{\otimes r}$ over $\mathcal{V}^{\otimes r}$ as above. We obtain a filtered bundle $\det(\mathcal{P}_*\mathcal{V})$ over $\det\mathcal{V} \subset \mathcal{V}^{\otimes r}$ by the canonical way.

We construct a filtered bundle over $\mathcal{P}^{(0)}_*(\mathcal{O}_X(*H))$ over $\mathcal{O}_X(*H)$. Let $P \in H$ and $(U_P, z_1, \ldots, z_n)$ be the admissible coordinate of P. For $\boldsymbol{a} \in \mathbb{R}^\Lambda$, we define

$$\mathcal{P}^{(0)}_{\boldsymbol{a}}(\mathcal{O}_X(*H))|_{U_P} := \mathcal{O}_X\left(\sum_{i=1}^{l(P)}[\kappa(\boldsymbol{a})_i]H_i\right)$$

here $\kappa(\boldsymbol{a})_i$ is the i-th component of $\kappa(\boldsymbol{a})$ and for $a \in \mathbb{R}$, $[a] := \max\{n \in \mathbb{Z} | n \leq a\}$. We then glue the filtered bundle above and obtain the filtered bundle $\mathcal{P}^{(0)}_*(\mathcal{O}_X(*H))$. Let $\mathcal{P}_*\mathcal{V}$ be a filtered bundle over $\mathcal{V}$. We have a filtered bundle $\mathcal{P}_*\mathcal{V}^\vee := \mathcal{H}om(\mathcal{P}_*\mathcal{V}, \mathcal{P}^{(0)}_*(\mathcal{O}_X(*H))$.

Induced bundles and filtrations

We use the same notation as the previous section.

Let $I \subset \Lambda$ be any subset and $\boldsymbol{\delta}_I \in \mathbb{R}^\Lambda$ be the element such that the j-th component is 0 if $j \in \Lambda \backslash I$ and 1 if $j \in \Lambda$. Let $H_I := \bigcap_{i \in I} H_i$ and $\partial H_I := H_I \backslash \big(\bigcup_{i \in \Lambda} H_i\big)$.

Let $\mathcal{P}_*\mathcal{V}$ be filtered bundle over (X, H). In this section, we introduce some subsheaves of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_I}(\boldsymbol{a} \in \mathbb{R}^\Lambda)$. We use these subsheaves to define Chern characters for $\mathcal{P}_*\mathcal{V}$ in the next section.

Let $i \in \Lambda$. Let $\boldsymbol{a} \in \mathbb{R}^\Lambda$ and for $a_i - 1 < b \leq a_i$, let $\boldsymbol{a}(b, i) := \boldsymbol{a} + (b - a_i)\boldsymbol{\delta}_i$. We want to introduce a filtration on $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_i}$. First, we define ${}^iF_b(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_i})$ as

$${}^iF_b(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_i}) := \mathcal{P}_{\boldsymbol{a}(b,i)}\mathcal{V}|_{H_i}\Big/\mathcal{P}_{\boldsymbol{a}(a_i-1,i)}\mathcal{V}|_{H_i}.$$

This is a locally free $\mathcal{O}_{H_i}$-module and it is a subbundle of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_i}$. Hence ${}^iF_*$ gives a increasing filtration on $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_i}$ indexed by $(a_i - 1, a_i]$.

For general $I \subset \Lambda$, we introduce a family of subbundle of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_I}$. Let $\boldsymbol{a}_I$ be the image of $\boldsymbol{a}$ of the natural projection $\mathbb{R}^\Lambda \to \mathbb{R}^I$. Let $(\boldsymbol{a}_I - \boldsymbol{\delta}_I, \boldsymbol{a}_I] := \prod_{i \in I}(a_i - 1, a_i]$. For any $\boldsymbol{b} \in (\boldsymbol{a}_I - \boldsymbol{\delta}_I, \boldsymbol{a}_I]$, we set

$$^I F_{\boldsymbol{b}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_I}) := \bigcap_{i \in I} {}^i F_{b_i}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_i}).$$

From the local description of filtered bundles, for any $P \in H_I$, there exists a neighborhood X_P of P in X and a non-canonical decomposition

$$\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{X_P \cap H_I} = \bigoplus_{\boldsymbol{b} \in (\boldsymbol{a}_I - \boldsymbol{\delta}_I, \boldsymbol{a}_I]} \mathcal{G}_{P,\boldsymbol{b}}$$

such that the following holds for any $\boldsymbol{c} \in (\boldsymbol{a}_I - \boldsymbol{\delta}_I, \boldsymbol{a}_I]$

$$^I F_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{X_P \cap H_I}) = \bigoplus_{\boldsymbol{b} \leq \boldsymbol{c}} \mathcal{G}_{P,\boldsymbol{b}}.$$

Hence for any $\boldsymbol{c} \in (\boldsymbol{a}_I - \boldsymbol{\delta}_I, \boldsymbol{a}_I]$, we obtain the following locally free $\mathcal{O}_{H_I}$-modules:

$$^I \mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}) := \frac{^I F_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_I})}{\sum_{\boldsymbol{b} \lneq \boldsymbol{c}} {}^I F_{\boldsymbol{b}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_I})}.$$

Here $(b_i) = \boldsymbol{b} \lneq \boldsymbol{c} = (c_i)$ means that $b_i \leq c_i$ for any i and $\boldsymbol{b} \neq \boldsymbol{c}$. We note that $^I \mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V})$ forms a subbundle of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_I}$ on the irreducible component of H_I.

First Chern class and Second Chern class for filtered bundles

We use the same notation as in the previous section.

In this section, we recall the definition of the first Chern class and the second Chern character for filtered bundles. Let $\mathcal{P}_*\mathcal{V}$ be a filtered bundle over (X, H). In Section 4.4.2, we recalled the definition of the first Chren class for filtered sheaves. Since filtered bundles are filtered sheaves, the first Chern class of filtered bundles is defined as follows.

$$c_1(\mathcal{P}_*\mathcal{V}) = c_1(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}) - \sum_{i \in \lambda} \sum_{a_i - 1 < b \leq a_i} b \cdot \mathrm{rank}^i \mathrm{Gr}^F_b(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_i}) \cdot [H_i] \in H^2(X, \mathbb{R}).$$

Let $\mathrm{Irr}(H_i \cap H_j)$ be the set of irreducible components of $H_i \cap H_j$. For $C \in \mathrm{Irr}(H_i \cap H_j)$, let $[C] \in H^4(X, \mathbb{R})$ be the induced cohomology class and let $^C \mathrm{Gr}^F_{(c_i,c_j)}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V})$ be the restriction of $^{(i,j)}\mathrm{Gr}^F_{(c_i,c_j)}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V})$ to C. Let $\iota_{i*} : H^2(H_i, \mathbb{R}) \to H^4(X, \mathbb{R})$ be the Gysin map induced by $\iota_i : H_i \to X$. The second Chern character for filtered bundles is defined as follows.

$$\begin{aligned}\mathrm{ch}_2(\mathcal{P}_*\mathcal{V}) :=& \mathrm{ch}_2(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}) - \sum_{i \in \Lambda} \sum_{a_i - 1 < b \leq a_i} b \cdot \iota_{i*}(c_1(^i \mathrm{Gr}^F_b(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{H_i}))) \\ &+ \frac{1}{2} \sum_{i \in \Lambda} \sum_{a_i - 1 < b \leq a_i} b^2 \cdot \mathrm{rank}(^i \mathrm{Gr}^F_b(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}))[H_i]^2 \\ &+ \frac{1}{2} \sum_{i,j \in \Lambda^2, i \neq j} \sum_{C \in \mathrm{Irr}(H_i \cap H_j)} \sum_{a_i - 1 < c_i \leq a_i, a_j - 1 < c_j \leq a_j} c_i \cdot c_j \mathrm{rank}^C \mathrm{Gr}^F_{(c_i,c_j)}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}) \cdot [C].\end{aligned}$$

4.4.4 Prolongation of vector bundles

Let X be a complex manifold and $H = \cup_{i \in \Lambda} H_i$ be a normal crossing hypersurface. Let $(E, \overline{\partial}_E)$ be a holomorphic vector bundle over $X \backslash H$ and h be a hermitian metric of E. We define a presheaf $\widetilde{\mathcal{P}^h_{\boldsymbol{a}}E}$ on X such that for an open set U of X, $\widetilde{\mathcal{P}^h_{\boldsymbol{a}}E}(U)$ is a set of holomorphic section of E on U which satisfies the following growing condition along $U \cap H$:

- Let $P \in H$ and $(U_P, z_1, \dots, z_n)$ be an admissible neighborhood of P such that $\overline{U}_P \subset U$. Let $\boldsymbol{c} := \kappa_P(\boldsymbol{a})$. A holomorphic section s of E on U is $s \in \mathcal{P}^h_{\boldsymbol{a}}E(U)$ when s satisfies the following estimate on U_P

$$|s|_h \leq O\Big(\prod_{i=1}^{c} |z_i|^{-c_i-\epsilon}\Big)$$

  for any $\epsilon \in \mathbb{R}_{>0}$.

We denote the sheafification of $\widetilde{\mathcal{P}^h_{\boldsymbol{a}}E}$ as $\mathcal{P}^h_{\boldsymbol{a}}E$. We obtain a $\mathcal{O}_X$-module $\mathcal{P}^h_{\boldsymbol{a}}E$ and we obtain a $\mathcal{O}_X(*H)$-module $\mathcal{P}^h_*E := \bigcup_{\boldsymbol{a}\in\mathbb{R}^\Lambda} \mathcal{P}^h_{\boldsymbol{a}}E$.

Definition 4.4.1. *Let $\mathcal{P}_*\mathcal{V}$ be a filtered bundle over (X, H). Let $(E, \overline{\partial}_E)$ be a holomorphic bundle obtained from the restriction of $\mathcal{V}$ to $X - H$. Let h be a hermitian metric of E. h is called adapted if $\mathcal{P}^h_*E = \mathcal{P}_*\mathcal{V}$ stands.*

We remark that in general, we do not know whether $\mathcal{P}^h_*E$ is locally free or not. However, it was proved in [M3, Theorem 21.3.1] that when the metric h is *acceptable* and $\det(E, \overline{\partial}_E, h)$ is flat, $\mathcal{P}^h_*E$ is locally free. We say that h is acceptable when the following condition holds:

- Let $P \in H$ and let $(U_P, z_1, \dots, z_n)$ be an admissible neighborhood of P. We regard as $U_P = \prod_{i=1}^n \{|z_i| < 1\}$. Let g_P be a Poincaré like metric on $U_P \backslash U_P \cap H$. The metric h is called acceptable around P when the curvature of the Chern connection is bounded with respect to g_P and h. h is called acceptable if it is acceptable around any $P \in H$.

4.4.5 Good filtered Higgs bundle

Throughout this section, we assume X to be a complex manifold and $H = \bigcup_{i\in\Lambda} H_i$ to be a simple normal crossing hypersurface of it.

Good set of Irregular values

Let $P \in H$. Let $(U_P, z_1, \dots, z_n)$ be an admissible coordinate around P. We denote the stalk of $\mathcal{O}_X(*H)$ at P as $\mathcal{O}_X(*H)_P$. Let $f \in \mathcal{O}_X(*H)_P$. If $\mathcal{O}_{X,P}$, we set $\mathrm{ord}(f) = (0, \dots 0) \in \mathbb{R}^{l(P)}$. If there exsits a $g \in \mathcal{O}_{X,P}$, $g(P) \neq 0$ and a $\boldsymbol{n} \in \mathbb{Z}^{l(P)}_{<0}$ such that $g = f \prod z_i^{-n_i}$, we set $\mathrm{ord}(f) = \boldsymbol{n}$. Otherwise, $\mathrm{ord}(f)$ is not defined. Note that when $\dim X = 1$ and when f has at least a simple pole at P, then $\mathrm{ord}(f)$ is the usual order.

For any $\mathfrak{a} \in \mathcal{O}_X(*H)_P/\mathcal{O}_{X,P}$, we take a lift $\tilde{\mathfrak{a}} \in \mathcal{O}_X(*H)_P$. If $\mathrm{ord}(\mathfrak{a})$ is defined, we set $\mathrm{ord}(\mathfrak{a}) := \mathrm{ord}(\tilde{\mathfrak{a}})$. Otherwise $\mathrm{ord}(\mathfrak{a})$ is not defined. $\mathrm{ord}(\mathfrak{a})$ does not depend on the lift.

Let $\mathcal{I}_P \subset \mathcal{O}_X(*H)_P/\mathcal{O}_{X,P}$ be finite subset. We say that $\mathcal{I}_P$ is called a good set of irregular values if

- $\mathrm{ord}(\mathfrak{a})$ is defined for any $\mathrm{ord}(\mathfrak{a}) \in \mathcal{I}_P$
- $\mathrm{ord}(\mathfrak{a} - \mathfrak{b})$ is defined for any $\mathrm{ord}(\mathfrak{a}), \mathrm{ord}(\mathfrak{b}) \in \mathcal{I}_P$
- $\{\mathrm{ord}(\mathfrak{a} - \mathfrak{b}) | \mathfrak{a}, \mathfrak{b} \in \mathcal{I}_P\}$ is totally orded with respect to the order $\leq_{\mathbb{Z}^{l(P)}}$.

Note that when $\dim X = 1$, then any finite subset of $\mathcal{O}_X(*H)_P/\mathcal{O}_{X,P}$ is a good set of irregular values.

Good filtered Higgs bundle

Let $(\mathcal{P}_*\mathcal{V}, \theta)$ be a filtered Higgs bundle. Let $P \in X$ and let $\mathcal{O}_{X,\widehat{P}}$ be the completion of the local ring $\mathcal{O}_{X,P}$ with respect to its maximal ideal.

We say that $(\mathcal{P}_*\mathcal{V}, \theta)$ is called unramifiedly good at P if there exisits a good set of irregular values $\mathcal{I}_P$ and exsits a decomposition of Higgs bunlde

$$(\mathcal{P}_*\mathcal{V}, \theta) \otimes \mathcal{O}_{X,\widehat{P}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}_P} (\mathcal{P}_*\mathcal{V}_{\mathfrak{a}}, \theta_{\mathfrak{a}})$$

such that $(\theta_{\mathfrak{a}} - d\tilde{\mathfrak{a}}\mathrm{Id}_{\mathcal{V}_{\mathfrak{a}}})\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{\mathfrak{a}} \subset \mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{\mathfrak{a}} \otimes \Omega^1_X(\log H)$ for evey $\boldsymbol{a} \in \mathbb{R}^\Lambda$. Here $\tilde{\mathfrak{a}}$ is the lift of $\mathfrak{a}$.

$(\mathcal{P}_*\mathcal{V}, \theta)$ is called good at P if there exists a neighborhood U_P and a covering map $\varphi_P : U'_P \to U_P$ ramified over $H \cap U_P$ such that $\varphi_P^*(\mathcal{P}_*\mathcal{V}, \theta)$ is unramified good at $\varphi_P^{-1}(P)$.

$(\mathcal{P}_*\mathcal{V}, \theta)$ is called good (resp. unramifiedly good) if it is good (resp. unramfiedly good) at any point of H.

4.4.6 Good Wild Harmonic Bundles

Local condition for Higgs fields

Let $U := \prod_{i=1}^n \{|z_i| < 1\}$ and $H_{U_i} := U \cap \{z_i = 0\}$ and $H_U := \bigcup_{i=1}^l H_{U_i}$. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on $U - H_U$. The Higgs field θ has an expression

$$\theta = \sum_{i=1}^{l} \frac{F_i}{z_i} dz_i + \sum_{i=l+1}^{n} G_i dz_i.$$

Let T be a formal variable. We have characteristic polynomials

$$\det(T - F_i(z)) = \sum_k A_{i,k}(z) T^k, \det(T - G_i(z)) = \sum_k B_{i,k}(z) T^k$$

where $A_{i,k}(z), B_{i,k}(z)$ are holomorphic functions on $U - H_U$.

Definition 4.4.2. *We say that θ is tame if $A_{i,k}(z), B_{i,k}(z)$ are holomorphic functions on U and if the restriction of $A_{i,k}$ to H_{U_i} are constant for any j and k.*

Definition 4.4.3.

- *We say that θ is unramfiedly good if there exists a good set of irregular value $\mathrm{Irr}(\theta) \subset M(U, H_U)/H(X)$ and a decomposition*

$$(E, \theta) = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\theta)} (E_{\mathfrak{a}}, \theta_{\mathfrak{a}})$$

  such that each $\theta_{\mathfrak{a}} - d\widetilde{\mathfrak{a}} \cdot \mathrm{Id}_{E_{\mathfrak{a}}}$ is tame. Here $\widetilde{\mathfrak{a}}$ is the lift of $\mathfrak{a}$.

- *For $e \in \mathbb{Z}_{>0}$, we define the covering map $\phi_e : U \to U$ as $\phi(z_1, \ldots, z_n) = (z_1^e, \ldots, z_l^e, z_{l+1}, \ldots, z_n)$. We say that θ is good if there exists a $e \in \mathbb{Z}_{>0}$ and the pullback of $(E, \overline{\partial}_E, \theta)$ by ϕ_e is unramifiedly good.*

Global condition of Higgs fields and Good Wild Harmonic bundles

Let X be a complex manifold and H be a normal crossing hypersurface. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on $X - H$.

Definition 4.4.4.

- *We say that θ is (unramifiedly) good at $P \in H$ if it is (unramifiedly) good on an admissible coordinate neighborhood of P.*
- *We say that θ is (unramifiedly) good on (X, H) if it is (unramifiedly) good for any $P \in H$.*

We next recall *good wild harmonic bundles*. Let h be a pluri-harmonic metric of $(E, \overline{\partial}_E, \theta)$ (i.e. $(E, \overline{\partial}_E, \theta, h)$ is a harmonic bundle on $X - H$).

Definition 4.4.5. *We say that $(E, \overline{\partial}_E, \theta, h)$ is a (unramifiedly) good wild harmonic bundle on (X, H) if θ is (unramifiedly) good on (X, H).*

4.4.7 Kobayashi-Hitchin Correspondence

Let X be a connected smooth projective variety and H be a simple normal crossing divisor. Let L be any ample line bundle.

In [M2, M3] Mochizuki proved that there is a one-on-one correspondence between μ_L-polystable good filtered Higgs bundles with vanishing Chern classes and good wild harmonic bundles. This correspondence is called Kobayashi-Hitchin Correspondence.

Proposition 4.4.2 ([M2, Proposition 13.6.1 and 13.6.4])**.** *Let (E, θ, h) be a good wild harmonic bundle on (X, H).*

- $(\mathcal{P}^h_* E, \theta)$ *is* μ_L*-polystable with* $\mu_L(\mathcal{P}^h_* E) = 0$.
- $c_1(\mathcal{P}^h_* E) = 0$ *and* $\int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V}) c_1(L)^{\dim X - 2} = 0$ *holds.*
- *Let* h' *be another pluri-harmonic metric of* (E, θ, h) *such that* $\mathcal{P}^{h'}_* E = \mathcal{P}^h_* E$. *Then there exists a decomposition of the Higgs bundle* $(E, \theta) = \oplus_i (E_i, \theta_i)$ *such that (i) the decomposition is orthogonal with respect to both* h *and* h', *(ii)* $h|_{E_i} = a_i h'|_{E_i}$ *for some* $a_i > 0$.

Theorem 4.4.1 ([M3, Theorem 2.23.]). *Let* $(\mathcal{P}_*\mathcal{V}, \theta)$ *be a good filtered Higgs bundle on* (X, H) *and* $(E, \overline{\partial}_E, \theta)$ *be the Higgs bundle on* $X \backslash H$ *which is the restriction of* $(\mathcal{P}_*\mathcal{V}, \theta)$.

Suppose that $(\mathcal{P}_*\mathcal{V}, \theta)$ *is* μ_L*-polystable and satisfies the following vanishing condition:*

$$\mu_L(\mathcal{P}_*\mathcal{V}) = 0, \int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V}) c_1(L)^{\dim X - 2} = 0. \tag{4.2}$$

Then there exists a pluri-harmonic metric h *for* $(E, \overline{\partial}_E, \theta)$ *such that* $(\mathcal{V}, \theta)|_{X \backslash H} \simeq (E, \theta)$ *extends to* $(\mathcal{P}_*\mathcal{V}, \theta) \simeq (\mathcal{P}^h_* E, \theta)$.

Remark 4.4.2. *We note that Theorem 4.4.1 was proved not only for the Higgs bundles but for all* λ*-flat bundles. The* $\lambda = 1$ *case was established in [M2].*

4.5 Good filtered Higgs bundles with skew-symmetric pairings

4.5.1 Pairings of filtered bundle

Throughout this section, we assume X to be a smooth projective variety and let $H = \bigcup_{i \in \Lambda} H_i$ be a normal crossing divisor of it, and L to be an ample line bundle on X. However, we only use this assumption in Section 4.1.4. The results in other sections can generalized for any complex manifold and normal crossing hypersurfaces.

Pairings of locally free $\mathcal{O}_X(*H)$-modules

Let $\mathcal{O}_X(*H)$ be the sheaf of meromorphic function on X whose poles are contained in H. We recall the pairings of $\mathcal{O}_X(*H)$-modules following [LM1].

Let $\mathcal{V}$ be a locally free $\mathcal{O}_X(*H)$-module of finite rank. Let $\mathcal{V}^\vee := \mathcal{H}om_{\mathcal{O}_X(*H)}(\mathcal{V}, \mathcal{O}_X(*H))$ be the dual of $\mathcal{V}$. The determinant bundle of $\mathcal{V}$ is denoted by $\det(\mathcal{V}) := \bigwedge^{\mathrm{rank}\mathcal{V}} \mathcal{V}$. There exists a natural isomorphism $\det(\mathcal{V}^\vee) \simeq \det(\mathcal{V})^\vee$. For a morphism $f : \mathcal{V}_1 \to \mathcal{V}_2$ of locally free $\mathcal{O}_X(*H)$-modules, we have the dual $f^\vee : \mathcal{V}_2^\vee \to \mathcal{V}_1^\vee$. If rank($\mathcal{V}_1$)=rank($\mathcal{V}_2$), then we have the induced morphism $\det(f) : \det(\mathcal{V}_1) \to \det(\mathcal{V}_2)$.

A pairing P of a pair of locally free $\mathcal{O}_X(*H)$-modules $\mathcal{V}_1$ and $\mathcal{V}_2$ is a morphism $P : \mathcal{V}_1 \otimes \mathcal{V}_2 \to \mathcal{O}_X(*D)$. It induces a morphism $\Psi_P : \mathcal{V}_1 \to \mathcal{V}_2^\vee$ by $\Psi_P(u)(v) := P(u, v)$. Let $\mathrm{ex} : \mathcal{V}_1 \otimes \mathcal{V}_2 \simeq \mathcal{V}_2 \otimes \mathcal{V}_1$ be the morphism defined by $\mathrm{ex}(u \otimes v) = v \otimes u$. We obtain a pairing $P \circ \mathrm{ex} : \mathcal{V}_2 \otimes \mathcal{V}_1 \to \mathcal{O}_X(*H)$. We have $\Psi_P^\vee = \Psi_{P \circ \mathrm{ex}}$. If rank $\mathcal{V}_1$=rank $\mathcal{V}_2$, we obtain the induced pairing $\det P : \det(\mathcal{V}_1) \otimes \det(\mathcal{V}_2) \to \mathcal{O}_X(*H)$. We have $\det(\Psi_P) = \Psi_{\det(P)}$.

A pairing P is called non-degenerate if Ψ_P is an isomorphism. It is equivalent to that $P \circ \mathrm{ex}$ is non-degenerate. It is also equivalent to be $\det P$ is non-degenerate. If P is non-degenerate, we obtain a pairing $P^\vee$ of $\mathcal{V}_2^\vee$ and $\mathcal{V}_1^\vee$ defined by $P \circ (\Psi_P^{-1} \otimes \Psi_{P \circ \mathrm{ex}})$.

A pairing P of locally free $\mathcal{O}_X(*H)$-module $\mathcal{V}$ is a morphism $P : \mathcal{V} \otimes \mathcal{V} \to \mathcal{O}_X(*H)$. It is called skew-symmetric if $P \circ \mathrm{ex} = -P$. Note that $\det(P)$ is natural defined in this case. If P is non-degenerate, then rank: $\mathcal{V}$ must be even and we have induced pairing $P^\vee$ of $\mathcal{V}^\vee$.

Pairings of filtered bundles

Let $\mathcal{P}_*\mathcal{V}_i$ $(i = 1, 2)$ be a filtered bundle on (X, H). A pairing P of $\mathcal{P}_*\mathcal{V}_1$ and $\mathcal{P}_*\mathcal{V}_2$ is a morphism between filtered bunlde

$$P : \mathcal{P}_*\mathcal{V}_1 \otimes \mathcal{P}_*\mathcal{V}_2 \to \mathcal{P}^{(0)}_*(\mathcal{O}_X(*H)).$$

We obtain a pairing $P \circ \mathrm{ex}$ of $\mathcal{P}_*\mathcal{V}_2$ and $\mathcal{P}_*\mathcal{V}_1$.

From the pairing P, we also obtain the following morphism

$$\Psi_P : \mathcal{P}_*\mathcal{V}_1 \to \mathcal{P}_*\mathcal{V}_2^\vee.$$

Definition 4.5.1. *P is called perfect if the morphism Ψ_P is an isomorphism of filtered bundles.*

Let $\mathcal{V}_i' \subset \mathcal{V}_i$ be a locally free $\mathcal{O}_X(*H)$-submodules. We also assume $\mathcal{V}_i'$ are saturated i.e. $\mathcal{V}_i/\mathcal{V}_i'$ are locally free. From a pairing P of $\mathcal{P}_*\mathcal{V}_1$ and $\mathcal{P}_*\mathcal{V}_2$, we have the induced pairing P' for $\mathcal{P}_*\mathcal{V}_1'$ and $\mathcal{P}_*\mathcal{V}_2'$. We have a sequence of sheaves:

$$\mathcal{V}_1' \xrightarrow{i_1} \mathcal{V}_1 \xrightarrow{\Psi_P} \mathcal{V}_2^\vee \xrightarrow{i_2^\vee} \mathcal{V}_2'^\vee$$

where i_1 is the canonical inclusion and $i_2^\vee$ is the dual of the canonical inclusion. Note that $\Psi_{P'} = i_2^\vee \circ \Psi_P \circ i_1$. Let $\mathcal{U}_1 := \ker(i_2^\vee \circ \Psi_P)$. It is a subsheaf of $\mathcal{V}_1$.

Lemma 4.5.1. *If P and P' are perfect, then we have the decomposition $\mathcal{V}_1 = \mathcal{V}_1' \oplus \mathcal{U}_1$.*

Proof. We have the following short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{V}_1' \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{V}_1/\mathcal{V}_1' \longrightarrow 0.$$

Since P and P' are non-degenerate, we have another short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{V}_1' \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{U}_1 \longrightarrow 0.$$

By the standard argument of sheaves, we have $\mathcal{U}_1 \simeq \mathcal{V}_1/\mathcal{V}_1'$. Hence we have $\mathcal{V}_1 = \mathcal{V}_1' \oplus \mathcal{U}_1$. □

Skew-symmetric pairings of filtered bundles

Let ω be a skew-symmetric pairing of a filtered bundle $\mathcal{P}_*\mathcal{V}$ on (X, H). Let $\mathcal{V}' \subset \mathcal{V}$ be a saturated locally free $\mathcal{O}_X(*H)$-submodule. Let $(\mathcal{V}')^{\perp\omega}$ be the kernel of the following composition:

$$\mathcal{V} \xrightarrow{\Psi_\omega} \mathcal{V}^\vee \xrightarrow{i^\vee} \mathcal{V}'^\vee$$

where $i^\vee$ is the dual of the canonical inclusion. Let ω' be the induced skew-symmetric pairing of $\mathcal{P}_*\mathcal{V}'$. The next Lemma is the special case of Lemma 4.5.1.

Lemma 4.5.2. *If ω and ω' are perfect, then we have the decomposition $\mathcal{V} = \mathcal{V}' \oplus (\mathcal{V}')^{\perp\omega}$.*

4.5.2 Skew-symmetric pairings of good filtered Higgs bundle

Throughout this section, we assume X to be a smooth projective variety and let $H = \bigcup_{i\in\Lambda} H_i$ be a normal crossing divisor of it, and L to be an ample line bundle on X.

Skew-symmetric pairings of Higgs bundle

Definition 4.5.2. *A skew-symmetric pairing ω on a good filtered Higgs bundle $(\mathcal{P}_*\mathcal{V}, \theta)$ over (X, H) is a skew-symmetric pairing ω of $\mathcal{P}_*\mathcal{V}$ such that $\omega(\theta \otimes \mathrm{Id}) = -\omega(\mathrm{Id} \otimes \theta)$.*

When $(\mathcal{P}_*\mathcal{V}, \theta)$ has a skew-symmetric pairing ω, we have an induced morphism $\Psi_\omega : (\mathcal{P}_*\mathcal{V}, \theta) \to (\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee)$ between good filtered Higgs bundles. We also obtain a symmetric pairing $\det(\omega)$ of $(\det(\mathcal{P}_*\mathcal{V}), \mathrm{tr}\theta)$.

Harmonic bundles with skew-symmetric structure

We use the same notation as the last section. Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle on (X, H). Let ω be a symplectic structure of the harmonic bundle $(E, \overline{\partial}_E, \theta, h)$. By Proposition 4.4.2, we obtain a μ_L-polystable good filtered Higgs bundle $(\mathcal{P}^h_*E, \theta)$ with vanishing Chern classes.

Lemma 4.5.3. *ω induces a perfect skew-symmetric pairing for the Higgs bundle $(\mathcal{P}^h_*E, \theta)$.*

Proof. Since ω is compatible with h, it induces an isomorphism $\Psi_\omega : \mathcal{P}^h_* E \to \mathcal{P}^{h^\vee}_* E^\vee$. Since $\mathcal{P}^{h^\vee}_* E^\vee$ is naturally isomorphic to $(\mathcal{P}^h_* E)^\vee$, ω induces a perfect pairing for $\mathcal{P}^h_* E$. □

As a consequence, we have the following.

Proposition 4.5.1. *Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle equipped with symplectic structure ω. Then $(\mathcal{P}^h_* E, \theta)$ is a μ_L-polystable good filtered Higgs bundle equipped with a perfect skew-symmetric pairing ω and satisfies the vanishing condition (4.2).*

4.5.3 Kobayashi-Hitchin correspondence with skew-symmetry

Throughout this section, we assume X to be a smooth projective variety and let $H = \bigcup_{i\in\Lambda} H_i$ be a normal crossing divisor of it, and L to be an ample line bundle on X.

Basic polystable object (1)

Let $(\mathcal{P}_*\mathcal{V}, \theta)$ be a stable good filtered Higgs bundle of degree 0 such that $(\mathcal{P}_*\mathcal{V}, \theta) \simeq (\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee)$. Let P be a pairing of a filtered bundle

$$P : \mathcal{P}_*\mathcal{V} \otimes \mathcal{P}_*\mathcal{V} \to \mathcal{P}^{(0)}_*(\mathcal{O}_X(*H))$$

such that it induces an isomorphism $\Psi_P : (\mathcal{P}_*\mathcal{V}, \theta) \to (\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee)$. If there is another pairing P' which induces an isomorphism $\Psi_{P'}$, then since a stable bundle is simple there exists an $\alpha \in \mathbb{C}$ such that $P' = \alpha P$.

Lemma 4.5.4. *Either one of $P \circ \mathrm{ex} = P$ or $P \circ \mathrm{ex} = -P$ holds.*

Proof. This was proved in [LM1, Lemma 3.19]. The claim follows from the fact that there exists a $\alpha \in \mathbb{C}$ such that $\Psi_P^\vee = \alpha\Psi_P$, $(\Psi_P^\vee)^\vee = \Psi_P$, $\Psi_{P\circ\mathrm{ex}} = \Psi_P^\vee$. □

Let $C_{\mathbb{C}^l}$ be a symmetric pairing of $\mathbb{C}^l$ defined by $C(\boldsymbol{x}, \boldsymbol{y}) := \sum_i x_i y_i$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^l$. Let $\omega_{\mathbb{C}^{2k}}$ be a skew-symmetric pairing of $\mathbb{C}^{2k}$ defined by $\omega_{\mathbb{C}^{2k}}(\boldsymbol{x}, \boldsymbol{y}) := \sum_i (x_{2i-1}y_{2i} - x_{2i-1}y_{2i})$. If P_1 is a symmetric pairing then $P_1 \otimes \omega_{\mathbb{C}^{2k}}$ is a skew-symmetric pairing for $(E, \theta) \otimes \mathbb{C}^{2k}$. If P_1 is skew-symmetric then $P_1 \otimes C_{\mathbb{C}^l}$ is a skew-symmetric pairing for $(\mathcal{P}_*\mathcal{V}, \theta) \otimes \mathbb{C}^l$.

Lemma 4.5.5. *Suppose that $(\mathcal{P}_*\mathcal{V}, \theta) \otimes \mathbb{C}^l$ is equipped with a perfect skew-symmetric pairing ω.*

- *If P_1 is symmetric, then l is an even number $2k$ and there exists an automorphism τ for $\mathbb{C}^{2k}$ such that $(Id \otimes \tau)^*\omega = P_1 \otimes \omega_{\mathbb{C}^{2k}}$.*
- *If P_1 is skew-symmetric then there exists an automorphism τ for $\mathbb{C}^l$ such that $(Id \otimes \tau)^*\omega = P_1 \otimes C_{\mathbb{C}^l}$.*

Proof. We only give the outline of the proof for the case when P is symmetric. The other case can be proved similarly.

Let $\{e_i\}_{i=1}^l$ be the canonical base of $\mathbb{C}^l$. Since ω is a perfect skew-symmetric pairing of $(\mathcal{P}_*\mathcal{V}, \theta) \otimes \mathbb{C}^l$, it induces an isomorphism $\Psi_\omega : (\mathcal{P}_*\mathcal{V}, \theta) \otimes \mathbb{C}^l \to (\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee) \otimes \mathbb{C}^l$. Let $\Psi_{\omega,ij}$ be the composition of

$$(\mathcal{P}_*\mathcal{V}, \theta) \otimes e_i \xrightarrow{i} (\mathcal{P}_*\mathcal{V}, \theta) \otimes \mathbb{C}^l \xrightarrow{\Psi_\omega} (\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee) \otimes \mathbb{C}^l \xrightarrow{pr_j} (\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee) \otimes e_j$$

where i is the inclusion and pr_j is the projection. Either one $\Psi_{\omega,ij} = 0$ or $\Psi_{\omega,ij} = \alpha_{ij}\Psi_{P_1}$ for a $\alpha_{ij} \in \mathbb{C}$ holds. Since ω is a perfect pairing, $(\alpha_{ij})_{i,j}$ is non-degenerate matrix and since ω is skew-symmetric and P is symmetric, $(\alpha_{ij})_{i,j}$ is a skew-symmetric matrix. Hence l is an even number $2k$ and there is an automorphism τ which we want. □

Lemma 4.5.6. *There is an unique harmonic metric h_0 on $\mathcal{V}|_{X/D}$ such that (1) it is adapted to $\mathcal{P}_*\mathcal{V}$ and (2) Ψ_{P_1} is isometric with respect to h_0 and $h_0^\vee$.*

Proof. By Theorem 4.2.1, we have a harmonic metric h on $\mathcal{V}|_{X/D}$ which is adapted to $\mathcal{P}_*\mathcal{V}$. Let $h^\vee$ be the induced harmonic metric of $\mathcal{V}^\vee|_{X\setminus D}$ by h, which is also adapted to $\mathcal{P}_*\mathcal{V}^\vee$. Since $\Psi_P : (\mathcal{P}_*\mathcal{V}, \theta) \to (\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee)$ is an isomorphism, $\Psi_P^*(h^\vee)$ is also a harmonic metric which is adapted to $\mathcal{P}_*\mathcal{V}$. Since the adapted harmonic metric for a stable Higgs bundle is unique up to positive constant, we have an $a > 0$ such that $\Psi_P^*(h^\vee) = a^2 h$. Set $h_0 := ah$ then we obtain the desired metric. The uniqueness is clear. □

Lemma 4.5.7.

- *For any hermitian metric $h_{\mathbb{C}^l}$ of $\mathbb{C}^l$, $h_0 \otimes h_{\mathbb{C}^l}$ is a harmonic metric of $\mathcal{V}|_{X/D} \otimes \mathbb{C}^l$ which is adapted to $\mathcal{P}_*\mathcal{V} \otimes \mathbb{C}^l$. Conversely, for any harmonic metric h on $\mathcal{V}|_{X/D} \otimes \mathbb{C}^l$ which is adapted to $\mathcal{P}_*\mathcal{V} \otimes \mathbb{C}^l$, there is a hermitian metric $h_{\mathbb{C}^l}$ of $\mathbb{C}^l$ such that $h = h_0 \otimes h_{\mathbb{C}^l}$.*
- *If P_1 is symmetric (resp. skew-symmetric), a harmonic metric $h_0 \otimes h_{\mathbb{C}^l}$ of $\mathcal{V}|_{X/D} \otimes \mathbb{C}^l$ is compatible with $P_1 \otimes \omega_{\mathbb{C}^l}$ (resp. $P_1 \otimes C_{\mathbb{C}^l}$) if and only if $h_{\mathbb{C}^l}$ is compatible with $\omega_{\mathbb{C}^l}$ (resp. $C_{\mathbb{C}^l}$).*

Proof. The first claim follows from the uniqueness of the harmonic metric to a parabolic structure. See [M3, Corollary 13.6.2].

The second claim follows from the following argument: Let $E_i (i = 1, 2)$ be complex vector bundles and $h_i (i = 1, 2)$ be hermitian metrics for E_i. Let $P_i (i = 1, 2)$ be pairings for E_i (i.e. P_i is a section of $E_i^\vee \otimes E_i^\vee$) and $\Psi_{P_i} : E_i \to E_i^\vee$ be the indued morphisms. Let $h_1 \otimes h_2$ be the hermitian metric of $E_1 \otimes E_2$ induced by h_i and $P_1 \otimes P_2$ be the pairing of $E_1 \otimes E_2$ induced by P_i. Let $u_i \otimes v_i (i = 1, 2)$ be sections of $E_1 \otimes E_2$. $h_i \otimes h_2$ and $P_1 \otimes P_2$ are defined as $h_i \otimes h_2(u_1 \otimes v_1, u_2 \otimes v_2) = h_1(u_1, u_2)h_2(v_1, v_2)$ and $P_1 \otimes P_2(u_1 \otimes v_1, u_2 \otimes v_2) = P_1(u_1, u_2)P_2(v_1, v_2)$. Hence $\Psi_{P_1 \otimes P_2} = \Psi_{P_1} \otimes \Psi_{P_2}$ and $(h_1 \otimes h_2)^\vee(\Psi_{P_1 \otimes P_2}(u_1 \otimes v_1), \Psi_{P_1 \otimes P_2}(u_2 \otimes v_2)) = h_1^\vee(\Psi_{P_1}(u_1), \Psi_{P_1}(u_2))h_2^\vee(\Psi_{P_2}(v_1), \Psi_{P_2}(v_2))$ holds. Once we apply this discussion to $h_0 \otimes h_{\mathbb{C}^l}$ and $P_1 \otimes \omega_{\mathbb{C}^l}$ or $P_1 \otimes C_{\mathbb{C}^l}$, the second claim follows. □

Basic polystable objects (2)

Let $(\mathcal{P}_*\mathcal{V}, \theta)$ be a stable good filtered Higgs bundle that satisfies the vanishing condition (4.2) and $(\mathcal{P}_*\mathcal{V}, \theta) \not\simeq (\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee)$. We set $\mathcal{P}_*\widetilde{\mathcal{V}} := \mathcal{P}_*\mathcal{V} \oplus \mathcal{P}_*\mathcal{V}^\vee$ and set $\widetilde{\theta} := \theta \oplus -\theta^\vee$. Then we obtain a Higgs bundle $(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\theta})$. We have a naturally defined perfect skew-symmetric pairing of $(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\theta})$,

$$\widetilde{\omega}_{(\mathcal{P}_*\mathcal{V},\theta)} : (\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\theta}) \otimes (\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\theta}) \to \mathcal{P}_*^{(0)}(\mathcal{O}_X(*H))$$

such that $\widetilde{\omega}_{(\mathcal{P}_*\mathcal{V},\theta)}((u_1, v_1^\vee), (u_2, v_2^\vee)) = v_1^\vee(u_2) - v_2^\vee(u_1)$ for any local section $(u_1, v_1^\vee), (u_2, v_2^\vee)$ of $\mathcal{P}_*\widetilde{\mathcal{V}}$. $(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\theta}, \widetilde{\omega}_{(\mathcal{P}_*\mathcal{V},\theta)})$ forms a Higgs bundle with a perfect skew-symmetric pairing.

Lemma 4.5.8. *Suppose $\big((\mathcal{P}_*\mathcal{V}, \theta) \otimes \mathbb{C}^{l_1}\big) \oplus \big((\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee) \otimes \mathbb{C}^{l_2}\big)$ is equipped with a perfect skew-symmetric pairing ω. Then we have $l_1 = l_2$ and there exists an isomorphism $(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\theta}) \otimes \mathbb{C}^{l_1} \simeq (\mathcal{P}_*\mathcal{V}, \theta) \otimes \mathbb{C}^{l_1} \oplus (\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee) \otimes \mathbb{C}^{l_2}$ such that under the isomorphism, $\widetilde{\omega}_{(\mathcal{P}_*\mathcal{V},\theta)} \otimes C_{\mathbb{C}^{l_1}} = \omega$ holds.*

Proof. We have one-dimensional subspaces $L_1 \subset \mathbb{C}^{l_1}$ and $L_2 \subset \mathbb{C}^{l_2}$ such that the restriction of ω to $\big((\mathcal{P}_*\mathcal{V}, \theta) \otimes L_1\big) \oplus \big((\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee) \otimes L_2\big)$ is not identically zero. We define $\Psi_{\omega,12}$ to be the composition of

$$(\mathcal{P}_*\mathcal{V}, \theta) \otimes L_1 \xrightarrow{i} \big((\mathcal{P}_*\mathcal{V}, \theta) \otimes L_1\big) \oplus \big((\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee) \otimes L_2\big)$$
$$\xrightarrow{\Psi_\omega} \big((\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee) \otimes L_1^\vee\big) \oplus \big((\mathcal{P}_*\mathcal{V}, \theta) \otimes L_2^\vee\big) \xrightarrow{pr_2} (\mathcal{P}_*\mathcal{V}, \theta) \otimes L_2^\vee$$

where i and pr_2 are the canonical inclusion and the canonical projection. We define $\Psi_{\omega,11}$,$\Psi_{\omega,21}$ and $\Psi_{\omega,22}$ in the same manner. Since $(\mathcal{P}_*\mathcal{V}, \theta) \not\simeq (\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee)$, we obtain $\Psi_{\omega,11} = 0, \Psi_{\omega,22} = 0$ and $\Psi_{\omega,12} = \alpha \mathrm{Id}_{(\mathcal{P}_*\mathcal{V},\theta)}, \Psi_{\omega,21} = \beta \mathrm{Id}_{(\mathcal{P}_*\mathcal{V},\theta)}$ for some $\alpha, \beta \in \mathbb{C}$. Since ω is a skew-symmetric pairing, we have $\beta = -\alpha$. Hence $\omega = \alpha \widetilde{\omega}_{(\mathcal{P}_*\mathcal{V},\theta)}$. In particular the restriction of ω to $\big((\mathcal{P}_*\mathcal{V}, \theta) \otimes L_1\big) \oplus \big((\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee) \otimes L_2\big)$ induces a perfect skew-symmetric pairing on it. Hence we obtain an orthonormal decomposition with respect to ω:

$$(\mathcal{P}_*\mathcal{V} \otimes \mathbb{C}^{l_1}) \oplus \big(\mathcal{P}_*\mathcal{V}^\vee \otimes \mathbb{C}^{l_2}\big) \simeq \big(\mathcal{P}_*\mathcal{V} \otimes L_1\big) \oplus \big(\mathcal{P}_*\mathcal{V}^\vee \otimes L_2\big) \oplus \mathcal{P}_*\mathcal{V}'.$$

It is preserved by the Higgs field and the induced Higgs field to $\mathcal{P}_*\mathcal{V}'$ is isomorphic to $\big((\mathcal{P}_*\mathcal{V}, \theta) \otimes \mathbb{C}^{l_1-1}\big) \oplus \big((\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee) \otimes \mathbb{C}^{l_2-1}\big)$. We obtain the claim by induction. □

By using $C_{\mathbb{C}^l}$, we can identify $\mathbb{C}^l$ and it's dual $(\mathbb{C}^l)^\vee$. Then we can induce a perfect skew-symmetric pairing $\widetilde{\omega}_{(\mathcal{P}_*\mathcal{V},\theta)} \otimes C_{\mathbb{C}^l}$ on

$$(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\theta}) \otimes \mathbb{C}^l = \big((\mathcal{P}_*\mathcal{V}, \theta) \otimes \mathbb{C}^l\big) \oplus \big((\mathcal{P}_*\mathcal{V}^\vee, -\theta^\vee) \otimes (\mathbb{C}^l)^\vee\big)$$

by the canonical way.

We obtain the induced harmonic metric $h_0^\vee$ on $\mathcal{V}^\vee|_{X/D}$ which is adapted to $\mathcal{P}_*\mathcal{V}^\vee$.

Lemma 4.5.9.

- *Let $h_{\mathbb{C}^l}$ be any hermitian metric on $\mathbb{C}^l$. Let $h^{\vee}_{\mathbb{C}^l}$ denote the induced hermitian metric on $(\mathbb{C}^l)^{\vee}$. Then, $(h_0 \otimes h_{\mathbb{C}^l}) \oplus (h_0^{\vee} \otimes h^{\vee}_{\mathbb{C}^l})$ is a harmonic metric of $(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\theta}) \otimes \mathbb{C}^l$ such that it is compatible with $\widetilde{\omega}_{(\mathcal{P}_*\mathcal{V},\theta)} \otimes C_{\mathbb{C}^l}$.*
- *Conversely, let h be any harmonic metric of $(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\theta}) \otimes \mathbb{C}^l$ which is compatible with $\widetilde{\omega}_{(\mathcal{P}_*\mathcal{V},\theta)} \otimes C_{\mathbb{C}^l}$. Then there exists a hermitian metric $h_{\mathbb{C}^l}$ of $\mathbb{C}^l$ such that $h = (h_0 \otimes h_{\mathbb{C}^l}) \oplus (h_0^{\vee} \otimes h^{\vee}_{\mathbb{C}^l})$.*

Proof. The compatibility of $(h_0 \otimes h_{\mathbb{C}^l}) \oplus (h_0^{\vee} \otimes h^{\vee}_{\mathbb{C}^l})$ with $\widetilde{\omega}_{(\mathcal{P}_*\mathcal{V},\theta)} \otimes C_{\mathbb{C}^l}$ follows from the argument in the second claim of Lemma 4.5.7. The second claim follows from [LM1, Lemma 3.25]. □

Polystable objects

Let $(\mathcal{P}_*\mathcal{V}, \theta)$ be a polystable good filtered Higgs bundle of degree 0 on X equipped with a perfect skew-symmetric pairing ω. Let

$$(\mathcal{P}_*\mathcal{V}, \theta) = \sum_i (\mathcal{P}_*\mathcal{V}_i, \theta_i) \otimes \mathbb{C}^{n(i)}$$

be the canonical decomposition. Since the perfect skew-symmetric pairing ω induces an isomorophism $(\mathcal{P}_*\mathcal{V}, \theta) \simeq (\mathcal{P}_*\mathcal{V}^{\vee}, -\theta^{\vee})$, each $(\mathcal{P}_*\mathcal{V}_i, \theta_i) \otimes \mathbb{C}^{n(i)}$ is a basic polystable object we observed above. Hence the next proposition is deduced from previous sections.

Proposition 4.5.2. *There exist stable Higgs bundles $(\mathcal{P}_*\mathcal{V}_i^{(0)}, \theta_i^{(0)})$ $(i = 1, \dots, p(0))$, $(\mathcal{P}_*\mathcal{V}_i^{(1)}, \theta_i^{(1)})$ $(i = 1, \dots, p(1))$ and $(\mathcal{P}_*\mathcal{V}_i^{(2)}, \theta_i^{(2)})$ $(i = 1, \dots, p(2))$ of degree 0 on X such that the following holds.*

- *$(\mathcal{P}_*\mathcal{V}_i^{(0)}, \theta_i^{(0)})$ is equipped with a symmetric pairing $P_i^{(0)}$.*
- *$(\mathcal{P}_*\mathcal{V}_i^{(1)}, \theta_i^{(1)})$ is equipped with a skew-symmetric pairing $P_i^{(1)}$.*
- *$(\mathcal{P}_*\mathcal{V}_i^{(2)}, \theta_i^{(2)}) \not\simeq (\mathcal{P}_*\mathcal{V}_i^{(2)}, -\theta_i^{(2)})^{\vee}$.*
- *There exists positive integers $l(a, i)$ and an isomorphism*

$$(\mathcal{P}_*\mathcal{V}, \theta) \simeq \bigoplus_{i=1}^{p(0)} (\mathcal{P}_*\mathcal{V}_i^{(0)}, \theta_i^{(0)}) \otimes \mathbb{C}^{2l(0,i)} \oplus \bigoplus_{i=1}^{p(1)} (\mathcal{P}_*\mathcal{V}_i^{(1)}, \theta_i^{(1)}) \otimes \mathbb{C}^{l(1,i)}$$
$$\oplus \bigoplus_{i=1}^{p(2)} \Big(\big((\mathcal{P}_*\mathcal{V}_i^{(2)}, \theta_i^{(2)}) \otimes \mathbb{C}^{l(2,i)}\big) \oplus \big((\mathcal{P}_*\mathcal{V}_i^{(2)}, -\theta_i^{(2)})^{\vee} \otimes (\mathbb{C}^{l(2,i)})^{\vee}\big) \Big).$$

  Under this isomorphism, ω is identified with the direct sum of $P_i^{(0)} \otimes \omega_{\mathbb{C}^{2l(0,i)}}$, $P_i^{(1)} \otimes C_{\mathbb{C}^{l(1,i)}}$ and $\widetilde{\omega}_{(E_i^{(2)}, \theta_i^{(2)})} \otimes C_{\mathbb{C}^{l(2,i)}}$

- *$(\mathcal{P}_*\mathcal{V}_i^{(a)}, \theta_i^{(a)}) \not\simeq (\mathcal{P}_*\mathcal{V}_j^{(a)}, \theta_j^{(a)})$ $(i \neq j)$ for a=0,1,2, and $(\mathcal{P}_*\mathcal{V}_i^{(2)}, \theta_i^{(2)}) \not\simeq (\mathcal{P}_*\mathcal{V}_j^{(2)}, -\theta_j^{(2)})^{\vee}$ for any i, j.*

Proof. It follows from Lemma 4.5.5 and Lemma 4.5.8. □

Let $h_i^{(a)}$ $(a = 0, 1)$ be the unique harmonic metrics of $(\mathcal{V}_i^{(a)}, \theta_i^{(a)})|_{X \setminus D}$ such that (i) $h_i^{(a)}$ is adapted to $(\mathcal{P}_*\mathcal{V}_i^{(a)}, \theta_i^{(a)})$, (ii) $\Psi_{P_i^{(a)}}$ is isomoteric with respect to $h_i^{(a)}$ and $(h_i^{(a)})^{\vee}$. Let $h_i^{(2)}$ be any harmonic metrics of $(\mathcal{V}_i^{(2)}, \theta_i^{(2)})|_{X \setminus D}$ which is adapted to $(\mathcal{P}_*\mathcal{V}_i^{(2)}, \theta_i^{(2)})$.

Proposition 4.5.3. *There exists a harmonic metric h of $(\mathcal{V}, \theta)|_{X \setminus D}$ such that* (i) *h is adapted to $\mathcal{P}_*\mathcal{V}$,* (ii) *it is compatible with ω. Moreover, we have the following.*

- *Let $h_{\mathbb{C}^{2l(0,i)}}$ be a hermitian metric of $\mathbb{C}^{2l(0,i)}$ compatible with $\omega_{\mathbb{C}^{2l(0,i)}}$. Let $h_{\mathbb{C}^{l(1,i)}}$ be a hermitian metric of $\mathbb{C}^{l(1,i)}$ compatible with $C_{\mathbb{C}^{l(1,i)}}$. Let $h_{\mathbb{C}^{l(2,i)}}$ be any hermitian metric on $\mathbb{C}^{l(2,i)}$. Then,*

$$\bigoplus_{i=1}^{p(0)} h_i^{(0)} \otimes h_{\mathbb{C}^{2l(0,i)}} \oplus \bigoplus_{i=1}^{p(1)} h_i^{(1)} \otimes h_{\mathbb{C}^{l(1,i)}} \oplus \bigoplus_{i=1}^{p(2)} \Big(\big((h_i^{(2)} \otimes h_{\mathbb{C}^{l(2,i)}} \big) \oplus \big((h_i^{(2)})^\vee \otimes (h_{\mathbb{C}^{l(2,i)}})^\vee \big) \Big) \tag{4.3}$$

  is a harmonic metric which satisfies the condition (i), (ii).

- *Conversely, if h is a harmonic metric of $(\mathcal{V}, \theta)|_{X\setminus D}$ which satisfies the condition (i) and (ii), then it has the form of (4.3).*

Proof. The first claim follows from Proposition 4.5.2. The second claim follows from Lemma 4.5.7 and Lemma 4.5.9. □

An equivalence

In this section, we state the Kobayashi-Hitchin correspondence with skew symmetry. Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle with symplectic structure ω. From section 4.5.2, we obtain a good filtered Higgs bundle $(\mathcal{P}^h_* E, \theta)$ satisfying the vanishing condition (4.2) equipped with a perfect skew-symmetric pairing. From section 4.5.3, we also have the converse. As a result, we have the following.

Theorem 4.5.1. *Let X be a smooth projective variety and H be a normal crossing divisor of X.*
The following objects are equivalent on (X, H)

- *Good wild harmonic bundles with a symplectic structure.*
- *Good filtered polystable Higgs bundles with a perfect skew-symmetric pairing satisfying the vanishing condition (4.2).*

Proof. In section 4.5.2, we proved that from a good wild harmonic bundle with a symplectic structure we obtain a good filtered Higgs bundle satisfying the vanishing condition (4.2) equipped with a perfect skew-symmetric pairing. We have the opposite side from section 4.5.3. □

The compact case is straightforward from Theorem 4.5.1. However, for the compact case, we do not have to assume X to be projective. In particular, the statement holds for arbitrary Kähler manifolds.

Corollary 4.5.1. *Let X be a compact Kähler manifold. The following objects are equivalent on X.*

- *Harmonic bundles with a symplectic structure.*
- *Polystable Higgs bundles with vanishing Chern classes with a perfect skew-symmetric pairing.*

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