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Osaka University
ON THE INTERMEDIATE COHOMOLOGY GROUP
OF A HOLOMORPHIC LINE BUNDLE OVER
A COMPLEX TORUS

Yozo MATSUSHIMA

(Received August 7, 1978)

Let \( E = V / L \) be a complex torus, where \( V \) is an \( n \)-dimensional complex
vector space and \( L \) a lattice of \( V \). Let \( H \) be a Hermitian form on \( V \) and \( A \) the imaginary part of \( H \). Then \( A \) is an \( \mathbb{R} \)-bilinear alternating form on \( V \). We say that \( H \) is a Riemann form of signature \((s, r)\) for the torus \( E \) if

(a) \( H \) is non-degenerate and of signature \((s, r)\);
(b) \( A \) is integral valued on the lattice \( L \).

To a Riemann form \( H \) we associate a factor

\[
\mathcal{J}_{H, \psi}(g, z) = \psi(g) e^{\frac{1}{2i} H(z, g) + \frac{1}{4i} H(g, g)}
\]

with \( g \in L, \ z \in V \), where \( e^{[\cdot]} = \exp 2\pi i \cdot \) and \( \psi \) is a map from \( L \) to \( \mathbb{C}^* = \{z \in \mathbb{C} | |z| = 1\} \) satisfying \( \psi(g + h) = \psi(g) \psi(h) e^{\frac{1}{2} A(g, h)} \); the function \( \psi \) being called a semi-character of \( L \) for \( A \).

The factor \( \mathcal{J}_{H, \psi} : L \times V \to \mathbb{C}^* \) satisfies the equation

\[
\mathcal{J}_{H, \psi}(g + f, z) = \mathcal{J}_{H, \psi}(g, h + z) \mathcal{J}_{H, \psi}(h, z),
\]

where \( g, h \in L, \ z \in V \).

Given the factor \( \mathcal{J}_{H, \psi} \) we define an action of the lattice group \( L \) on \( V \times C \) by the rule

\[
(z, \xi) \cdot g = (z + g, \mathcal{J}_{H, \psi}(g, z) \xi),
\]

where \( z \in V, \ \xi \in C \) and \( g \in L \). The action of \( L \) on \( V \times C \) is free and the quotient of \( V \times C \) by \( L \) has a natural structure of a holomorphic line bundle over the complex torus \( E = V / L \). We shall denote this line bundle by \( F(H, \psi) \).

The following vanishing theorem for the cohomology of \( F(H, \psi) \) is well-known [2, 4]: If \( H \) is a Riemann form of signature \((s, r)\), then we have

\[
H^\bullet(E, F(H, \psi)) = 0
\]
for $q=r$ (for a proof see Appendix 2 of this paper).

In particular if $r=0$, namely if $H$ is positive definite, $H^q(E, F(H, \psi))=0$ except for $q=0$ and $H^0(E, F(H, \psi))$ is identified with the space of all holomorphic theta functions on $V$ for the factor $J_H, \psi$. Replacing $H$ by a suitable positive integer multiple of $H$, if necessary, these theta functions define a holomorphic imbedding of $E$ into a complex projective space. A complex torus which admits a positive definite Riemann form is called an abelian variety. There exist complex tori which are not abelian varieties but which admit Riemann forms of signature $(s, r)$ with $s>0$ and $r>0$ (see Appendix 1). For such a complex torus $E$, there exists no non-trivial theta function. However, there exists the non-trivial intermediate cohomology group $H^r(E, F(H, \psi))$ with $0<r<n$.

The purpose of this paper is to give an interpretation of the intermediate cohomology group $H^r(E, F(H, \psi))$. Namely we associate to a Riemann form $H$ of signature $(s, r)$ a family of polarized abelian varieties $(E_b, H_b)$ parametrized by elements $b$ of the Hermitian symmetric space $\mathcal{B}=U(H)/K$, where $U(H)$ is the unitary group of the Hermitian form $H$ and $K$ is a maximal compact subgroup of $U(H)$. Here $E_b$ is of the form $E_b=V_b/L$, where $V_b$ is an $n$-dimensional complex vector space with the same underlying real vector space as $V$ and with a complex structure $J_b$ distinct from that of $V$ and parametrized by $b \in \mathcal{B}$ and $H_b$ is a positive definite Riemann form for $E_b$ whose imaginary part is equal to $A$. We then assign a family of line bundles $F(H_b, \psi)$ over $E_b$ for each $b$. Finally we shall show that there exists a canonically defined isomorphism from $H^r(E, F(H, \psi))$ to $H^0(E_b, F_b, \psi))$ for each $b$. We also see that there exists a family $\Phi_b$ of differentiable imbedding of $E$ into a complex projective space which is partially holomorphic and partially antiholomorphic.

It should be mentioned that C.L. Siegel [3] has associated to an indefinite quadratic form a family of theta series parametrized by a symmetric space. It is possible to interpret Siegel's family of theta series as a subfamily of the family $\{H^0(E_b, F(H_b, \psi))\}$ of theta functions attached to a certain complex torus $E$ and a Riemann form $H$ related with the given indefinite quadratic form.

1. A Riemann form of signature $(s, r)$ and a family of polarized abelian varieties

Let $E=V/L$ be a complex torus, where $V$ is an $n$-dimensional complex vector space and $L$ a lattice of $V$. We shall denote by $W$ the underlying $2n$-dimensional real vector space of $V$ and by $J$ the complex structure of $W$ defining the complex vector space $V$.

Let $H$ be a non-degenerate Hermitian form on $V$ of signature $(s, r)$, where $s+r=n$. We denote by $A$ the imaginary part of $H$. Then we have

$$H(u, v) = A(Ju, v) + iA(u, v), \quad u, v \in W.$$
We assume that the alternating $\mathbf{R}$-bilinear form $A$ to be integral valued on $L \times L$ and we call $H$ a Riemann form of signature $(s, r)$ for the complex tours $E$.

We shall denote by $U(H)$ the unitary group of the Hermitian form $H$. A basis $B = \{v_1, \ldots, v_n\}$ of $V$ is said to be a privileged basis for $H$ if the matrix of $H$ with respect to the basis $B$ is of the form

$$1_s \cdot r = \begin{pmatrix} 1_s & 0 \\ 0 & -1_r \end{pmatrix}$$

where $1_s$ and $1_r$ denote the unit matrix of size $s$ and $r$ respectively.

The group $U(H)$ acts simply transitively on the set of all privileged bases for $H$. We denote by $U(H)$ the unitary group of the Hermitian form $H$. We shall denote by $V_1(B)$ and $V_2(B)$ the subspaces of $V$ spanned by $\{v_1, \ldots, v_l\}$ and $\{v_{s+1}, \ldots, v_n\}$ respectively. Then we have

$$W = V_1(B) \oplus V_2(B).$$

We say that two privileged bases $B$ and $B'$ are equivalent, $B \sim B'$, if $V_i(B) = V_i(B')$ for $i=1, 2$. We shall denote by $\mathcal{B}$ the set of equivalence classes of privileged bases for $H$. Then the group $U(H)$ acts transitively on $\mathcal{B}$ and $\mathcal{B}$ is identified with the Hermitian symmetric space $U(H)/K$, where $K$ is a maximal compact subgroup of $U(H)$.

Let $b \in \mathcal{B}$ and let $B$ be a privileged basis representing $b$. We define a linear transformation $J_b$ of $W$ by requiring

$$J_b = \begin{cases} J & \text{on } V_1(B) \\ -J & \text{on } V_2(B). \end{cases}$$

We have $J_b^2 = -1$ and hence $J_b$ defines a complex structure on $W$. We shall denote by $W$ the complex vector space defined by $W$ and $J_b$.

Define the symbol $\varepsilon_k(k=1, 2, \ldots, n)$ by

$$\varepsilon_k = \begin{cases} 1, & k \in [1, s] \\ -1, & k \in [s+1, n]. \end{cases}$$

If $B = \{v_1, \ldots, v_n\}$ is a privileged basis representing $b$, then we have

$$J_b v_k = \varepsilon_k J v_k$$

We also have $H(v_k, v_j) = \varepsilon_k \cdot \delta_{kj}$ and since $H(v_k, v_j) = A(J v_k, v_j) + iA(v_k, v_j)$, we get $A(v_k, v_j) = 0$ and $A(J v_k, v_j) = \varepsilon_k \cdot \delta_{kj}$. It follows from these that the decomposition (2) is orthogonal for $A$ and also for $H$. We have also $A(J_b u, J_b v) = A(u, v)$ for $u, v \in W$. For let $u = u_1 + u_2$, $v = v_1 + v_2$ with $u_i, v_i \in V_i(B)$ and $u_2, v_2 \in V_2(B)$. Then $J u_i = J u_i - J u_2$ and $J v = J v_1 - J v_2$ and $J u_i, J v_1 \in V_1(B)$ and $J u_2, J v_2 \in V_2(B)$. Hence $A(J_b u, J_b v) = A(J u_1, J v_1) + A(J u_2, J v_2) = A(u_1, v_1) + A(u_2, v_2)$. We can then
define a Hermitian form $H_b$ on the complex vector space $V_b$ by

$$H_b(u, v) = A(J_b u, v) + iA(u, v).$$

Then the imaginary part of $H_b$ is $\mathcal{A}$ and we have $H_b(v, v_j) = A(J_b v_j, v_j) + iA(v, v_j) = \delta_{ij} = \delta_{bk}$. This means that $B$ is an orthonormal basis of $V_b$ for the Hermitian form $H_b$ and in particular $H_b$ is positive definite and the decomposition (2) of $W$ is also orthogonal for $H_b$.

Let now

$$E_b = V_b/L.$$  

Then $H_b$ is a positive definite Riemann form for $E_b$ and hence $E_b$ is an abelian variety. Thus we have associated to a complex torus $E$ and a Riemann form $H$ of signature $(s, r)$ a family of polarized abelian varieties $(E_b, H_b)$ parametrized by $b \in \mathcal{B} = U(H)/K$.

We need the following lemma in the next section.

**Lemma.** We have

$$H(u, v) = H_b(u, v) \text{ for } u \in V_1(B)$$

and

$$H(u, v) = -H_b(v, u) \text{ for } u \in V_2(B).$$

For we have $H_b(u, v) = A(J_b u, v) + iA(u, v)$ and $J_b = J$ or $J_b = -J$ according as $u \in V_1(B)$ or $u \in V_2(B)$. 

**2. The cohomology group $H'(E, F(H, \psi))$**

We associate to the Riemann form $H$ of signature $(s, r)$ for $E$ the factor $J_H, \psi$ defined by (1) and the line bundle $F(H, \psi)$ over $E$. For the cohomology groups of $F(H, \psi)$ we have the following theorem.

**Theorem 1.** (i) We have $H^q(E, F(H, \psi)) = 0$ for $q \neq r$.

(ii) Let $(z_1, \cdots, z_n)$ be the coordinates of the complex vector space $V$ determined by a privileged basis $B$ of $V$ for the Hermitian form $H$. Then $H^r(E, F(H, \psi))$ is identified with the complex vector space of all $C^\infty$ functions $f$ on $V$ satisfying the following conditions:

1) $f$ is a differentiable theta function for the factor $J_H, \psi$; namely we have

$$f(z + g) = J_H, \psi(z, g) \cdot f(z), \ z \in V, \ g \in L.$$  

2) $\frac{\partial f}{\partial \bar{z}_k} = 0$ for $k \in [1, s]$ 

and

$$\frac{\partial f}{\partial \bar{z}_{s+j}} + \pi \bar{z}_{s+j} f = 0 \text{ for } j \in [1, r].$$
The assertion (i) in Theorem 1 is a well-known vanishing theorem due to Mumford. We shall give a proof of Theorem 1 in the Appendix 2 based on the harmonic theory.

We denote by $H(B)$ the space of $C^\infty$ functions $f$ on $W$ satisfying the above conditions (1) and (2) to make explicit its dependence of the condition (2) on the choice of the privileged basis $B$. We show that if $B$ and $B'$ are equivalent, then we have $H(B)=H(B')$. In fact, let $(z'_1, \ldots, z'_s)$ be the coordinates of $V$ determined by $B'$. Then we have

$$z'_i = \sum_{j=1}^s a_{ij} z_j \quad (i=1, \ldots, s)$$

and

$$z'_{s+i} = \sum_{j=1}^s b_{ij} z_{s+j} \quad (i=1, \ldots, r)$$

where the matrices $(a_{ij})$ and $(b_{ij})$ are both unitary. We get

$$\frac{\partial f}{\partial \bar{z}_k} = \sum_{i=1}^s a_{ik} \frac{\partial f}{\partial \bar{z}_i} \quad (k=1, \ldots, s)$$

and

$$\frac{\partial f}{\partial \bar{z}_{s+k}} + \pi \bar{z}_{s+k} f = \sum_{i=1}^s b_{ik} \frac{\partial f}{\partial \bar{z}_{s+i}} + \pi \left( \sum_{i=1}^s \bar{b}_{ki} \bar{z}_{s+i} \right) f,$$

where $(b'_{ki})$ is the inverse matrix of $(b_{ki})$. Since $(b_{ki})$ is unitary, we have $(b'_{ki}) = (b_{ki})$ and hence $\bar{b}'_{ki} = b_{ik}$. Hence we get

$$\frac{\partial f}{\partial \bar{z}_{s+k}} + \pi \bar{z}_{s+k} f = \sum_{i=1}^s b_{ik} \left( \frac{\partial f}{\partial \bar{z}_{s+i}} + \pi \bar{z}_{s+i} f \right)$$

From (*) and (**) we get $H(B)=H(B')$. Hence we can denote the space of $C^\infty$ functions $f$ satisfying (1) and (2) by $H(b)$, $b \in \mathcal{B}$.

Consider now the family of polarized abelian varieties $(E_b, H_b)$ defined in §1. We have the factor $J_{H_b, \psi} : L \times V \to \mathbb{C}^*$ defined by

$$J_{H_b, \psi}(g, u) = \psi(g) \varepsilon \left[ \frac{1}{2i} H_b(u, g) + \frac{1}{4i} H_b(g, u) \right]$$

where $g \in L$ and $u \in V$. This is because the imaginary part of $H_b$ is equal to $A$ for any $b$. Let $F(H_b, \psi)$ be the line bundle over $E_b$ associated with the factor $J_{H_b, \psi}$. Since $H_b$ is positive definite, we have $H^q(E_b, F(H_b, \psi)) = 0$ for $q \neq 0$ and $H^0(E_b, F(H_b, \psi))$ is identified with the complex vector space of all holomorphic theta functions on $V_b$ for the factor $J_{H_b, \psi}$.

Let $p_i : W \to V_i(B)$ ($i=1, 2$) be the projection of $W$ onto $V_i(B)$ with respect
to the decomposition (2) of $W$ and let

$$\phi_b(u) = \exp \left[ -\pi H_b(p_3(u), p_2(u)) \right].$$

We have

$$\phi_b(u+g) = \phi_b(u) \exp L(u, g), \ u \in W, \ g \in L,$$

where

$$L(u, g) = -\pi[H_b(p_3(u), p_2(g)) + H_b(p_2(u), p_2(g)) + H_b(p_2(g), p_2(g))].$$

Let $\theta$ be a holomorphic theta function on $V_b$ for the factor $J_{\mathcal{H}b, \psi}$ and let

$$f = \phi_b \cdot \theta.$$

We show that the function $f$ satisfies the conditions (1) and (2) in Theorem 1, i.e. $f \in H(b)$. We have

$$f(u+g) = f(u) \cdot \varphi(g) \exp \left[ L(u, g) + \pi H_b(u, g) + \frac{\pi}{2} H_b(g, g) \right].$$

Since the decomposition (2) is orthogonal for $H_b$ we get

$$\pi H_b(u, g) + \frac{\pi}{2} H_b(g, g) = \pi H_b(p_3(u), p_2(g)) + \pi H_b(p_2(u), p_2(g)) + \frac{\pi}{2} H_b(p_2(g), p_2(g))$$

$$+ \frac{\pi}{2} H_b(p_2(g), p_2(g))$$

and hence

$$L(u, g) + \pi H_b(u, g) + \frac{\pi}{2} H_b(g, g)$$

$$= \pi[H_b(p_2(u), p_2(g)) - H_b(p_2(g), p_2(u)) + \frac{\pi}{2} [H_b(p_2(g), p_2(g)) - H_b(p_2(g), p_2(g))].$$

From Lemma at the end of §1 and from the orthogonality of the decomposition (2) for $H$ we see that the left hand side of the above equality is equal to $\pi H(u, g) + \frac{\pi}{2} H(g, g)$. Hence we get

$$f(u+g) = f(u) \cdot \varphi(g) \exp \left[ \pi H(u, g) + \frac{\pi}{2} H(g, g) \right] = f(u) J_{\mathcal{H}_b, \psi}(g, u)$$

which shows that $f$ is a differentiable theta function for the factor $J_{\mathcal{H}_b, \psi}$.

Now let $B$ be any privileged basis representing $b$ and let $(z_1, \ldots, z_n)$ be the coordinates of $V$ determined by $B$. Then $B$ is also an orthonormal basis of $V_b$ for the Hermitian form $H_b$ and let $(w_1, \ldots, w_n)$ be the coordinates of $V_b$ determined by $B$. Then as functions on $W$ we have

$$z_i = w_i \quad \text{for} \quad i \in [1, s],$$

$$\bar{z}_{s+i} = \bar{v}_{s+i} \quad \text{for} \quad i \in [1, r].$$

Since $\theta$ is a holomorphic function on $V$, we have

$$\frac{\partial \theta}{\partial w_k} = 0, \quad \text{for} \quad k \in [1, n].$$
and hence
\[ \frac{\partial \theta}{\partial \bar{z}_i} = 0, \quad i \in [1, s]; \quad \frac{\partial \theta}{\partial z_{s+i}} = 0, \quad i \in [1, r], \]

If \( u = \sum_{i=1}^{s} z_i v_i \), then \( p_2(u) = \sum_{i=1}^{s} z_{s+i} v_{s+i} \) and hence \( H_b(p_2(u), p_2(u)) = \sum_{i=1}^{s} |z_{s+i}|^2 \) and so
\[ \phi_b = \exp \left[ -\pi \sum_{i=1}^{s} |z_{s+i}|^2 \right]. \]

We see easily that we have \( \frac{\partial f}{\partial \bar{z}_i} = 0 \) for \( i \in [1, s] \) and \( \frac{\partial f}{\partial z_{s+i}} + \pi z_{s+i} f = 0 \) and hence \( f \) belongs to \( H(b) \). Analogously we can see that if \( f \) is a function belonging to \( H(b) \), then the function \( \theta \) defined by \( \theta(u) = f(u) \cdot \phi_b(u)^{-1} \) is a holomorphic theta function on \( V_b \) for the factor \( J_{H, \psi} \) and moreover the map \( f \to \theta \) defines a bijection of \( H(b) \) onto the space \( H^q(E_b, F(H_b, \psi)) \) of holomorphic theta functions on \( V_b \) for the factor \( J_{H, \psi} \). Since \( H(b) \) is canonically isomorphic to \( H'(E, F(H, \psi)) \) by Theorem 1, we obtain the following theorem.

**Theorem 2.** Let \( H \) be a Riemann form of signature \( (s, r) \) for a complex torus \( E \) and let \( F(H, \psi) \) be the holomorphic line bundle over \( E \) associated with the factor \( J_{H, \psi} \) defined by (1). Let \( (E_b, H_b) \) and \( (F(H_b, \psi)) \) be the family of polarized abelian varieties and the family of line bundles over each \( E_b \) parametrized by \( b \in \mathbb{B} \). Then there exists a canonical isomorphism of \( H'(E, F(H, \psi)) \) onto \( H^q(E_b, F(H_b, \psi)) \).

In particular, we have
\[ \dim H'(E, F(H, \psi)) = \dim H^q(E_b, F(H_b, \psi)) \]

and since the imaginary part of \( H_b \) is equal to the imaginary part \( A \) of \( H \), we have \( \dim H^q(E_b, F(H_b, \psi)) = e_1, \ldots, e_n \), where \( e_1, \ldots, e_n \) are the elementary divisors of the integral valued alternating form \( A \) on \( L \times L \). Thus we get also
\[ \dim H'(E, F(H, \psi)) = e_1 \cdots e_n. \]

Let \( N+1 = \dim H'(E, F(H, \psi)) \) and let \( (f_0, f_1, \ldots, f_N) \) be a basis of the complex vector space \( H(b) \) which is canonically isomorphic to \( H'(E, F(H, \psi)) \). The map \( u \to [f_0(u): \cdots : f_N(u)] \) defines a differentiable map \( \Phi \) from \( W \) into the complex projective space \( P^N \). Since each \( f_i \) is a differentiable theta function on \( W \) for the factor \( J_{H, \psi} \), the map \( \Phi \) defines a map \( \Phi \) from \( E = V/L \) into \( P^N \).

Let \( \theta_i = \phi_b \cdot f_i \) for \( i \in [0, N] \). Then we have:
\[ [\theta_0(u): \theta_1(u): \cdots : \theta_N(u)] = [f_0(u): f_1(u): \cdots : f_N(u)]. \]

It follows from this that \( \Phi \) defines a holomorphic map from \( E_b = V_b/L \) to \( P^N \). We
may assume without loss of generality that $\Phi$ is a holomorphic imbedding (this can be achieved by replacing $H$ by $3H$ and $\psi$ by $\psi^3$). Then $\Phi$ defines a differentiable imbedding of $E$ into $\mathbb{P}^N$. Let $(z_1, \ldots, z_n)$ be the coordinates on $V$ corresponding to a privileged basis of $V$ for $H$. Then these coordinates define local coordinates of the complex torus $E$ at each point of $E$. Since $\Phi$ is holomorphic as a map from $E$ into $\mathbb{P}^N$, we see that $\Phi$ is holomorphic in $z_1, \ldots, z_s$ and anti-holomorphic in $z_{s+1}, \ldots, z_n$. Thus we get the following theorem.

**Theorem 3.** Let $H$ be a Riemann form of signature $(s, r)$ for a complex torus $E$. Then the cohomology group $H^r(E, F(3H, \psi^3))$ of the holomorphic line bundle $F(3H, \psi^3)$ defines a differentiable imbedding of $E$ into the complex projective space $\mathbb{P}^N$ with $N+1=\dim H^r(E, F(3H, \psi^3))$ which is holomorphic in $z_1, \ldots, z_s$ and anti-holomorphic in $z_{s+1}, \ldots, z_n$, where $(z_1, \ldots, z_n)$ are the coordinates of the complex vector space $V$ determined by a privileged basis for $H$.

**Appendix 1.** We give here an example of a complex 2-torus which is not an abelian variety and which admits a Riemann form or signature $(1,1)$. Let

$$
\omega_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \omega_3 = \begin{pmatrix} i\sqrt{2} \\ i\sqrt{3} \end{pmatrix}, \quad \omega_4 = \begin{pmatrix} i\sqrt{3} \\ -i\sqrt{5} \end{pmatrix}.
$$

These vectors are linearly independent over $\mathbb{R}$ and they generate a lattice $L$ of $\mathbb{C}^2$.

The matrix $J_0$ of the complex structure of $\mathbb{C}^2$ with respect to the basis \{${\omega_1, \omega_2, \omega_3, \omega_4}$\} of $\mathbb{C}^2$ over $\mathbb{R}$ is of the form

$$
J_0 = \begin{pmatrix} 0 & J_1 \\ J_2 & 0 \end{pmatrix}
$$

where

$$
J_1 = \begin{pmatrix} -\sqrt{2} & -\sqrt{3} \\ \sqrt{3} & \sqrt{5} \end{pmatrix}
$$

and

$$
J_2 = \frac{1}{d} \begin{pmatrix} -\sqrt{5} & -\sqrt{3} \\ -\sqrt{3} & \sqrt{2} \end{pmatrix}, \quad d = -\sqrt{10} - 3.
$$

Let $A$ be an alternating $\mathbb{R}$-bilinear form on $\mathbb{C}^2 \times \mathbb{C}^2$ which is integral valued on $L \times L$ and let $A_0$ be the matrix of $A$ with respect to the basis \{${\omega_1}$\} and write

$$
A_0 = \begin{pmatrix} P_1 & P_2 \\ -P_2 & P_3 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix},
$$

where $p$ and $q$ are integers and $P_2$ is an integral $2 \times 2$ matrix.
The alternating form $A$ is the imaginary part of a Riemann form if and only if the $\mathbb{R}$-bilinear form $S$ on $\mathbb{C}^2 \times \mathbb{C}^2$ defined by $S(u, v) = A(iu, v)$ is symmetric and non-degenerate. Let $S_0$ be the matrix of $S$ with respect to $\{\omega_i\}$. Then we have $S_0 = J_0 \cdot A_0$. We see easily that the condition that $S_0$ is symmetric is equivalent to the set of the following three conditions: (a) $P_1 J_1 = -J_2 P_2$; (b) $P_2 J_2$ is symmetric; (c) $P_2 J_1$ is symmetric. The conditions (b) and (c) are both equivalent to the single condition that $P_2$ is to be of the form

$$P_2 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad a \in \mathbb{Z}.$$ 

The condition (a) is equivalent to the condition $pd = q$, where $d = -\sqrt{10} - 3$ and $p$ and $q$ are integers and so (a) is equivalent to the condition $p = q = 0$.

Thus we reached the conclusion that $S_0$ is symmetric if and only if $A_0$ is the form

$$(*) \quad A_0 = \begin{pmatrix} 0 & a_1 \\ -a_2 & 0 \end{pmatrix}$$

where $a \neq 0$ is an integer and $1_2$ denote the $2 \times 2$ unit matrix. Then $S_0$ takes the form

$$S_0 = \begin{pmatrix} -a J_2 & 0 \\ 0 & a J_1 \end{pmatrix}$$

and $S_0$ is a non-singular matrix. However $S_0$ is not definite because the symmetric matrix $J_1$ is not definite. Thus $A$ can be the imaginary part of a Riemann form if and only if $A_0$ is of the form $(*)$ and when this is the case, the corresponding Riemann form is not definite but of signature $(1,1)$. Hence $E = \mathbb{C}^2/L$ provides an example of a complex torus which is not an abelian variety and which admits a Riemann form of signature $(1,1)$.

**Appendix 2.** Since the second assertion in Theorem 1 is an essential part of this article we give a proof of Theorem 1 in this appendix.

Let $H$ be a Riemann form of signature $(s, r)$ for a complex torus $E = V/L$ and $J_{H, \psi}$ the factor defined by (1) and $F(H, \psi)$ the holomorphic line bundle associated with $J_{H, \psi}$. Let $D'$ be the vector space of all $F(H, \psi)$-valued differential forms of type $(0, q)$ on $E$. Then the cohomology group $H(E, F(H, \psi))$ of $E$ with coefficient in the sheaf of germs of holomorphic sections of $F(H, \psi)$ is isomorphic to the cohomology group of the complex $D = \bigoplus_{q=0} D'$, where the coboundary operator is given by $d''(or \partial)$. On the other hand, there exists a canonical identification of an $F(H, \psi)$-valued $(0, q)$-form $\alpha$ on $E$ with a $(0, q)$-form $\varphi$ on $V$ (of class $C^\infty$) satisfying the condition that
for $g \in L$, where $T_g$ denotes the translation of $V$ by $g$. Then $d''\varphi$ also satisfies the same condition (*) and $d''\alpha$ is identified with $d''\varphi$. Denote by $A^q$ the vector space of all $(0, q)$-form on $V$ (of class $C^\infty$) satisfying the condition (*). Then the cohomology group $H(E, F(H, \varphi))$ is isomorphic to the cohomology group of the complex $A=\sum_{q=0}^{s} A^q$, where the coboundary operator is given by $d''$. Notice that $A^0$ is the vector space of all differentiable theta functions on $V$. Let $(x_1, \ldots, x_n)$ be coordinates on $V$. Then a $(0, q)$-form is expressed uniquely in the form

$$\varphi = \frac{1}{q!} \sum_{J} \varphi_J dz_J,$$

where $J=(j_1, \ldots, j_q)$ is a multi-index and each $\varphi_J$ is alternating in the indices and $dz_J = dz_j_1 \wedge \cdots \wedge dz_j_q$. Since $dz_j$ is invariant by translation, $\varphi$ satisfies the condition (*) if and only if each component $\varphi_J$ belongs to $A^0$.

**Lemma 1.** Let $f, g \in A^0$ and let $\langle f, g \rangle$ be defined by

$$\langle f, g \rangle(u) = f(u)g(u) \exp \left[ -\pi H(u, u) \right]$$

for $u \in V$. Then the function $\langle f, g \rangle$ is invariant by the translation $T_g$ for any $g \in L$.

We can verify the lemma by a straightforward computation.

We may consider $\langle f, g \rangle$ as function on the torus $E=V/L$.

**Corollary of Lemma 1.** If $f, g \in A^0$, then

$$|f(u)| |g(u)| \leq C \exp \pi H(u, u)$$

for any $u \in V$, where $C$ is a positive constant.

Let us choose a positive definite Hermitian form $G$ and let

$$G = \sum_{i,j} g_{ij} z_i \bar{z}_j.$$

Let

$$dV = \left( \frac{i}{2} \right)^n \det (g_{ij}) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

the volume element on $V$ determined by $G$. The volume element $dV$ is invariant by translation and so it defines a translation invariant volume element $dv$ on $E$ such that $\pi^*dv=dV$, where $\pi: V \to E$ is the canonical projection. We define the inner product $(f, g)$, where $f, g \in A^0$, by
\[(f, g) = \int_P \langle f, g \rangle dV,\]

where \(P\) is a fundamental parallelepiped for the lattice \(L\), or equivalently by

\[(f, g) = \int_E \langle f, g \rangle dv\]

regarding \(\langle f, g \rangle\) as function on \(E\).

Let us write

\[H = \sum_{i,j} h_{ij} \bar{z}_i \bar{z}_j\]

and introduce covariant derivations \(D_i, D'_i\) \((i=1, \ldots, n)\) by the formula

\[\begin{align*}
(D(f))(z) &= \frac{\partial f}{\partial z_i}(z) - \pi(\sum_{j} h_{ij} \bar{z}_j)f(z), \\
(D'_i f)(z) &= \frac{\partial f}{\partial \bar{z}_i}(z).
\end{align*}\]

We can show without difficulty that if \(f \in \mathcal{A}^0\), then we have

\[D_i f, D'_i f \in \mathcal{A}^0\]

for \(i=1, \ldots, n\). We have also the following formulas:

\[\begin{align*}
\langle D_i f, g \rangle + \langle f, D'_i g \rangle &= \frac{\partial}{\partial z_i} \langle f, g \rangle, \\
\langle D'_i f, g \rangle + \langle f, D'_i g \rangle &= \frac{\partial}{\partial \bar{z}_i} \langle f, g \rangle.
\end{align*}\]

Integralating both sides of the equalities and using the Green's theorem we obtain

\[\begin{align*}
(D_i f, g) + (f, D'_i g) &= 0, \\
(D'_i f, g) + (f, D'_i g) &= 0,
\end{align*}\]

where \(f, g \in \mathcal{A}^0\) and \(i \in [1, n]\).

Denote by \(g^{ij}\) the \((i, j)\)-entry of the inverse matrix of the Hermitian matrix \((g_{ij})\) and let

\[g^{IJ} = g^{i_{I}j_{I}} \cdots g^{i_{J}j_{J}}\]

where \(I = (i_1, \ldots, i_q)\) and \(J = (j_1, \ldots, j_q)\).

For \(\varphi, \psi \in \mathcal{A}^q\), we define the function \(\langle \varphi, \psi \rangle\) by

\[\langle \varphi, \psi \rangle = \frac{1}{q!} \sum_{I,J} g^{I,J} \langle \varphi_I, \psi_J \rangle\]

Then \(\langle \varphi, \psi \rangle\) is invariant by the translation \(T_g(g \in L)\) and we define the inner
product $\langle \varphi, \psi \rangle$ by

$$\langle \varphi, \psi \rangle = \int_{\mathcal{P}} \langle \varphi, \psi \rangle dV = \int_{\mathcal{H}} \langle \varphi, \psi \rangle dv.$$ 

There exists the adjoint operator $b$ for the operator $d''$: $A^q \to A^{q+1}$ so that we have

$$(d'' \varphi, \psi) = (\varphi, b \psi)$$

for $\varphi \in A^q$ and $\psi \in A^{q+1}$.

We define the Laplacian $\Box$ by

$$\Box = -d''b + bd''$$

Then $\Box$ is an operator from $A^q$ to $A^q$ for all $q$ and a $(0, q)$-form $\varphi \in A^q$ is said to be harmonic if $\Box \varphi = 0$. Each element of the cohomology group $H^q(A)$ of the complex $A$ is represented by a unique harmonic form. In this sense we can say that each element of the cohomology group $H^q(E, F(H, \psi))$ is represented by a unique harmonic form. Thus we may identify $H^q(E, F(H, \psi))$ with the vector space of all harmonic forms in $A^q$. We now introduce the following notation. For $I=(i_1, \ldots, i_{q+1})$, $I_u$ will denote the multi-index $(i_1, \ldots, i_u, \ldots, i_{q+1})$, where the index $i_u$ under $\wedge$ is omitted. We also introduce the operator $D'^i$ by

$$D'^i = \sum_{j} g^{ij} D_j'$$

We can prove the following three formulas.

A) Let $\varphi \in A^q$. Then the components $(d'' \varphi)_I$, $I=(i_1, \ldots, i_{q+1})$ of $d'' \varphi \in A^{q+1}$ is given by the formula

$$(d'' \varphi)_I = \sum_{s=1}^{q+1} (-1)^{s+1} D'_s \varphi_{I_u}$$

B) Let $\psi \in A^{q+1}$. Then the component $(b \psi)_J$, $J=(j_1, \ldots, j_q)$, of $b \psi$ is given by the formula

$$(b \psi)_J = -\sum_{j=1}^{q} D'^i \psi_{jJ},$$

where $jJ=(j_1, j_2, \ldots, j_q)$.

C) Let $\varphi \in A^q$. Then the component $(\Box \varphi)_I$ of $\Box \varphi \in A^q$ is given by the formula

$$(\Box \varphi)_I = -\sum_{i=1}^{q} D'^i D'_i \varphi + \pi \sum_{s=1}^{q} (-1)^{s+1} \sum_{i=1}^{q} g^{i\hat{i} h_{\hat{i}u}} \varphi_{iI_u}$$

where $iI_u=(i, i, \ldots, \hat{i}_u, \ldots, i_q)$.

We omit the proof of these formulas. Similar formulas had been proved in [1] in a somewhat different context, but the proof can be carried out quite
similarly.

Up to this point the choices of the coordinates \((z_1, \ldots, z_n)\) and the positive definite Hermitian form \(G = \sum_{i,j} g_{ij} z_i \overline{z}_j\) are arbitrary. From now on we choose privileged coordinates \((z_1, \ldots, z_n)\) for the Hermitian form \(H\) so that we have

\[
H = \sum_{i=1}^n |z_i|^2 - \sum_{i=1}^s |z_{s+i}|^2
\]

and hence we have \(h_{ij} = 0\) for \(i \neq j\) and \(H_{ii} = \varepsilon_i\) (the symbol \(\varepsilon_i\) being defined in §1). We choose \(G\) such that

\[
G = \frac{1}{a} (|z_1|^2 + \cdots + |z_s|^2) + |z_{s+1}|^2 + \cdots + |z_n|^2,
\]

where \(a > 0\) (cf. [4]). Then we have \(g^{ij} = 0\) for \(i \neq j\) and

\[
g^{ii} = \begin{cases} a & \text{for } i \in [1, s] \\ 1 & \text{for } i \in [s+1, n]. \end{cases}
\]

Then we have

\[
\sum_k g^{ik} h_{kj} = \begin{cases} 0, & i \neq j \\ a, & i = j \text{ and } i \in [1, s] \\ -1, & i = j \text{ and } i \in [s+1, n]. \end{cases}
\]

Let

\[
\alpha_i = \begin{cases} a & \text{for } i \in [1, s] \\ -1 & \text{for } i \in [s+1, n]. \end{cases}
\]

Then we have \(\sum_{i=1}^s (\sum_{k=1}^s g^{ik} h_{ki}) \varphi_{i k} = (-1)^{n-1} \alpha_i \varphi_i\) and we get

\[
(\Box \varphi)_i = -(\sum_i D^{ii} D^i') \varphi_i + \pi(\sum_{k=1}^s \alpha_k) \varphi_i,
\]

where

\[
D^{ii} = g^{ii} D^i' \quad \text{(not summed)}.
\]

From this we obtain

\[
((\Box \varphi)_i, \varphi_i) = \sum_{i=1}^s g^{ii}(D^i' \varphi_i, D^i' \varphi_i) + \pi(\sum_{k=1}^s \alpha_k)(\varphi_i, \varphi_i).
\]

Since the first term of the right hand side is non-negative, we get

\[
((\Box \varphi)_i, \varphi_i) \geq \pi \cdot \alpha(I) \cdot (\varphi_i, \varphi_i).
\]

where we put

\[
\alpha(I) = \sum_{k=1}^s \alpha_k.
\]
Let us denote by \( N \) (resp. \( M \)) the number of indices \( i_u \) such that \( i_u \leq s \) (resp. \( i_u > s \)). Then by the definition of \( \alpha_u \), we have

\[
\alpha(I) = a \cdot N - M.
\]

For the multi-index \( I \) we may assume that these \( q \) indices are distinct, otherwise we get \( \varphi_I = 0 \). Suppose \( q > r \). Then at least one of the indices \( i_u \) must be less than or equal to \( s \) and hence \( N \geq 1 \). Then we get

\[
\alpha(I) \geq a - r
\]

Choose \( a \) such that \( a > r \). Then we have \( \alpha(I) > 0 \) for \( q > r \).

Suppose the \( \Box \varphi = 0 \), where \( \varphi \in A^q \) with \( q > r \). Then

\[
0 = (\Box \varphi_I, \varphi_I) \geq \pi \alpha(I) \cdot (\varphi_I, \varphi_I)
\]

with \( \alpha(I) > 0 \). Hence we must have \( (\varphi_I, \varphi_I) = 0 \) and hence \( \varphi_I = 0 \) for any \( I \) and this means \( \varphi = 0 \). This shows that \( H^q(A) = 0 \) and hence \( H^q(E, F(H, \psi)) = 0 \) for \( q > r \).

On the other hand, by the Serre duality, we have

\[
H^q(E, F(H, \psi)) \cong H^{n-q}(E, K \otimes F^*)
\]

where \( K \) is the canonical line bundle of \( E \) and \( F^* \) is the dual of \( F(H, \psi) \). It is easy to see that \( F^* \cong F(-H, \varphi^{-1}) \) and \( -H \) is of signature \((r, s)\). Moreover since \( E \) is a complex torus, \( K \) is a trivial bundle and so we get

\[
H^q(E, F(H, \psi)) \cong H^{n-q}(E, F(-H, \varphi^{-1})).
\]

Since \( -H \) is of signature \((r, s)\), we get from what we have already proved that \( H^{n-q}(E, F(-H, \varphi^{-1})) = 0 \) whenever \( n-q > s \) or whenever \( n-s = r > q \). Thus we get \( H^q(E, F(H, \psi)) = 0 \) for \( q < r \) and these prove the first assertion in Theorem 1.

Consider now that case \( q = r \), \( \varphi \in A^r \) and \( \Box \varphi = 0 \). Even in this case we get \( \alpha(I) \geq a-r > 0 \) except in the case where all of the \( r \) indices in \( I \) are greater than \( s \), namely except in the case where \( I \) is a permutation of \((s+1, \ldots, n)\). Then we get \( \varphi_I = 0 \) for each \( I \) which is not a permutation of \((s+1, \ldots, n)\) and \( \varphi \) is of the form

\[
(\ast\ast) \quad \varphi = f d\bar{z}_{s+1} \land \cdots \land d\bar{z}_n,
\]

where \( f = \varphi_{s+1, \ldots, n} \).

Conversely let \( \varphi \) be a \((0, r)\)-form on \( V \) of the type \((\ast\ast)\) belonging to \( A^r \). Then \( f \in A^0 \) and we have

\[
d'' \varphi = 0 \Rightarrow \frac{\partial f}{\partial \bar{z}_i} = 0 \quad \text{for} \quad i \in [1, s]
\]

and
\[(b\varphi)_I = -\sum_{i=1}^{s} g_{ii} D'_i \varphi_{ii}\]

where \(I = (i_1, \ldots, i_{s+1})\). If \(I = (s+1, \ldots, \hat{u}, \ldots, n)\) for some \(u\), \((i, I)\) cannot be a permutation of \((s+1, \ldots, n)\) and \(\varphi_{ii} = 0\) and hence \((b\varphi)_i = 0\). If \(I = (s+1, \ldots, \hat{u}, \ldots, n)\) for some \(u\), then

\[(b\varphi)_I = \pm g^{uu} D'_u f\]

Therefore we have

\[b\varphi = 0 \iff D'_u f = 0 \quad \text{for} \quad u = s+1, \ldots, n.\]

It follows from our definition of the operator \(D'_u\) and from the fact \(h_{ij} = \delta_{ij} \cdot \varepsilon_j\), we see that

\[D'_u f = \frac{\partial f}{\partial z_u} + \pi \bar{z}_u f.\]

We have thus proved that the space of harmonic \((0, r)\)-form \(\varphi\) in \(A'\) consists of all the \((0, r)\)-form \(\varphi\) on \(V\) of the form

\[\varphi = f d\bar{z}_{s+1} \wedge \cdots \wedge d\bar{z}_n,\]

where

1) \(f\) is a differentiable theta function for the factor \(J_{H, \psi}\),

2) \(\frac{\partial f}{\partial \bar{z}_i} = 0\) for \(i \in [1, s]\)

and

\[\frac{\partial f}{\partial z_i} + \pi \bar{z}_i f = 0 \quad \text{for} \quad i \in [s+1, n].\]

Then we can identify the cohomology group \(H'(A)\) with the vector space of functions \(f\) satisfying the conditions 1) and 2) and this proves the second assertion in Theorem 1.

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