<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>On extending property on direct sums of uniform modules</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Harada, Manabu; Oshiro, Kiyoichi</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 18(3) P.767–P.785</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1981</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/10326">https://doi.org/10.18910/10326</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/10326</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
</tbody>
</table>
First we take a right artinian ring $R$. Then every injective $R$-module $E$ is a direct sum of indecomposable modules. Further for every simple submodule $S$ of $E$, there exists a direct summand of $E$ whose socle is equal to $S$. Let $\sum \oplus S_\sigma$ be a decomposition of the socle of $E$. Then we have a decomposition of $E$ by indecomposable modules $E_\sigma$ such that $E = \sum \oplus E_\sigma$ and the socle of $E_\sigma$ is $S_\sigma$. We shall call the first property and the second property the extending property of simple module and of decomposition, respectively. These concepts are dual to those of lifting properties mentioned in [7].

We shall study the above properties on direct sums of completely indecomposable modules with certain condition over an arbitrary ring. We shall give characterizations of those properties in terms of endomorphisms over direct summands and show that quasi-injective modules and generalized uniserial rings [15] are related to those properties. Our results are dual or similar to those in [9] and are applied to the study of QF-2 rings in [8].

1. Notations

Throughout this paper $R$ is a ring with identity and every $R$-module is a unitary right $R$-module. For an $R$-module $M$, we denote its socle and its injective envelope by $S(M)$ and $E(M)$, respectively. For a submodule $N$ of $M$, we use the symbol $N \subseteq_e M$ to indicate that $M$ is an essential extension of $N$.

In [9], the first author has studied direct sums of hollow modules by introducing the lifting property of simple module and that of decomposition. In order to deal with their dual properties, we must consider the dual condition to (E-I) in [9]:

(M-I) Every monomorphism of an $R$-module into itself is isomorphic.

If a uniform $R$-module $M$ satisfies (M-I), then $\text{End}_R(M)$ is a local ring, namely $M$ is completely indecomposable. In particular, indecomposable quasi-injective $R$-modules are completely indecomposable modules with (M-I). Artinian $R$-modules clearly satisfy (M-I).
For a set \( \{M_a\}_I \) of \( R \)-modules with \( (M-I) \), we define a partial order \( \leq^* \) in the set as follows: If \( M_a \cong M_\beta \), we put \( M_a \equiv M_\beta \). If there exists a monomorphism of \( M_a \) to \( M_\beta \), we define \( M_a \leq^* M_\beta \).

Let \( \{N_a\}_I \) be an independent set of submodules of an \( R \)-module \( M \). \( \sum_r \oplus N_a \) is said to be a locally direct summand of \( M \) [10] if for any finite subset \( J \) of \( I \), \( \sum_r \oplus N_\beta \) is a direct summand of \( M \).

A set \( \{M_a\}_I \) which consists of completely indecomposable \( R \)-modules is called a locally (resp. semi-) T-nilpotent set if for any family of countable non-isomorphisms \( \{ f_i : M_{i_n} \to M_{i_{n+1}} \mid n \geq 1 \} \) (resp. \( i_n \neq i_{n'} \) for \( n \neq n' \)) and any \( x \) in \( M_{i_1} \), there exists an integer \( m \) depending on \( x \) such that \( f_{i_n} f_{i_{n-1}} \cdots f_1(x) = 0 \) (5). It is known in [10] that \( \{M_a\}_I \) is locally semi-T-nilpotent if and only if, for any independent set \( \{N_\beta\}_I \) of submodules of \( M = \sum_r \oplus M_a \), \( \sum_r \oplus N_\beta \) is a direct summand of \( M \) if it is a locally direct summand of \( M \).

For given \( R \)-modules \( M \) and \( M' \) and submodules \( N \subseteq M \) and \( N' \subseteq M' \), we say \( \text{Hom}_R(N, N') \) is extended to \( \text{Hom}_M^M, M' \) if every element in \( \text{Hom}_N^N, N' \) is extended to one in \( \text{Hom}_M^M, M' \). Similarly we say \( \text{Aut}_R(N) \) is extended to \( \text{Aut}_M^M, M' \) provided every automorphism of \( N \) is extended to one of \( M \).

Let \( M = \sum_r \oplus M_a \). If \( N \) is a submodule of \( M \) such that \( N \subseteq M \) and \( N' \subseteq M' \), we say \( \text{Hom}_R(N, N') \) is extended to \( \text{Hom}_R(M, M') \) if every element in \( \text{Hom}_R(N, N') \) is extended to one in \( \text{Hom}_R(M, M') \). Similarly we say \( \text{Aut}_R(N) \) is extended to \( \text{Aut}_R(M) \) provided every automorphism of \( N \) is extended to one of \( M \).

Let \( M = \sum_r \oplus M_a \). If \( N \) is a submodule of \( M \) such that \( N \subseteq \sum_r \oplus M_a \) for some finite subset \( \{\alpha_1, \ldots, \alpha_n\} \) in \( I \), we say \( N \) is finitely contained in the direct sum (with respect to the decomposition \( M = \sum_r \oplus M_a \)) and we briefly write it as f.c. module. Let \( x \) be a non-zero element in \( M \) and express it in the direct sum as \( x = x_{\alpha_1} + \cdots + x_{\alpha_n} \) where \( 0 \neq x_{\alpha_i} \subseteq M_{\alpha_i} \). Then we say \( x \) has the length \( n \) for \( M = \sum_r \oplus M_a \). In the expression of \( x \) it may happen that all annihilators \( (0 : x_{\alpha_i}) \) are the same. If \( x \) is written in this form, we say \( x \) is a smooth element for \( M = \sum_r \oplus M_a \). We omit the word 'for \( M = \sum_r \oplus M_a \)' in the definition if no confusions arise. We denote the set of all smooth elements of \( M \) by \( S(M = \sum_r \oplus M_a) \). We say a submodule of \( M \) is smooth if every non-zero elements in it is smooth.

We can easily verify the following facts about smooth elements for \( M = \sum_r \oplus M_a \).

1) Every element in \( (\cup M_a) - \{0\} \) is smooth with the length 1.

2) Let \( \{1, \ldots, n\} \subseteq I \) and \( 0 \neq x_i \subseteq M_i \) for \( i = 1, \ldots, n \). If \( f_i : x_i R \to x_{i+1} R \) is a monomorphism for \( i = 1, \ldots, n \), then \( \{z + f_i(z) + \cdots + f_{2}f_{1} \cdots f_{i}(z) \mid z \subseteq x_i R\} \) is a smooth submodule of \( M \).

3) If \( x \) is a smooth element with the length \( n \), so is every non-zero element in \( x R \).
4) For $0 \neq x$ in $M$, there exists $r$ in $R$ such that $xr \neq 0$ is smooth.
5) If $xR$ is a simple submodule of $M$, then it is smooth.

2. Extending property of submodule

First we shall consider the following special case.

Proposition 1. Let $\{E_a\}$ be a set of indecomposable injective $R$-modules and $E=\sum_a E_a$. Then

1) Each $E_a$ is a uniform module satisfying (M-I).
2) Let $\{N_\beta\}$ be an independent set of f.c. uniform modules of $E$. Then there exists a set $\{F_\beta\}_I$ of direct summands of $E$ such that $N_\beta \subseteq F_\beta$ and $\sum_\beta F_\beta$ is a locally direct summand.
3) If $x$ is a smooth element of $E$, then there exists a direct summand $F$ of $E$ such that $F$ is indecomposable and $xR \subseteq F$.

Proof. 1) is clear. 2) Since $N_\beta$ is f.c., we can take $F_\beta=E(N_\beta)$ in $E$ for each $\beta$. Then $\{F_\beta\}_I$ is a desired set. Similarly we can show 3).

In this section we shall study some properties in the above proposition on a more general case, namely on direct sums of completely indecomposable modules. For this purpose we introduce several properties.

Let $M$ be an $R$-module and $N$ a submodule of $M$. We say $N$ is essentially extended to a direct summand of $M$ if there exists a direct summand of $M$ which contains $N$ as an essential submodule. We say $M$ has the extending property of simple (resp. uniform) module provided every simple (resp. uniform) submodule of $M$ is essentially extended to a direct summand of $M$. When $M=\sum_I \oplus M_\alpha$, we say $M$ has the extending property of cyclic smooth module for $M=\sum_I \oplus M_\alpha$ if every cyclic smooth submodule of $M$ is essentially extended to a direct summand of $M$. We also omit the word 'for $M=\sum_I \oplus M_\alpha$' in the above. Similarly we can define the extending property of smooth module, the extending property of f.c. uniform module and the extending property of uniform and smooth module, etc.

Now, we are concerned with those properties on direct sums of completely indecomposable $R$-modules. Therefore, from now on, we assume $\{M_\alpha\}_I$ is a set of completely indecomposable $R$-modules and $M=\sum_I \oplus M_\alpha$; whence each $M_\alpha$ has the exchange property [16].

We often take a subset $\mathcal{E}$ of $S(M=\sum_I \oplus M_\alpha)$ which satisfies the following condition:

(*) For any $0 \neq x \in M$, there exists $r$ in $R$ such that $xr \in \mathcal{E}$.

For example, $S(M=\sum_I \oplus M_\alpha)$ itself satisfies the condition (*). In the case of $S(M) \subseteq M$, the set of all non-zero elements $x$ in $M$ such that $xR$ is simple
satisfies the condition (*).

**Proposition 2.** Let $\mathcal{F}$ be a subset of $S(M=\sum\oplus M_\alpha)$ satisfying (*) and let $\beta \in I$. If every cyclic submodule of $M$ generated by an element in $\mathcal{F} \cap M_\beta$ is essentially extended to a direct summand of $M$, then $M_\beta$ is uniform.

Proof. Contrary to the assertion, assume that $M_\beta$ is not uniform and take non-zero elements $x_1, x_2$ in $M_\beta$ such that $x_1 R \cap x_2 R = 0$. Since $\mathcal{F}$ satisfies (*), there exists $r$ in $R$ such that $x_1 r, x_2 r \in \mathcal{F}$. By the assumption, we can take a direct summand $N$ of $M$ such that $x_1 r R \subseteq N$. Since $N$ is a direct summand of $M$, there exists a direct summand $N' \leq \oplus N$ which is isomorphic to some member in $\{M_\alpha\}$ by [1]. Inasmuch as $N'$ has the exchange property, we have

$$M = N' \oplus \sum_{\tau} \oplus M_\tau$$

for some $J \subseteq I$. Then we see from $N' \cap M_\beta \neq 0$ that $I - J = \{\beta\}$. Let $x_2 = y + z$ where $y \in N'$ and $z \in \sum_{\tau} \oplus M_\tau$. Since $x_2 \in M_\beta$, $y$ must be non-zero. Hence there exists $s \in R$ such that $0 \neq y s \in x_1 r R$ since $x_1 r R \subseteq N$. Then $x_2 s - y s \in M_\beta$ and $z s \in \sum_{\tau} \oplus M_\tau$, whence $x_2 s - y s = 0$. However $x_1 r \cap x_2 R = 0$ shows $y s = 0$, a contradiction.

**Corollary 3.** Let $\mathcal{F}$ be as in Proposition 2. If every cyclic $R$-submodule of $M$ generated by an element in $\mathcal{F}$ is essentially extended to a direct summand of $M$, then each $M_\alpha$ is uniform.

**Proposition 4.** Let $x \in S(M=\sum \oplus M_\alpha)$. If $N$ is a direct summand of $M$ such that $x R \subseteq e N$, then $N$ is completely indecomposable.

Proof. We express $x$ in $M=\sum \oplus M_\alpha$ as $x = x_1 + \cdots + x_n$ where $0 \neq x_i \in M_i$, $i = 1, \ldots, n$. Since $N \leq \oplus M$, there exists a direct summand $N' \leq \oplus N$ which is isomorphic to some member in $\{M_\alpha\}$ by again [1]. It is sufficient to show $N' = N$. Since $N'$ has the exchange property,

$$M = N' \oplus \sum_{\tau} \oplus M_\tau$$

(1)

for some $J \subseteq I$. Since $N' \cap x R \neq 0$, there exists $i \in \{1, \ldots, n\}$ which does not lie in $J$. Without loss of generality we can assume $i = 1$. Then $I - J = \{1\}$. Now, let $N = N' \oplus N''$ and assume $N'' \neq 0$. We take $0 \neq z$ in $N''$ and express it in (1) as $z = y + p + q$ where $y \in N'$, $p \in M_2 \oplus \cdots \oplus M_\alpha$ and $q \in \sum_{\tau} \oplus M_\tau (K = J - \{2, \ldots, \tau\})$. Since $x R \subseteq e N$, we can easily take $r$ in $R$ such that $0 \neq z r \in x R$ and $y r \in x R$. Since $z r - y r \in x R$, $z r - y r = x t$ for some $t \in R$. Noting that $x t = (p + q) r$, $x t \in M_2 \oplus \cdots \oplus M_\alpha$ and $p r \in M_2 \oplus \cdots \oplus M_\alpha$, we see $q r = 0$ and hence $x t = p r$. Since $x$
is a smooth element with the length \( n \), so is \( xt \) if \( xt \neq 0 \). Therefore the fact \( xt = pr \in M_2 \oplus \cdots \oplus M_\ell \) implies \( xt = 0 \), whence \( xz = yr \). However \( N' \cap N'' = 0 \) says that \( xz = 0 \), a contradiction. Thus we must have \( N' = N \).

**Theorem 5.** Let \( \mathcal{F} \) be as in Proposition 2 and assume if \( \sum_i x_i \in \mathcal{F} \), \( x_i + x_j \in \mathcal{F} \) for each \( i, j \), where \( x_i \in M_i \). Then the following conditions are equivalent:

1) Every cyclic submodule of \( M \) generated by an element in \( \mathcal{F} \) is essentially extended to a direct summand of \( M \).

2) For any pair \( i, j \) in \( I \), every cyclic submodule of \( M \) generated by an element in \( \mathcal{F} \cap (M_i \oplus M_j) \) is essentially extended to a direct summand of \( M_i \oplus M_j \).

3) i) Each \( M_i \) is uniform.

ii) For any pair \( i, j \) in \( I \) and any non-zero elements \( x_i \in M_i \) and \( x_j \in M_j \) such that \( x_i + x_j \in \mathcal{F} \), there exists a monomorphism \( g \) of either \( M_i \) into \( M_j \) or \( M_j \) into \( M_i \) such that \( g(x_i) = x_j \) or \( g(x_j) = x_i \).

**Proof.** In view of these conditions we may only show 1) \( \Rightarrow \) 3).

1) \( \Rightarrow \) 3). By Proposition 2, each \( M_i \) is uniform. Let \( M_1, M_2 \) be two members in \( \{ M_i \} \) and let \( 0 \neq x_i \in M_i \), \( i = 1, 2 \) such that \( x = x_1 + x_2 \in \mathcal{F} \). By 1), there exists a direct summand \( M' \) of \( M \) such that \( xR \subseteq M' \). Then, by Proposition 4, \( M' \) is a completely indecomposable and uniform module. Using the exchange property of \( M' \) and the fact \( xR \subseteq x_1 R \oplus x_2 R \), we have

\[
M = M' \oplus \sum_{i=1}^{n} \oplus M_i \quad \text{or} \quad M = M' \oplus \sum_{j=1}^{n} \oplus M_j.
\]

In the former case let \( \psi : M = M' \oplus \sum_{i=1}^{n} \oplus M_i \rightarrow M_2 \) be the projection. Then it is easy to see \( \psi(x_1) = -x_2 \). Since \( x_1 R \subseteq M_1 \) and \( x \in \mathcal{F} \), \( -\psi|M_1 \) is a monomorphism of \( M_1 \) into \( M_2 \) satisfying \( -\psi(x_1) = x_2 \). Similarly we have a monomorphism \( g \) of \( M_2 \) into \( M_1 \) with \( g(x_2) = x_1 \) in the latter case.

3) \( \Rightarrow \) 1). Let \( x \in \mathcal{F} \). If \( x \) lies in some \( M_i \), then we have indeed \( xR \subseteq e_i M_i \). Thus assume that the length \( n \) of \( x \) is not 1, and let \( x = x_1 + \cdots + x_n \in M_1 \oplus \cdots \oplus M_n \) where \( x_i \in M_i \). By 3), there exists either a monomorphism \( h_{ij} : M_i \rightarrow M_j \) or \( h_{ji} : M_j \rightarrow M_i \) satisfying \( h_{ij}(x_i) = x_j \) or \( h_{ji}(x_j) = x_i \) for each pair \( i, j \) in \( \{1, \ldots, n\} \). As a result we can take \( k \in \{1, \ldots, n\} \) and monomorphisms \( h_{ki} : M_k \rightarrow M_i \) for all \( j \neq k \) satisfying \( h_{ki}(x_k) = x_j \). We may assume \( k = 1 \). We put

\[
M'_1 = \{ x + h_{12}(x) + \cdots + h_{1n}(x) \mid x \in M_1 \}.
\]

Then \( M'_1 \oplus M_2 \oplus \cdots \oplus M_n = M_1 \oplus M_2 \oplus \cdots \oplus M_n \) and moreover it follows from \( x_i R \subseteq M_i \) that \( xR \subseteq M_1 \).

**Remark.** In the case where \( M \) has the essential socle and \( S = \{ x \in M \mid xR \) is simple\}, the above proposition is dual to [9, Theorem 2].

**Corollary 6.** If \( M \) has the extending property of cyclic smooth module for
$M = \sum \oplus M_\alpha$, then it has also the extending property of cyclic smooth module for any other decomposition of $M$ by completely indecomposable modules.

Corollary 7. Let $\mathcal{E}$ be as in Theorem 5 and assume that each $M_\alpha$ satisfies (M-1). Then the following conditions are equivalent:

1) Every cyclic submodule of $M$ generated by an element in $\mathcal{E}$ is essentially extended to a direct summand of $M$.

2) i) For $M_\beta \in \{M_\alpha\}$, let $\{M_i\}_{K(\beta)}$ be the set of all $M_i \in \{M_\alpha\}_{I-\{\beta\}}$ such that $M_\alpha \oplus M_i$ has a smooth element with the length 2. Then the relation $\leq^*$ is linear in $\{M_\beta\} \cup \{M_i\}_{K(\beta)}$ for any $\beta \in I$.

ii) For any pair $M_\alpha \leq^* M_\beta$ and any $x$ in $\mathcal{E} \cap (M_\alpha \oplus M_\beta)$ with the length 2, say $x = x_\alpha + x_\beta$, there exists a monomorphism $f: M_\alpha \rightarrow M_\beta$ satisfying $f(x_\alpha) = x_\beta$.

Corollary 8 (cf. [9, Corollary of Theorem 2]). Assume each $M_\alpha$ satisfies the condition (M-1). Then the following conditions are equivalent:

1) $M$ has the extending property of simple module.

2) For any $\beta$ in $I$, the relation $\leq^*$ is linear in the subset $\{M_i\}_{K(\beta)}$ of all $M_i$ in $\{M_\alpha\}$, such that $S(M_\alpha) \approx S(M_\beta)$, and $\text{Hom}_R(S(M_\alpha), S(M_\beta))$ is extended to $\text{Hom}_R(M_\alpha, M_\beta)$ for any pair $\alpha, \beta$ in $I$.

Proof. We can assume $S(M) \subseteq \mathcal{E}$, whence this is immediate by Theorem 5.

We can obtain similarly to Theorem 5.

Theorem 9. We assume each $M_\alpha$ is uniform. Then the following conditions are equivalent:

1) $M$ has the extending property of uniform and smooth module.

2) For any pair $\alpha, \beta$ in $I$ and any monomorphism $f$ of a submodule $A_\alpha$ of $M_\alpha$ to $M_\beta$ is extended to a monomorphism of $M_\alpha$ to $M_\beta$ or $f^{-1}$ is extended to one of $M_\beta$ to $M_\alpha$.

Proof. If $U$ is a uniform and smooth module, then there exists $\{\alpha_1, \ldots, \alpha_n\} \subseteq I$ for which $U \subseteq M_{\alpha_1} \oplus \cdots \oplus M_{\alpha_n}$ and every non-zero element in $U$ has the length $n$. Noting this fact, the proof is done by the same argument as in the proof of Theorem 5.

Theorem 10. We assume each $M_\alpha$ is uniform. Then the following conditions are equivalent:

1) $M$ has the extending property of f.c. uniform module.

2) For any pair $\alpha, \beta$ in $I$, any homomorphism $f$ of a submodule $A_\alpha$ in $M_\alpha$ to $M_\beta$ is extended to an element in $\text{Hom}_R(M_\alpha, M_\beta)$ or $f^{-1}$ is extended to one in $\text{Hom}_R(M_\beta, M_\alpha)$, provided $\ker f = 0$.

Proof. 1) $\Rightarrow$ 2). Using the same notations as in the proof of Theorem 5, we have
DIRECT SUMS OF UNIFORM MODULES

\[ M = M' \oplus \bigoplus_{i \in [a]} \oplus M_a \] or \[ M = M' \oplus \bigoplus_{j \in [b]} \oplus M_a \]

where \( \{x+f(x)| x \in M_a\} \subseteq M' \oplus M \). If \( \ker f \neq 0 \), \( M' \supseteq \ker f \). Hence, we have the former decomposition. Let \( \pi : M \to M_2 \) be the projection on this decomposition. Then \( -\pi | M_1 \) is an extension of \( f \). If \( \ker f = 0 \), then the assertion follows from Theorem 9.

2) \( \Rightarrow \) 1). Let \( N \) be a f.c. uniform submodule in \( M \), say \( N \subseteq \bigoplus_{i=1}^n \oplus M_{a_i} \). Let \( \pi_a : M \to M_a \) be the projection. Since \( \cap \ker \pi_{a_i} | N = 0 \) and \( N \) is uniform, there exists \( i \) such that \( \ker \pi_{a_i} | N = 0 \). Hence, by 2) there exists \( j \) say \( 1 \) such that \( N = \{x+f_1(x)+\cdots+f_n(x)| x \in \pi_{a_1}(N)\} \) where \( f: M_{a_1} \to M_{a_1} \) is an extension of \( \pi_{a_1}(\pi_{a_1} | N)^{-1} \). Put \( M' = \{y+f_1(y)+\cdots+f_n(y)| y \in M_{a_1}\} \). Then \( \bigoplus_{i=1}^n \oplus M_{a_i} = M' \oplus \bigoplus_{i=1}^n \oplus M_{a_i} \) and \( N \subseteq M' \).

**Corollary 11.** Let \( T \) be a completely indecomposable and uniform module. Then \( T \) is quasi-injective if and only if \( T \oplus T \) has the extending property of uniform module and \( T \) satisfies (M-I).

**Proof.** This is clear from Theorem 10 and [4], [11].

**Theorem 12.** Assume each \( M_a \) is uniform and for any pair \( \alpha, \beta \) in \( I \), every monomorphism of \( M_\alpha \) to \( M_\beta \) is isomorphic. Then the following conditions are equivalent:

1) \( M \) has the extending property of uniform module.

2) For any \( \alpha \) in \( I \) and a submodule \( A_\alpha \) of \( M_\alpha \), every homomorphism of \( A_\alpha \) to \( \bigoplus_{\alpha \in I} \oplus M_\alpha \) is extended to an element in \( \text{Hom}_K(M_{\alpha}, \bigoplus_{\alpha \in I} \oplus M_{\alpha}) \).

**Proof.** For \( \alpha \in I \), \( \pi_\alpha \) denotes the projection \( M = \bigoplus_{\alpha \in I} \oplus M_\alpha \to M_\alpha \).

1) \( \Rightarrow \) 2). Let \( A_\alpha \) be a submodule of \( M_\alpha \) and \( f \) a homomorphism of \( A_\alpha \) to \( \bigoplus_{\alpha \in I} \oplus M_\alpha \). Putting \( N = \{x+f(x)| x \in A_\alpha\} \), \( N \) is uniform; whence we can easily take \( \{\alpha_1, \ldots, \alpha_n\} \subseteq I \) such that \( N \cap S(M = \bigoplus_{\alpha \in I} \oplus M_\alpha) = \{x+\pi_{a_1}f(x)+\cdots+\pi_{a_n}f(x)| 0 \neq x \in A_\alpha\} \) and all elements in \( N \cap S(M = \bigoplus_{\alpha \in I} \oplus M_\alpha) \) has the length \( n+1 \) (cf. the proof 2) \( \Rightarrow \) 1) in Theorem 10). Put \( f_1 = \sum_{i=1}^n \pi_{a_i}f \) and \( f_2 = \sum_{k \in K} \pi_{a_k}f \) where \( K = I - \{\alpha, \alpha_1, \ldots, \alpha_n\} \). Then \( f = f_1 + f_2 \). Since \( \pi_{a_i}f \) is monomorphic for all \( i = 1, \ldots, n \), \( f_1 \) can be extended to an element \( f_1' \) in \( \text{Hom}_K(M_{a_1}, \bigoplus_{\alpha \in I} \oplus M_{\alpha}) \) by Theorem 9 and the assumption that every monomorphism of \( M_{a_1} \) to \( M_{a_1} \), is isomorphic for any pair \( \alpha_i, \alpha_j \) in \( \{\alpha_1, \ldots, \alpha_n\} \). On the other hand, \( \pi_{a_k}f \) is non-monomorphic for all \( \beta \in K \), and hence by the same argument as in the proof of Theorem 10, \( f_2 \) can be also extended to an element \( f_2' \) in \( \text{Hom}_K(M_{a_k}, \bigoplus_{\alpha \in I} \oplus M_{\alpha}) \). Thus \( f_1' + f_2' \) is a desired extension of \( f \).
2) \implies 1). Let \( N (\neq 0) \) be a uniform submodule of \( M \). Then there exist \( \{\alpha_1, \ldots, \alpha_j\} \subseteq I \) for which 
\[ N \cap S(M=\sum_{i=1}^{j} \bigoplus M_\alpha) = \{\pi_\alpha + \cdots + \pi_{\alpha_j}(\alpha) \mid |x| \in N \} \]
and all elements in \( N \cap S(M=\sum_{i=1}^{j} \bigoplus M_\alpha) \) have the length \( n \). Take \( \alpha \) in \( \{\alpha_1, \ldots, \alpha_j\} \). Then the mapping \( f: \pi_\alpha(N) \mapsto \sum_{i=1}^{j} \bigoplus M_\beta \) given by \( f(\pi_\alpha(x)) = \sum_{i=1}^{j} \pi_\beta(x) \) is homomorphism and \( N = \{x+f(x) \mid x \in \pi_\alpha(N)\} \). Let \( f': M_\alpha \mapsto \sum_{i=1}^{j} \bigoplus M_\beta \) be an extension of \( f \) and put \( N' = \{y+f(y) \mid y \in M_\alpha\} \). Then we can see \( N \subseteq N' \). 

**Theorem 13** (cf. [9, Theorem 1]). Assume each \( M_\alpha \) is uniform. Then the following conditions are equivalent:

1) For any independent family \( \{N_\beta\} \) of direct summands of \( M \) which are uniform, \( N = \sum_{\beta} \bigoplus N_\beta \) is a locally direct summand.

2) Every monomorphism of \( M_\alpha \) to \( M_\beta \) is isomorphic for any pair \( \alpha, \beta \) in \( I \).

Proof. 1) \(\implies\) 2). We assume \( f: M_\alpha \mapsto M_\beta \) is monomorphic where \( \alpha, \beta \in I \) and \( \alpha \neq \beta \). Put \( M'_\alpha = \{x+f(x) \mid x \in M_\alpha\} \). Then \( M'_\alpha \bigcap M_\beta = 0 \) and \( M'_\alpha \bigoplus M_\beta = M_\alpha \bigoplus M_\beta \), from which we see \( M'_\alpha \cong M_\alpha \) and \( M'_\beta \) is a direct summand of \( M_\alpha \bigoplus M_\beta \). Further \( M'_\alpha \bigcap M_\alpha = 0 \). Hence, by 1) \( M'_\alpha \bigoplus M_\alpha = M_\alpha \bigoplus M_\beta \) and hence \( f \) is epimorphic.

2) \(\implies\) 1). We note that if \( M_\alpha \cong M_\beta \) for \( \alpha \neq \beta \), then \( M_\alpha \) satisfies (M-I). To show 1), we may show the following: If \( \{N_1, \ldots, N_n\} \) is an independent set of direct summands of \( M \) which are uniform then \( N_1 \bigoplus \cdots \bigoplus N_n \) is also a direct summand of \( M \). If \( n=1 \), this is clear. Assume \( n>1 \) and \( N_1 \bigoplus \cdots \bigoplus N_{n-1} \) is a direct summand of \( M \). Since each \( N_i \) is a direct summand of \( M \) and is uniform, it is isomorphic to some member in \( \{M_\alpha\} \) by [1]. Since \( N_1 \bigoplus \cdots \bigoplus N_{n-1} \) has the exchange property, we have

\[ M = N_1 \bigoplus \cdots \bigoplus N_{n-1} \bigoplus \sum_{j} \bigoplus M_\gamma \]

for some \( J \subseteq I \). Since \( N_n \) has the exchange property, we have either \( M = N_1 \bigoplus \cdots \bigoplus N_{n-1} \bigoplus \bigoplus M_\gamma \) for some \( k \) or \( M = \sum_{i=1}^{k} \bigoplus N_i \bigoplus \sum_{j} \bigoplus M_\gamma \) for some \( \sigma \in J \). We have done in the latter case. In the former case \( N_k \cong N_n \). Let \( \pi_\sigma: N_1 \bigoplus \cdots \bigoplus N_{n-1} \bigoplus \sum_{j} \bigoplus M_\gamma \to M_\gamma \) be the projection. Noting \( N_n \cong N_1 \bigoplus \cdots \bigoplus N_{n-1} \bigoplus \bigoplus M_\gamma \), we can easily see that there exists \( 0 \neq x \in N_n \) and some \( \rho \) in \( J \) for which \( \pi_\rho | xR \) is monomorphic. Since \( xR \subseteq N_n \), it follows that \( \pi_\rho | N_n \) is monomorphic. Since \( N_k \cong N_n \) for some \( \alpha \) in \( J \), \( \pi_\rho | N_n \) is isomorphic by 2). Thus \( M = N_n \bigoplus \ker \pi_\rho = N_1 \bigoplus \cdots \bigoplus N_\alpha \bigoplus \sum_{j} \bigoplus M_\beta \).

**Corollary 14.** Assume each \( M_\alpha \) is uniform. Then the following conditions are equivalent:

1) If \( N \) is in Theorem 13, then \( N \) is a direct summand of \( M \).
2) \( \{M_\alpha\}_I \) is a locally semi-\( T \)-nilpotent set and 2) of Theorem 13 holds.

Proof. Use the same argument as in [9].

**Theorem 15.** Assume \( \{M_\alpha\}_I \) is a locally semi-\( T \)-nilpotent set and every monomorphism of \( M_\alpha \) to \( M_\beta \) is isomorphic for any pair \( \alpha, \beta \) in \( I \). Then the following conditions are equivalent:

1) \( M \) has the extending property of submodule.
2) i) Each \( M_\alpha \) uniform.
   ii) For any subset \( J \) of \( I \) and any submodule \( A \) of \( \sum_j \oplus M_\beta \), \( \text{Hom}_R(A, \sum_j \oplus M_\alpha) \) is extended to \( \text{Hom}_R(\sum_j \oplus M_\beta, \sum_j \oplus M_\alpha) \).

Proof. 1)\( \Rightarrow \)2). By Corollary 3, each \( M_\alpha \) is uniform. Let \( J \) be a subset and \( A \) a submodule of \( \sum_j \oplus M_\beta \). Put \( P=\sum_j \oplus M_\beta \) and \( Q=\sum_j \oplus M_\alpha \). To show \( \text{Hom}_R(A, Q) \) is extended to \( \text{Hom}_R(P, Q) \), we may assume \( A \subseteq P \). Now, by 1), there exists a direct summand \( B \) of \( M \) such that \( \{x+f(x) \mid x \in A\} \subseteq B \). Then \( B \cap Q=0 \) and \( B \oplus Q \subseteq M \). Since \( B \subseteq M \), \( B \) is also a direct sum of completely indecomposable modules by [6], [12]. Hence we see \( M=B \oplus Q \) by Corollary 14. Let \( \pi: M=B \oplus Q \rightarrow Q \) be the projection. Then we see \( -\pi|P \) is an extension of \( f \).

2)\( \Rightarrow \)1). Let \( A (\neq 0) \) be a submodule of \( M \). By Zorn's lemma, we can take \( J \subseteq I \) such that \( A \cap \sum_j \oplus M_\alpha = 0 \) and \( A \cap \sum_k \oplus M_\gamma = 0 \) for any \( K \supseteq I-J \). Again we put \( P=\sum_j \oplus M_\beta \) and \( Q=\sum_j \oplus M_\alpha \). Noting each \( M_\alpha \) is uniform, as is easily seen, \( A \oplus Q \subseteq M \). Let \( \pi \) and \( \pi' \) be projections: \( M=\sum_j \oplus M_\alpha \rightarrow P \) and \( M=\sum_j \oplus M_\alpha \rightarrow Q \), respectively. Then we see \( \pi(A) \subseteq P \) since \( \pi(A) \oplus Q=A \oplus Q \subseteq M \). Since \( A \cap Q=0 \), the mapping \( f: \pi(A) \rightarrow Q \) given by \( f(\pi(a))=\pi'(a) \) is a homomorphism. Using 2), \( f \) is extended to a homomorphism \( f' \) of \( P \) to \( Q \). Put \( A' = \{x+f(x) \mid x \in P\} \). Then \( A \subseteq A' \subseteq M \). Moreover \( \pi(A) \subseteq P \) shows \( A \subseteq A' \).

**Theorem 16.** Assume that i) each \( M_\alpha \) satisfies (M-I), ii) every monomorphism of \( M_\alpha \) to \( M_\beta \) is isomorphic for any pair \( \alpha, \beta \) in \( I \) and iii) \( M \) has the extending property of cyclic smooth module. Then every submodule of \( M \) which is isomorphic to some member of \( \{M_\alpha\}_I \) is a direct summand of \( M \). Therefore the following conditions are equivalent by [7, Lemma 2] (cf. [17]):

1) \( \{M_\alpha\}_I \) is a locally semi-\( T \)-nilpotent set.
2) \( M \) has the exchange property.

Proof. Let \( N \) be a submodule of \( M \) which is isomorphic to some member in \( \{M_\alpha\}_I \). By iii) and Corollary 3, each \( M_\alpha \) is uniform and so is \( N \). Let \( x \in N \cap S(M=\sum_j \oplus M_\alpha) \). Then, by iii) and Proposition 4, there exists a direct sum-
mand \( N' \) of \( M \) such that \( xR \subseteq eN' \) and \( N' \) is completely indecomposable. Since \( N' \) has the exchange property,
\[
M = N' \oplus \sum_{J} M_{\beta}
\]
for some \( J \subseteq I \). Let \( \pi: M = N' \oplus \sum_{J} M_{\beta} \to N' \) be the projection. Noting \( xR \subseteq eN \) and \( xR \subseteq eN' \), we can easily verify \( \pi|N \) is monomorphic and \( N \cap \sum_{J} M_{\beta} = 0 \). Here, using i) and ii) we see \( \pi|N \) is isomorphic. Thus we have \( M = N \oplus \sum_{J} M_{\beta} \).

3. Extending property of decomposition

Let \( M \) be an \( R \)-module and \( \{N_{\beta}\}_{\beta} \) be an independent set of submodules of \( M \). We say \( \sum_{J} N_{\beta} \) is extended to a decomposition of \( M \) if there exists a decomposition \( M = \sum_{J} N_{\beta} \oplus M' \) such that \( N_{\beta} \subseteq eN' \) for all \( \beta \in J \). For a submodule \( N \) of \( M \), we say \( M \) has the extending property of decomposition of \( N \) if every decomposition of \( N \) is extended to a decomposition of \( M \). Further we say \( M \) has the extending property of direct sum of uniform modules if every direct sum of uniform submodules of \( M \) is extended to a decomposition of \( M \). Similarly we can define the phrase of the extending property of direct sum of submodules. When \( M = \sum_{J} \oplus M_{\alpha} \), we say \( M \) has the extending property of direct sum of cyclic smooth modules for \( M = \sum_{J} \oplus M_{\alpha} \) if every direct sum of cyclic smooth submodules is extended to a decomposition of \( M \). We also omit the word 'for \( M = \sum_{J} \oplus M_{\alpha} \)' in the above if no confusions arise. We can also define the phrases of the extending property of direct sum of uniform and smooth modules and the extending property of direct sum of f.c. uniform modules, etc.

In this section we also assume \( \{M_{\alpha}\}_{\alpha} \) is a set of completely indecomposable \( R \)-modules and put \( M = \sum_{J} \oplus M_{\alpha} \).

**Lemma 17.** Let \( \mathcal{F} \) be as in Proposition 2. If, for any \( \{x_{\beta}\}_{\beta} \subseteq \mathcal{F} \) such that \( \{x_{\beta}R\}_{\beta} \) is independent \( \sum_{J} x_{\beta} R \) is extended to a decomposition of \( M \), then every cyclic submodule of \( M \) generated by an element in \( \mathcal{F} \) is essentially extended to a direct summand of \( M \).

**Proof.** Let \( x \in \mathcal{F} \). Since \( \mathcal{F} \) satisfies the condition (\( \ast \)) in § 2, we can take \( \{x_{\beta}\}_{\beta} \subseteq \mathcal{F} \) such that \( x \in \{x_{\beta}\}_{\beta} \), \( \{x_{\beta}R\}_{\beta} \) is independent and \( \sum_{J} \oplus x_{\beta} R \subseteq eM \). Therefore, using the assumption, we can easily see that \( xR \) is essentially extended to a direct summand of \( M \).

**Theorem 18.** We assume \( \{M_{\alpha}\}_{\alpha} \) is a locally semi-T-nilpotent set and each
$M_a$ satisfies (M-I). Then, for a subset $\mathcal{F}$ of $S (M=\sum_{\lambda} \oplus M_\alpha)$ satisfying the conditions in Theorem 5, the following conditions are equivalent:

1) For any subset $\{x_\beta\}_J$ of $\mathcal{F}$ such that $\{x_\beta R\}_J$ is independent, $\sum_{\lambda} \oplus x_\beta R$ is extended to a decomposition of $M$.

2) i) Each $M_a$ is uniform.

ii) If $\mathcal{F}$ contains $x$ with the length 2 expressed as $x=x_1+x_2$ where $x_i \in M_i$, then there exists an isomorphism $f: M_i \cong M_\beta$ with $f(x_1)=x_2$.

3) i) Every cyclic submodule of $M$ generated by an element in $\mathcal{F}$ is essentially extended to a direct summand of $M$.

ii) Every monomorphism of $M_a$ to $M_\beta$ is isomorphic for any pair $\alpha$, $\beta$ in $I$.

Proof. 1) $\Rightarrow$ 2). By Corollary 3 and Lemma 17, each $M_a$ is uniform. Let $M_i \subseteq \{M_a\}_I$ and assume that there exists $M_i \subseteq \{M_a\}_I$ such that $M_i \oplus M_1$ contains a smooth element with the length 2 and denote the set of all such $M_i$ and $M_1$ by $\{M_i\}_K$. Then by Corollary 7, $\lhd^*$ is linear in $\{M_i\}_K$. For each $i \in K-\{1\}$, we can choose $y_i \in \mathcal{F} \cap (M_i \oplus M_1)$ with the length 2 since $\mathcal{F}$ satisfies (*). Let $y_i=x_i+x_i$ where $x_i \in M_1$ and $x_i \in M_i$. Further we take $0=x_\gamma \in M_\gamma \cap \mathcal{F}$ for all $\gamma \in J=I-K$. Then $\{y_i\}_K \cup \{x_i\}_J \subseteq \mathcal{F}$ and $\{y_i R\}_K \cup \{x_i R\}_J$ is independent. Since each $M_a$ is uniform, this shows $\sum_{\lambda} \oplus y_i R \oplus \sum_{\lambda} \oplus x_i R \subseteq M$.

By 1), we have a decomposition $M=\sum_{\lambda} \oplus F_{I} \oplus \sum_{\lambda} \oplus F_{\gamma}$ such that $y_i R \subseteq F_{i}$ for all $i \in K$ and $x_i R \subseteq F_{\gamma}$ for all $\gamma \in J$. Then each $F_{i}$ and each $F_{\gamma}$ are completely indecomposable by Proposition 4 and Lemma 17.

Now, noting $y_i R \subseteq F_{i}$ for all $i \in K$, we see from the choice of $\{M_i\}_K$ that, for any $i \in K$, $F_{i}$ is not isomorphic to any member of $\{M_i\}_J$. Hence there exists one to one mapping via isomorphism between the set $\{M_i\}_J$ and $\{F_{i}\}_K$ by Krull-Remak-Schmidt-Azumaya's theorem [1]. Let $\pi: M=\sum_{\lambda} \oplus M_a \rightarrow M_1$ be the projection. Since $\pi | y_i R$ for all $i \in K$ is monomorphic, we see from $y_i R \subseteq F_{i}$ that $\pi | F_{i}$ are monomorphic for all $i \in K$. Consequently $M_i$ is the largest with the relation $\lhd^*$. As a result, we see that if $x$ is an element in $\mathcal{F}$ with the length 2, say $x=x_a+x_\beta$, then $M_a \cong M_\beta$ and there exists an isomorphism $f: M_a \cong M_\beta$ satisfying $f(x_a)=x_\beta$ by Corollary 7 and the condition (M-I).

2) $\Rightarrow$ 1). For any $x \in \mathcal{F}$, $x R$ is essentially extended to a direct summand of $M$ by Theorem 5. Further if $f$ is a monomorphism of $M_a$ to $M_\beta$, then there exists an element in $\mathcal{F} \cap (M_a \oplus M_\beta)$ with the length 2, whence $M_a \cong M_\beta$ by ii) and therefore $f$ is isomorphic by (M-I). Now, let $\{x_\beta\}_J$ be a subset of $\mathcal{F}$ such that $\{x_\beta R\}_J$ is independent. To show that $\sum_{\lambda} \oplus x_\beta R$ can be extended to a decomposition of $M$, we can assume $\sum_{\lambda} \oplus x_\beta R \subseteq M$. Let $N_\beta$ be a direct summand of $M$ such that $x_\beta R \subseteq N_\beta$ for all $\beta \in J$. Then $\{N_\beta\}_J$ is independent since
\{x_\beta R\}_J \text{ is so and } x_\beta R \subseteq e N_\beta \text{ for all } \beta \text{ in } J. \text{ Therefore, by Corollary 14, } \\
\sum \bigoplus N_\beta \cong \bigoplus M, \text{ from which we have } M = \sum \bigoplus N_\beta.

1), 2) \implies 3). \text{ i) follows from Lemma 17 and ii) follows from 2).}

3) \implies 1). \text{ Let } \{x_\beta R\}_J \text{ be a subset of } \mathcal{F} \text{ such that } \{x_\beta R\}_J \text{ is independent. By i), there exists a direct summand } N_\beta \text{ of } M \text{ such that } x_\beta R \subseteq e N_\beta \text{ for all } \beta \text{ in } J. \text{ Since } \{x_\beta R\}_J \text{ is independent, so is } \{N_\beta\}_J. \text{ Hence } \sum \bigoplus N_\beta \text{ is a direct summand of } M \text{ by Corollary 14, from which we see that } \sum \bigoplus x_\beta R \text{ is extended to a decomposition of } M.

\textbf{Corollary 19.} \text{ We assume } \{M_\alpha\}_I \text{ is a locally semi-T-nilpotent set and each } M_\alpha \text{ satisfies (M-1). Then}

1) \text{ } M \text{ has the extending property of direct sum of cyclic smooth modules if and only if it has the extending property of cyclic smooth modules and every monomorphism of } M_\alpha \text{ to } M_\beta \text{ is isomorphic for any pair } \alpha, \beta \text{ in } I.

2) \text{ If } M \text{ has the extending property of direct sum of cyclic smooth modules, it has also the extending property of direct sum of cyclic smooth modules for any other decomposition of } M \text{ by completely indecomposable modules.}

3) \text{ Let } M_1, M_2 \in \{M_\alpha\}_I \text{ and } 0 \neq x_i \in M_i, i=1, 2. \text{ If } x_1 + x_2 \text{ is smooth, then } \text{Aut}_R(x_1 R) \text{ is extended to } \text{Aut}_R(M_i) \text{ for } i=1, 2.

\textbf{Proof.} \text{ If we take } \mathcal{F} = S(M = \sum \bigoplus M_\alpha) \text{ in Theorem 18, 1) and 2) are clear.}

3). \text{ By Theorem 18, there exists an isomorphism } f \text{ of } M_1 \text{ to } M_2 \text{ with } f(x_1) = x_2. \text{ Let } g \in \text{Aut}_R(x_1 R). \text{ Then } g(x_1) + x_2 \text{ is also a smooth element and hence, by again Theorem 18, there exists an isomorphism } h \text{ of } M_1 \text{ to } M_2 \text{ with } h(g(x_1)) = x_2. \text{ Then } h^{-1}f \in \text{Aut}_R(M_1) \text{ and } h^{-1}f | x_1 R = g.

\textbf{Corollary 20.} \text{ We assume } \{M_\alpha\}_I \text{ is a locally semi-T-nilpotent set and each } M_\alpha \text{ is uniform and satisfies (M-1). Then the following conditions are equivalent:}

1) \text{ } M \text{ has the extending property of decomposition of } S(M).

2) \text{ For any pair } \alpha \text{ and } \beta \text{ in } I, \text{ if } S(M_\alpha) \cong S(M_\beta), \text{ then } M_\alpha \cong M_\beta. \text{ And } \text{End}_R(S(M_\alpha)) \text{ is extended to } \text{End}_R(M_\alpha) \text{ for any } \alpha \text{ in } I.

3) \text{ } M \text{ has the extending property of simple module and every monomorphism of } M_\alpha \text{ to } M_\beta \text{ is isomorphic for any pair } \alpha, \beta \text{ in } I.

\textbf{Proof.} \text{ We can assume } M_i \supseteq S(M), \text{ and hence take } \mathcal{F} = \{x \in M \mid x R \text{ is simple}\} \text{ in Theorem 18.}

\textbf{Theorem 21.} \text{ Assume } \{M_\alpha\}_I \text{ is a locally semi-T-nilpotent set and each } M_\alpha \text{ is uniform and satisfies (M-1). Then the following conditions are equivalent:}

1) \text{ } M \text{ has the extending property of direct sum of uniform and smooth modules.
2) For any pair $\alpha, \beta$ in $I$, any monomorphism of any submodule $A_{\alpha}$ of $M_{\alpha}$ to $M_{\beta}$ is extended to an isomorphism of $M_{\alpha}$ to $M_{\beta}$.

3) i) $M$ has the extending property of uniform and smooth module.

ii) Every monomorphism of $M_{\alpha}$ to $M_{\beta}$ is isomorphic for any pair $\alpha, \beta$ in $I$.

Proof. 1)$\Rightarrow$ i) of 3) is clear. Hence we have 1)$\Rightarrow$2)$\Rightarrow$3) by Theorems 9 and 18. 3)$\Rightarrow$1) is shown by the same argument as in the proof of 2)$\Rightarrow$1) in Theorem 18.

**Theorem 22.** We assume each $M_{\alpha}$ is uniform and satisfies (M-I). Then the following conditions are equivalent:

1) $M$ has the extending property of direct sum of f.c. uniform modules.

2) i) $\{M_{\alpha}\}_{\alpha}$ is a locally semi-T-nilpotent set.

ii) Every monomorphism of $M_{\alpha}$ to $M_{\beta}$ is isomorphic for any pair $\alpha, \beta$ in $I$.

iii) For any pair $\alpha, \beta$ in $I$ and any submodule $A_{\alpha}$ of $M_{\alpha}$, $\text{Hom}_{R}(A_{\alpha}, M_{\beta})$ is extended to $\text{Hom}_{R}(M_{\alpha}, M_{\beta})$.

3) i) $\{M_{\alpha}\}_{\alpha}$ is a locally semi-T-nilpotent set.

ii) Every monomorphism of $M_{\alpha}$ to $M_{\beta}$ is isomorphic for any pair $\alpha, \beta$ in $I$.

iii) $M$ has the extending property of c.f. uniform module.

Proof. 1)$\Rightarrow$2), 3). Clearly iii) of 3) holds and hence iii) of 2) follows from Theorem 10. ii) of 2) follows from Theorem 13, whence to show the rest that $\{M_{\alpha}\}_{\alpha}$ is locally semi-T-nilpotent, we may show the following: Let $\{M_{\alpha}\}_{\alpha}$ and $\{f_{\alpha}: M_{\alpha}\rightarrow M_{\alpha+1}|_{i\geq 1}\}$ a family of non-monomorphisms. Then, for any $x$ in $M_{\alpha}$, there exists integer $n$ (depending on $x$) such that $f_{n}f_{n-1}\cdots f_{1}(x)=0$.

To verify this fact, put $M_{\alpha}^{{1}}=\{x+f(x)|x\in M_{\alpha}\}$ for $i\geq 1$. Then $M_{\alpha}^{{1}}\oplus M_{\alpha+1}^{{1}}=M_{\alpha}\oplus M_{\alpha+1}$ for all $i\geq 1$ and moreover $\{M_{\alpha}^{{i}}|_{i\geq 1}\}$ is independent. Since $M_{\alpha}^{{1}}\cap M_{\alpha}=0$ for all $i\geq 1$, we see $\sum_{i=1}^{\infty}M_{\alpha}^{{i}}\subseteq \sum_{i=1}^{\infty}M_{\alpha}$, whence by 1) we see $\sum_{i=1}^{\infty}M_{\alpha}^{{i}}=\sum_{i=1}^{\infty}M_{\alpha}$. This fact implies that if $x$ in $M_{\alpha}$, then there exists $n$ such that $f_{n}f_{n-1}\cdots f_{1}(x)=0$.

2) $\iff$ 3) follows from Theorem 10.

3)$\Rightarrow$1) is shown by the same argument as in the proof of 3)$\Rightarrow$1) in Theorem 18.

Similarly we obtain

**Theorem 23.** We assume each $M_{\alpha}$ is uniform and satisfies (M-I). Then the following conditions are equivalent:

1) $M$ has the extending property of direct sum of uniform modules.
2) i) \(\{M_\alpha\}_{\alpha\in I}\) is a locally semi-T-nilpotent set.
    ii) Every monomorphism of \(M_\alpha\) to \(M_\beta\) is isomorphic for any pair \(\alpha, \beta\) in \(I\).
    iii) For any \(\alpha\) in \(I\) and any submodule \(A_\alpha\) of \(M_\alpha\), \(\text{Hom}_R(A_\alpha, \sum_{\beta\neq \alpha} M_\beta)\)
    is extended to \(\text{Hom}_R(M_\alpha, \sum_{\beta\neq \alpha} M_\beta)\).

3) i) \(\{M_\alpha\}_{\alpha\in I}\) is a locally semi-T-nilpotent set.
    ii) Every monomorphism of \(M_\alpha\) to \(M_\beta\) is isomorphic for any pair \(\alpha, \beta\) in \(I\).
    iii) \(M\) has the extending property of uniform module.

Proof. We can show this by the same argument as in the proof of Theorem 22 (use Theorem 12 instead of Theorem 10).

Further we have the following theorem.

**Theorem 24.** The following conditions are equivalent:

1) \(M\) has the extending property of direct sum of submodule.
2) i) Each \(M_\alpha\) is uniform.
    ii) Every monomorphism of \(M_\alpha\) to \(M_\beta\) is isomorphic for any pair \(\alpha, \beta\) in \(I\).
    iii) \(\{M_\alpha\}_{\alpha\in I}\) is a locally semi-T-nilpotent set.
    iv) For any \(J\subseteq I\) and any submodule \(A \subseteq \sum_{\beta\in J} M_\beta\), \(\text{Hom}_R(A, \sum_{\beta\in J} M_\beta)\)
    is extended to \(\text{Hom}_R(\sum_{\beta\in J} M_\beta, \sum_{\beta\in J} M_\alpha)\).

3) i) Every monomorphism of \(M_\alpha\) to \(M_\beta\) is isomorphic for any pair \(\alpha, \beta\) in \(I\),
    ii) \(\{M_\alpha\}_{\alpha\in I}\) is a locally semi-T-nilpotent set.
    iii) \(M\) has the extending property of submodule.

Proof. 1)\(\Rightarrow\)2). iii) follows from Corollary 3. So, i) and ii) follow from Corollary 14 and hence we know iv) by Theorem 15.

2)\(\leftrightarrow\)3) follows from Theorem 15.

3)\(\Rightarrow\)1). Let \(\{N_\beta\}_{\beta\in J}\) be an independent set of submodules of \(M\). By iii), there exists \(N_\beta\subseteq \oplus M\) such that \(N_\beta \subseteq N_\beta\) for all \(\beta\) in \(J\). Since \(N_\beta\subseteq \oplus M\) and ii) holds, each \(N_\beta\) is a direct sum of completely indecomposable \(R\)-modules by [6], [12]. Hence \(\sum_{\beta\in J} N_\beta\) is extended to a decomposition of \(M\).

4. Applications

We start with

**Proposition 25.** If \(M\) is a quasi-injective \(R\)-module, then \(M\) has the extending property of submodule. If we assume further that \(M\) is a direct sum of completely indecomposable modules, say \(M = \sum_{\alpha\in I} M_\alpha\), then \(M\) has the extending property of direct sum of submodules.
Proof. Let $N$ be a submodule of $M$. Then $E(M) = E(N) \oplus K$ for some $K$. Since $M$ is quasi-injective, $M = (E(N) \cap M) \oplus (K \cap M)$ and then $N \subseteq E(N) \cap M$. Next, since $M$ is quasi-injective, $M$ has the exchange property by [3]. Hence we see from [6], [12] that $\{M_\alpha\}_I$ is locally semi-T-nilpotent. For any pair $\alpha, \beta$ in $I$, we show that every monomorphism $f: M_\alpha \to M_\beta$ is isomorphic. Let $\rho$ be an isomorphism of $E(M_\alpha)$ to $E(M_\beta)$ which is an extension of $f$. Since $M_\alpha \oplus M_\beta$ is quasi-injective, $\rho^{-1}(M_\beta) \subseteq M_\alpha$ by [11]. Hence $M_\beta \subseteq \rho(M_\alpha) = f(M_\alpha)$ and hence $f$ is indeed isomorphic. Now, let $\{N_\beta\}_J$ be an independent set of submodules of $M$. Then, there exists a direct summand $N_\beta$ of $M$ such that $N_\beta \subseteq M_\beta$ for all $\beta$ in $J$. Since $\{M_\alpha\}_I$ is locally semi-T-nilpotent and $N_\beta \subseteq \bigoplus M$, each $N_\beta$ is a direct sum of completely indecomposable modules. Consequently we see from Corollary 14 that $\sum J \oplus N_\beta$ is a direct summand of $M$; whence $\sum J \oplus N_\beta$ is extended to a decomposition of $M$.

**Theorem 26.** Let $\{M_\alpha\}_I$ be a set of completely indecomposable $R$-modules and put $M = \sum \bigoplus M_\alpha$. Then the following conditions are equivalent:

1) $M$ is quasi-injective.
2) $M \oplus M$ has the extending property of direct sum of submodules.

Proof. 1) $\Rightarrow$ 2). Since $M$ is quasi-injective, $M \oplus M$ is also quasi-injective by [4], [11]. Hence 2) holds by Proposition 25.

2) $\Rightarrow$ 1) is easily seen from Theorem 24.

**Corollary 27.** Let $T$ be a completely indecomposable $R$-module and consider $M = \sum \bigoplus M_\alpha$ where $M_\alpha \cong T$ for all $\alpha$ in $I$; $|I| \geq 2$. Then $M$ is quasi-injective if and only if $M$ has the extending property of direct sum of submodules.

Proof. If the cardinal $|I| = \infty$, then $M \oplus M \cong M$; whence the statement follows from Theorem 26. If $|I| < \infty$, then $M$ is quasi-injective if and only if $T$ is quasi-injective. Hence in this case, the proof also follows from Theorem 26.

**Theorem 28.** Let $\{M_\alpha\}_I$ be a set of indecomposable quasi-injective $R$-modules and $M = \sum \bigoplus M_\alpha$. Consider the following conditions:

1) $M$ is quasi-injective.
2) For any pair $\alpha, \beta$ in $I$, if $E(M_\alpha) \cong E(M_\beta)$, then $M_\alpha \cong M_\beta$.
3) $M$ has the extending property of direct sum of submodules.

Then we have 1) $\Rightarrow$ 3) $\Rightarrow$ 2), and in the case when $M$ is non-singular, all conditions are equivalent.

Proof. 1) $\Rightarrow$ 3) follows from Proposition 25.

3) $\Rightarrow$ 2). Let $\rho$ be any isomorphism of $E(M_\alpha)$ to $E(M_\beta)$ for pair $\alpha, \beta$ in $I$. 

---

*Please note: The above text is a transcription of the provided document.*
Then \( \rho^{-1}(M_\beta) \cap M_\alpha = 0 \). Hence, there exists \( f: M_\alpha \to M_\beta \) extended from \( \rho | \rho^{-1}(M_\beta) \cap M_\alpha \) and \( f \) is isomorphic by Theorem 22. Assuming \( M \) is non-singular, we show (2) \( \Rightarrow \) (1). By [13], we see that \( \sum \bigoplus E(M_\alpha) \) is quasi-injective and every non-zero homomorphism of \( E(M_\alpha) \) to \( E(M_\beta) \) is isomorphic for any pair \( \alpha, \beta \) in \( I \); so \( \text{Hom}_R(E(M_\alpha), E(M_\beta)) = 0 \) if \( E(M_\alpha) \cong E(M_\beta) \). Further if \( \alpha, \beta \in I \) and \( f: E(M_\alpha) \to E(M_\beta) \) is isomorphic, then \( f(M_\alpha) \subseteq M_\beta \). For, by (2), there exists an isomorphism \( g: M_\alpha \to M_\beta \). Let \( \rho: E(M_\alpha) \to E(M_\beta) \) be an extension of \( g \). Since \( M_\alpha \) is quasi-injective, \( \rho^{-1}(M_\beta) \subseteq M_\alpha \) by [11]. This indeed implies \( f(M_\alpha) \subseteq M_\beta \). Thus \( M \) is quasi-injective by [4] and [11].

**Proposition 29.** Let \( \{M_\alpha \} I \) be a set of uniform and completely indecomposable \( R \)-modules with \( (M-I) \) and the cardinal \( | I | = 1 \), and put \( M = \sum \bigoplus M_\alpha \). We assume \( E(M_\alpha) \cong E(M_\beta) \) for all \( \alpha, \beta \) in \( I \) and \( M \) has the extending property of direct sum of f.c. uniform modules. Then if either \( | I | < \infty \) or \( R \) is right Noetherian, \( M \) is quasi-injective.

**Proof.** Let \( \rho \) be any isomorphism of \( E(M_1) \) to \( E(M_2) \) for \( 1, 2 \) in \( I \). Then there exists an isomorphism \( f: M_1 \to M_2 \) such that \( f = \rho \) on \( \rho^{-1}(M_2 \cap M_1) \) (see proof of Theorem 28). Let \( h \in \text{Hom}_R(A, M_1) \) for \( A \subseteq M_1 \). Then \( fh \) is extended to a homomorphism \( t: M_1 \to M_2 \) by Theorem 22. Hence, \( f^{-1}t \) is an extension of \( h \) and so \( M_1 \) is quasi-injective. Further, since \( f^{-1}\rho \in \text{End}_R(E(M_2)) \), \( M_2 = f(\rho^{-1}(\rho(M_1))) = \rho(M_1) \) by [11]. If \( | I | < \infty \) or \( R \) is Noetherian, \( E(M) = \sum \bigoplus E(M_\alpha) \). Hence, \( M \) is quasi-injective by [4] and [11].

**Corollary 30.** Let \( \{M_\alpha \} I \) and \( M \) be as above. We assume further each \( M_\alpha \) is non-singular and \( M \) has the extending property of f.c. uniform modules. Then if either \( | I | < \infty \) or \( R \) is right Noetherian, \( \sum \bigoplus M_\alpha \) is quasi-injective, where \( I' \) is a subset of \( I \) such that for \( \alpha \) in \( I' \) there exists \( \rho(\alpha) \neq \alpha \) in \( I \) with \( E(M_\alpha) \cong E(M_\rho(\alpha)) \).

**Proof.** Since \( M_\alpha \) is non-singular, \( \text{Hom}_R(E(M_\alpha), E(M_\beta)) = 0 \) if \( E(M_\alpha) \cong E(M_\beta) \). Hence, we have the corollary by Proposition 29 and [4], [11].

Especially, when \( R \) is a Dedekind domain, we have

**Theorem 31.** Let \( R \) be a Dedekind domain and \( M \) an \( R \)-module. Then \( M \) has the extending property of direct sum of uniform modules if and only if either

1) \( M \) is quasi-injective or
2) \( M = K \oplus E \), where \( E \) is torsion and injective and \( K \subseteq Q = E(R) \).

If \( M \) has the extending property of direct sum of uniform modules, then \( M = \sum \bigoplus M_\alpha \) with \( M_\alpha \) uniform, since \( R \) is Noetherian. We shall complete the proof by making use of elementary properties of abelian groups as follows:
Lemma 32. Let $K$ be an $R$-submodule of $Q$ and $\{f_i: K \rightarrow E(P_i)\}^{(I_i)}$ a set of homomorphisms. We assume $P_i \neq P_j$ if $i \neq j$ and for some $a \neq 0$ in $K$, $f_i(a) = 0$ for almost all $i$. Then $\{f_i\}$ is summable, i.e., for any $x$ in $K$, $f_i(x) = 0$ for almost all $i$, where $P_i$ is a non-zero prime ideal, $E(P_i) = E(R/P_i)$ and $E(P_i)^{(I_i)}$ is a direct sum of $I_i$-copies of $E(P_i)$ (cf. [14]).

Proof. We may assume $a = 1$ and $K \supseteq R$. Then $f_i \in \text{Hom}_R(K/R, E(P_i))^{(I_i)}$ for almost all $i$. Hence, since $K/R = \bigoplus_{i}^r (q$-primary part of $K/R)$, $\{f_i\}$ is summable.

Lemma 33. Let $K$ be as above and $L$ an $R$-submodule of $E(P)$. If $\text{Hom}_R(A, L)$ is extended to $\text{Hom}_R(K, L)$ for any submodule $A$ in $K$, then $L = E(P)$.

Proof. We may assume $K \supseteq R$. We assume $L \neq E(P)$. Then $L = \overline{p}nR$, where $E(P) = \bigcup p^{-n}R$ and $\overline{p}^{-n} \in Q/R$. Put $A = p^*R \supseteq B = p^*nR$. Then the natural epimorphism $f: A \rightarrow A/B = L$ in $\text{Hom}_R(A, L)$. Let $g \in \text{Hom}_R(R, L)$ be an extension of $f$. Then $R = \ker g + A$. Hence, $R_p = (\ker g)_p + A_p = (\ker g)_p$. Since $(\ker g) \cap A = B$, $A_p = B_p$, a contradiction. Hence, $L = E(P)$.

The following lemma is similar to Theorem 22.

Lemma 34. Let $T = \sum T_\alpha$ be any decomposition of an $R$-module and let each $T_\alpha$ be uniform. If $T$ has the extending property of direct sum of uniform modules, then any element in $\text{Hom}_R(A_\alpha, T_\beta)$ is extended to $\text{Hom}_R(T_\alpha, T_\beta)$ for any pair $\alpha, \beta$ in $I$ and for any submodule $A_\alpha$ in $T_\alpha$.

Proof. Let $f$ be in $\text{Hom}_R(A_1, T_2)$. Put $A(f) = \{a + f(a) | a \in A_1\}$ and consider an essential submodule $A(f) + T_2 \bigoplus \sum_{I \supseteq \{1, 2\}} T_\alpha$ of $T$. Then there exists a decomposition $T = \sum T_\alpha$ such that $S_{\alpha} \supseteq T_\alpha$ for $\alpha \neq 1$ and $S_1 \supseteq A(f)$. Since $T_\alpha$ is a direct summand of $T$, $S_\alpha = T_\alpha$ for $\alpha \neq 1$. Thus, $T = S_1 \bigoplus \sum_{I \supseteq \{1\}} T_\alpha$ and so $\pi | T_1$ is an extension of $f$, where $\pi: T \rightarrow T_2$ is the projection on the decomposition.

Proof of Theorem 31. We assume $M$ has the extending property of direct sum of uniform modules. Then $M = \sum_{I_1} M_\alpha \bigoplus \sum_{I_2} M_\beta$, where $M_\alpha$ (resp. $M_\beta$) is a torsion-free (resp. torsion) uniform submodule of $M$. We may assume $M_\alpha \subseteq Q$ for all $\alpha$ in $I_1$. First we assume $|I_1| \geq 2$. Take $0 \neq x$ in $M_{a_1} \cap M_{a_2}$. For any $m \neq 0 \in R$ a homomorphism $f: xmR \rightarrow xR$ by setting $f(xmr) = xr$ is extended to $g: M_{a_1} \rightarrow M_\alpha$ by Lemma 34. Hence, $xR(1/m) \subseteq M_\alpha(1/m) = g(M_{a_1}) \subseteq M_{a_2}$. Therefore, $M_{a_2} = Q$ and so each $M_\alpha = Q$ for $\alpha$ in $I_1$. Furthermore we know $M_\beta = E(P)$ for any $\beta$ in $I_2$ by Lemmas 32 and 33. Hence, $M$ is injective. If $|I_1| = 1$ and $M_{a_1} = Q$, $M$ is also injective as above. Next, assume $|I_1| = 1$ and $M_{a_1} \subseteq Q$. Then $M$ is of the form 2). Finally, we assume $M$ is torsion.
Each \( M_\beta \) is a completely indecomposable with \((M-I)\). Hence, \( M \) is quasi-injective by Proposition 29, the fact: \( \text{Hom}_R(E(P), E(Q))=0 \) if \( P \neq Q \) and [3], [9]. Conversely, if \( M \) is quasi-injective, \( M \) has the extending property of direct sum of submodules by Proposition 25. Let \( M=K\bigoplus \sum_{\beta} E(P)^{(\ell_\beta)} \) as in 2). We assume \( N=N_1\bigoplus \sum_{\ell_\beta} \oplus N_\beta \) be an essential submodule of \( M \), where \( N_1 \) (resp. \( N_\beta \)) is torsion-free (resp. torsion) and uniform. Let \( \pi \) and \( \pi_\ell \) be the projection of \( M \) onto \( K \) and \( E(P)^{(\ell_\beta)} \), respectively. Then \( \pi|N_1 \) is isomorphic. Put \( f_\ell = \pi_\ell (\pi|N_1)^{-1}: \pi(N_1)\rightarrow E(P)^{(\ell_\beta)} \). Then \( \{f_\ell\} \) is summable. Since \( E(P)^{(\ell_\beta)} \) is injective, we obtain an extension \( g_\ell \in \text{Hom}_R(K, E(P)^{(\ell_\beta)}) \) of \( f_\ell \). Then \( \{g_\ell\} \) is also summable by Lemma 32. Put \( K'=\{x+\sum_\ell g_\ell(x)|x\in K\}\subseteq M \). Then \( M=K'\bigoplus \sum_{\ell} E(P)^{(\ell_\beta)} \) and \( K'\supseteq N_1, \sum_\ell E(P)^{(\ell_\beta)} \supseteq \sum_{\ell_\beta} \oplus N_\beta \) (cf. The proof of Theorem 10). Hence, \( N_1\bigoplus \sum_\ell \oplus N_\beta \) is extended to a decomposition of \( M \) by proposition 25.

**Theorem 35.** Let \( R \) be a left perfect ring [2]. Then the following conditions are equivalent.

1) Every direct sum of completely indecomposable uniform \( R \)-module has the extending property of simple module.

2) For every completely indecomposable submodules \( U_1 \) and \( U_2 \) of an indecomposable injective module \( E \), there exists an automorphism \( f \) of \( E \) such that either \( f(U_1)\subseteq U_2 \) with \( f|S(E)=f \) or \( f(U_2)\subseteq U_1 \) with \( f|S(E)=f^{-1} \) for any \( f\in \text{End}_R(S(E)) \).

Proof. 1)\( \Rightarrow \)2) is clear from Theorem 5. Since every uniform module is embedded in an indecomposable injective module, 2)\( \Rightarrow \)1) also follows from Theorem 5.

We end this paper with the following theorem.

**Theorem 36.** Let \( R \) be a right and left artinian ring. Then the following conditions are equivalent:

1) \( R \) is a generalized uniserial ring [15].

2) Every right \( R \)-module, as well as every left \( R \)-module, has the extending property of simple module.

Proof. 1)\( \Rightarrow \)2). Every module \( M \) is a direct sum of uniserial modules by [15]. Hence, we know 1)\( \Rightarrow \)2) by Theorem 35.

2)\( \Rightarrow \)1). Let \( e \) be a primitive idempotent. Since \( eR/eA \) is indecomposable for every right ideal \( eA \), \( S(eR/eA) \) is simple by 2). Hence, \( R \) is a right generalized uniserial ring. By the same argument, we see \( R \) is also uniserial.
DIRECT SUMS OF UNIFORM MODULES

References


Manabu Harada
Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558
Japan

Kiyoichi Oshiro
Department of Mathematics
Yamaguchi University
Oaza, Yoshida, Yamaguchi 753
Japan