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ON EXTENDING PROPERTY ON DIRECT SUMS OF UNIFORM MODULES

Dedicated to Professor Kentaro Murata on his 60th birthday

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First we take a right artinian ring R. Then every injective R-module E is a direct sum of indecomposable modules. Further for every simple submodule S of E, there exists a direct summand of E whose socle is equal to S. Let $\sum_I \oplus S_\alpha$ be a decomposition of the socle of E. Then we have a decomposition of E by indecomposable modules E_α such that $E=\sum_I \oplus E_\alpha$ and the socle of E_α is S_α. We shall call the first property and the second propert the extending property of simple module and of decomposition, respectively. These concepts are dual to those of lifting properties mentioned in [7].

We shall study the above properties on direct sums of completely indecomposable modules with certain condition over an arbitiary ring. We shall give characterizations of those properties in terms of endomorphisms over direct summands and show that quasi-injective modules and generalized uniserial rings [15] are related to those properties. Our results are dual or similar to those in [9] and are applied to the study of QF-2 rings in [8].

1. Notations

Throughout this paper R is a ring with identity and every R-module is a unitary right R-module. For an R-module M, we denote its socle and its injective envelope by $S(M)$ and $E(M)$, respectively. For a submodule N of M, we use the symbol $N\subseteq_e M$ to indicate that M is an essential extension of N.

In [9], the first author has studied direct sums of hollow modules by introducing the lifting property of simple module and that of decomposition. In order to deal with their dual properties, we must consider the dual condition to (E-I) in [9]:

(M-I) *Every monomorphism of an R-module into itself is isomorphic.*

If a uniform R-module M satisfies (M-I), then $\mathrm{End}_R(M)$ is a local ring, namely M is *completely indecomposable*. In particular, indecomposable quasi-injective R-modules are completely indecomposable modules with (M-I). Artinian R-modules clearly satisfy (M-I).

For a set $\{M_\alpha\}_I$ of R-modules with (M-I), we define a *partial order* $\leqslant^*$ in the set as follows: If $M_\alpha \approx M_\beta$, we put $M_\alpha \equiv M_\beta$. If there exists a monomorphism of M_α to M_β, we define $M_\alpha \leqslant^* M_\beta$.

Let $\{N_\alpha\}_I$ be an independent set of submodules of an R-module M. $\sum_I \oplus N_\alpha$ is said to be *a locally direct summand* of M [10] if for any finite subset J of I, $\sum_J \oplus N_\beta$ is a direct summand of M.

A set $\{M_\alpha\}_I$ which consists of completely indecomposable R-modules is called a *locally* (resp. *semi-*) *T-nilpotent set* if for any family of countable non-isomorphisms $\{f_{i_n}: M_{i_n} \to M_{i_{n+1}} \mid n \geq 1\}$ (resp. $i_n \neq i_{n'}$ for $n \neq n'$) and any x in M_{i_1}, there exists an integer m depending on x such that $f_{i_m} f_{i_{m-1}} \cdots f_{i_1}(x) = 0$ ([5]). It is known in [10] that $\{M_\alpha\}_I$ is locally semi-T-nilpotent if and only if, for any independent set $\{N_\beta\}_J$ of submodules of $M = \sum_I \oplus M_\alpha$, $\sum_J \oplus N_\beta$ is a direct summand of M if it is a locally direct summand of M.

For given R-modules M and M' and submodules $N \subseteq M$ and $N' \subseteq M'$, we say $\mathrm{Hom}_R(N, N')$ is *extended to* $\mathrm{Hom}_R(M, M')$ if every element in $\mathrm{Hom}_R(N, N')$ is extended to one in $\mathrm{Hom}_R(M, M')$. Similarly we say $\mathrm{Aut}_R(N)$ is extended to $\mathrm{Aut}_R(M)$ provided every automorphism of N is extended to one of M.

Let $M = \sum_I \oplus M_\alpha$. If N is a submodule of M such that N is contained in $\sum_{i=1}^n \oplus M_{\alpha_i}$ for some finite subset $\{\alpha_1, \cdots, \alpha_n\}$ in I, we say N is *finitely contained in the direct sum* (with respect to the decomposition $M = \sum_I \oplus M_\alpha$) and we briefly write it as f.c. module. Let x be a non-zero element in M and express it in the direct sum as $x = x_{\alpha_1} + \cdots + x_{\alpha_n}$ where $0 \neq x_{\alpha_i} \in M_{\alpha_i}$. Then we say x has *the length n* for $M = \sum_I \oplus M_\alpha$. In the expression of x it may happen that all annihilators $(0: x_{\alpha_i})$ are the same. If x is written in this form, we say x is *a smooth element* for $M = \sum_I \oplus M_\alpha$. We omit the word 'for $M = \sum_I \oplus M_\alpha$' in the definition if no confusions arise. We denote the set of all smooth elements of M by $\mathcal{S}(M = \sum_I \oplus M_\alpha)$. We say a submodule of M is *smooth* if every non-zero elements in it is smooth.

We can easily verify the following facts about smooth elements for $M = \sum_I \oplus M_\alpha$.

1) *Every element in* $(\bigcup_I M_\alpha) - \{0\}$ *is smooth with the length* 1.

2) *Let* $\{1, \cdots, n\} \subseteq I$ *and* $0 \neq x_i \in M_i$ *for* $i = 1, \cdots, n$. *If* $f_i: x_i R \to x_{i+1} R$ *is a monomorphism for* $i = 1, \cdots, n$, *then* $\{z + f_1(z) + \cdots + f_n f_{n-1} \cdots f_1(z) \mid z \in x_1 R\}$ *is a smooth submodule of* M.

3) *If x is a smooth element with the length n, so is every non-zero element in xR.*

4) *For* $0 \neq x$ *in* M, *there exists* r *in* R *such that* $xr \neq 0$ *is smooth.*

5) *If* xR *is a simple submodule of* M, *then it is smooth.*

2. Extending property of submodule

First we shall consider the following special case.

Proposition 1. *Let* $\{E_\alpha\}_I$ *be a set of indecomposable injective R-modules and* $E=\sum_I \oplus E_\alpha$. *Then*

1) *Each* E_α *is a uniform module satisfying* (M-I).

2) *Let* $\{N_\beta\}$ *be an independent set of f.c. uniform modules of* E. *Then there exists a set* $\{F_\beta\}_J$ *of direct summands of* E *such that* $N_\beta \subseteq_e F_\beta$ *and* $\sum_J \oplus F_\beta$ *is a locally direct summand.*

3) *If* x *is a smooth element of* E, *then there exists a direct summand* F *of* E *such that* F *is indecomposable and* $xR \subseteq_e F$.

Proof. 1) is clear. 2) Since N_β is f.c., we can take $F_\beta = E(N_\beta)$ in E for each β. Then $\{F_\beta\}_J$ is a desired set. Similarly we can show 3).

In this section we shall study some properties in the above proposition on a more general case, namely on direct sums of completely indecomposable modules. For this purpose we introduce several properties.

Let M be an R-module and N a subouule of M. We say N is *essentially extended to a direct summand of* M if there exists a direct summand of M which contains N as an essential submodule. We say M has *the extending property of simple* (resp. *uniform*) *module* provided every simple (resp. uniform) submodule of M is essentially extended to a direct summand of M. When $M=\sum_I \oplus M_\alpha$, we say M has *the extending property of cyclic smooth module for* $M=\sum_I \oplus M_\alpha$ if every cyclic smooth submodule of M is essentially extended to a direct summand of M. We also omit the word '*for* $M=\sum_I \oplus M_\alpha$' in the above. Similarly we can define *the extending property of smooth module, the extending property of f.c. uniform module* and *the extending property of uniform and smooth module,* etc.

Now, we are concerned with those properties on direct sums of completely indecomposable R-modules. Therefore, from now on, we assume $\{M_\alpha\}_I$ is *a set of completely indecomposable R-modules and* $M=\sum_I \oplus M_\alpha$; whence each M_α has the exchange property [16].

We often take a subset $\mathcal{F}$ of $\mathcal{S}(M=\sum_I \oplus M_\alpha)$ which satisfies the following condition:

(*) *For any* $0 \neq x \in M$, *there exists* r *in* R *such that* $xr \in \mathcal{F}$.

For example, $\mathcal{S}(M=\sum_I \oplus M_\alpha)$ itself satisfies the condition (*). In the case of $S(M) \subseteq_e M$, the set of all non-zero elements x in M such that xR is simple satisfies the condition (*).

Proposition 2. *Let $\mathcal{F}$ be a subset of $\mathcal{S}(M=\sum_I \oplus M_\alpha)$ satisfying (*) and let $\beta \in I$. If every cyclic submodule of M generated by an element in $\mathcal{F} \cap M_\beta$ is essentially extended to a direct summand of M, then M_β is uniform.*

Proof. Contrary to the assertion, assume that M_β is not uniform and take non-zero elements x_1, x_2 in M_β such that $x_1R \cap x_2R=0$. Since $\mathcal{F}$ satisfies (*), there exists r in R such that $x_1r \in \mathcal{F}$. By the assumption, we can take a direct summand N of M such that $x_1rR \subseteq_e N$. Since N is a direct summand of M, there exists a direct summand $N' \langle \oplus N$ which is isomorphic to some member in $\{M_\alpha\}_I$ by [1]. Inasmuch as N' has the exchange property, we have

$$M = N' \oplus \sum_J \oplus M_\gamma$$

for some $J \subseteq I$. Then we see from $N' \cap M_\beta \neq 0$ that $I-J=\{\beta\}$. Let $x_2=y+z$ where $y \in N'$ and $z \in \sum_J \oplus M_\gamma$. Since $x_2 \in M_\beta$, y must be non-zero. Hence there exists $s \in R$ such that $0 \neq ys \in x_1rR$ since $x_1rR \subseteq_e N$. Then $x_2s-ys \in M_\beta$ and $zs \in \sum_J \oplus M_\gamma$, whence $x_2s-ys=0$. However $x_1R \cap x_2R=0$ shows $ys=0$, a contradiction.

Corollary 3. *Let $\mathcal{F}$ be as in Proposition 2. If every cyclic R-submodule of M generated by an element in $\mathcal{F}$ is essentially extended to a direct summand of M, then each M_α is uniform.*

Proposition 4. *Let $x \in \mathcal{S}(M=\sum_I \oplus M_\alpha)$. If N is a direct summand of M such that $xR \subseteq_e N$. then N is completely indecomposable.*

Proof. We express x in $M=\sum_I \oplus M_\alpha$ as $x=x_1+\cdots+x_n$ where $0 \neq x_i \in M_i$, $i=1, \cdots, n$. Since $N \langle \oplus M$, there exists a direct summand $N' \langle \oplus N$ which is isomorphic to some member in $\{M_\alpha\}_I$ by again [1]. It is sufficient to show $N'=N$. Since N' has the exchange property,

$$M = N' \oplus \sum_J \oplus M_\beta \tag{\#}$$

for some $J \subseteq I$. Since $N' \cap xR \neq 0$, there exists $i \in \{1, \cdots, n\}$ which does not lie in J. Without loss of generality we can assume $i=1$. Then $I-J=\{1\}$. Now, let $N=N' \oplus N''$ and assume $N'' \neq 0$. We take $0 \neq z$ in N'' and express it in (#) as $z=y+p+q$ where $y \in N'$, $p \in M_2 \oplus \cdots \oplus M_n$ and $q \in \sum_K \oplus M_\gamma$ $(K=J-\{2, \cdots, n\})$. Since $xR \subseteq_e N$, we can easily take r in R such that $0 \neq zr \in xR$ and $yr \in xR$. Since $zr-yr \in xR$, $zr-yr=xt$ for some $t \in R$. Noting that $xt=(p+q)r$, $xt \in M_1 \oplus \cdots \oplus M_n$ and $pr \in M_2 \oplus \cdots \oplus M_n$, we see $qr=0$ and hence $xt=pr$. Since x

is a smooth element with the length n, so is xt if $xt \neq 0$, Therefore the fact $xt = pr \in M_2 \oplus \cdots \oplus M_n$ implies $xt=0$, whence $zr=yr$. However $N' \cap N''=0$ says that $zr=0$, a contradiction. Thus we must have $N'=N$.

Theorem 5. *Let $\mathcal{F}$ be as in Proposition 2 and assume if $\sum_{i=1}^{m} x_i \in \mathcal{F}$, $x_i+x_j \in \mathcal{F}$ for each i, j, where $x_i \in M_i$. Then the following conditions are equivalent:*

1) *Every cyclic submodule of M generated by an element in $\mathcal{F}$ is essentially extended to a direct summand of M.*

2) *For any pair α and β in I, every cyclic submodule of M generated by an element in $\mathcal{F} \cap (M_\alpha \oplus M_\beta)$ is essentially extended to a direct summand of $M_\alpha \oplus M_\beta$.*

3) i) *Eahc M_α is uniform.*

ii) *For any pair α and β in I and any non-zero elements $x_\alpha \in M_\alpha$ and $x_\beta \in M_\beta$ such that $x_\alpha + x_\beta \in \mathcal{F}$, there exists a monomorphism g of either M_α into M_β or M_β into M_α such that $g(x_\alpha)=x_\beta$ or $g(x_\beta)=x_\alpha$.*

Proof. In view of these conditions we may only show 1)$\Leftrightarrow$3)

1)$\Rightarrow$3). By Proposition 2, each M_α is uniform. Let M_1, M_2 be two members in $\{M_\alpha\}_I$ and let $0 \neq x_i \in M_i$, $i=1, 2$ such that $x=x_1+x_2 \in \mathcal{F}$. By 1), there exists a direct summand M' of M such that $xR \subseteq_e M'$. Then, by Proposition 4, M' is a completely indecomposable and uniform module. Using the exchange property of M' and the fact $xR \subseteq x_1R \oplus x_2R$, we have

$$M = M' \oplus \sum_{I-\{1\}} \oplus M_\alpha \quad \text{or} \quad M = M' \oplus \sum_{I-\{2\}} \oplus M_\beta .$$

In the former case let ψ: $M = M' \oplus \sum_{I-\{1\}} \oplus M_\alpha \to M_2$ be the projection. Then it is easy to see $\psi(x_1) = -x_2$. Since $x_iR \subseteq_e M_i$ and $x \in \mathcal{F}$, $-\psi | M_1$ is a monomorphism of M_1 into M_2 satisfying $-\psi(x_1)=x_2$. Similarly we have a monomorphism g of M_2 into M_1 with $g(x_2)=x_1$ in the latter case.

3)$\Rightarrow$1). Let $x \in \mathcal{F}$. If x lies in some M_α, then we have indeed $xR \subseteq_e M_\alpha \langle \oplus M$. Thus assume that the length n of x is not 1, and let $x=x_1+\cdots+x_n \in M_1 \oplus \cdots \oplus M_n$ where $x_i \in M_i$. By 3), there exists either a monomorphism h_{ij}: $M_i \to M_j$ or h_{ji}: $M_j \to M_i$ satisfying $h_{ij}(x_i)=x_j$ or $h_{ji}(x_j)=x_i$ for each pair i, j in $\{1, \cdots, n\}$. As a result we can take $k \in \{1, \cdots, n\}$ and monomorphisms h_{kj}: $M_k \to M_j$ for all $j \neq k$ satisfying $h_{kj}(x_k)=x_j$. We may assume $k=1$. We put

$$M_1' = \{z+h_{12}(z)+\cdots+h_{1n}(z) \mid z \in M_1\} .$$

Then $M_1' \oplus M_2 \oplus \cdots \oplus M_n = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ and moreover it follows from $x_1R \subseteq_e M_1$ that $xR \subseteq_e M_1$.

REMARK. In the case where M has the essential socle and $\mathcal{S} = \{x \in M \mid xR$ is simple$\}$, the above proposition is dual to [9, Theorem 2].

Corollary 6. *If M has the extending propety of cyclic smooth module for*

$M=\sum\oplus M_\alpha$, then it has also the extending property of cyclic smooth module for any other decomposition of M by completely indecomposable modules.

Corollary 7. *Let $\mathfrak{F}$ be as in Theorem 5 and assume that each M_α satisfies* (M-I). *Then the following conditions are equivalent:*

1) *Every cyclic submodule of M generated by an element in $\mathfrak{F}$ is essentially extended to a direct summand of M.*

2) i) *For $M_\beta\in\{M_\alpha\}_I$, let $\{M_i\}_{K(\beta)}$ be the set of all $M_i\in\{M_\alpha\}_{I-\{\beta\}}$ such that $M_\alpha\oplus M_i$ has a smooth element with the length 2. Then the relation $\leqslant^*$ is linear in $\{M_\beta\}\cup\{M_i\}_{K(\beta)}$ for any $\beta\in I$.*

ii) *For any pair $M_\alpha\leqslant^* M_\beta$ and any x in $\mathfrak{F}\cap(M_\alpha\oplus M_\beta)$ with the length 2, say $x=x_\alpha+x_\beta$, there exists a monomorphism $f\colon M_\alpha\to M_\beta$ satisfying $f(x_\alpha)=x_\beta$.*

Corollary 8 (cf. [9, Corollary of Theorem 2]). *Assume each M_α satisfies the condition* (M-I). *Then the following conditions are equivalent:*

1) *M has the extending property of simple module.*

2) *For any β in I, the relation $\leqslant^*$ is linear in the subset $\{M_i\}_{K(\beta)}$ of all M_i in $\{M_\alpha\}_I$ such that $S(M_\alpha)\approx S(M_\beta)$, and* $\mathrm{Hom}_R(S(M_\alpha), S(M_\beta))$ *is extended to* $\mathrm{Hom}_R(M_\alpha, M_\beta)$ *for any pair α, β in I.*

Proof. We can assume $S(M)\subseteq_e M$, whence this is immediate by Theorem 5.

We can obtain similarly to Theorem 5

Theorem 9. *We assume each M_α is uniform. Then the following conditions are equivalent:*

1) *M has the extending property of uniform and smooth module.*

2) *For any pair α, β in I and any monomorphism f of a submodule A_α of M_α to M_β is extended to a monomorphism of M_α to M_β or f^{-1} is extended to one of M_β to M_α.*

Proof. If U is a uniform and smooth module, then there exists $\{\alpha_1, \cdots, \alpha_n\}\subseteq I$ for which $U\subseteq M_{\alpha_1}\oplus\cdots\oplus M_{\alpha_n}$ and every non-zero element in U has the length n. Noting this fact, the proof is done by the same argument as in the proof of Theorem 5.

Theorem 10. *We assume each M_α is uniform. Then the following conditions are equivalent:*

1) *M has the extending property of f.c. uniform module.*

2) *For any pair α, β in I, any homomorphism f of a submodule A_α in M_α to M_β is extended to an element in* $\mathrm{Hom}_R(M_\alpha, M_\beta)$ *or f^{-1} is extended to hne in* $\mathrm{Hom}_R(M_\beta, M_\alpha)$, *provided* $\ker f=0$.

Proof. 1)$\Rightarrow$2). Using the same notations as in the proof of Theorem 5, we have

$$M = M'\oplus \sum_{I-\{1\}} \oplus M_\alpha \quad \text{or} \quad M = M'\oplus \sum_{I-\{2\}} \oplus M_\alpha$$

where $\{x+f(x)\,|\,x\in M_1\}\subseteq_e M'\langle\oplus M$. If $\ker f \neq 0$, $M'\supseteq \ker f$. Hence, we have the former decomposition. Let $\pi\colon M\to M_2$ be the projection on this decomposition. Then $-\pi|M_1$ is an extension of f. If $\ker f=0$, then the assertion follows from Theorem 9.

2)$\Rightarrow$1). Let N be a f.c. uniform submodule in M, say $N\subseteq\sum_{i=1}^{n}\oplus M_{\alpha_i}$. Let $\pi_\alpha\colon M\to M_\alpha$ be the projection. Since $\cap \ker \pi_{\alpha_i}|N=0$ and N is uniform, there exists i such that $\ker \pi_{\alpha_i}|N=0$. Hence, by 2) there exists j say 1 such that $N=\{x+f_1(x)+\cdots+f_n(x)\,|\,x\in\pi_{\alpha_1}(N)\}$ where $f\colon M_{\alpha_1}\to M_{\alpha_i}$ is an extension of $\pi_{\alpha_i}(\pi_{\alpha_1}|N)^{-1}$. Put $M'=\{y+f_2(y)+\cdots+f_n(y)\,|\,y\in M_{\alpha_1}\}$. Then $\sum_{i=1}^{n}\oplus M_{\alpha_i}=M'\oplus\sum_{i=2}^{n}\oplus M_{\alpha_i}$ and $N\subseteq_e M'$.

Corollary 11. *Let T be a completely indecomposable and uniform module. Then T is quasi-injective if and only if $T\oplus T$ has the extending property of uniform module and T satisfies* (M-I).

Proof. This is clear from Theorem 10 and [4], [11].

Theorem 12. *Assume each M_α is uniform and for any pair α, β in I, every monomorphism of M_α to M_β is isomorphic. Then the following conditions are equivalent*:

1) *M has the extending property of uniform module.*

2) *For any α in I and a submodule A_α of M_α, every homomorphism of A_α to $\sum_{I-\{\alpha\}}\oplus M_\beta$ is extended to an element in* $\mathrm{Hom}_R(M_\alpha, \sum_{I-\{\alpha\}}\oplus M_\beta)$.

Proof. For $\alpha\in I$, π_α denotes the projection $M=\sum_I\oplus M_\alpha\to M_\alpha$.

1)$\Rightarrow$2). Let A_α be a submodule of M_α and f a homomorphism of A_α to $\sum_{I-\{\alpha\}}\oplus M_\alpha$. Putting $N=\{x+f(x)\,|\,x\in A_\alpha\}$, N is uniform; whence we can easily take $\{\alpha_1, \cdots, \alpha_n\}\subseteq I$ such that $N\cap\mathcal{S}(M=\sum_I\oplus M_\alpha)=\{x+\pi_{\alpha_1}f(x)+\cdots+\pi_{\alpha_n}(f(x)\,|\,0\neq x\in A_\alpha\}$ and all elements in $N\cap\mathcal{S}(M=\sum_I\oplus M_\alpha)$ has the length $n+1$ (cf. the proof 2)$\Rightarrow$1) in Theorem 10). Put $f_1=\sum_{i=1}^{n}\pi_{\alpha_i}f$ and $f_2=\sum_K\pi_\beta f$ where $K=I-\{\alpha, \alpha_1, \cdots, \alpha_n\}$. Then $f=f_1+f_2$. Since $\pi_{\alpha_i}f$ is monomorphic for all $i=1, \cdots, n$, f_1 can be extended to an element f_1' in $\mathrm{Hom}_R(M_\alpha, \sum_{i=1}^{n}\oplus M_{\alpha_i})$ by Theorem 9 and the assumption that every monomorphism of M_{α_i} to M_{α_j} is isomorphic for any pair α_i, α_j in $\{\alpha_1, \cdots, \alpha_n\}$. On the other hand, $\pi_\beta f$ is non-monomorphic for all $\beta\in K$, and hence by the same argument as in the proof of Theorem 10, f_2 can be also extended to an element f_2' in $\mathrm{Hom}_R(M_\alpha, \sum_{I-\{\alpha\}}\oplus M_\beta)$. Thus $f_1'+f_2'$ is a desired extension of f.

2)$\Rightarrow$1). Let $N\,(\neq 0)$ be a uniform submodule of M. Then there exist $\{\alpha_1, \cdots, \alpha_n\}\subseteq I$ for which $N\cap \mathcal{S}(M=\sum_I \oplus M_\alpha)=\{\pi_{\alpha_1}+\cdots+\pi_{\alpha_n}(x)\,|\,0\neq x\in N\}$ and all elements in $N\cap \mathcal{S}(M=\sum_I \oplus M_\alpha)$ have the length n. Take α in $\{\alpha_1, \cdots, \alpha_n\}$. Then the mapping $f\colon \pi_\alpha(N)\to \sum_{I-\{\alpha\}} \oplus M_\beta$ given by $f(\pi_\alpha(x))=\sum_{I-\{\alpha\}} \pi_\beta(x)$ is homomorphism and $N=\{x+f(x)\,|\,x\in\pi_\alpha(N)\}$. Let $f'\colon M_\alpha\to \sum_{I-\{\alpha\}} \oplus M_\beta$ be an extension of f and put $N'=\{y+f(y)\,|\,y\in M_\alpha\}$. Then we can see $N\subseteq_e N'\langle\oplus M$.

Theorem 13 (cf. [9, Theorem 1]). *Assume each M_α is uniform. Then the following conditions are equivalent.*

1) *For any independent family $\{N_\beta\}_J$ of direct summands of M which are uniform, $N=\sum_J \oplus N_\beta$ is a locally direct summand.*

2) *Every monomorphism of M_α to M_β is isomorphic for any pair α and β in I*

Proof. 1)$\Rightarrow$2). We assume $f\colon M_\alpha\to M_\beta$ is monomorphic where $\alpha, \beta\in I$ and $\alpha\neq\beta$. Put $M'_\alpha=\{x+f(x)\,|\,x\in M_\alpha\}$. Then $M'_\alpha\cap M_\beta=0$ and $M'_\alpha\oplus M_\beta=M_\alpha\oplus M_\beta$, from which we see $M'_\alpha\approx M_\alpha$ and M'_α is a direct summand of $M_\alpha\oplus M_\beta$. Further $M'_\alpha\cap M_\alpha=0$. Hence, by 1) $M'_\alpha\oplus M_\alpha=M_\alpha\oplus M_\beta$ and hence f is epimorphic.

2)$\Rightarrow$1). We note that if $M_\alpha\approx M_\beta$ for $\alpha\neq\beta$, then M_α satisfies (M-I). To show 1), we may show the following: If $\{N_1, \cdots, N_n\}$ is an independent set of direct summands of M which are uniform then $N_1\oplus\cdots\oplus N_n$ is also a direct summand of M. If $n=1$, this is clear. Assume $n>1$ and $N_1\oplus\cdots\oplus N_{n-1}$ is a direct summand of M. Since each N_i is a direct summand of M and is uniform, it is isomorphic to some member in $\{M_\alpha\}_I$ by [1]. Since $N_1\oplus\cdots\oplus N_{n-1}$ has the exchange property, we have

$$M = N_1\oplus\cdots\oplus N_{n-1}\oplus\sum_J\oplus M_\gamma$$

for some $J\subseteq I$. Since N_n has the exchange property, we have either $M=N_1\oplus\cdots\oplus N_{k-1}\oplus N_{k+1}\oplus\cdots\oplus N_{n-1}\oplus N_n\oplus\sum_J\oplus M_\gamma$ for some k or $M=\sum_{i=1}^n\oplus N_i\oplus\sum_{J-\{\sigma\}}\oplus M_\gamma$ for some $\sigma\in J$. We have done in the latter case. In the former case $N_k\approx N_n$. Let $\pi_\gamma\colon N_1\oplus\cdots\oplus N_{n-1}\oplus\sum_J\oplus M_\gamma\to M_\gamma$ be the projection. Noting $N_n\nsubseteq N_1\oplus\cdots\oplus N_{n-1}$, we can easily see that there exists $0\neq x\in N_n$ and some ρ in J for which $\pi_\rho\,|\,xR$ is monomorphic. Since $xR\subseteq_e N_n$, it follows that $\pi_\rho\,|\,N_n$ is monomorphic. Since $N_n\approx N_k\approx M_\alpha$ for some α in J, $\pi_\rho\,|\,N_n$ is isomorphic by 2). Thus $M=N_n\oplus\ker\pi_\rho=N_1\oplus\cdots\oplus N_n\oplus\sum_{J-\{\rho\}}\oplus M_\beta$.

Corollary 14. *Assume each M_α is uniform. Then the following conditions are equivalent*:

1) *If N is in Theorem* 13, *then N is a direct summand of M.*

2) $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent set and* 2) *of Theorem* 13 *holds.*

Proof. Use the same argument as in [9].

Theorem 15. *Assume* $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent set and every monomorphism of* M_α *to* M_β *is isomorphic for any pair* α, β *in* I. *Then the following conditions are equivalent*:

1) M *has the extending property of submodule.*

2) i) *Each* M_α *uniform.*

ii) *For any subset* J *of* I *and any submodule* A *of* $\sum_J \oplus M_\beta$, $\mathrm{Hom}_R(A, \sum_{I-J}\oplus M_\alpha)$ *is extended to* $\mathrm{Hom}_R(\sum_J \oplus M_\beta, \sum_{I-J}\oplus M_\alpha)$.

Proof. 1)$\Rightarrow$2). By Corollary 3, each M_α is uniform. Let J be a subset and A a submodule of $\sum_J\oplus M_\beta$. Put $P=\sum_J\oplus M_\beta$ and $Q=\sum_{I-J}\oplus M_\alpha$. To show $\mathrm{Hom}_R(A, Q)$ is extended to $\mathrm{Hom}_R(P, Q)$, we may assume $A\subseteq_e P$. Now, by 1), there exists a direct summand B of M such that $\{x+f(x)\,|\,x\in A\}\subseteq_e B$. Then $B\cap Q=0$ and $B\oplus Q\subseteq_e M$. Since $B\langle\oplus M$, B is also a direct sum of completely indecomposable modules by [6], [12]. Hence we see $M=B\oplus Q$ by Corollary 14. Let π: $M=B\oplus Q\to Q$ be the projection. Then we see $-\pi|P$ is an extension of f.

2)$\Rightarrow$1). Let A ($\neq 0$) be a submodule of M. By Zorn's lemma, we can take $J\subseteq I$ such that $A\cap\sum_{I-J}\oplus M_\alpha=0$ and $A\cap\sum_K\oplus M_\gamma\neq 0$ for any $K\supsetneqq I-J$. Again we put $P=\sum_I\oplus M_\alpha$ and $Q=\sum_{I-J}\oplus M_\alpha$. Noting each M_α is uniform, as is easily seen, $A\oplus Q\subseteq_e M$. Let π and π' be projections: $M=\sum_I\oplus M_\alpha\to P$ and $M=\sum_I\oplus M_\alpha\to Q$, respectively. Then we see $\pi(A)\subseteq_e P$ since $\pi(A)\oplus Q=A\oplus Q\subseteq_e M$. Since $A\cap Q=0$, the mapping f: $\pi(A)\to Q$ given by $f(\pi(a))=\pi'(a)$ is a homomorphism. Using 2), f is extended to a homomorphism f' of P to Q. Put $A'=\{x+f(x)\,|\,x\in P\}$. Then $A\subseteq A'\langle\oplus M$. Moreover $\pi(A)\subseteq_e P$ shows $A\subseteq_e A'$.

Theorem 16. *Assume that* i) *each* M_α *satisfies* (M-I), ii) *every monomorphism of* M_α *to* M_β *is isomorphic for any pair* α, β *in* I *and* iii) M *has the extending property of cyclic smooth module. Then every submodule of* M *which is isomorphic to some member of* $\{M_\alpha\}_I$ *is a direct summand of* M. *Therefore the following conditions are equivalent by* [7, Lemma 2] (cf. [17]):

1) $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent set.*

2) M *has the exchange property.*

Proof. Let N be a submodule of M which is isomorphic to some member in $\{M_\alpha\}_I$. By iii) and Corollary 3, each M_α is uniform and so is N. Let $x\in N\cap S(M=\sum_I\oplus M_\alpha)$. Then, by iii) and Proposition 4, there exists a direct summand N' of M such that $xR\subseteq_e N'$ and N' is completely indecomposable. Since N' has the exchange property,

$$M=N'\oplus\sum_J\oplus M_\beta$$

for some $J\subseteq I$. Let π: $M=N'\oplus\sum_J\oplus M_\beta\to N'$ be the projection. Noting $xR\subseteq_e N$ and $xR\subseteq_e N'$, we can easily verify $\pi|N$ is monomorphic and $N\cap\sum_J\oplus M_\beta=0$. Here, using i) and ii) we see $\pi|N$ is isomorphic. Thus we have $M=N\oplus\sum_J\oplus M_\beta$.

3. Extending property of decomposition

Let M be an R-module and $\{N_\beta\}_I$ be an independent set of submodules of M. We say $\sum_I\oplus N_\beta$ is *extended to a decomposition* of M if there exists a decomposition $M=\sum_J\oplus N'_\beta\oplus M'$ such that $N_\beta\subseteq_e N'_\beta$ for all $\beta\in J$. For a submodule N of M, we say M has *the extending property* of *decomposition* of N if every decomposition of N is extended to a decomposition of M. Further we say M has *the extending property of direct sum of uniform modules* if every direct sum of uniform submodules of M is extended to a decomposition of M. Similarly we can define the phrase of the extending property of direct sum of submodules. When $M=\sum_I\oplus M_\alpha$, we say M has *the extending property of direct sum of cyclic smooth modules* for $M=\sum_I\oplus M_\alpha$ if every direct sum of cyclic smooth submodules is extended to a decomposition of M. We also omit the word '*for* $M=\sum_I\oplus M_\alpha$' in the above if no confusions arise. We can also define the phrases of the extending property of direct sum of uniform and smooth modules and the extending property of direct sum of f.c. uniform modules, etc.

In this section we also assume $\{M_\alpha\}_I$ is a set of completely indecomposable R-modules and put $M=\sum_I\oplus M_\alpha$.

Lemma 17. *Let $\mathfrak{F}$ be as in Proposition 2. If, for any $\{x_\beta\}_J\subseteq\mathfrak{F}$ such that $\{x_\beta R\}_J$ is independent $\sum_J\oplus x_\beta R$ is extended to a decomposition of M, then every cyclic submodule of M generated by an element in $\mathfrak{F}$ is essentially extended to a direct summand of M.*

Proof. Let $x\in\mathfrak{F}$. Since $\mathfrak{F}$ satisfies the condition (*) in §2, we can take $\{x_\beta\}_J\subseteq\mathfrak{F}$ such that $x\in\{x_\beta\}_J$, $\{x_\beta R\}_J$ is independent and $\sum_J\oplus x_\beta R\subseteq_e M$. Therefore, using the assumption, we can easily see that xR is essentially extended to a direct summand of M.

Theorem 18. *We assume $\{M_\alpha\}_I$ is a locally semi-T-nilpotent set and each*

M_α satisfies (M-I). *Then, for a subset $\mathcal{F}$ of $\mathcal{S}$ $(M=\sum_I \oplus M_\alpha)$ satisfying the conditions in Theorem* 5, *the following conditions are equivalent*:

1) *For any subset $\{x_\beta\}_J$ of $\mathcal{F}$ such that $\{x_\beta R\}_J$ is independent, $\sum_J \oplus x_\beta R$ is extended to a decomposition of M.*

2) i) *Each M_α is uniform.*

ii) *If $\mathcal{F}$ contains x with the length* 2 *expressed as $x=x_1+x_2$ where $x_i \in M_i$, then there exists an isomorphism f: $M_1 \approx M_2$ with $f(x_1)=x_2$.*

3) i) *Every cyclic submodule of M generated by an element in $\mathcal{F}$ is essentially extended to a direct summand of M.*

ii) *Every monomorphism of M_α to M_β is isomorphic for any pair α, β in I.*

Proof. 1)$\Rightarrow$2). By Corollary 3 and Lemma 17, each M_α is uniform. Let $M_1 \in \{M_\alpha\}_I$ and assume that there exists $M_i \in \{M_\alpha\}_{I-\{1\}}$ such that $M_1 \oplus M_i$ contains a smooth element with the length 2 and denote the set of all such M_i and M_1 by $\{M_i\}_K$. Then by Corollary 7, $\leqslant^*$ is linear in $\{M_i\}_K$. For each $i \in K-\{1\}$, we can choose $y_i \in \mathcal{F} \cap (M_1 \oplus M_i)$ with the length 2 since $\mathcal{F}$ satisfies $(*)$. Let $y_i=z_i+x_i$ where $z_i \in M_1$ and $x_i \in M_i$. Further we take $0 \neq x_\gamma \in M_\gamma \cap \mathcal{F}$ for all $\gamma \in J=I-K$. Then $\{y_i\}_K \cup \{x_\gamma\}_J \subseteq \mathcal{F}$ and $\{y_i R\}_K \cup \{x_\gamma R\}_J$ is independent. Since each M_α is uniform, this shows $\sum_K \oplus y_i R \oplus \sum_J \oplus x_\gamma R \subseteq_e M$. By 1), we have a decomposition $M=\sum_K \oplus F_i \oplus \sum_J \oplus F_\gamma$ such that $y_i R \subseteq_e F_i$ for all $i \in K$ and $x_\gamma R \subseteq_e F_\gamma$ for all $\gamma \in J$. Then each F_i and each F_γ are completely indecomposable by Proposition 4 and Lemma 17.

Now, noting $y_i R \subseteq_e F_i$ for all $i \in K$, we see from the choise of $\{M_i\}_K$ that, for any $i \in K$, F_i is not isomorphic to any member of $\{M_\gamma\}_J$. Hence there exists one to one mapping via isomorphism between the set $\{M_i\}_K$ and $\{F_i\}_K$ by Krull-Remak-Schmidt-Azumaya's theorem [1]. Let π: $M=\sum_I \oplus M_\alpha \to M_1$ be the projection. Since $\pi \mid y_i R$ for all $i \in K$ is monomorphic, we see from $y_i R \subseteq_e F_i$ that $\pi \mid F_i$ are monomorphic for all $i \in K$. Consequently M_1 is the largest with the relation $\leqslant^*$. As a result, we see that if x is an element in $\mathcal{F}$ with the length 2, say $x=x_\alpha+x_\beta$, then $M_\alpha \approx M_\beta$ and there exists an isomorphism f: $M_\alpha \approx M_\beta$ satisfying $f(x_\alpha)=x_\beta$ by Corollary 7 and the condition (M-I).

2)$\Rightarrow$1). For any $x \in \mathcal{F}$, xR is essentially extended to a direct sumand of M by Theorem 5. Further if f is a monomorphism of M_α to M_β, then there exists an element in $\mathcal{F} \cap (M_\alpha \oplus M_\beta)$ with the length 2, whence $M_\alpha \approx M_\beta$ by ii) and therefore f is isomorphic by (M-I). Now, let $\{x_\beta\}_J$ be a subset of $\mathcal{F}$ such that $\{x_\beta R\}_J$ is independent. To show that $\sum_J \oplus x_\beta R$ can be extended to a decomposition of M, we can assume $\sum_J \oplus x_\beta R \subseteq_e M$. Let N_β be a direct summand of M such that $x_\beta R \subseteq_e N_\beta$ for all $\beta \in J$. Then $\{N_\beta\}_J$ is independent since

$\{x_\beta R\}_J$ is so and $x_\beta R \subseteq_e N_\beta$ for all β in J. Therefore, by Corollary 14, $\sum_J \oplus N_\beta < \oplus M$, from which we have $M = \sum_J \oplus N_\beta$.

1), 2)$\Rightarrow$3). i) follows from Lemma 17 and ii) follows from 2).

3)$\Rightarrow$1). Let $\{x_\beta\}_J$ be a subset of $\mathcal{F}$ such that $\{x_\beta R\}_J$ is independent. By i), there exists a direct summand N_β of M such that $x_\beta R \subseteq_e N_\beta$ for all β in J. Since $\{x_\beta R\}_J$ is independent, so is $\{N_\beta\}_J$. Hence $\sum_J \oplus N_\beta$ is a direct summand of M by Corollary 14, from which we see that $\sum_J \oplus x_\beta R$ is extended to a decomposition of M.

Corollary 19. *We assume $\{M_\alpha\}_I$ is a locally semi-T-nilpotent set and each M_α satisfies* (M-I). *Then*

1) *M has the extending property of direct sum of cyclic smooth modules if and only if it has the extending property of cyclic smooth module and every monomorphism of M_α to M_β is isomorphic for any pair α, β in I.*

2) *If M has the extending property of direct sum of cyclic smooth modules, it has also the extending property of direct sum of cyclic smooth modules for any other decomposition of M by completely indecomposable modules.*

3) *Let M_1, $M_2 \in \{M_\alpha\}_I$ and $0 \neq x_i \in M_i$, $i=1, 2$. If $x_1 + x_2$ is smooth, then* $\mathrm{Aut}_R(x_i R)$ *is extended to* $\mathrm{Aut}_R(M_i)$ *for $i=1, 2$.*

Proof. If we take $\mathcal{F} = \mathcal{S}$ $(M = \sum_I \oplus M_\alpha)$ in Theorem 18, 1) and 2) are clear.

3). By Theorem 18, there exists an isomorphism f of M_1 to M_2 with $f(x_1) = x_2$. Let $g \in \mathrm{Aut}_R(x_1 R)$. Then $g(x_1) + x_2$ is also a smooth element and hence, by again Theorem 18, there exists an isomorphism h of M_1 to M_2 with $h(g(x_1)) = x_2$. Then $h^{-1} f \in \mathrm{Aut}_R(M_1)$ and $h^{-1} f \mid x_1 R = g$.

Corollary 20. *We assume $\{M_\alpha\}_I$ is a locally semi-T-nilpotent set and each M_α is uniform and satisfies* (M-I). *Then the following conditions are equivalent*:

1) *M has the extending property of decomposition of* $\mathrm{S}(M)$.

2) *For any pair α and β in I, if* $\mathrm{S}(M_\alpha) \approx \mathrm{S}(M_\beta)$, *then $M_\alpha \approx M_\beta$. And* $\mathrm{End}_R(S(M_\alpha))$ *is extended to* $\mathrm{End}_R(M_\alpha)$ *for any α in I.*

3) *M has the extending property of simple module and every monomorphism of M_α to M_β is isomorphic for any pair α, β in I.*

Proof. We can assume $M_e \supseteq S(M)$, and hence take $\mathcal{F} = \{x \in M \mid xR$ is simple$\}$ in Theorem 18.

Theorem 21. *Assume $\{M_\alpha\}_I$ is a locally semi-T-nilpotent set and each M_α is uniform and satisfies* (M-I). *Then the following conditions are equivalent*:

1) *M has the extending property of direct sum of uniform and smooth modules.*

2) *For any pair* α, β *in* I, *any monomorphism of any submodule* A_α *of* M_α *to* M_β *is extended to an isomorphism of* M_α *to* M_β,

3) i) *M has the extending property of uniform and smooth module.*

ii) *Every monomorphism of* M_α *to* M_β *is isomorphic for any pair* α, β *in* I.

Proof. 1)$\Rightarrow$i) of 3) is clear. Hence we have 1)$\Leftrightarrow$2)$\Rightarrow$3) by Theorems 9 and 18. 3)$\Rightarrow$1) is shown by the same argument as in the proof of of 3)$\Rightarrow$1) in Theorem 18.

Theorem 22. *We assume each* M_α *is uniform and satisfies* (M-I). *Then the following conditions are equivalent*:

1) *M has the extending property of direct sum of f.c. uniform modules.*

2) i) $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent set.*

ii) *Every monomorphism of* M_α *to* M_β *is isomorphic for any pair* α, β *in* I.

iii) *For any pair* α, β *in* I *and any submodule* A_α *of* M_α, $\mathrm{Hom}_R(A_\alpha, M_\beta)$ *is extended to* $\mathrm{Hom}_R(M_\alpha, M_\beta)$.

3) i) $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent set.*

ii) *Every monomorphism of* M_α *to* M_β *is isomorphic for any pair* α, β *in* I.

iii) *M has the extending property of c.f. uniform module.*

Proof. 1)$\Rightarrow$2), 3). Clearly iii) of 3) holds and hence iii) of 2) follows from Theorem 10. ii) of 2) follows from Theorem 13, whence to show the rest that $\{M_\alpha\}_I$ is locally semi-T-nilpotent, we may show the following: Let $\{M\}_{i=1}^{\infty}\subseteq\{M_\alpha\}_I$ and $\{f_i\colon M_i\to M_{i+1}\,|\,i\geq 1\}$ a family of non-monomorphisms. Then, for any x in M_1, there exists integer n (depending on x) such that $f_n f_{n-1}\cdots f_1(x)=0$.

To verify this fact, put $M'_i=\{x+f(x)\,|\,x\in M_i\}$ for $i\geq 1$. Then $M'_i\oplus M_{i+1}=M_i\oplus M_{i+1}$ for all $i\geq 1$ and moreover $\{M'_i\,|\,i\geq 1\}$ is independent. Since $M'_i\cap M_i\neq 0$ for all $i\geq 1$, we see $\sum_{i=1}^{\infty}\oplus M'_i\subseteq_e\sum_{i=1}^{\infty}\oplus M_i$, whence by 1) we see $\sum_{i=1}^{\infty}\oplus M'_i=\sum_{i=1}^{\infty}\oplus M_i$. This fact implies that if x in M_1, then there exists n such that $f_n f_{n-1}\cdots f_1(x)=0$.

2)$\Leftrightarrow$3) follows from Theorem 10.

3)$\Rightarrow$1) is shown by the same argument as in the proof of 3)$\Rightarrow$1) in Theorem 18.

Similarly we obtain

Theorem 23. *We assume each* M_α *is uniform and satisfies* (M-I). *Then the following conditions are equivalent*:

1) *M has the extending property of direct sum of uniform modules.*

2) i) $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent set.*

ii) *Every monomorphism of* M_α *to* M_β *is isomorphic for any pair* α, β *in* I.

iii) *For any* α *in* I *and any submodule* A_α *of* M_α, $\mathrm{Hom}_R(A_\alpha, \sum_{I-\{\alpha\}} \oplus M_\beta)$ *is extended to* $\mathrm{Hom}_R(M_\alpha, \sum_{I-\{\alpha\}} \oplus M_\beta)$.

3) i) $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent set.*

ii) *Every monomorphism of* M_α *to* M_β *is isomorphic for any pair* α, β *in* I.

iii) M *has the extending property of uniform module.*

Proof. We can show this by the same argument as in the proof of Theorem 22 (use Theorem 12 instead of Theorem 10).

Further we have the following theorem.

Theorem 24. *The following conditions are equivalent:*

1) M *has the extending property of direct sum of submodules.*

2) i) *Each* M_α *is uniform.*

ii) *Every monomorphism of* M_α *to* M_β *is isomorphic for any pair* α, β *in* I.

iii) $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent set.*

iv) *For any* $J \subseteq I$ *and any submodule* $A \subseteq \sum_J \oplus M_\beta$, $\mathrm{Hom}_R(A, \sum_{I-J} \oplus M_\alpha)$ *is extended to* $\mathrm{Hom}_R(\sum_J \oplus M_\beta, \sum_{I-J} \oplus M_\alpha)$.

3) i) *Every monomorphism of* M_α *to* M_β *is isomorphic for any pair* α, β *in* I,

ii) $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent set.*

iii) M *has the extending property of submodule.*

Proof. 1)$\Rightarrow$2). iii) follows from Corollary 3. So, i) and ii) follow from Corollary 14 and hence we know iv) by Theorem 15.

2)$\Leftrightarrow$3) follows from Theorem 15.

3)$\Rightarrow$1). Let $\{N_\beta\}_J$ be an independent set of submodules of M. By iii), there exists $N'_\beta \langle \oplus M$ such that $N_\beta \subseteq_e N'_\beta$ for all β in J. Since $N'_\beta \langle \oplus M$ and ii) holds, each N'_β is a direct sum of completely indecomposable R-modules by [6], [12]. Hence $\sum_J \oplus N_\beta \langle \oplus M$ by Corollary 14. Hence $\sum_J \oplus N_\beta$ is extended to a decomposition of M.

4. Applications

We start with

Proposition 25. *If* M *is a quasi-injective* R*-module, then* M *has the extending property of submodule. If we assume futher that* M *is a direct sum of completely indecomposable modules, say* $M = \sum_I \oplus M_\alpha$, *then* M *has the extending property of direct sum of submodules.*

Proof. Let N be a submodule of M. Then $E(M)=E(N)\oplus K$ for some K. Since M is quasi-injective, $M=(E(N)\cap M)\oplus(K\cap M)$ and then $N\subseteq_e E(N)\cap M$. Next, since M is quasi-injective, M has the exchange property by [3]. Hence we see from [6], [12] that $\{M_\alpha\}_I$ is locally semi-T-nilpotent. For any pair α, β in I, we show that every monomorphism f: $M_\alpha \to M_\beta$ is isomorphic. Let ρ be an isomorphism of $E(M_\alpha)$ to $E(M_\beta)$ which is an extension of f. Since $M_\alpha \oplus M_\beta$ is quasi-injective, $\rho^{-1}(M_\beta)\subseteq M_\alpha$ by [11]. Hence $M_\beta \subseteq \rho(M_\alpha)=f(M_\alpha)$ and hence f is indeed isomorphic. Now, let $\{N_\beta\}_J$ be an independent set of submodules of M. Then, there exists a direct summand N'_β of M such that $N_\beta \subseteq_e N'_\beta$ for all β in J. Since $\{M_\alpha\}_I$ is locally semi-T-nilpotent and $N'_\beta <\oplus M$, each N'_β is a direct sum of completely indecomposable modules. Consequently we see from Corollary 14 that $\sum_J \oplus N_\beta$ is a direct summand of M; whence $\sum_J \oplus N_\beta$ is extended to a decomposition of M.

Theorem 26. *Let $\{M_\alpha\}_I$ be a set of completely indecomposable R-modules and put $M=\sum_I \oplus M_\alpha$. Then the following conditions are equivalent:*

1) *M is quasi-injective*
2) *$M\oplus M$ has the extending property of direct sum of submodules.*

Proof. 1)$\Rightarrow$2). Since M is quasi-injective, $M\oplus M$ is also quasi-injective by [4], [11]. Hence 2) holds by Proposition 25.

2)$\Rightarrow$1) is easily seen from Theorem 24.

Corollary 27. *Let T be a completely indecomposable R-module and consider $M=\sum_I \oplus M_\alpha$ where $M_\alpha \approx T$ for all α in I; $|I|\geqslant 2$. Then M is quasi-injective if and only if M has the extending property of direct sum of submodules.*

Proof. If the cardinal $|I|=\infty$, then $M\oplus M\approx M$; whence the statement follows from Theorem 26. If $|I|<\infty$, then M is quasi-injective if and only if T is quasi-injective. Hence in this case, the proof also follows from Theorem 26.

Theorem 28. *Let $\{M_\alpha\}_I$ be a set of indecomposable quasi-injective R-modules and $M=\sum_I \oplus M_\alpha$. Consider the following conditions:*

1) *M is quasi-injective.*
2) *For any pair α, β in I,* if *$E(M_\alpha)\approx E(M_\beta)$, then $M_\alpha \approx M_\beta$.*
3) *M has the extending property of direct sum of submodules.*

Then we have 1)$\Rightarrow$3)$\Rightarrow$2)*, and in the case when M is non-singular, all conditions are equivalent.*

Proof. 1)$\Rightarrow$3) follows from Proposition 25.

3)$\Rightarrow$2). Let ρ be any isomorphism of $E(M_\alpha)$ to $E(M_\beta)$ for pair α, β in I.

Then $\rho^{-1}(M_\beta)\cap M_\alpha \neq 0$. Hence, there exists f: $M_\alpha \rightarrow M_\beta$ extended from $\rho|\rho^{-1}(M_\beta)\cap M_\alpha$ and f is isomorphic by Theorem 22. Assuming M is non-singular, we show 2)$\Rightarrow$1). By [13], we see that $\sum_I \oplus E(M_\alpha)$ is quasi-injective and every non-zero homomorphism of $E(M_\alpha)$ to $E(M_\beta)$ is isomorphic for any pair α, β in I; so $\mathrm{Hom}_R(E(M_\alpha), E(M_\beta))=0$ if $E(M_\alpha)\not\approx E(M_\beta)$. Further if α, β in I and f: $E(M_\alpha)\rightarrow E(M_\beta)$ is isomorphic, then $f(M_\alpha)\subseteq M_\beta$. For, by 2), there exists an isomorphism g: $M_\alpha\rightarrow M_\beta$. Let ρ: $E(M_\alpha)\rightarrow E(M_\beta)$ be an extension of g. Since M_α is quasi-injective, $\rho^{-1}f(M_\alpha)\subseteq M_\alpha$ by [11]. This indeed implies $f(M_\alpha)\subseteq M_\beta$. Thus M is quasi-injective by [4] and [11].

Proposition 29. *Let $\{M_\alpha\}_I$ be a set of uniform and completely indecomposable R-modules with* (M-I) *and the cardinal $|I|\neq 1$, and put $M=\sum_I\oplus M_\alpha$. We assume $E(M_\alpha)\approx E(M_\beta)$ for all α, β in I and M has the extending property of direct sum of f.c. uniform modules. Then if either $|I|<\infty$ or R is right Noethrian, M is quasi-injective.*

Proof. Let ρ be any isomorphism of $E(M_1)$ to $E(M_2)$ for 1, 2 in I. Then there exists an isomorphism f: $M_1\rightarrow M_2$ such that $f=\rho$ on $\rho^{-1}(M_2\cap M_1)$ (see proof of Theorem 28). Let $h\in \mathrm{Hom}_R(A, M_1)$ for $A\subseteq M_1$. Then fh is extended to a homomorphism t: $M_1\rightarrow M_2$ by Theorem 22. Hence, $f^{-1}t$ is an extension of h and so M_1 is quasi-injective. Further, since $f^{-1}\rho\in \mathrm{End}_R(E(M_2))$, $M_2=f(\rho^{-1}(\rho(M_1)))=\rho(M_1)$ by [11]. If $|I|<\infty$ or R is Noetherian, $E(M)=\sum_I\oplus E(M_\alpha)$. Hence, M is quasi-injective by [4] and [11].

Corollary 30. *Let $\{M_\alpha\}_I$ and M be as above. We assume further each M_α is non-singular and M has the extending property of f.c. uniform modules. Then if either $|I|<\infty$ or R is right Notherian, $\sum_{I'}\oplus M_\alpha$ is quasi-injective, where I' is a subset of I such that for α in I' there exists $\rho(\alpha)\neq\alpha$ in I with $E(M_\alpha)\approx E(M_{\rho(\alpha)})$.*

Proof. Since M_α is non-singular, $\mathrm{Hom}_R(E(M_\alpha), E(M_\beta))=0$ if $E(M_\alpha)\not\approx E(M_\beta)$. Hence, we have the corollary by Proposition 29 and [4], [11].

Especially, when R is a Dedekind domain, we have

Theorem 31. *Let R be a Dedekind domain and M an R-module. Then M has the extending property of direct sum of uniform modules if and only if either*

1) *M is quasi-injective or*
2) *$M=K\oplus E$, where E is torsion and injective and $K\subsetneqq Q=E(R)$.*

If M has the extending property of direct sum of uniform modules, then $M=\sum_I\oplus M_\alpha$ with M_α uniform, since R is Notherian. We shall complete the proof by making use of elementary properties of abelian groups as follows:

Lemma 32. *Let K be an R-submodule of Q and $\{f_i\colon K\to E(P_i)^{(I_i)}\}$ a set of homomorphisms. We assume $P_i\neq P_j$ if $i\neq j$ and for some $a\neq 0$ in K, $f_i(a)=0$ for almost all i. Then $\{f_i\}$ is summable, i.e., for any x in K, $f_i(x)=0$ for almost all i, where P_i is a non-zero prime ideal, $E(P_i)=E(R/P_i)$ and $E(P_i)^{(I_i)}$ is a direct sum of I_i-copies of $E(P_i)$* (cf. [14]).

Proof. We may asume $a=1$ and $K\supseteq R$. Then $f_i\in \mathrm{Hom}_R(K/R, E(P_i)^{(I_i)})$ for almost all i. Hence, since $K/R=\sum_q\oplus$(q-primary part of K/R), $\{f_i\}$ is summable.

Lemma 33. *Let K be as above and L an R-submodule of $E(P)$. If $\mathrm{Hom}_R(A, L)$ is extended to $\mathrm{Hom}_R(K, L)$ for any submodule A in K, then $L=E(P)$.*

Proof. We may assume $K\supseteq R$. We assume $L\neq E(P)$. Then $L=\overline{p^{-n}}R$, where $E(P)=\cup\,\overline{p^{-m}}R$ and $\overline{p^{-m}}\in Q/R$. Put $A=p^nR\supseteq B=p^{2n}R$. Then the natural epimorphism $f\colon A\to A/B\approx L$ is in $\mathrm{Hom}_R(A, L)$. Let $g\in\mathrm{Hom}_R(R, L)$ be an extension of f. Then $R=\ker g+A$. Hence, $R_P=(\ker g)_P+A_P=(\ker g)_P$. Since $(\ker g)\cap A=B$, $A_P=B_P$, a contradiction. Hence, $L=E(P)$.

The following lemma is similar to Theorem 22.

Lemma 34. *Let $T=\sum_I\oplus T_\alpha$ be any decomposition of an R-module and let each T_α be uniform. If T has the extending property of direct sum of f.c. uniform modules, then any element in $\mathrm{Hom}_R(A_\alpha, T_\beta)$ is extended to $\mathrm{Hom}_R(T_\alpha, T_\beta)$ for any pair α, β in I and for any submodule A_α in T_α.*

Proof. Let f be in $\mathrm{Hom}_R(A_1, T_2)$. Put $A(f)=\{a+f(a)\,|\,a\in A_1\}$ and consider an essential submodule $A(f)\oplus T_2\oplus \sum_{I-\{1,2\}}\oplus T_\alpha$ of T. Then there exists a decomsition $T=\sum_I\oplus S_\alpha$ such that $S_\alpha\,{}_e\!\supseteq T_\alpha$ for $\alpha\neq 1$ and $S_1\,{}_e\!\supseteq A(f)$. Since T_α is a direct summand of T, $S_\alpha=T_\alpha$ for $\alpha\neq 1$. Thus, $T=S_1\oplus\sum_{I-\{1\}}\oplus T_\alpha$ and so $\pi\,|\,T_1$ is an extension of f, where $\pi\colon T\to T_2$ is the projection on the decomposition.

Proof of Theorem 31. We assume M has the extending property of direct sum of uniform modules. Then $M=\sum_{I_1}\oplus M_\alpha\oplus\sum_{I_2}\oplus M_\beta$, where M_α (resp. M_β) is a torsion-free (resp. torsion) uniform submodule of M. We may assume $M_\alpha\subseteq Q$ for all α in I_1. First we assume $|I_1|\geq 2$. Take $0\neq x$ in $M_{\alpha_1}\cap M_{\alpha_2}$. For any $m\neq 0\in R$ a homomorphism $f\colon xmR\to xR$ by setting $f(xmr)=xr$ is extended to $g\colon M_{\alpha_1}\to M_{\alpha_2}$ by Lemma 34. Hence, $xR(1/m)\subseteq M_{\alpha_1}(1/m)=g(M_{\alpha_1})\subseteq M_{\alpha_2}$. Therefore, $M_{\alpha_2}=Q$ and so each $M_\alpha=Q$ for α in I_1. Furthermore we know $M_\beta=E(P)$ for any β in I_2 by Lemmas 32 and 33. Hence, M is injective. If $|I_1|=1$ and $M_{\alpha_1}=Q$, M is also injective as above. Next, assume $|I_1|=1$ and $M_{\alpha_1}\subsetneqq Q$. Then M is of the form 2). Finally, we assume M is torsion.

Each M_β is a completely indecomposable with (M-I). Hence, M is quasi-injective by Proposition 29, the fact: $\mathrm{Hom}_R(E(P), E(Q))=0$ if $P\neq Q$ and [3], [9]. Conversely, if M is quasi-injective, M has the extending property of direct sum of submodules by Proposition 25. Let $M=K\oplus\sum_P\oplus E(P)^{(I_P)}$ as in 2). We assume $N=N_1\oplus\sum_{I_2}\oplus N_\beta$ be an essential submodule of M, where N_1 (resp. N_β) is torsion-free (resp. torsion) and uniform. Let π and π_P be the projection of M onto K and $E(P)^{(I_P)}$, respectively. Then $\pi|N_1$ is isomorphic. Put $f_P=\pi_P(\pi|N_1)^{-1}$: $\pi(N_1)\to E(P)^{(I_P)}$. Then $\{f_P\}$ is summable. Since $E(P)^{(I_P)}$ is injective, we obtain an extension $g_P\in\mathrm{Hom}_R(K, E(P)^{(I_P)})$ of f_P. Then $\{g_P\}$ is also summable by Lemma 32. Put $K'=\{x+\sum_P g_P(x)\,|\,x\in K\}\subseteq M$. Then $M=K'\oplus\sum_P\oplus E(P)^{(I_P)}$ and $K'\supseteq N_1$, $\sum_P\oplus E(P)^{(I_P)}{}_e\supseteq\sum_{I_2}\oplus N_\beta$ (cf. The proof of Theorem 10). Hence, $N_1\oplus\sum_{I_2}\oplus N_\beta$ is extended to a decomposition of M by proposition 25.

Theorem 35. *Let R be a left perfect ring* [2]. *Then the following conditions are equivalent.*

1) *Every direct sum of completely indecomposable uniform R-module has the extending property of simple module.*

2) *For every completely indecomposable submodules U_1 and U_2 of an indecomposable injective module E, there exists an automorphism $\tilde{f}$ of E such that either $\tilde{f}(U_1)\subseteq U_2$ with $\tilde{f}\,|\,S(E)=f$ or $\tilde{f}(U_2)\subseteq U_1$ with $f\,|\,S(E)=f^{-1}$ for any $f\in\mathrm{End}_R(S(E))$.*

Proof. 1)$\Rightarrow$2) is clear from Theorem 5. Since every uniform module is embedded in an indecomposable injective module, 2)$\Rightarrow$1) also follows from Theorem 5.

We end this paper with the following theorem.

Theorem 36. *Let R be a right and left artinian ring. Then the following conditions are equivalent:*

1) *R is a generalized uniserial ring* [15].

2) *Every right R-module, as well as every left R-module, has the extending property of simple module.*

Proof. 1)$\Rightarrow$2). Every module M is a direct sum of uniserial modules by [15]. Hence, we know 1)$\Rightarrow$2) by Theorem 35.

2)$\Rightarrow$1). Let e be a primitive idempotent. Since eR/eA is indecomposable for every right ideal eA, $S(eR/eA)$ is simple by 2). Hence, R is a right generalized uniserial ring. By the same argument, we see R is also left uniserial.

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