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Author(s)	Aomoto, Kazuhiko
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ON A THETA PRODUCT FORMULA FOR THE SYMMETRIC A-TYPE CONNECTION FUNCTION

Kazuhiko AOMOTO

1. Introduction

In this note we are concerned about a formula which gives a product expression for a sum of theta rational functions. This sum has already appeared in the connection formulae among symmetric A-type Jackson integrals (See [1], [2]).

Let $q \in C$, $|q| < 1$ be the elliptic modulus. We shall use frequently the Jacobi elliptic theta function $\theta(u) = (u)_\infty (q/u)_\infty (q)_\infty$, where $(u)_\infty = \Pi_{\nu=0}^\infty (1 - q^\nu u)$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta, \gamma$ and γ' be complex numbers such that $\alpha_j = \alpha_1 + (j-1)(\gamma' - \gamma)$ and $\gamma + \gamma' = 1$. The symmetric group of n-th degree $\mathscr{G}_n$ acts on a function $f(t)$ on the n dimensional algebraic torus $(C^*)^n$ as $\sigma f(t) = f(\sigma^{-1}(t)) = f(t_{\sigma(1)}, \cdots, t_{\sigma(n)})$ for $t = (t_1, \cdots, t_n) \in (C^*)^n$.

Let $\{U_\sigma(t)\}_{\sigma \in \mathscr{G}_n}$ be the theta rational functions on $(C^*)^n$ defined as follows,

$$(1.1) \qquad U_\sigma(t) = \Pi_{\substack{1 \le i < j \le n \\ \sigma^{-1}(i) > \sigma^{-1}(j)}} \left(\frac{t_j}{t_i}\right)^{\gamma - \gamma'} \frac{\theta(q^\gamma t_j/t_i)}{\theta(q^{\gamma'} t_j/t_i)}.$$

These are pseudo-constants and define one-cocycle of $\mathscr{G}_n$ with values in C^*,

$$(1.2) \qquad U_{\sigma\sigma'}(t) = U_{\sigma'}(t) \cdot \sigma U_\sigma(t) \qquad \text{and} \qquad U_e(t) = 1$$

for all $\sigma, \sigma' \in \mathscr{G}_n$ (e denotes the identity).

Let $\varphi(x), x = (x_1, \cdots, x_n) \in (C^*)^n$, be the theta rational function

$$(1.3) \qquad \varphi(x) = \Pi_{j=1}^n x_j^{\alpha_j} \frac{\theta(q^{\alpha_j + \cdots + \alpha_n + \gamma + 1} x_j/x_{j-1})}{\theta(q^{\gamma+1} x_j/x_{j-1})}$$

for $x_0 = q^\gamma$. Consider the generalized alternating sum with the weight $\{U_\sigma^{-1}(x)\}_{\sigma \in \mathscr{G}_n}$ as follows,

$$(1.4) \qquad \tilde{\varphi}(x) = \sum_{\sigma \in \mathscr{G}_n} \sigma \varphi(x) \cdot \text{sgn}(\sigma) \cdot U_\sigma(x)^{-1}.$$

It has the equivariant property

(1.5) $\sigma\tilde{\varphi}(x) = U_\sigma(x) \cdot \tilde{\varphi}(x) \cdot \operatorname{sgn}\sigma$ for $\sigma \in \mathscr{G}_n$.

We want to show that $\tilde{\varphi}(x)$ can be expressed as a product of theta monomials. More precisely we can prove the following Theorem.

Theorem.

$$(1.6) \qquad \tilde{\varphi}(x) = \Pi_{j=1}^n q^{-(j-1)^2\gamma} \frac{\theta(q^{\alpha_j + \cdots + \alpha_n + 1})}{\theta(q^{\alpha_1 + 1 - (n+j-2)\gamma})}$$

$$\cdot \Pi_{j=1}^n x_j^{\alpha_1 - 2(j-1)\gamma} \frac{\theta(q^{\alpha_1 + 1 - (n-1)\gamma} x_j)}{\theta(q x_j)} \cdot \Pi_{1 \le i < j \le n} \frac{\theta(q x_j/x_i)}{\theta(q^{1+\gamma} x_j/x_i)}.$$

This formula has been stated as a conjecture and has been proved in case of $n = 2$ and 3 in [3]. It can be regarded as an elliptic version of the one concerning Hall-Littlewood polynomials stated in [9], p 104. We shall give elsewhere an application of it to establising the explicit connection formulae for general symmetric A-type Jackson integrals relevant to Yang-Baxter equation (See for relevant subjects [2], [10], [11], and [12]).

2. Proof of Theorem.

We denote by $\varphi^*(x)$ the function of the right hand side of (1.6). If $n = 1$, $\tilde{\varphi}(x)$ reduces to $x_1^{\alpha_1} \dfrac{\theta(q^{\alpha_1 + 1} x_1)}{\theta(q x_1)}$ which coincides with $\varphi^*(x)$. So the Theorem holds. We assume now $n \ge 2$. Suppose that the formula (1.6) is true for $n \le N - 1$. We must prove it for $n = N$. We denote by σ_r the permutation: $(t_1, \cdots, t_n) \to (t_2, \cdots, t_r, t_1, t_{r+1}, \cdots, t_n)$ so that $\sigma^{-1}(1) = 2, \cdots, \sigma^{-1}(r-1) = r$, $\sigma^{-1}(r) = 1$, $\sigma^{-1}(j) = j$ for $j \ge r + 1$. Then $\tilde{\varphi}(x)$ can be described as

$$(2.1) \qquad \tilde{\varphi}(x) = \sum_{r=1}^N \sum_{\sigma' \in \mathscr{G}_{N-1}} \sigma_r \sigma' \varphi(x) \cdot \operatorname{sgn}(\sigma_r \sigma') \cdot U_{\sigma_r \sigma}^{-1}(x)$$

$$= \sum_{r=1}^N (-1)^{r-1} U_{\sigma_r}^{-1}(x) \tilde{\varphi}_r(x),$$

$$\tilde{\varphi}_r(x) = \sum_{\sigma' \in \mathscr{G}_{N-1}} \sigma' \varphi(\sigma_r^{-1}(x)) \cdot \operatorname{sgn}(\sigma') \cdot U_\sigma^{-1}(\sigma_r^{-1}(x)),$$

where $\mathscr{G}_{N-1}$ denotes the symmetric group of $(N-1)$th degree consisting of the permutations which fix the argument 1. $\tilde{\varphi}_r(x)$ equals

(2.2)
$$\tilde{\varphi}_r(x) = x_r^{\alpha_1} \frac{\theta(q^{\alpha_1 + \cdots + \alpha_N + 1} x_r)}{\theta(qx_r)} \cdot \sum_{\sigma' \in \mathscr{S}_N} \mathrm{sgn}\,(\sigma') U_{\sigma'}^{-1}(x')$$

$$\sigma' \{ \Pi_{j=2}^N x_j'^{\alpha_j} \frac{\theta(q^{\alpha_j + \cdots + \alpha_N + \gamma + 1} x_j'/x_1')}{\theta(q^{\gamma+1} x_j'/x_1')} \},$$

for $x_1' = x_r, x_2' = x_1, \cdots, x_r' = x_{r-1}, x_j' = x_j$ for $j \geq r+1$.   We can now apply the formula (1.6) for $n = N-1$ by replacing $\alpha_1, \cdots, \alpha_{N-1}$ and $x_1, \cdots, x_{N-1}$ by $\alpha_2, \cdots, \alpha_N$ and $q^\gamma x_2'/x_1', \cdots, q^\gamma x_N'/x_1'$ respectively (Remark that $\alpha_j = \alpha_2 + (j-2)(\gamma' - \gamma)$).   Hence we have

(2.3)
$$\tilde{\varphi}_r(x) = x_r^{\alpha_1} \frac{\theta(q^{\alpha_1 + \cdots + \alpha_N + 1} x_r)}{\theta(qx_r)}$$

$$\cdot \Pi_{j=1}^{r-1} (q^{-\gamma} x_r)^{j-1} x_j^{\alpha_2 - 2(j-1)\gamma} \frac{\theta(q^{\alpha_2 + 1 - (N-2)\gamma + \gamma} x_j/x_r)}{\theta(q^{1+\gamma} x_j/x_r)}$$

$$\cdot \Pi_{j=r+1}^N (q^{-\gamma} x_r)^{j-2} x_j^{\alpha_2 - 2(j-2)\gamma} \frac{\theta(q^{\alpha_2 + 1 - (N-2)\gamma + \gamma} x_j/x_r)}{\theta(q^{1+\gamma} x_j/x_r)}$$

$$\cdot \Pi_{j=2}^N q^{-(j-2)^2 \gamma} \frac{\theta(q^{\alpha_j + \cdots + \alpha_N + 1})}{\theta(q^{\alpha_2 + 1 - (N+j-4)\gamma})} \cdot \Pi_{\substack{1 \leq i < j \leq N \\ i,j \neq r}} \frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma} x_j/x_i)}.$$

Since $U_{\sigma_r}(x) = \Pi_{1 \leq i \leq r} \left(\frac{x_r}{x_i}\right)^{\gamma - \gamma'} \frac{\theta(q^\gamma x_r/x_i)}{\theta(q^{\gamma'} x_r/x_i)}$, we have

(2.4)
$$U_{\sigma_r}^{-1}(x) \Pi_{\substack{1 \leq j \leq N \\ j \neq r}} \frac{1}{\theta(q^{1+\gamma} x_j/x_r)} \Pi_{\substack{1 \leq i < j \leq N \\ i,j \neq r}} \frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma} x_j/x_i)}$$

$$= (-1)^{r-1} \Pi_{i=1}^{r-1} \left(\frac{x_r}{x_i}\right)^{-2\gamma} \Pi_{\substack{1 \leq j \leq N \\ j \neq r}} \theta(qx_j/x_r)^{-1} \Pi_{1 \leq i < j \leq N} \frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma} x_j/x_i)}.$$

Hence $\tilde{\varphi}(x)$ can be simplified to

(2.5)
$$\tilde{\varphi}(x) = - \{ \Pi_{j=2}^N q^{-(j-2)^2 \gamma} \frac{\theta(q^{\alpha_j + \cdots + \alpha_N + 1})}{\theta(q^{1 + \alpha_2 - (N+j-4)\gamma})} \cdot \Pi_{j=1}^N x_j^{\alpha_1 + 1 - 2(j-1)\gamma}$$

$$\Pi_{1 \leq i < j \leq N} \frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma} x_j/x_i)} \} \cdot \{ \sum_{r=1}^N \frac{\theta(q^{\alpha_1 + \cdots + \alpha_N + 1} x_r)}{\theta(x_r)} (q^{-\gamma} x_r)^{\frac{(N-1)(N-2)}{2}}$$

$$\Pi_{\substack{1 \leq j \leq N \\ j \neq r}} \frac{\theta(q^{\alpha_1 + 2 - (N-1)\gamma} x_j/x_r)}{\theta(qx_j/x_r)}.$$

We now use the following Lemma.

Lemma. *We put* $f(u) = \Pi_{j=1}^N \theta(q^{\alpha_1+1-(N-1)\gamma} x_j u)$ *and* $\delta = \dfrac{(N-1)(N-2)}{2}$. *Then*

$f(u)$ *can be described as an interpolation formula expressed by elliptic theta functions at the points* $u = q/x_r$,

$$(2.6) \qquad f(u) = \sum_{r=1}^N \frac{\theta(q^{\alpha_1+\cdots+\alpha_N+1-\delta} x_r u)}{\theta(q^{\alpha_1+\cdots+\alpha_N+2-\delta})} f(q x_r^{-1}) \Pi_{\substack{1 \le j \le N \\ j \ne r}} \frac{\theta(x_j u)}{\theta(q x_j/x_r)}.$$

Proof. We denote by $f^*(u)$ the right hand side of (2.6). Remark that the theta polynomials $f(u)$ and $\theta(q^{\alpha_1+\cdots+\alpha_N+1-\delta} x_r u) \Pi_{\substack{1 \le j \le N \\ j \ne r}} \theta(x_j u)$ both satisfy the same quasi-periodicity for the shift $u \to qu : f(qu) = (-1)^N q^{\dfrac{-N\alpha_1-N+N(N-1)\gamma}{x_1 \cdots x_N}} u^{-N} f(u)$, since

$\alpha_1 + \cdots + \alpha_N = N\alpha_1 + \dfrac{N(N-1)}{2} - N(N-1)\gamma$. Hence $f(u)$ and $f^*(u)$ have the same

quasiperiodicity. On the other hand, we have $f(q/x_r) = f^*(q/x_r)$, $1 \le r \le N$. Hence $f^*(u) - f(u)$ must be divided out by the product $\Pi_{j=1}^N \theta(x_j u) : f^*(u) - f(u) = g(u) \Pi_{j=1}^N \theta(x_j u)$, where $g(u)$ denotes a theta polynomial satisfying the multiplicative property with constant multiplier,

$$(2.7) \qquad g(qu) = q^{-N\alpha_1-N+N(N-1)\gamma} g(u).$$

$g(u)$ having a Laurent expansion $g(u) = \sum_{-\infty}^{+\infty} c_m u^m$, (2.7) implies that the coefficients c_m vanish except probably for one, say c_k ($k \in \mathbf{Z}$) such that $k = -N\alpha_1 - N + N(N-1)\gamma$. Since α_1 is a general complex number, this equality is impossible. Hence c_k also vanishes. $g(u)$ vanishes identically., i.e., $f^*(u) = f(u)$.

Corollary. *When we put* $u = 1$, *then*

$$(2.8) \qquad \sum_{r=1}^N \frac{\theta(q^{\alpha_1+\cdots+\alpha_N+1-\delta} x_r) \theta(q^{\alpha_1+2-(N-1)\gamma})}{\theta(q^{\alpha_1+\cdots+\alpha_N+2-\delta})} \Pi_{\substack{1 \le j \le N \\ j \ne r}} \{ \theta(q^{\alpha_1+2-(N-1)\gamma} x_j/x_r)$$

$$\Pi_{\substack{1 \le j \le N \\ j \ne r}} \frac{\theta(x_j)}{\theta(q x_j/x_r)} \} = \Pi_{j=1}^N \theta(q^{\alpha_1+1-(N-1)\gamma} x_j) \qquad for \qquad \delta = \frac{(N-1)(N-2)}{2},$$

or equivalently

$$(2.9) \qquad \sum_{r=1}^N (q^{-1} x_r)^\delta \frac{\theta(q^{\alpha_1+\cdots+\alpha_N+1} x_r) \theta(q^{\alpha_1+2-(N-1)\gamma})}{\theta(q^{\alpha_1+\cdots+\alpha_N+2}) \theta(x_r)}$$

$$\Pi_{\substack{1 \le j \le N \\ j \ne r}} \frac{\theta(q^{\alpha_1+2-(N-1)\gamma} x_j/x_r)}{\theta(q x_j/x_r)} = \Pi_{j=1}^N \frac{\theta(q^{\alpha_1+1-(N-1)\gamma} x_j)}{\theta(x_j)}.$$

We now return to the proof of the Theorem. By applying the formula (2.9) to the RHS of (2.5), we have finally

$$(2.10) \qquad \tilde{\varphi}(x) = -\frac{\theta(q^{\alpha_1 + \cdots + \alpha_N + 2})}{\theta(q^{\alpha_1 + 2 - (N-1)\gamma})} q^{(1-\gamma)\delta} \cdot \Pi_{j=2}^{N} \{ q^{-(j-2)^2\gamma}$$

$$\frac{\theta(q^{\alpha_j + \cdots + \alpha_N + 1})}{\theta(q^{\alpha_1 + 2 - (N+j-2)\gamma})} \} \cdot \Pi_{j=1}^{N} x_j^{\alpha_1 + 1 - 2(j-1)\gamma} \frac{\theta(q^{\alpha_1 + 1 - (N-1)\gamma} x_j)}{\theta(x_j)}$$

$$\Pi_{1 \le i < j \le N} \frac{\theta(q x_j / x_i)}{\theta(q^{1+\gamma} x_j / x_i)} = \varphi^*(x). \qquad\qquad\qquad \text{Q.E.D.}$$

References

[1] K. Aomoto: *On connection coefficients for q-difference system of A-type Jackson integral* (dedicated to Prof. R. Askey on occasion of his 60th birthday), SIAM J. Math. Anal., **25** (1994), 256–273.

[2] K. Aomoto and Y. Kato: *Connection formula of symmetric A-type Jackson integrals*, Duke Math. J., **74** (1994), 129–143.

[3] K. Aomoto: *2 conjectural formulae for symmetric A-type Jackson integrals*, preprint, 1992.

[4] R. Askey: *Some basic hypergeometric extensions of integrals of Selberg and Andrews*, SIAM J., Math. Anal., **11** (1980), 938–951.

[5] R. Evans: *Multidimensional q-beta integrals*, ibid., **23** (1992), 758–765.

[6] L. Habsieger: *Une q-intégrale de Selberg et Askey*, ibid., **19** (1988), 1475–1489.

[7] H. Hancock: Lectures on the theory of elliptic functions, Dover, 1958.

[8] K. Kadell: *A proof of Askey's conjectured q-analogue of Selberg's integral and a conjecture of Morris*, SIAM J., Math. Anal., **19** (1989), 969–986.

[9] I. G. Macdonald: Symmetric functions and Hall polynomials, Clarendon Press, Oxford, 1979.

[10] A. Matsuo: *Quantum algebra structure of certain Jackson integrals*, Commu. Math. Phys., **157** (1993), 479–498.

[11] K. Mimachi: *Connection problem in holonomic q-difference system associated with a Jackson integral of Jordan Pochhammer type*, Nagoya Math. J., **116** (1989), 149–161.

[12] A. Varchenko: *Quantized Knizhnik-Zamolodchikov equations, quantum Yang-Baxter, and difference equations for q-hypergeometric functions*, preprint, 1993.

Dept. of Math.
Nagoya University
Furo-cho 1, Chikusa-ku
Nagoya, Japan