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ON A THETA PRODUCT FORMULA FOR THE SYMMETRIC A-TYPE CONNECTION FUNCTION

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1. Introduction

In this note we are concerned about a formula which gives a product expression for a sum of theta rational functions. This sum has already appeared in the connection formulae among symmetric A-type Jackson integrals (See [1], [2]).

Let $q \in \mathbb{C}$, $|q| < 1$ be the elliptic modulus. We shall use frequently the Jacobi elliptic theta function $\theta(u) = (u)_\infty (q/u)_\infty (q)_\infty$, where $(u)_\infty = \prod_{v=0}^{\infty} (1 - q^v u)$. Let $\alpha_1, \alpha_2, \dots, \alpha_n, \beta, \gamma$ and γ' be complex numbers such that $\alpha_j = \alpha_1 + (j-1)(\gamma' - \gamma)$ and $\gamma + \gamma' = 1$. The symmetric group of n -th degree \mathcal{G}_n acts on a function $f(t)$ on the n dimensional algebraic torus $(\mathbb{C}^*)^n$ as $\sigma f(t) = f(\sigma^{-1}(t)) = f(t_{\sigma(1)}, \dots, t_{\sigma(n)})$ for $t = (t_1, \dots, t_n) \in (\mathbb{C}^*)^n$.

Let $\{U_\sigma(t)\}_{\sigma \in \mathcal{G}_n}$ be the theta rational functions on $(\mathbb{C}^*)^n$ defined as follows,

$$(1.1) \quad U_\sigma(t) = \prod_{\substack{1 \leq i < j \leq n \\ \sigma^{-1}(i) > \sigma^{-1}(j)}} \binom{t_j}{t_i}^{\gamma - \gamma'} \frac{\theta(q^\gamma t_j / t_i)}{\theta(q^\gamma t_j / t_i)}$$

These are pseudo-constants and define one-cocycle of \mathcal{G}_n with values in \mathbb{C}^* ,

$$(1.2) \quad U_{\sigma\sigma'}(t) = U_{\sigma'}(t) \cdot \sigma U_\sigma(t) \quad \text{and} \quad U_e(t) = 1$$

for all $\sigma, \sigma' \in \mathcal{G}_n$ (e denotes the identity).

Let $\varphi(x), x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n$, be the theta rational function

$$(1.3) \quad \varphi(x) = \prod_{j=1}^n x_j^{\alpha_j} \frac{\theta(q^{\alpha_j + \dots + \alpha_n + \gamma + 1} x_j / x_{j-1})}{\theta(q^{\gamma+1} x_j / x_{j-1})}$$

for $x_0 = q^\gamma$. Consider the generalized alternating sum with the weight $\{U_\sigma^{-1}(x)\}_{\sigma \in \mathcal{G}_n}$ as follows,

$$(1.4) \quad \tilde{\varphi}(x) = \sum_{\sigma \in \mathcal{G}_n} \sigma \varphi(x) \cdot \text{sgn}(\sigma) \cdot U_\sigma(x)^{-1}.$$

It has the equivariant property

$$(1.5) \quad \sigma \tilde{\varphi}(x) = U_\sigma(x) \cdot \tilde{\varphi}(x) \cdot \operatorname{sgn} \sigma \quad \text{for} \quad \sigma \in \mathcal{S}_n.$$

We want to show that $\tilde{\varphi}(x)$ can be expressed as a product of theta monomials. More precisely we can prove the following Theorem.

Theorem.

$$(1.6) \quad \tilde{\varphi}(x) = \prod_{j=1}^n q^{-(j-1)^2 \gamma} \frac{\theta(q^{\alpha_j + \dots + \alpha_n + 1})}{\theta(q^{\alpha_1 + 1 - (n+j-2)\gamma})} \\ \cdot \prod_{j=1}^n x_j^{\alpha_1 - 2(j-1)\gamma} \frac{\theta(q^{\alpha_1 + 1 - (n-1)\gamma} x_j)}{\theta(q x_j)} \cdot \prod_{1 \leq i < j \leq n} \frac{\theta(q x_j / x_i)}{\theta(q^{1+\gamma} x_j / x_i)}.$$

This formula has been stated as a conjecture and has been proved in case of $n=2$ and 3 in [3]. It can be regarded as an elliptic version of the one concerning Hall-Littlewood polynomials stated in [9], p 104. We shall give elsewhere an application of it to establishing the explicit connection formulae for general symmetric A-type Jackson integrals relevant to Yang-Baxter equation (See for relevant subjects [2], [10], [11], and [12]).

2. Proof of Theorem.

We denote by $\varphi^*(x)$ the function of the right hand side of (1.6). If $n=1$, $\tilde{\varphi}(x)$ reduces to $x_1^{\alpha_1} \frac{\theta(q^{\alpha_1+1} x_1)}{\theta(q x_1)}$ which coincides with $\varphi^*(x)$. So the Theorem holds. We assume now $n \geq 2$. Suppose that the formula (1.6) is true for $n \leq N-1$. We must prove it for $n=N$. We denote by σ_r the permutation: $(t_1, \dots, t_n) \rightarrow (t_2, \dots, t_r, t_1, t_{r+1}, \dots, t_n)$ so that $\sigma_r^{-1}(1)=2, \dots, \sigma_r^{-1}(r-1)=r, \sigma_r^{-1}(r)=1, \sigma_r^{-1}(j)=j$ for $j \geq r+1$. Then $\tilde{\varphi}(x)$ can be described as

$$(2.1) \quad \tilde{\varphi}(x) = \sum_{r=1}^N \sum_{\sigma' \in \mathcal{S}_{N-1}} \sigma_r \sigma' \varphi(x) \cdot \operatorname{sgn}(\sigma_r \sigma') \cdot U_{\sigma_r \sigma'}^{-1}(x) \\ = \sum_{r=1}^N (-1)^{r-1} U_{\sigma_r}^{-1}(x) \tilde{\varphi}_r(x), \\ \tilde{\varphi}_r(x) = \sum_{\sigma' \in \mathcal{S}_{N-1}} \sigma' \varphi(\sigma_r^{-1}(x)) \cdot \operatorname{sgn}(\sigma') \cdot U_{\sigma'}^{-1}(\sigma_r^{-1}(x)),$$

where \mathcal{S}_{N-1} denotes the symmetric group of $(N-1)$ th degree consisting of the permutations which fix the argument 1. $\tilde{\varphi}_r(x)$ equals

$$(2.2) \quad \tilde{\varphi}_r(x) = x_r^{\alpha_1} \frac{\theta(q^{\alpha_1 + \dots + \alpha_N + 1} x_r)}{\theta(qx_r)} \cdot \sum_{\sigma' \in \mathfrak{S}_N} \text{sgn}(\sigma') U_{\sigma'}^{-1}(x')$$

$$\sigma' \left\{ \prod_{j=2}^N x_j^{\alpha_j} \frac{\theta(q^{\alpha_j + \dots + \alpha_N + \gamma + 1} x_j'/x_1')}{\theta(q^{\gamma+1} x_j'/x_1')} \right\},$$

for $x'_1 = x_r, x'_2 = x_1, \dots, x'_r = x_{r-1}, x'_j = x_j$ for $j \geq r+1$. We can now apply the formula (1.6) for $n=N-1$ by replacing $\alpha_1, \dots, \alpha_{N-1}$ and x_1, \dots, x_{N-1} by $\alpha_2, \dots, \alpha_N$ and $q^\gamma x'_2/x'_1, \dots, q^\gamma x'_N/x'_1$ respectively (Remark that $\alpha_j = \alpha_2 + (j-2)(\gamma - \gamma)$). Hence we have

$$(2.3) \quad \tilde{\varphi}_r(x) = x_r^{\alpha_1} \frac{\theta(q^{\alpha_1 + \dots + \alpha_N + 1} x_r)}{\theta(qx_r)}$$

$$\cdot \prod_{j=1}^{r-1} (q^{-\gamma} x_r)^{j-1} x_j^{\alpha_2 - 2(j-1)\gamma} \frac{\theta(q^{\alpha_2 + 1 - (N-2)\gamma + \gamma} x_j/x_r)}{\theta(q^{1+\gamma} x_j/x_r)}$$

$$\cdot \prod_{j=r+1}^N (q^{-\gamma} x_r)^{j-2} x_j^{\alpha_2 - 2(j-2)\gamma} \frac{\theta(q^{\alpha_2 + 1 - (N-2)\gamma + \gamma} x_j/x_r)}{\theta(q^{1+\gamma} x_j/x_r)}$$

$$\cdot \prod_{j=2}^N q^{-(j-2)2\gamma} \frac{\theta(q^{\alpha_j + \dots + \alpha_N + 1})}{\theta(q^{\alpha_2 + 1 - (N+j-4)\gamma})} \cdot \prod_{\substack{1 \leq i < j \leq N \\ i, j \neq r}} \frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma} x_j/x_i)}.$$

Since $U_{\sigma_r}(x) = \prod_{1 \leq i \leq r} \left(\frac{x_r}{x_i}\right)^{\gamma - \gamma} \frac{\theta(q^\gamma x_r/x_i)}{\theta(q^\gamma x_r/x_i)}$, we have

$$(2.4) \quad U_{\sigma_r}^{-1}(x) \prod_{\substack{1 \leq j \leq N \\ j \neq r}} \frac{1}{\theta(q^{1+\gamma} x_j/x_r)} \prod_{\substack{1 \leq i < j \leq N \\ i, j \neq r}} \frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma} x_j/x_i)}$$

$$= (-1)^{r-1} \prod_{i=1}^{r-1} \left(\frac{x_r}{x_i}\right)^{-2\gamma} \prod_{\substack{1 \leq j \leq N \\ j \neq r}} \theta(qx_j/x_r)^{-1} \prod_{1 \leq i < j \leq N} \frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma} x_j/x_i)}.$$

Hence $\tilde{\varphi}(x)$ can be simplified to

$$(2.5) \quad \tilde{\varphi}(x) = - \left\{ \prod_{j=2}^N q^{-(j-2)2\gamma} \frac{\theta(q^{\alpha_j + \dots + \alpha_N + 1})}{\theta(q^{1+\alpha_2 - (N+j-4)\gamma})} \cdot \prod_{j=1}^N x_j^{\alpha_1 + 1 - 2(j-1)\gamma} \right.$$

$$\left. \prod_{1 \leq i < j \leq N} \frac{\theta(qx_j/x_i)}{\theta(q^{1+\gamma} x_j/x_i)} \right\} \cdot \left\{ \sum_{r=1}^N \frac{\theta(q^{\alpha_1 + \dots + \alpha_N + 1} x_r)}{\theta(x_r)} (q^{-\gamma} x_r)^{\frac{(N-1)(N-2)}{2}} \right.$$

$$\left. \prod_{\substack{1 \leq j \leq N \\ j \neq r}} \frac{\theta(q^{\alpha_1 + 2 - (N-1)\gamma} x_j/x_r)}{\theta(qx_j/x_r)} \right\}.$$

We now use the following Lemma.

Lemma. We put $f(u) = \prod_{j=1}^N \theta(q^{\alpha_1+1-(N-1)\gamma} x_j u)$ and $\delta = \frac{(N-1)(N-2)}{2}$. Then $f(u)$ can be described as an interpolation formula expressed by elliptic theta functions at the points $u = q/x_r$,

$$(2.6) \quad f(u) = \sum_{r=1}^N \frac{\theta(q^{\alpha_1+\dots+\alpha_N+1-\delta} x_r u)}{\theta(q^{\alpha_1+\dots+\alpha_N+2-\delta})} f(qx_r^{-1}) \prod_{\substack{1 \leq j \leq N \\ j \neq r}} \frac{\theta(x_j u)}{\theta(qx_j/x_r)}.$$

Proof. We denote by $f^*(u)$ the right hand side of (2.6). Remark that the theta polynomials $f(u)$ and $\theta(q^{\alpha_1+\dots+\alpha_N+1-\delta} x_r u) \prod_{\substack{1 \leq j \leq N \\ j \neq r}} \theta(x_j u)$ both satisfy the same quasi-periodicity for the shift $u \rightarrow qu: f(qu) = (-1)^{Nq} \frac{-N\alpha_1 - N + N(N-1)\gamma}{x_1 \cdots x_N} u^{-N} f(u)$, since

$\alpha_1 + \dots + \alpha_N = N\alpha_1 + \frac{N(N-1)}{2} - N(N-1)\gamma$. Hence $f(u)$ and $f^*(u)$ have the same quasiperiodicity. On the other hand, we have $f(q/x_r) = f^*(q/x_r)$, $1 \leq r \leq N$. Hence $f^*(u) - f(u)$ must be divided out by the product $\prod_{j=1}^N \theta(x_j u): f^*(u) - f(u) = g(u) \prod_{j=1}^N \theta(x_j u)$, where $g(u)$ denotes a theta polynomial satisfying the multiplicative property with constant multiplier,

$$(2.7) \quad g(qu) = q^{-N\alpha_1 - N + N(N-1)\gamma} g(u).$$

$g(u)$ having a Laurent expansion $g(u) = \sum_{-\infty}^{+\infty} c_m u^m$, (2.7) implies that the coefficients c_m vanish except probably for one, say c_k ($k \in \mathbf{Z}$) such that $k = -N\alpha_1 - N + N(N-1)\gamma$. Since α_1 is a general complex number, this equality is impossible. Hence c_k also vanishes. $g(u)$ vanishes identically, i.e., $f^*(u) = f(u)$.

Corollary. When we put $u = 1$, then

$$(2.8) \quad \sum_{r=1}^N \frac{\theta(q^{\alpha_1+\dots+\alpha_N+1-\delta} x_r) \theta(q^{\alpha_1+2-(N-1)\gamma})}{\theta(q^{\alpha_1+\dots+\alpha_N+2-\delta})} \prod_{\substack{1 \leq j \leq N \\ j \neq r}} \{ \theta(q^{\alpha_1+2-(N-1)\gamma} x_j/x_r) \\ \prod_{\substack{1 \leq j \leq N \\ j \neq r}} \frac{\theta(x_j)}{\theta(qx_j/x_r)} \} = \prod_{j=1}^N \theta(q^{\alpha_1+1-(N-1)\gamma} x_j) \quad \text{for} \quad \delta = \frac{(N-1)(N-2)}{2},$$

or equivalently

$$(2.9) \quad \sum_{r=1}^N (q^{-1} x_r)^\delta \frac{\theta(q^{\alpha_1+\dots+\alpha_N+1} x_r) \theta(q^{\alpha_1+2-(N-1)\gamma})}{\theta(q^{\alpha_1+\dots+\alpha_N+2}) \theta(x_r)} \\ \prod_{\substack{1 \leq j \leq N \\ j \neq r}} \frac{\theta(q^{\alpha_1+2-(N-1)\gamma} x_j/x_r)}{\theta(qx_j/x_r)} = \prod_{j=1}^N \frac{\theta(q^{\alpha_1+1-(N-1)\gamma} x_j)}{\theta(x_j)}.$$

We now return to the proof of the Theorem. By applying the formula (2.9) to the RHS of (2.5), we have finally

$$(2.10) \quad \tilde{\varphi}(x) = \frac{\theta(q^{\alpha_1 + \dots + \alpha_N + 2})}{\theta(q^{\alpha_1 + 2 - (N-1)\gamma})} q^{(1-\gamma)\delta} \cdot \prod_{j=2}^N \{q^{-(j-2)2\gamma}$$

$$\frac{\theta(q^{\alpha_j + \dots + \alpha_N + 1})}{\theta(q^{\alpha_1 + 2 - (N+j-2)\gamma})\} \cdot \prod_{j=1}^N x_j^{\alpha_j + 1 - 2(j-1)\gamma} \frac{\theta(q^{\alpha_1 + 1 - (N-1)\gamma} x_j)}{\theta(x_j)}$$

$$\prod_{1 \leq i < j \leq N} \frac{\theta(q x_j / x_i)}{\theta(q^{1+\gamma} x_j / x_i)} = \varphi^*(x). \quad \text{Q.E.D.}$$

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