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# A SPLITTING THEOREM FOR BLOCKS

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Let $F$ be an algebraically closed field of prime characteristic $p$, let $G$ be a finite group, and let $H$ be a normal subgroup of $G$ such that $G/H$ is a $p$-group. Moreover, let $B$ be a block of the group algebra $FH$ of $H$ over $F$.

By Osima's theorem, there is a unique block $A$ of $FG$ covering $B$. We are interested in the structure of $A$. As usual, the general case reduces to the special one where $B$ is $G$-stable. Thus we assume in the following that $B$ is $G$-stable and denote by $P$ a defect group of $A$. Then $Q:=P\cap H$ is a defect group of $B$, and $G=PH$ (see [3, V]).

If $P$ is abelian then the character theory of $A$ is described in a paper by R. Knörr [5]. We are interested in the structure of $A$ as a ring under the additional hypothesis that $Q$ has a complement in $P$. We prove that such a splitting of defect groups implies a splitting of blocks:

**Theorem.** *Let $F$ be an algebraically closed field of prime characteristic $p$, and let $H$ be a normal subgroup of a finite group $G$ such that the factor group $G/H$ is a $p$-group. Let $B$ be a $G$-stable block of $FH$, and let $A$ be the unique block of $FG$ covering $B$. Suppose that $A$ has an abelian defect group $P$ and that $Q:=P\cap H$ has a complement $R$ in $P$. Then $A$ and the tensor product $FR\otimes_F B$ are isomorphic $F$-algebras.*

Proof. As observed above, we have $G=PH=RH$ and $R\cap H=1$. We consider the group algebra $FG$ as a crossed product of $FH$ with $G/H\cong R$, as usual (see [6] for crossed products). Since $1_A=1_B$ for the block idempotents $1_A$ and $1_B$ of $A$ and $B$, respectively, the block $A=1_AFG=1_BFG$ then becomes a crossed product of $1_BFH=B$ with $G/H\cong R$.

Arguing by induction on $q:=|G:H|=|R|$ we may assume that $G/H\cong R$ is cyclic. We write $R=\langle r\rangle$. Then it suffices to show that the center $ZA$ of $A$ contains a graded unit $x$ of $A$ of degree $r$ and order $q$; for, in that case, we will have $A=\oplus_{i=0}^{q-1}x^iB\cong FR\otimes_F B$.

From the main result in [5], we obtain $k(A)=q\cdot k(B)$, where $k(A)$ is the number of all irreducible complex characters of $G$ in $A$. Hence $\dim ZA=q\dim ZB$. On the other hand, $ZA$ is contained in the centralizer $C_A(B)=:C$ of $B$ in $A$ which is an algebra graded by $R$, with 1-component $C_1 = ZB$. We want to show that in our situation $C$ is a crossed product of $ZB$ with $R$. Thus we look at the subgroup

$$G[B] := \{g \in G : C_{gH} C_{g^{-1}H} = C_1\}$$

of $G$. This subgroup plays an important role in Dade's theory of block extensions ([1] and [2]).

Let $\mathscr{A}$ be a root of $A$ in $FC_G(P)$. Then $\mathscr{A}$ has defect group $P$, and $a := \mathscr{A}^{C_G(Q)}$ is a well-defined block of $FC_G(Q)$ with defect group $P$. Since $a^G = A$ we have $\mathrm{Br}_Q(1_A)1_a \neq 0$, and since $C_G(Q) = C_G(Q) \cap PH = PC_H(Q)$ we have $1_a \in FC_H(Q)$ by Osima's theorem. We choose a block $b$ of $FC_H(Q)$ covered by $a$ such that $\mathrm{Br}_Q(1_B)1_b \neq 0$. Then $b$ is a block of $FC_H(Q)$ with defect group $Q$ such that $b^H = B$.

Let $C_G(Q)_b$, $N_G(Q)_b$, $N_H(Q)_b$ denote the stabilizers of $b$ in $C_G(Q)$, $N_G(Q)$, $N_H(Q)$, respectively. Since $a$ is the unique block of $FC_G(Q)$ covering $b$ by Osima's theorem, it follows from Fong's theorems [3, V Theorems 3.12 and 3.14] that $P$ is conjugate in $C_G(Q)$ to a subgroup of $C_G(Q)_b$. But $C_G(Q) = PC_H(Q)$, so $P$ is conjugate in $C_H(Q)$ to a subgroup of $C_G(Q)_b$, which means that $P \subseteq C_G(Q)_b$. Thus $C_G(Q)_b = PC_H(Q) = C_G(Q)$.

In [2], Dade has defined a natural bilinear map

$$\omega : N_H(Q)_b / C_H(Q) \times C_G(Q)_b / C_H(Q) \to F^{\times}$$

and shown that $G[B] = C_G(Q)_\omega H$ where

$$C_G(Q)_\omega := \{g \in C_G(Q)_b : \omega(N_H(Q)_b / C_H(Q), gC_H(Q)) = 1\}$$

(see [2, (0.3b) and Corollary 12.6]). By definition, $C_G(Q)_b / C_G(Q)_\omega$ is isomorphic to a subgroup of $\mathrm{Hom}(N_H(Q)_b / C_H(Q), F^{\times})$ and thus a $p'$-group (see [2, (11.13)]). On the other hand, in our situation $C_G(Q)_b / C_H(Q) \cong C_G(Q)_b H / H$ is a $p$-group. Thus $C_G(Q)_\omega = C_G(Q)_b = C_G(Q)$ and

$$G = PH = C_G(Q)_b H = C_G(Q)_\omega H = G[B].$$

It follows easily that $C$ is a crossed product of the local algebra $ZB$ with $R$ (see [6, p.149]); in particular, $\dim C = q \dim ZB = \dim ZA$. Since $ZA \subseteq C$ we conclude that $ZA = C$.

The inertial group $N_G(P)_{\mathscr{A}}$ acts on $P$, and $Q = P \cap H$ is an $N_G(P)_{\mathscr{A}}$-stable subgroup of $P$. Since $N_G(P)_{\mathscr{A}} / C_G(P)$ is a $p'$-group, Maschke's theorem [4, Theorem 3.3.2] implies that $Q$ has an $N_G(P)_{\mathscr{A}}$-stable complement in $P$. We may assume that our notation is such that $R$ is $N_G(P)_{\mathscr{A}}$-stable. Since $G/H \cong R$ is abelian we obtain $[R, N_G(P)_{\mathscr{A}}] \subseteq R \cap H = 1$. Thus $R \subseteq C_P(N_G(P)_{\mathscr{A}})$.

Let $\alpha = \mathscr{A}^{C_G(R)}$. By Watanabe's result [8, Theorem 2 (ii)], the map

$$f : ZA \to Z\alpha, \quad z \mapsto \mathrm{Br}_R(z)1_\alpha$$

is an isomorphism of $F$-algebras. But we have $C_G(R) = R \times C_H(R)$; in particular,

$1_\alpha \in FC_H(R)$. This implies that $f$ is $R$-graded.

There is a unique block $\beta$ of $FC_H(R)$ covered by $\alpha$, and we have $\alpha \cong FR\otimes_F\beta$ by multiplication; in particular, $Z\alpha \cong FR\otimes_F Z\beta$ by multiplication. Obviously, $r1_\beta$ is a graded unit of degree $r$ and order $q$ in $Z\alpha$. Thus $x:=f^{-1}(r1_\beta)$ is a graded unit of degree $r$ and order $q$ in $ZA$, and we are done.

REMARKS. (i) The condition that $Q$ has a complement in $P$ is essential. Take $G$ the cyclic group of order $p^2$ and $H$ its subgroup of index $p$, for instance.

(ii) The condition that $P$ is abelian is also essential. Take $G$ the extra-special group of order $p^3$ of exponent $p$ and $H$ its subgroup of index $p$ for $p$ odd, for example.

(iii) The theorem above is related to the main result of [7] where a similar splitting of blocks occurs.

(iv) It seems likely that our result holds also when the field $F$ is replaced by a suitable complete discrete valuation ring $\mathcal{O}$. However, since Watanabe's result, on which we lean heavily, does not immediately lift to $\mathcal{O}$, a different proof would have to be found.

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