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COHOMOLOGY OF THE FUNDAMENTAL GROUPS OF TOROIDAL GROUPS

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Abstract

We construct an isomorphism between the $\bar{\partial}$-cohomology and the cohomology of the fundamental groups of toroidal groups, and get a standard form of p-cocycles, which was given by Vogt [6] in case of 1-cocycles. Using differential forms via the above isomorphism enables us to obtain new results in higher dimensional cases. An explicit isomorphism between the Čech cohomology and the cohomology of the fundamental groups of complex tori is given in [5] (p.14). Our results give a generalization of this isomorphism to toroidal groups.

1. Introduction

Kazama and the second named author [3] calculated the $\bar{\partial}$-cohomology of toroidal groups using the Fourier expansions of $\bar{\partial}$-closed forms and characterized the toroidal groups of cohomologically finite type, namely those having finite-dimensional cohomology groups. For this purpose, they proved that a toroidal group is of cohomologically finite type if and only if any $\bar{\partial}$-closed $(0, p)$-forms on it are $\bar{\partial}$-cohomologous to constant $(0, p)$-forms. Further, they proved that a toroidal group which is not of cohomologically finite type has non-Hausdorff topology.

On the other hand, Vogt [6] obtained a standard form (in Theorem 2.2) of the 1-cocycles of the cohomology of the fundamental groups of toroidal groups by solving difference equations.

In this paper we construct an isomorphism between the $\bar{\partial}$-cohomology and the cohomology of the fundamental groups of toroidal groups, and get a standard form of the p-cocycles for all $p \geq 1$ (Theorem 2.2). This gives another proof of the above result of Vogt in case of 1-cocycles and its generalization to the case of p-cocycles. Using differential forms via the isomorphism enables us to odtain the new results in higher dimensional cases. By the isomorphism, any $\bar{\partial}$-closed $(0, p)$-forms on toroidal groups correspond to p-cocycles which are cohomologous to the standard p-cocycles. We hope this correspondence could shed some light on the study of cocycles in toroidal groups which have infinite dimensional cohomology groups. An explicit isomorphism between the Čech cohomology and the cohomology of the fundamental groups of com-

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plex tori is given in [5] (p.14), and this is valid for toroidal groups. Since our results include a method of constructing an isomorphism between the $\overline{\partial}$-cohomology and the cohomology of the fundamental groups of complex tori, these are regarded as another approach for the cohomlogy of complex tori.

2. Definition and statement of results

At first, for a toroidal group X of complex dimension n, we recall the definition of the cohomology groups $H^p(\pi_1(X), H^0(\mathbb{C}^n, \mathcal{O}))$ of $\pi_1(X)$ with values in the additive group $H^0(\mathbb{C}^n, \mathcal{O})$ of the holomorphic functions on $\mathbb{C}^n$, according to [5] (p.22). We simply call them the cohomology of the fundamental group of a toroidal group X.

DEFINITION 1. A connected complex Lie group X is called a toroidal group if every holomorphic function on X is constant.

Let n be the complex dimension of X. Since any toroidal group of complex dimension n is a complex abelian Lie group, there exists a discrete subgroup Γ of $\mathbb{C}^n$ such that X is isomorphic onto the quotient group $\mathbb{C}^n/\Gamma$ and Γ is the fundamental group of X. Let $r = \operatorname{rank}\Gamma$, then $n + 1 \leq r \leq 2n$ and there exists a basis $\lambda_1, \dots, \lambda_r$ for Γ. We write $\Gamma = \mathbb{Z}\{\lambda_1, \dots, \lambda_r\}$. The matrix $P = [\lambda_1, \dots, \lambda_r]$ is called a period matrix for $\mathbb{C}^n/\Gamma$.

DEFINITION 2. A toroidal group $\mathbb{C}^n/\Gamma$ is called of type q $(1 \leq q \leq n)$ if $\operatorname{rank}\Gamma = n + q$.

Let $H = H^0(\mathbb{C}^n, \mathcal{O})$, $C^0(\Gamma, H) := H$, and $C^p(\Gamma, H) := \{f \mid f : \Gamma^p \to H\}$, where $\Gamma^p = \Gamma \times \cdots \times \Gamma$ (p-times). The coboundary $\delta : C^p(\Gamma, H) \to C^{p+1}(\Gamma, H)$ is defined as follows:

$$\begin{aligned} \delta f(\lambda)(z) &= f(z + \lambda) - f(z), && \text{for} \quad p = 0, \\ \delta f(\lambda_0, \dots, \lambda_p)(z) &= f(\lambda_1, \dots, \lambda_p)(z + \lambda_0) \\ &\quad + \sum_{i=0}^{p-1} (-1)^{i+1} f(\lambda_0, \dots, \lambda_i + \lambda_{i+1}, \dots, \lambda_p)(z) \\ &\quad + (-1)^{p+1} f(\lambda_0, \dots, \lambda_{p-1})(z), && \text{for} \quad p > 0, \end{aligned} \tag{2.1}$$

where $\lambda, \lambda_0, \dots, \lambda_p \in \Gamma$. Put

$$Z^p(\Gamma, H) := \operatorname{Ker}(\delta), \quad \text{for} \quad p \geq 0,$$

$$B^p(\Gamma, H) := \operatorname{Im}(\delta), \quad H^p(\Gamma, H) := Z^p(\Gamma, H)/B^p(\Gamma, H), \quad \text{for} \quad p > 0,$$

and

$$H^0(\Gamma, H) = Z^0(\Gamma, H) = H^0(\mathbb{C}^n/\Gamma, \mathcal{O}).$$

DEFINITION 3. We call $H^p(\Gamma, H)$ the cohomology of Γ with values in H.

Let $\mathcal{C}^{r,s}$ be the sheaf of germs of $C^\infty(r, s)$-forms on a given complex manifold. Put $D^{r,s} = H^0(\mathbb{C}^n, \mathcal{C}^{r,s})$. Then $C^p(\Gamma, D^{r,s})$, for $p \geq 0$, and the coboundary δ are similary defined. Then $\overline{\partial}\colon C^p(\Gamma, D^{r,s}) \to C^p(\Gamma, D^{r,s+1})$ and $\partial/\partial z_i\colon C^p(\Gamma, D^{r,s}) \to C^p(\Gamma, D^{r,s})$ are defined as follows:

$$(\overline{\partial} f)(\lambda_1, \dots, \lambda_p)(z) := \overline{\partial}(f(\lambda_1, \dots, \lambda_p)(z)),$$

and

$$\left(\frac{\partial}{\partial z_i} f\right)(\lambda_1, \dots, \lambda_p)(z) := \frac{\partial}{\partial z_i}(f(\lambda_1, \dots, \lambda_p)(z)),$$

for $f \in C^p(\Gamma, D^{r,s})$ and $\lambda_1, \dots, \lambda_p \in \Gamma$. Then

$$\overline{\partial}\delta = \delta\overline{\partial} \quad \text{and} \quad \frac{\partial}{\partial z_i}\delta = \delta\frac{\partial}{\partial z_i}. \tag{2.2}$$

By a suitable linear change of $\mathbb{C}^n$, $\mathbb{C}^n/\Gamma$ has a period matrix

$$P = [I_n, V], \tag{2.3}$$

where $I_n = [e_1, \dots, e_n]$ is the $n \times n$ unit matrix and

$$V = [v_{ij};\ 1 \leq i \leq n,\ 1 \leq j \leq q] = [v_1, \dots, v_q] \tag{2.4}$$

is a $n \times q$ matrix. We can assume

$$\det \operatorname{Im}[v_{ij};\ 1 \leq i,\ j \leq q] \neq 0. \tag{2.5}$$

Put $v_i = \sqrt{-1}e_i$ $(q+1 \leq i \leq n)$, $[v_1, \dots, v_n] = [v_{ij};\ 1 \leq i \leq n,\ 1 \leq j \leq n]$ and $\beta_i = \operatorname{Im} v_i$ $(1 \leq i \leq n)$, then $\beta_1, \dots, \beta_n$ are $\mathbb{C}$-linearly independent. Define two coordinates $z_1, \dots, z_n$ and $t_1, \dots, t_{2n}$ in $\mathbb{C}^n$, where $t_i \in \mathbb{R}$ $(1 \leq i \leq 2n)$ as follows,

$$\begin{aligned} z &= z_1\beta_1 + \cdots + z_n\beta_n \\ &= t_1e_1 + \cdots + t_ne_n + t_{n+1}v_1 + \cdots + t_{2n}v_n. \end{aligned} \tag{2.6}$$

Hereafter, we write $(z_1, \dots, z_n)$ instead of $z_1\beta_1 + \cdots + z_n\beta_n$ and ${}^t(t_1, \dots, t_{2n})$ instead of $t_1e_1 + \cdots + t_ne_n + t_{n+1}v_1 + \cdots + t_{2n}v_n$.

We denote by $Z_{\overline{\partial}}(M, \mathcal{C}^{r,s})$ the space of the $\overline{\partial}$-closed $C^{\infty}(r, s)$-forms on a complex manifold M and by $B_{\overline{\partial}}(M, \mathcal{C}^{r,s})$ the space of the $\overline{\partial}$-exact $C^{\infty}(r, s)$-forms on a complex manifold M. Then by the Dolbeault theorem, we have

$$H^{p}(M, \mathcal{O}) \cong H^{0,p}_{\overline{\partial}}(M) = \frac{Z_{\overline{\partial}}(M, \mathcal{C}^{0,p})}{B_{\overline{\partial}}(M, \mathcal{C}^{0,p})}. \tag{2.7}$$

Let

$$\varphi = \frac{1}{p!} \sum_{1 \leq \alpha_1, \ldots, \alpha_p \leq n} \varphi_{\alpha_1 \cdots \alpha_p} d\overline{z}_{\alpha_1} \wedge \cdots \wedge d\overline{z}_{\alpha_p} \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p}).$$

Looking at φ as an element of $C^0(\Gamma, D^{0,p})$, $\delta\varphi(\lambda)(z) = \varphi(z+\lambda) - \varphi(z) = 0$, for any $\lambda \in \Gamma$. Put

$$\varphi^{(0)} = \varphi.$$

We define $\varphi^{(k)} \in Z^k(\Gamma, D^{0,p-k})$, satisfying $\varphi^{(k)}(\lambda_1, \ldots, \lambda_k) \in Z_{\overline{\partial}}(\mathbb{C}^n, \mathcal{C}^{0,p-k})$, for each $\lambda_1, \ldots, \lambda_k \in \Gamma$, inductively on k $(1 \leq k \leq p)$. Since $\varphi^{(0)}$ is a $\overline{\partial}$-closed form on $\mathbb{C}^n$, there exists $\Phi^{(0)} \in C^0(\Gamma, D^{0,p-1})$ satisfying

$$\varphi^{(0)} = \overline{\partial}\Phi^{(0)}. \tag{2.8}$$

Put

$$\varphi^{(1)} := \delta\Phi^{(0)}. \tag{2.9}$$

Then clearly $\delta\varphi^{(1)} = 0$, and

$$\overline{\partial}\varphi^{(1)}(\lambda) = \delta(\overline{\partial}\Phi^{(0)})(\lambda) = \delta\varphi^{(0)}(\lambda) = 0, \quad \text{for} \quad \lambda \in \Gamma. \tag{2.10}$$

Hence, for each $\lambda \in \Gamma$, $\varphi^{(1)}(\lambda) \in Z_{\overline{\partial}}(\mathbb{C}^n, \mathcal{C}^{0,p-1})$. Assume we get $\varphi^{(k)} \in Z^k(\Gamma, D^{0,p-k})$, $(1 \leq k < p)$ satisfying $\varphi^{(k)}(\lambda_1, \ldots, \lambda_k) \in Z_{\overline{\partial}}(\mathbb{C}^n, \mathcal{C}^{0,p-k})$, for each $\lambda_1, \ldots, \lambda_k \in \Gamma$. Then there exists $\Phi^{(k)} \in C^k(\Gamma, D^{0,p-k-1})$ satisfying

$$\varphi^{(k)}(\lambda_1, \ldots, \lambda_k) = \overline{\partial}\Phi^{(k)}(\lambda_1, \ldots, \lambda_k). \tag{2.11}$$

Put

$$\varphi^{(k+1)} := \delta\Phi^{(k)}, \tag{2.12}$$

then

$$\overline{\partial}\varphi^{(k+1)}(\lambda_1, \ldots, \lambda_{k+1}) = \delta\varphi^{(k)}(\lambda_1, \ldots, \lambda_{k+1}) = 0. \tag{2.13}$$

Thus we get $\varphi^{(k)} \in Z^k(\Gamma, D^{0,p-k})$ satisfying $\varphi^{(k)}(\lambda,\dots,\lambda_k) \in Z_{\bar\partial}(\mathbb{C}^n, \mathcal{C}^{0,p-k})$, $1 \le k \le p$. Since $\varphi^{(p)}(\lambda_1,\dots,\lambda_p) \in Z_{\bar\partial}(\mathbb{C}^n, \mathcal{C}^{0,0})$, $\varphi^{(p)} \in Z^p(\Gamma, H)$. We denote by Π the mapping $Z_{\bar\partial}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p}) \to Z^p(\Gamma, H)$ such that

$$\Pi(\varphi) = \varphi, \quad \text{for} \quad p = 0, \qquad = \varphi^{(p)}, \quad \text{for} \quad p > 0. \tag{2.14}$$

Note that the definition of Π depends on the choice of $\Phi^{(k)}$ $(0 \le k < p)$. But we have the following

Theorem 2.1. *Let $\mathbb{C}^n/\Gamma$ be a toridal group. Then the mapping Π defined by* (2.14) *induces an isomorphism for each p,*

$$H^{0,p}_{\bar\partial}(\mathbb{C}^n/\Gamma) \cong H^p(\Gamma, H).$$

Then we get the following

Theorem 2.2. *Let $\mathbb{C}^n/\Gamma$ be a toroidal group of type q. Then every $f^{(p)} \in Z^p(\Gamma, H)$ is δ-cohomologous to $a^{(p)} \in Z^p(\Gamma, H)$ satisfying for all $\lambda_1, \dots, \lambda_p \in \Gamma$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$,*

$$a(\lambda_1, \dots, \lambda_p)(z) = a(\lambda_1, \dots, \lambda_p)(0, \dots, 0, z_{q+1}, \dots, z_n).$$

3. Proof of Theorem 2.1

Before proving Thorem 2.1, we note some facts about cohomology of Γ with values in $D^{r,s} = H^0(\mathbb{C}^n, \mathcal{C}^{r,s})$. Let $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Gamma$ be the projection, $\{V_i\}$ be an open coverings of $\mathbb{C}^n/\Gamma$ and $\{W_i\}$ an open subsets of $\mathbb{C}^n$ such that

$$\pi_i = \pi|W_i : W_i \to V_i \quad \text{is a homeomorphism and} \qquad \pi^{-1}(V_i) = \sum_{\lambda\in\Gamma}(W_i + \lambda) \quad \text{is a disjoint union.} \tag{3.1}$$

At first, we show the following

Proposition 3.1. *Let $\mathbb{C}^n/\Gamma$ be a toroidal group. Then*

$$H^p(\Gamma, D^{r,s}) = 0, \quad p > 0.$$

Proof. We take coverings $\{V_i\}$ and $\{W_i\}$ as in (3.1). Let $f \in Z^p(\Gamma, D^{r,s})$. For any $z \in \pi^{-1}(V_i)$, there exists a unique $\lambda_0 \in \Gamma$ such that $z \in W_i + \lambda_0$ and $\pi_i^{-1}(\pi(z)) = z - \lambda_0$. Put for $z \in \pi^{-1}(V_i)$

$$g_i(\lambda_1, \dots, \lambda_{p-1})(z) := f(\lambda_0, \lambda_1, \dots, \lambda_{p-1})(\pi_i^{-1}(\pi(z))).$$

Putting $\pi_i^{-1}(\pi(z))$ in place of z in (2.1), since $\delta f = 0$, we get for any $z \in \pi^{-1}(V_i)$,

$$
\begin{aligned}
f(\lambda_1, \dots, \lambda_p)(z) &= \sum_{j=0}^{p-1} (-1)^j f(\lambda_0, \dots, \lambda_j + \lambda_{j+1}, \dots, \lambda_p)(\pi_i^{-1}(\pi(z))) \\
&\quad + (-1)^p f(\lambda_0, \dots, \lambda_{p-1})(\pi_i^{-1}(\pi(z))) \\
&= \delta g_i(\lambda_1, \dots, \lambda_p)(z).
\end{aligned} \tag{3.2}
$$

Let $\{\gamma_i\}$ be a partition of unity subordinate to the covering $\{V_i\}$. Put for any $z \in \mathbb{C}^n$,

$$g(\lambda_1, \dots, \lambda_{p-1})(z) := \sum_i \gamma_i(\pi(z)) g_i(\lambda_1, \dots, \lambda_{p-1})(z).$$

Then

$$f(\lambda_1, \dots, \lambda_p)(z) = \delta g(\lambda_1, \dots, \lambda_p)(z).$$

Thus the proof is completed. □

Then we begin to prove Theorem 2.1. For $p > 1$, we need the following

Lemma 3.2. *Let $\mathbb{C}^n/\Gamma$ be a toroidal group and $\varphi, \varphi' \in Z_{\bar\partial}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p})$, $p > 1$. Suppose two forms $\varphi^{(k)}$ and $\varphi'^{(k)}$ in $Z^k(\Gamma, D^{0,p-k})$ are constructed from φ and φ' respectively as in* (2.11) *and* (2.12) *for $1 \le k < p$.*

If φ and φ' are $\bar\partial$-cohomologous, then there exists $\rho^{(k-1)} \in C^{k-1}(\Gamma, D^{0,p-k-1})$ satisfying

$$\varphi^{(k)} - \varphi'^{(k)} = \bar\partial \delta \rho^{(k-1)}. \tag{3.3}$$

Proof. For $\varphi^{(k)}$ and $\varphi'^{(k)}$, there exist $\Phi^{(k)}$ and $\Phi'^{(k)}$ in $C^k(\Gamma, D^{0,p-k-1})$, respectively which satisfy (2.11) and (2.12), $0 \le k < p$. By the assumption, there exists $\psi \in H^0(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p-1})$ such that $\varphi - \varphi' = \bar\partial \psi$. We shall prove the lemma by induction on k. For $k = 0$, there exists $\rho^{(0)} \in C^0(\Gamma, D^{0,p-2})$ satisfying

$$\Phi^{(0)} - \Phi'^{(0)} = \psi + \bar\partial \rho^{(0)}. \tag{3.4}$$

Then

$$\varphi^{(1)} - \varphi'^{(1)} = \delta \bar\partial \rho^{(0)} = \bar\partial \delta \rho^{(0)}.$$

Thus the lemma is proved for $k = 1$. Suppose there exists $\rho^{(k-2)} \in C^{k-2}(\Gamma, D^{0,p-k})$ satisfying

$$\varphi^{(k-1)} - \varphi'^{(k-1)} = \bar\partial \delta \rho^{(k-2)}, \tag{3.5}$$

for $k > 1$. Then there exists $\rho^{(k-1)} \in C^{k-1}(\Gamma, D^{0,p-k-1})$ satisfying

$$\Phi^{(k-1)} - \Phi'^{(k-1)} = \delta\rho^{(k-2)} + \bar{\partial}\rho^{(k-1)}. \tag{3.6}$$

Then we get

$$\varphi^{(k)} - \varphi'^{(k)} = \bar{\partial}\delta\rho^{(k-1)}.$$

Hence the lemma is proved for k and the proof is completed. □

Then we show the following

Proposition 3.3. *Let $\mathbb{C}^n/\Gamma$ be a toroidal group. Then the mapping Π defined by* (2.14) *induces a homomorphism*

$$\Pi : H^{0,p}_{\bar{\partial}}(\mathbb{C}^n/\Gamma) \to H^p(\Gamma, H). \tag{3.7}$$

Proof. We may assume $p > 0$. Let φ and φ' are $\bar{\partial}$-cohomologous in $Z_{\bar{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p})$. Suppose $\varphi^{(k)}$ and $\varphi'^{(k)}$ are constructed as in (2.11) and (2.12) by $\Phi^{(k)}$ and $\Phi'^{(k)}$ respectively, $0 \le k < p$. For $p = 1$, there exists $\psi \in H^0(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,0})$ satisfying

$$\varphi - \varphi' = \bar{\partial}\psi.$$

Since

$$\varphi = \bar{\partial}\Phi^{(0)} \quad \text{and} \quad \varphi' = \bar{\partial}\Phi'^{(0)},$$
$$h := \Phi^{(0)} - \Phi'^{(0)} - \psi \in C^0(\Gamma, H).$$

Hence

$$\varphi^{(1)} - \varphi'^{(1)} = \delta\Phi^{(0)} - \delta\Phi'^{(0)} = \delta h.$$

Accordingly Π defines a homomorphism for $p = 1$. Next we consider the case $p > 1$. By Lemma 3.2, there exists $\rho^{(p-2)} \in C^{p-2}(\Gamma, D^{0,0})$ satisfying

$$\varphi^{(p-1)} - \varphi'^{(p-1)} = \bar{\partial}\delta\rho^{(p-2)}.$$

Then there exists $h \in C^{p-1}(\Gamma, H)$ such that

$$\Phi^{(p-1)} - \Phi'^{(p-1)} = \delta\rho^{(p-2)} + h.$$

Hence

$$\varphi^{(p)} - \varphi'^{(p)} = \delta h.$$

This means Π induces a homomorphism (3.7) and the proof is completed. □

Next we show the following

Proposition 3.4. *Let $\mathbb{C}^n/\Gamma$ be a toroidal group. Then the homomorphism Π is injective.*

Proof. We may assume $p \geq 1$. Let $\varphi, \varphi' \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p})$. Suppose $\varphi^{(k)}$ and $\varphi'^{(k)}$ are constructed as in (2.11) and (2.12) by $\Phi^{(k)}$ and $\Phi'^{(k)}$ respectively, $0 \leq k < p$. Assume $\varphi^{(p)} := \Pi\varphi$ and $\varphi'^{(p)} := \Pi\varphi'$ are δ-cohomologous. Then there exists $h \in C^{p-1}(\Gamma, H)$ such that

$$\varphi^{(p)} - \varphi'^{(p)} = \delta h.$$

Then there exists $\sigma^{(p-1)} \in Z^{p-1}(\Gamma, D^{0,0})$ such that

$$\Phi^{(p-1)} - \Phi'^{(p-1)} = h + \sigma^{(p-1)}.$$

Then

$$\varphi^{(p-1)} - \varphi'^{(p-1)} = \overline{\partial}\sigma^{(p-1)}.$$

In case $p = 1$, $\sigma^{(0)} \in Z^0(\Gamma, D^{0,0}) = H^0(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,0})$. Note that $\varphi = \varphi^{(0)}$ and $\varphi' = \varphi'^{(0)}$. Hence φ and φ' are $\overline{\partial}$-cohomologous. If $p > 1$, by Proposition 3.1, there exist $\eta^{(p-2)} \in C^{p-2}(\Gamma, D^{0,0})$ such that $\sigma^{(p-1)} = \delta\eta^{(p-2)}$. Then there exists $\sigma^{(p-2)} \in Z^{p-2}(\Gamma, D^{0,1})$ such that

$$\Phi^{(p-2)} - \Phi'^{(p-2)} = \overline{\partial}\eta^{(p-2)} + \sigma^{(p-2)},$$

and

$$\varphi^{(p-2)} - \varphi'^{(p-2)} = \overline{\partial}\sigma^{(p-2)}.$$

Continuing this way, we have

$$\varphi - \varphi' = \overline{\partial}\sigma^{(0)}.$$

Since $\sigma^{(0)} \in Z^0(\Gamma, D^{0,p-1}) = H^0(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p-1})$, φ and φ' are $\overline{\partial}$-cohomologous, and the proof of the proposition is completed. □

To complete the proof of Theorem 2.1, we need to show the following

Proposition 3.5. *Let $\mathbb{C}^n/\Gamma$ be a toroidal group. Then the homomorphism Π is surjective.*

Proof. Let $f \in Z^p(\Gamma, H)$. Then by Proposition 3.1, there exists $\Phi^{(p-1)} \in C^{p-1}(\Gamma, D^{0,0})$ such that

$$f = \delta\Phi^{(p-1)}.$$

Put

$$\varphi^{(p-1)} := \overline{\partial}\Phi^{(p-1)} \in C^{p-1}(\Gamma, D^{0,1}).$$

Then $\delta\varphi^{(p-1)} = 0$, hence there exists $\Phi^{(p-2)} \in C^{p-2}(\Gamma, D^{0,1})$ satifying

$$\varphi^{(p-1)} = \delta\Phi^{(p-2)}, \quad \text{if} \quad p > 1.$$

Continuing this way, we get $\varphi^{(0)} \in C^0(\Gamma, D^{0,p})$ and $\Phi^{(0)} \in C^0(\Gamma, D^{0,p-1})$ satisfying

$$\varphi^{(0)} = \overline{\partial}\Phi^{(0)} \quad \text{and} \quad \delta\varphi^{(0)} = 0.$$

Then $\varphi^{(0)} \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p})$. By the construction we see $\Pi\varphi^{(0)} = f$. Hence Π is surjective, and the proof is completed. □

From Propositions 3.3, 3.4 and 3.5, the proof of Theorem 2.1 is completed.

4. $\overline{\partial}$-cohomology of toroidal groups

In this section, we summarize some facts about $\overline{\partial}$-cohomology of toroidal groups ([3] and [4]) to prove Theorem 2.2.

Let $\mathbb{C}^n/\Gamma$ be a toroidal group of type q with the period matrix $P = [I_n, V]$. From (2.6), for $i = 1, \dots, n$

$$t_i = \frac{1}{2\sqrt{-1}}\left(-\sum_{j=1}^{n} \overline{v}_{ij} z_j + \sum_{j=1}^{n} v_{ij}\overline{z}_j\right), \quad \text{and} \quad t_{n+i} = \frac{1}{2\sqrt{-1}}(z_i - \overline{z}_i). \tag{4.1}$$

Hence

$$\begin{aligned} \frac{\partial}{\partial z_i} &= -\frac{1}{2\sqrt{-1}}\left(\sum_{j=1}^{n} \overline{v}_{ji}\frac{\partial}{\partial t_j} - \frac{\partial}{\partial t_{n+i}}\right) \quad (1 \le i \le q) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial t_i} - \sqrt{-1}\frac{\partial}{\partial t_{n+i}}\right) \quad (q+1 \le i \le n), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \frac{\partial}{\partial \overline{z}_i} &= \frac{1}{2\sqrt{-1}}\left(\sum_{j=1}^{n} v_{ji}\frac{\partial}{\partial t_j} - \frac{\partial}{\partial t_{n+i}}\right) \quad (1 \le i \le q) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial t_i} + \sqrt{-1}\frac{\partial}{\partial t_{n+i}}\right) \quad (q+1 \le i \le n). \end{aligned} \tag{4.3}$$

Let

$$f(t)=\sum_{m\in\mathbb{Z}^{n+q}} f^m(t)=\sum_{m\in\mathbb{Z}^{n+q}} c^m(t^{2''})\exp(2\pi\sqrt{-1}\langle m,\tilde{t}\rangle)$$

be a C^∞-function on $\mathbb{C}^n/\Gamma$, where $t={}^t(t_1,\dots,t_{2n})$, $\tilde{t}={}^t(t_1,\dots,t_{n+q})$, and $t^{2''}={}^t(t_{n+q+1},\dots,t_{2n})$. For each $m\in\mathbb{Z}^{n+q}$, we put

$$K_{m,i}:={}^t m v_i - m_{n+i}\quad(1\le i\le q),\quad\text{and}\quad K_m:=\mathrm{Max}\{|K_{m,i}|;1\le i\le q\}. \tag{4.4}$$

If $m\neq 0$

(4.5)

$$\begin{aligned}\frac{\partial f^m(t)}{\partial z_i}&=-\pi\overline{K}_{m,i}c^m(t^{2''})\exp(2\pi\sqrt{-1}\langle m,\tilde{t}\rangle) && (1\le i\le q)\\ &=\sqrt{-1}\left(\pi m_i c^m(t^{2''})-\frac{1}{2}\frac{\partial c^m(t^{2''})}{\partial t_{n+i}}\right)\exp(2\pi\sqrt{-1}\langle m,\tilde{t}\rangle) && (q+1\le i\le n),\end{aligned}$$

and

(4.6)

$$\begin{aligned}\frac{\partial f^m(t)}{\partial \overline{z}_i}&=\pi K_{m,i}c^m(t^{2''})\exp(2\pi\sqrt{-1}\langle m,\tilde{t}\rangle) && (1\le i\le q)\\ &=\sqrt{-1}\left(\pi m_i c^m(t^{2''})+\frac{1}{2}\frac{\partial c^m(t^{2''})}{\partial t_{n+i}}\right)\exp(2\pi\sqrt{-1}\langle m,\tilde{t}\rangle) && (q+1\le i\le n).\end{aligned}$$

In case $m=0$,

$$\frac{\partial f^0(t)}{\partial z_i}=\frac{1}{2\sqrt{-1}}\frac{\partial c^0(t^{2''})}{\partial t_{n+i}},\quad\text{and}\quad\frac{\partial f^0(t)}{\partial \overline{z}_i}=-\frac{1}{2\sqrt{-1}}\frac{\partial c^0(t^{2''})}{\partial t_{n+i}}. \tag{4.7}$$

We have the following ([3], p.95)

Proposition 4.1. *A complex abelian Lie group $\mathbb{C}^n/\Gamma$ with a period matrix $P=[I_n,V]$ is a toroidal group if and only if*

$$K_m>0\quad\textit{for any}\quad m\in\mathbb{Z}^{n+q}\setminus\{0\}.$$

For each open subset $U\subset\mathbb{C}^n/\Gamma$

$$\mathcal{F}(U):=\left\{f\,|\,f:C^\infty\text{ in }U\text{ and }\frac{\partial f}{\partial\overline{z}_i}=0,\text{ for }i=q+1,\dots,n\right\}. \tag{4.8}$$

We denote by $\mathcal{F}$ the sheaf defined by the presheaves $\{\mathcal{F}(U)\}$. Suppose $f(t)\in H^0(\mathbb{C}^n/\Gamma,\mathcal{F})$, then from (4.6) and (4.8) we can write

$$f(t)=\sum_{m\in\mathbb{Z}^{n+q}} c^m\exp\left(-2\pi\sum_{i=q+1}^{n} m_i t_{n+i}\right)\exp(2\pi\sqrt{-1}\langle m,t'\rangle), \tag{4.9}$$

where c^m are constants satisfying for any real number $R > 0$ and any positive integer k,

$$C(k, R) := \sup\{|c^m| \, \|m'\|^k R^{\|m''\|}; m \in \mathbb{Z}^{n+q}\} < \infty, \tag{4.10}$$

where, $\|m'\| = \max\{|m_i|, |m_{n+i}|; 1 \le i \le q\}$, and $\|m''\| = \max\{|m_j|; q+1 \le j \le n\}$. (4.10) is obtained similarly to Lemma 7 in [2] (cf. Proposition 2.1 in [4]).

Conversely, a function satisfying (4.9) and (4.10) is in $H^0(\mathbb{C}^n/\Gamma, \mathcal{F})$. Let $\mathcal{F}^{r,s}$ the sheaf of (r, s)-forms with coefficients in $\mathcal{F}$, $Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{r,s})$ the space of the $\overline{\partial}$-closed $\mathcal{F}^{r,s}$-forms on $\mathbb{C}^n/\Gamma$ and $B_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{r,s})$ the space of the $\overline{\partial}$-exact $\mathcal{F}^{r,s}$-forms. Then by [4] (Lemma 2.2), we have

$$H^p(\mathbb{C}^n/\Gamma, \mathcal{O}) \cong H_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p}) := \frac{Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})}{B_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})}. \tag{4.11}$$

Suppose $\varphi = \sum_{m\in\mathbb{Z}^{n+q}} \varphi^m$ be a $\overline{\partial}$-closed $\mathcal{F}^{0,p}$-form. Then there exist a unique constant $(0, p)$-form

$$\chi = \frac{1}{p!} \sum_{1\le\alpha_1,\dots,\alpha_p\le q} c_{\alpha_1\cdots\alpha_p} d\overline{z}_{\alpha_1} \wedge \cdots \wedge d\overline{z}_{\alpha_p},$$

and $\mathcal{F}^{(0,p-1)}$-forms ψ^m satisfying

$$\varphi = \chi + \sum_{m\in\mathbb{Z}^{n+q}} \overline{\partial}\psi^m,$$

where for each $m \in \mathbb{Z}^{n+q}\setminus\{0\}$, $\varphi^m = \overline{\partial}\psi^m$ and for $m = 0$, $\varphi^0 = \chi + \overline{\partial}\psi^0$. We note that $\sum_{m\in\mathbb{Z}^{n+q}} \psi^m$ does not converge generally. Then we have the following (Theorem 4.3 in [1])

Theorem 4.2. *Let $\mathbb{C}^n/\Gamma$ be a toroidal group of type q with a period matrix $P = [I_n, V]$. Then the following conditions are equivalent.*
(1) *There exists $a > 0$ satisfying*

$$\sup_{m\in\mathbb{Z}^{n+q}\setminus\{0\}} \frac{\exp(-a\|m^*\|)}{K_m} < \infty,$$

where $\|m^*\| = \max\{|m_i|; 1 \le i \le n\}$.
(2)

$$\begin{aligned} H^p(\mathbb{C}^n/\Gamma, \mathcal{O}) &\cong \bigwedge^p \mathbb{C}\{d\overline{z}_1, \dots, d\overline{z}_q\}, \quad (1 \le p \le q) \\ &= 0, \quad (p > q). \end{aligned}$$

5. Proof of Theorem 2.2

Suppose $1 \le p \le q \le n$ are positive integers. Let

$$\Omega^p = \frac{1}{p!} \sum_{1\le i_1,\dots,i_p\le q} \Omega^p_{i_1\cdots i_p}\, dz_{i_1} \wedge \cdots \wedge dz_{i_p}$$

be a holomorphic p-form on $\mathbb{C}^n$. Put

$$d_q\Omega^p := \frac{1}{p!} \sum_{1\le i_1,\dots,i_p\le q} \sum_{i=1}^{q} \frac{\partial \Omega^p_{i_1\cdots i_p}}{\partial z_i}\, dz_i \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p}.$$

Then we have the following

Lemma 5.1. *Suppose $d_q\Omega^p = 0$. Then there exists a holomorphic $(p-1)$-form Ω^{p-1} on $\mathbb{C}^n$ satisfying*

$$d_q\Omega^{p-1} = \Omega^p.$$

Proof. Let $l := \max\left\{i_k \;\middle|\; 1 \le i_k \le q,\ 1 \le k \le p,\ \Omega^p_{i_1\cdots i_p} \not\equiv 0\right\}$. We shall prove this lemma by induction on l. If $l = p$, we have

$$\Omega^p = \Omega^p_{12\cdots p}\, dz_1 \wedge \cdots \wedge dz_p.$$

Then

$$d_q\Omega^p = \sum_{i=p+1}^{q} \frac{\partial \Omega^p_{12\cdots p}}{\partial z_i}\, dz_i \wedge dz_1 \wedge \cdots \wedge dz_p = 0.$$

Hence, for $p+1 \le i \le q$, we have $\partial\Omega^p_{12\cdots p}\big/\partial z_i = 0$. Then

$$\Omega^p_{12\cdots p}(z) = \Omega^p_{12\cdots p}(z_1, \dots, z_p, 0, \dots, 0, z_{q+1}, \dots, z_n).$$

Put

$$\Omega^{p-1}_{23\cdots p}(z) := \int_0^{z_1} \Omega^p_{12\cdots p}(\xi, z_2, \dots, z_p, 0, \dots, 0, z_{q+1}, \dots, z_n)\, d\xi,$$

and $\Omega^{p-1} := \Omega^{p-1}_{23\cdots p}(z)\, dz_2 \wedge \cdots \wedge dz_p$. Then

$$d_q\Omega^{p-1} = \Omega^p.$$

Assume that the lemma holds for $l-1$, $l > p$ and

$$\Omega^p = \frac{1}{p!} \sum_{1\le i_1,\dots,i_p\le l} \Omega^p_{i_1\cdots i_p}\, dz_{i_1} \wedge \cdots \wedge dz_{i_p}$$

is a holomorphic form on $\mathbb{C}^n$ such that $d_q\Omega^p = 0$. Since for $l+1 \le i \le q$, $\partial\Omega^p_{i_1\cdots i_p}\big/\partial z_i = 0$, we have

$$\Omega^p_{i_1\cdots i_p}(z) = \Omega^p_{i_1\cdots i_p}(z_1, \ldots, z_l, 0, \ldots, 0, z_{q+1}, \ldots, z_n).$$

Put

$$\omega_{i_2\cdots i_p}(z) := \int_0^{z_l} \Omega^p_{li_2\cdots i_p}(z_1, \ldots, z_{l-1}, \xi, 0, \ldots, 0, z_{q+1}, \ldots, z_n)\, d\xi,$$

and

$$\omega := \frac{1}{(p-1)!} \sum_{1\le i_2,\ldots,i_p\le l-1} \omega_{i_2\cdots i_p}(z)\, dz_{i_2} \wedge \cdots \wedge dz_{i_p}.$$

Then

$$\Omega^p - d_q\omega = \frac{1}{p!} \sum_{1\le i_1,\ldots,i_p\le l-1} \left(\Omega^p_{i_1\cdots i_p} - p\frac{\partial\omega_{i_2\cdots i_p}(z)}{\partial z_{i_1}}\right) dz_{i_1} \wedge \cdots \wedge dz_{i_p}.$$

If we put $\Omega'^p := \Omega^p - d_q\omega$, we have $d_q\Omega'^p = d_q\Omega^p = 0$. Then by the induction hypothesis, we have Ω'^{p-1} satisfying $\Omega'^p = d_q\Omega'^{p-1}$. Hence $\Omega^p = d_q(\Omega'^{p-1} + \omega)$ and the lemma is proved. □

Next, we show Π in (2.14) defines a mapping $Z_{\bar\partial}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p}) \to Z^p(\Gamma, H)$ which induces an isomorphism for each $p \ge 1$,

$$H_{\bar\partial}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p}) \cong H^p(\Gamma, H).$$

As in (2.7) and (4.11), we have two isomorphisms

$$I : H^p(\mathbb{C}^n/\Gamma, \mathcal{O}) \cong H^{0,p}_{\bar\partial}(\mathbb{C}^n/\Gamma)$$

and

$$J : H^p(\mathbb{C}^n/\Gamma, \mathcal{O}) \cong H_{\bar\partial}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p}).$$

Let $\{V_i\}$ be a Leray covering of $\mathbb{C}^n/\Gamma$ for $\mathcal{O}$. Then $H^p(\mathbb{C}^p/\Gamma, \mathcal{O})$ is isomorphic onto the Čech cohomology

$$H^p(\{V_i\}, \mathcal{O}) = \frac{Z^p(\{V_i\}, \mathcal{O})}{B^p(\{V_i\}, \mathcal{O})}.$$

We have the following (cf. Proposition 4 in [7] and Lemma 2.4 in [4])

Lemma 5.2. *Suppose* $f^p \in Z^p(\{V_i\}, \mathcal{O})$, $\varphi \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p})$ *and* $\hat{\varphi} \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$ *satisfy*

$$I([f^p]) = [\varphi] \quad \textit{and} \quad J([f^p]) = [\hat{\varphi}], \quad \textit{for} \quad p \geq 1, \tag{5.1}$$

where we denote by $[\omega]$ *the cohomology class of a cocycle* ω. *Then there exists* $\psi \in H^0(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p-1})$ *satisfying*

$$\varphi = \hat{\varphi} + \overline{\partial}\psi. \tag{5.2}$$

In particular, if $\hat{\varphi} \in B_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p})$, *then* $\hat{\varphi} \in B_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$.

Proof. We also use the symbol δ as the coboundary for the Čech cohomology $H^p(\{V_i\}, \mathcal{S})$, where $\mathcal{S} = \mathcal{O}, \mathcal{C}^{0,r}$ or $\mathcal{F}^{0,s}$, $r, s \geq 0$. For $f^p \in Z^p(\{V_i\}, \mathcal{O})$, there exist $\varphi^{p-1} \in C^{p-1}(\{V_i\}, \mathcal{C}^{0,0})$ and in case $p > 1$, a sequence of cochains $\varphi^k \in C^k(\{V_i\}, \mathcal{C}^{0,p-k-1})$, $(k = p-1, p-2, \ldots, 2, 1, 0)$ such that

$$f^p = \delta\varphi^{p-1}, \quad \text{and} \quad \overline{\partial}\varphi^k = \delta\varphi^{k-1}. \tag{5.3}$$

Since $\overline{\partial}\varphi^0$ is a $\overline{\partial}$-closed $C^\infty(0, p)$-form on $\mathbb{C}^n/\Gamma$, we get

$$I([f^p]) = [\overline{\partial}\varphi^0]. \tag{5.4}$$

Similary there exist $\hat{\varphi}^{p-1} \in C^{p-1}(\{V_i\}, \mathcal{F}^{0,0})$ and in case $p > 1$, a sequence of cochains $\hat{\varphi}^k \in C^k(\{V_i\}, \mathcal{F}^{0,p-k-1})$, $(k = p-1, p-2, \ldots, 2, 1, 0)$ such that

$$f^p = \delta\hat{\varphi}^{p-1}, \quad \text{and} \quad \overline{\partial}\hat{\varphi}^k = \delta\hat{\varphi}^{k-1}. \tag{5.5}$$

Then

$$J([f^p]) = [\overline{\partial}\hat{\varphi}^0]. \tag{5.6}$$

From (5.3) and (5.5), in case $p = 1$,

$$\varphi^0 - \hat{\varphi}^0 \in H^0(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,0}). \tag{5.7}$$

In case $p > 1$, we get a sequence of cochains $\psi^k \in C^k(\{V_i\}, \mathcal{C}^{0,p-k-2})$, $(k = p-2, p-3, \ldots, 1, 0)$ such that

$$\varphi^{p-1} - \hat{\varphi}^{p-1} = \delta\psi^{p-2}, \quad \text{and} \quad \varphi^k - \hat{\varphi}^k - \overline{\partial}\psi^k = \delta\psi^{k-1} \quad (k > 0). \tag{5.8}$$

Then

$$\overline{\partial}\varphi^1 - \overline{\partial}\hat{\varphi}^1 = \delta\overline{\partial}\psi^0, \quad \text{and} \quad \varphi^0 - \hat{\varphi}^0 - \overline{\partial}\psi^0 \in H^0(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p-1}). \tag{5.9}$$

From (5.7) and (5.9), there exists $\psi' \in H^0(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p-1})$ such that

$$\overline{\partial}\varphi^0 - \overline{\partial}\hat{\varphi}^0 = \overline{\partial}\psi'.$$

Combining this with (5.4) and (5.6), for any $\varphi \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p})$ and $\hat{\varphi} \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$ satisfying (5.1), there exists $\psi \in H^0(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p-1})$ which satisfies (5.2). Next, suppose $\hat{\varphi} \in B_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p})$. We take f^p and φ satisfying (5.1), then by (5.2), $\varphi \in B_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p})$. Hence $f^p \in B^p(\{V_i\}, \mathcal{O})$ and $\hat{\varphi} \in B_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$. Thus the lemma is proved. □

Then we have the following

Lemma 5.3. *Π in* (2.14) *defines a mapping $Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p}) \to Z^p(\Gamma, H)$ which induces an isomorphism for each $p \geq 1$,*

$$H_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p}) \cong H^p(\Gamma, H).$$

Proof. Suppose $\hat{\varphi} \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$. Since $\hat{\varphi} \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p})$, $\Pi\hat{\varphi} \in Z^p(\Gamma, H)$ is defined. Let $f \in Z^p(\Gamma, H)$. Then from Proposition 3.5, there exists $\varphi \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p})$ such that $\Pi\varphi = f$. By Lemma 5.2, there exist $\hat{\varphi} \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$ and $\psi \in H^0(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p-1})$ such that $\varphi = \hat{\varphi} + \overline{\partial}\psi$. Then $\Pi\varphi$ and $\Pi\hat{\varphi}$ are δ-cohomologous in $Z^p(\Gamma, H)$. Hence Π defines a surjective homomorphism $H_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p}) \to H^p(\Gamma, H)$. Suppose $\Pi\hat{\varphi}$ and $\Pi\hat{\varphi}'$ are δ-cohomologous, where $\hat{\varphi}, \hat{\varphi}' \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$. Then by Proposition 3.4, there exists $\psi \in H^0(\mathbb{C}^n/\Gamma, \mathcal{C}^{0,p-1})$ such that $\hat{\varphi} - \hat{\varphi}' = \overline{\partial}\psi$. Then by Lemma 5.2, there exists $\hat{\psi} \in H^0(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p-1})$ such that $\hat{\varphi} - \hat{\varphi}' = \overline{\partial}\hat{\psi}$. Hence the above homomorphism is injective, and the lemma is proved. □

Hereafter we write $\varphi \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$ instead of $\hat{\varphi}$. We put $F^{r,s} = H^0(\mathbb{C}^n, \pi^*\mathcal{F}^{r,s})$. Then $C^p(\Gamma, F^{r,s})$, for $p \geq 0$, and the coboundary δ are defined similarly to §2. Let $\varphi \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$. Since $H^p(\mathbb{C}^n, \pi^*\mathcal{F}^{r,s}) = 0$, for $p \geq 1$, we can construct $\varphi^{(k)} \in Z^k(\Gamma, F^{0,p-k})$ for $0 \leq k \leq p$, using $\Phi^{(k)} \in C^0(\Gamma, F^{0,p-k-1})$ as in (2.11) and (2.12). Namely

$$\varphi^{(0)} = \varphi = \overline{\partial}\Phi^{(0)}, \quad \varphi^{(1)} = \delta\Phi^{(0)}, \tag{5.10}$$

$$\varphi^{(k)} = \overline{\partial}\Phi^{(k)}, \quad \text{and} \quad \varphi^{(k+1)} = \delta\Phi^{(k)}, \quad \text{for} \quad 1 \leq k \leq p-1. \tag{5.11}$$

Then $[\Pi\varphi] = [\varphi^{(p)}]$. For

$$\varphi = \frac{1}{p!} \sum_{1 \leq \alpha_1, \ldots, \alpha_p \leq q} \varphi_{\alpha_1 \cdots \alpha_p}\, d\overline{z}_{\alpha_1} \wedge \cdots \wedge d\overline{z}_{\alpha_p} \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p}),$$

put

$$\varphi_i = \frac{\partial\varphi}{\partial z_i} := \frac{1}{p!}\sum_{1\le\alpha_1,\dots,\alpha_p\le q}\frac{\partial\varphi_{\alpha_1\cdots\alpha_p}}{\partial z_i}\,d\bar{z}_{\alpha_1}\wedge\cdots\wedge d\bar{z}_{\alpha_p}.$$

Then we have the following

Lemma 5.4. *Let $\varphi\in Z_{\bar\partial}(\mathbb{C}^n/\Gamma,\mathcal{F}^{0,p})$, and $\varphi^{(p)}\in Z^p(\Gamma,H)$. Then*

$$\varphi_i\in Z_{\bar\partial}(\mathbb{C}^n/\Gamma,\mathcal{F}^{0,p}),\quad \textit{and}\quad [\Pi\varphi_i]=\left[\frac{\partial\varphi^{(p)}}{\partial z_i}\right],\quad \textit{for}\quad 1\le i\le q.$$

Proof. Since

$$\bar\partial\varphi_i=\bar\partial\left(\frac{\partial\varphi}{\partial z_i}\right)=\frac{\partial}{\partial z_i}(\bar\partial\varphi)=0,$$

we have $\varphi_i\in Z_{\bar\partial}(\mathbb{C}^n/\Gamma,\mathcal{F}^{0,p})$. From (5.10)

$$\varphi_i=\frac{\partial\varphi}{\partial z_i}=\bar\partial\frac{\partial\Phi^{(0)}}{\partial z_i},\quad\text{and}\quad \varphi_i^{(1)}:=\delta\frac{\partial\Phi^{(0)}}{\partial z_i}=\frac{\partial\varphi^{(1)}}{\partial z_i}. \tag{5.12}$$

Continuing this for (5.11), we have

$$\varphi_i^{(k+1)}:=\delta\frac{\partial\Phi^{(k)}}{\partial z_i}=\frac{\partial\varphi^{(k+1)}}{\partial z_i},\quad\text{for}\quad 0\le k<p. \tag{5.13}$$

Hence $[\Pi\varphi_i]=\left[\varphi_i^{(p)}\right]=[\partial\varphi^{(p)}/\partial z_i]$ and the lemma is proved. □

In the proof of Lemma 5.4, $\varphi^{(p)}$ and $\varphi_i^{(p)}$ are defined by $\Phi^{(k)}$ $(0\le k<p)$ and $\varphi_i^{(p)}=\partial\varphi^{(p)}/\partial z_i$. Hereafter we identify $\varphi_i^{(p)}$ with $\partial\varphi^{(p)}/\partial z_i$. Then we have the following

Proposition 5.5. *Let $\mathbb{C}^n/\Gamma$ be a toroidal group of type q. Suppose $\varphi\in Z_{\bar\partial}(\mathbb{C}^n/\Gamma,\mathcal{F}^{0,p})$. Then $\varphi_i\in B_{\bar\partial}(\mathbb{C}^n/\Gamma,\mathcal{F}^{0,p})$, for $1\le i\le q$. Hence $\varphi_i^{(p)}\in B^p(\Gamma,H)$.*

Proof. Put

$$\varphi=\frac{1}{p!}\sum_{1\le\alpha_1,\dots,\alpha_p\le q}\varphi_{\alpha_1\cdots\alpha_p}\,d\bar{z}_{\alpha_1}\wedge\cdots\wedge d\bar{z}_{\alpha_p}\in Z_{\bar\partial}(\mathbb{C}^n/\Gamma,\mathcal{F}^{0,p}),$$

$$\varphi_{\alpha_1\cdots\alpha_p}:=\sum_{m\in\mathbb{Z}^{n+q}}\varphi^m_{\alpha_1\cdots\alpha_p},\quad \psi^m_{\alpha_1\cdots\alpha_{p-1}}:=\frac{\varphi^m_{i(m)\alpha_1\cdots\alpha_{p-1}}}{\pi K_{m,i(m)}}\quad (m\ne 0),$$

and

$$\psi^m := \frac{1}{(p-1)!} \sum_{1 \le \alpha_1, \dots, \alpha_{p-1} \le q} \psi^m_{\alpha_1 \cdots \alpha_{p-1}} \, d\bar{z}_{\alpha_1} \wedge \cdots \wedge d\bar{z}_{\alpha_{p-1}},$$

where $|K_{m,i(m)}| = K_m$. Then by (4.6)

$$\varphi^m = \bar{\partial}\psi^m, \quad \text{for each} \quad m \neq 0.$$

Let

(5.14)
$$\psi^{i,m}_{\alpha_1 \cdots \alpha_{p-1}} := -\frac{\overline{K}_{m,i}\varphi^m_{i(m)\alpha_1 \cdots \alpha_{p-1}}}{K_{m,i(m)}}, \quad m \neq 0, \quad \text{and}$$
$$\psi^m_i = \frac{1}{(p-1)!} \sum_{1 \le \alpha_1, \dots, \alpha_{p-1} \le q} \psi^{i,m}_{\alpha_1 \cdots \alpha_{p-1}} \, d\bar{z}_{\alpha_1} \wedge \cdots \wedge d\bar{z}_{\alpha_{p-1}}.$$

Then by (4.5)

$$\frac{\partial \varphi^m}{\partial z_i} = \bar{\partial}\psi^m_i,$$

and applying (5.14) to (4.10), we see $\psi_i := \sum_{m \neq 0} \psi^m_i$ converges. Hence

(5.15) $$\varphi_i = \bar{\partial}\psi_i.$$

Then $\varphi^{(p)}_i$ is δ-exact, and there exists $h^{(p-1)}_i \in C^{p-1}(\Gamma, H)$ such that $\varphi^{(p)}_i = \delta h^{(p-1)}_i$. Hence the proposition is proved. □

Further we have the following

Proposition 5.6. *Let $\mathbb{C}^n/\Gamma$ be a toroidal group of type q. Suppose $\varphi \in Z_{\bar{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$.*

Then, there exists $h^{(p-1)}_i \in C^{p-1}(\Gamma, H)$ such that

$$\varphi^{(p)}_i = \delta h^{(p-1)}_i \quad \textit{and} \quad \frac{\partial h^{(p-1)}_i}{\partial z_j} = \frac{\partial h^{(p-1)}_j}{\partial z_i}, \quad \textit{for} \quad 1 \le i, j \le q.$$

To prove Proposition 5.6, we need some facts. Let $H^{r,0}$ be the space of holomorphic r-forms on $\mathbb{C}^n$. Then $C^p(\Gamma, H^{r,0})$ and the coboundary δ are defined similarly to §2.

Lemma 5.7. *Let $\mathbb{C}^n/\Gamma$ be a toroidal group of type q. Suppose $\varphi \in Z_{\overline{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$, and $p > 1$. Then, there exist*

$$\Omega^{(p-1)} = \sum_{i=1}^{q} h_i^{(p-1)}\, dz_i \in C^{p-1}(\Gamma, H^{1,0})$$

and

$$\Omega^{(p-2)} = \frac{1}{2} \sum_{1 \le i, j \le q} h_{ij}^{(p-2)}\, dz_i \wedge dz_j \in C^{p-2}(\Gamma, H^{2,0})$$

such that

$$\varphi_i^{(p)} = \delta h_i^{(p-1)}, \quad \textit{for} \quad 1 \le i \le q, \quad \textit{and} \quad d_q \Omega^{(p-1)} = \delta \Omega^{(p-2)}.$$

Proof. Let $1 \le i \le q$. From Proposition 5.6, we have $\psi_i \in H^0(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p-1})$ satisfying (5.14) and (5.15). Further, we have $\Phi^{(k)} \in C^0(\Gamma, F^{0,p-k-1})$, $(0 \le k \le p-1)$ satisfying (5.10) and (5.11). Then

$$\varphi_i = \overline{\partial} \frac{\partial \Phi^{(0)}}{\partial z_i} = \overline{\partial} \psi_i. \tag{5.16}$$

Since $p > 1$, there exists $\rho_i^{(0)} \in C^0(\Gamma, F^{0,p-2})$ such that

$$\frac{\partial \Phi^{(0)}}{\partial z_i} - \psi_i = \overline{\partial} \rho_i^{(0)}. \tag{5.17}$$

Hence

$$\varphi_i^{(1)} = \delta \frac{\partial \Phi^{(0)}}{\partial z_i} = \delta \overline{\partial} \rho_i^{(0)} = \overline{\partial} \frac{\partial \Phi^{(1)}}{\partial z_i}. \tag{5.18}$$

In case $p > 2$, there exist $\rho_i^{(1)} \in C^1(\Gamma, F^{0,p-3})$ such that

$$\frac{\partial \Phi^{(1)}}{\partial z_i} - \delta \rho_i^{(0)} = \overline{\partial} \rho_i^{(1)}. \tag{5.19}$$

Suppose we have $\rho_i^{(k-1)} \in C^{k-1}(\Gamma, F^{0,p-k-1})$ for $\rho_i^{(k-2)} \in C^{k-2}(\Gamma, F^{0,p-k})$, $p > 2$ and $2 \le k \le p-1$ such that

$$\frac{\partial \Phi^{(k-1)}}{\partial z_i} - \delta \rho_i^{(k-2)} = \overline{\partial} \rho_i^{(k-1)}.$$

Then

$$\varphi_i^{(k)} = \delta \frac{\partial \Phi^{(k-1)}}{\partial z_i} = \delta \overline{\partial} \rho_i^{(k-1)} = \overline{\partial} \frac{\partial \Phi^{(k)}}{\partial z_i}.$$

Hence if $p > 3$ and $k < p-1$ there exists $\rho_i^{(k)} \in C^k(\Gamma, F^{0,p-k-2})$ for $\rho_i^{(k-1)} \in C^{k-1}(\Gamma, F^{0,p-k-1})$ such that

$$\frac{\partial \Phi^{(k)}}{\partial z_i} - \delta \rho_i^{(k-1)} = \overline{\partial} \rho_i^{(k)}. \tag{5.20}$$

From (5.19) and (5.20) we see inductively that (5.20) holds for $1 \le k \le p-2$. Since

$$\varphi_i^{(p-1)} = \delta \frac{\partial \Phi^{(p-2)}}{\partial z_i} = \delta \overline{\partial} \rho_i^{(p-2)} = \overline{\partial} \frac{\partial \Phi^{(p-1)}}{\partial z_i},$$

there exists $h_i^{(p-1)} \in C^{p-1}(\Gamma, H)$ such that

$$\frac{\partial \Phi^{(p-1)}}{\partial z_i} - \delta \rho_i^{(p-2)} = h_i^{(p-1)}, \quad \text{and} \quad \varphi_i^{(p)} = \delta h_i^{(p-1)}, \quad \text{for} \quad p > 3. \tag{5.21}$$

From (5.18) and (5.19), (5.21) is valid for $p > 1$. Then

$$\frac{\partial h_j^{(p-1)}}{\partial z_i} - \frac{\partial h_i^{(p-1)}}{\partial z_j} = -\delta \left(\frac{\partial \rho_j^{(p-2)}}{\partial z_i} - \frac{\partial \rho_i^{(p-2)}}{\partial z_j} \right). \tag{5.22}$$

Next, we shall describe $\partial \psi_i / \partial z_j$ $(1 \le i, j \le q)$, using (5.14). Put

$$\begin{gathered} \psi^{ji,m}_{\alpha_1 \cdots \alpha_{p-1}} := \frac{\partial \psi^{i,m}_{\alpha_1 \cdots \alpha_{p-1}}}{\partial z_j} = \pi \frac{\overline{K}_{m,j} \overline{K}_{m,i} \varphi^m_{i(m)\alpha_1 \cdots \alpha_{p-1}}}{K_{m,i(m)}}, \quad m \neq 0, \quad \textit{and} \\ \psi^m_{ji} = \frac{1}{(p-1)!} \sum_{1 \le \alpha_1, \ldots, \alpha_{p-1} \le q} \psi^{ji,m}_{\alpha_1 \cdots \alpha_{p-1}} \, d\overline{z}_{\alpha_1} \wedge \cdots \wedge d\overline{z}_{\alpha_{p-1}}. \end{gathered} \tag{5.23}$$

Then applying (5.23) to (4.10), we see $\psi_{ji} := \sum_{m \neq 0} \psi^m_{ji}$ converges and

$$\frac{\partial \psi_i}{\partial z_j} = \psi_{ji}.$$

From (5.23), we have $\psi^m_{ji} = \psi^m_{ij}$ for $1 \le i, j \le q$. Hence $\psi_{ji} = \psi_{ij}$ and

$$\frac{\partial \psi_i}{\partial z_j} - \frac{\partial \psi_j}{\partial z_i} = 0. \tag{5.24}$$

Hence

$$\bar{\partial}\left(\frac{\partial \rho_j^{(0)}}{\partial z_i} - \frac{\partial \rho_i^{(0)}}{\partial z_j}\right) = 0. \tag{5.25}$$

Then there exists $h_{ij}^{(0)} \in C^0(\Gamma, H)$ and $\rho_{ij}^{(0)} \in C^0(\Gamma, F^{0,p-3})$ $(p > 2)$ such that

$$\begin{aligned} \frac{\partial \rho_j^{(0)}}{\partial z_i} - \frac{\partial \rho_i^{(0)}}{\partial z_j} &= -h_{ij}^{(0)}, \quad \text{for} \quad p = 2 \\ &= \bar{\partial}\rho_{ij}^{(0)}, \quad \text{for} \quad p > 2. \end{aligned} \tag{5.26}$$

Further from (5.20), we have

$$\bar{\partial}\left(\frac{\partial \rho_j^{(k)}}{\partial z_i} - \frac{\partial \rho_i^{(k)}}{\partial z_j}\right) = -\delta\left(\frac{\partial \rho_j^{(k-1)}}{\partial z_i} - \frac{\partial \rho_i^{(k-1)}}{\partial z_j}\right) \quad (1 \le k < p-1). \tag{5.27}$$

In case $p > 2$, from (5.26) and (5.27), we have

$$\bar{\partial}\left(\frac{\partial \rho_j^{(1)}}{\partial z_i} - \frac{\partial \rho_i^{(1)}}{\partial z_j}\right) = -\delta\bar{\partial}\rho_{ij}^{(0)}. \tag{5.28}$$

Hence if $p > 3$, there exists $\rho_{ij}^{(1)} \in C^1(\Gamma, F^{0,p-4})$ such that

$$\frac{\partial \rho_j^{(1)}}{\partial z_i} - \frac{\partial \rho_i^{(1)}}{\partial z_j} = -\delta\rho_{ij}^{(0)} + \bar{\partial}\rho_{ij}^{(1)}. \tag{5.29}$$

Combining (5.29) with (5.27), inductively on k, we get $\rho_{ij}^{(k)} \in C^k(\Gamma, F^{0,p-k-3})$ $(1 \le k < p-2)$ such that

$$\frac{\partial \rho_j^{(k)}}{\partial z_i} - \frac{\partial \rho_i^{(k)}}{\partial z_j} = -\delta\rho_{ij}^{(k-1)} + \bar{\partial}\rho_{ij}^{(k)}, \quad \text{for} \quad p > 3. \tag{5.30}$$

From (5.27) and (5.30), there exists $h_{ij}^{(p-2)} \in C^{p-2}(\Gamma, H)$ such that

$$\frac{\partial \rho_j^{(p-2)}}{\partial z_i} - \frac{\partial \rho_i^{(p-2)}}{\partial z_j} = -\delta\rho_{ij}^{(p-3)} - h_{ij}^{(p-2)}, \quad \text{for} \quad p > 3. \tag{5.31}$$

From (5.28), (5.31) is valid for $p = 3$. In case $p > 2$, from (5.22) and (5.31), we get

$$\frac{\partial h_j^{(p-1)}}{\partial z_i} - \frac{\partial h_i^{(p-1)}}{\partial z_j} = \delta h_{ij}^{(p-2)}. \tag{5.32}$$

From (5.22) and (5.26), (5.32) is valid for $p = 2$. Put

$$\Omega^{(p-1)} := \sum_{i=1}^{q} h_i^{(p-1)}\, dz_i, \quad \text{and} \quad \Omega^{(p-2)} := \frac{1}{2} \sum_{1 \le i, j \le q} h_{ij}^{(p-2)}\, dz_i \wedge dz_j.$$

Then (5.32) means that

$$d_q \Omega^{(p-1)} = \delta \Omega^{(p-2)},$$

and the lemma is proved. $\square$

Lemma 5.8. *Let $\mathbb{C}^n/\Gamma$ be a toroidal group of type q and $p > 1$. Suppose $\varphi \in Z_{\bar\partial}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$. Then there exist*

$$h_{i_1 \cdots i_r}^{(p-r)} \in C^{p-r}(\Gamma, H) \quad (1 \le i_1, \ldots, i_r \le q, \text{ and } r = 1, \ldots, p),$$

which are skew-symmetric in all indices, such that if we put

$$\Omega^{(p-r)} = \frac{1}{r!} \sum_{1 \le i_1, \ldots, i_r \le q} h_{i_1 \cdots i_r}^{(p-r)}\, dz_{i_1} \wedge \cdots \wedge dz_{i_r} \in C^{p-r}(\Gamma, H^{r,0}),$$

then

(5.33) $$\varphi_i^{(p)} = \delta h_i^{(p-1)}, \quad i = 1, \ldots, q \quad (\text{in case } r = 1),$$

(5.34) $$d_q \Omega^{(p-r+1)} = \delta \Omega^{(p-r)}, \quad r = 2, \ldots, p$$

and

(5.35) $$d_q \Omega^{(0)} = 0 \quad (\text{in case } r = p).$$

Proof. By Lemma 5.7, we have $h_i^{(p-1)} \in C^{p-1}(\Gamma, H)$ satisfying (5.33). Suppose $h_{i_1 \cdots i_{r-1}}^{(p-r+1)} \in C^{p-r+1}(\Gamma, H)$ is given ($r \ge 2$). We claim that for each r ($r \ge 2$), there exist $\rho_{i_1 \cdots i_{r-1}}^{(s)} \in C^s(\Gamma, F^{0,p-s-r})$, for $1 \le i_1, \ldots, i_{r-1} \le q$ and $s = 0, \ldots, p-r$, which are skew-symmetric in all indices, satisfying

(5.36) $$\sum_{k=1}^{r} (-1)^{k-1} \frac{\partial h_{i_1 \cdots \hat{i}_k \cdots i_r}^{(p-r+1)}}{\partial z_{i_k}} = -\delta \left(\sum_{k=1}^{r} (-1)^{k-1} \frac{\partial \rho_{i_1 \cdots \hat{i}_k \cdots i_r}^{(p-r)}}{\partial z_{i_k}} \right),$$

(5.37) $$\bar\partial \left(\sum_{k=1}^{r} (-1)^{k-1} \frac{\partial \rho_{i_1 \cdots \hat{i}_k \cdots i_r}^{(0)}}{\partial z_{i_k}} \right) = 0,$$

and

$$(5.38)\quad \overline{\partial}\left(\sum_{k=1}^{r}(-1)^{k-1}\frac{\partial\rho^{(s)}_{i_1\cdots\hat{i}_k\cdots i_r}}{\partial z_{i_k}}\right)=-\delta\left(\sum_{k=1}^{r}(-1)^{k-1}\frac{\partial\rho^{(s-1)}_{i_1\cdots\hat{i}_k\cdots i_r}}{\partial z_{i_k}}\right),\quad \text{for}\quad 1\le s\le p-r.$$

In case $r=2$, in the proof of Lemma 5.7, we have $\rho^{(s)}_i\in C^s(\Gamma,\ F^{0,p-s-2})$ for $1\le i\le q$, and $s=0,\dots,p-2$ satisfying (5.22), (5.25), and (5.27). These conditions correspond to our claims. Suppose (5.36), (5.37) and (5.38) hold for r ($r\ge 2$). Let $p>r$. By (5.37), there exists $\rho^{(0)}_{i_1\cdots i_r}\in C^0(\Gamma,\ F^{0,p-r-1})$ such that

$$(5.39)\quad \sum_{k=1}^{r}(-1)^{k-1}\frac{\partial\rho^{(0)}_{i_1\cdots\hat{i}_k\cdots i_r}}{\partial z_{i_k}}=\overline{\partial}\rho^{(0)}_{i_1\cdots i_r}.$$

Then by (5.38)

$$(5.40)\quad \overline{\partial}\left(\sum_{k=1}^{r}(-1)^{k-1}\frac{\partial\rho^{(1)}_{i_1\cdots\hat{i}_k\cdots i_r}}{\partial z_{i_k}}\right)=-\delta\overline{\partial}\rho^{(0)}_{i_1\cdots i_r}.$$

Hence, if $p>r+1$, there exists $\rho^{(1)}_{i_1\cdots i_r}\in C^1(\Gamma,\ F^{0,p-r-2})$ such that

$$(5.41)\quad \sum_{k=1}^{r}(-1)^{k-1}\frac{\partial\rho^{(1)}_{i_1\cdots\hat{i}_k\cdots i_r}}{\partial z_{i_k}}=-\delta\rho^{(0)}_{i_1\cdots i_r}+\overline{\partial}\rho^{(1)}_{i_1\cdots i_r}.$$

Continuing this, using (5.38), we get $\rho^{(s)}_{i_1\cdots i_r}\in C^s(\Gamma,F^{0,p-r-s-1})$, for $s=1,\dots,p-r-1$, such that

$$(5.42)\quad \sum_{k=1}^{r}(-1)^{k-1}\frac{\partial\rho^{(s)}_{i_1\cdots\hat{i}_k\cdots i_r}}{\partial z_{i_k}}=-\delta\rho^{(s-1)}_{i_1\cdots i_r}+\overline{\partial}\rho^{(s)}_{i_1\cdots i_r}.$$

From (5.38) and (5.42), there exists $h^{(p-r)}_{i_1\cdots i_r}\in C^{p-r}(\Gamma,\ H)$, such that

$$(5.43)\quad \sum_{k=1}^{r}(-1)^{k-1}\frac{\partial\rho^{(p-r)}_{i_1\cdots\hat{i}_k\cdots i_r}}{\partial z_{i_k}}=-\delta\rho^{(p-r-1)}_{i_1\cdots i_r}-h^{(p-r)}_{i_1\cdots i_r},\quad \text{for}\quad (p>r+1).$$

From (5.40), (5.43) is valid for $p>r$. From (5.36) and (5.43), we have

$$(5.44)\quad \sum_{k=1}^{r}(-1)^{k-1}\frac{\partial h^{(p-r+1)}_{i_1\cdots\hat{i}_k\cdots i_r}}{\partial z_{i_k}}=\delta h^{(p-r)}_{i_1\cdots i_r}\quad (p>r).$$

Namely if $h^{(p-r+1)}_{i_1\cdots i_{r-1}} \in C^{p-r+1}(\Gamma, H)$ is given with (5.36), (5.37) and (5.38), then we can construct $h^{(p-r)}_{i_1\cdots i_r} \in C^{p-r}(\Gamma, H)$ satisfying (5.44). Then (5.34) holds for r ($r \geq 2$). Next we show (5.36), (5.37) and (5.38) hold for $r+1$. From (5.43)

$$\sum_{k=1}^{r+1}(-1)^{k-1}\frac{\partial h^{(p-r)}_{i_1\cdots\hat{i}_k\cdots i_{r+1}}}{\partial z_{i_k}} = -\delta\sum_{k=1}^{r+1}(-1)^{k-1}\frac{\partial \rho^{(p-r-1)}_{i_1\cdots\hat{i}_k\cdots i_{r+1}}}{\partial z_{i_k}}, \quad \text{for} \quad p > r. \tag{5.45}$$

On the other hand, from (5.39)

$$\bar{\partial}\left(\sum_{k=1}^{r+1}(-1)^{k-1}\frac{\partial \rho^{(0)}_{i_1\cdots\hat{i}_k\cdots i_{r+1}}}{\partial z_{i_k}}\right) = 0. \tag{5.46}$$

Further from (5.42), for $s = 1, \ldots, p-r-1$

$$\bar{\partial}\left(\sum_{k=1}^{r+1}(-1)^{k-1}\frac{\partial \rho^{(s)}_{i_1\cdots\hat{i}_k\cdots i_{r+1}}}{\partial z_{i_k}}\right) = \delta\left(\sum_{k=1}^{r+1}(-1)^{k-1}\frac{\partial \rho^{(s-1)}_{i_1\cdots\hat{i}_k\cdots i_{r+1}}}{\partial z_{i_k}}\right). \tag{5.47}$$

Thus we have $h^{(p-r)}_{i_1\cdots i_r}$ with (5.36), (5.37) and (5.38) for $r+1$. Then similary to getting (5.43), we can construct $h^{p-r-1}_{i_1\cdots i_{r+1}}$ which satisfies (5.44) in case $r+1$. Hence (5.34) holds for $2 \leq r \leq p$. Next we show (5.35). Since $r = p$, from (5.37), we have

$$\sum_{k=1}^{p}(-1)^{k-1}\frac{\partial \rho^{(0)}_{i_1\cdots\hat{i}_k\cdots i_p}}{\partial z_{i_k}} = h^{(0)}_{i_1\cdots i_p}.$$

Hence

$$\sum_{k=1}^{p+1}(-1)^{k-1}\frac{\partial h^{(0)}_{i_1\cdots\hat{i}_k\cdots i_{p+1}}}{\partial z_{i_k}} = 0.$$

Thus (5.35) holds for $p > 1$, and the lemma is proved. □

Then we begin to prove Proposision 5.6. In case $p = 1$, from (5.16), $h^{(0)}_i := \partial\Phi^{(0)}/\partial z_i - \psi_i \in C^0(\Gamma, H)$. Then $\varphi^{(1)}_i = \delta h^{(0)}_i$ and

$$\frac{\partial h^{(0)}_i}{\partial z_j} - \frac{\partial h^{(0)}_j}{\partial z_i} = \frac{\partial \psi_i}{\partial z_j} - \frac{\partial \psi_j}{\partial z_i} = 0.$$

Hence the proposition holds for $p = 1$. In case $p > 1$, by Lemma 5.8, we have holomorphic r-forms $\Omega^{(p-r)}$ ($r = 1, \ldots, p$) satisfying (5.33), (5.34) and (5.35). Hence by Lemma 5.1, there exists a holomorphic $(p-1)$-form $\Theta^{(0)} \in C^0(\Gamma, H^{p-1,0})$ such that

$$d_q\Theta^{(0)} = \Omega^{(0)}.$$

Then

$$d_q(\Omega^{(1)} - \delta\Theta^{(0)}) = 0.$$

Hence, there exists $\Theta^{(1)} \in C^1(\Gamma, H^{p-2,0})$ such that

$$d_q(\Omega^{(2)} - \delta\Theta^{(1)}) = 0.$$

Continuing this, we get $\Theta^{(p-2)} \in C^{p-2}(\Gamma, H^{1,0})$ such that

$$d_q(\Omega^{(p-1)} - \delta\Theta^{(p-2)}) = 0.$$

Put

$$\hat{\Omega}^{(p-1)} := \Omega^{(p-1)} - \delta\Theta^{(p-2)} = \sum_{i=1}^{q} \hat{h}_i^{(p-1)}\, dz_i.$$

Then

$$\delta\hat{\Omega}^{(p-1)} = \delta\Omega^{(p-1)}.$$

Hence

$$\varphi_i^{(p)} = \delta\hat{h}_i^{(p-1)}, \quad \text{and} \quad \frac{\partial \hat{h}_i^{(p-1)}}{\partial z_j} = \frac{\partial \hat{h}_j^{(p-1)}}{\partial z_i}.$$

Hence the proof of the proposition is completed. □

Proof of Theorem 2.2. By Lemma 5.3, for every $f^{(p)} \in Z^p(\Gamma, H)$, there exists $\varphi \in Z_{\bar{\partial}}(\mathbb{C}^n/\Gamma, \mathcal{F}^{0,p})$ such that $\varphi^{(p)}$ and $f^{(p)}$ are δ-cohomologous. It suffices to prove the theorem for $\varphi^{(p)}$. Then for each $1 \leq i \leq q$, by Proposition 5.6 there exist $h_i^{(p-1)} \in C^{p-1}(\Gamma, H)$ such that

$$\varphi_i^{(p)} = \delta h_i^{(p-1)} \quad \text{and} \quad \frac{\partial h_i^{(p-1)}}{\partial z_j} = \frac{\partial h_j^{(p-1)}}{\partial z_i}.$$

Put

$$\Omega^{(p-1)} = \sum_{i=1}^{q} h_i^{(p-1)}\, dz_i,$$

then $d_q\Omega^{(p-1)} = 0$. Hence by Lemma 5.1, there exists $h^{(p-1)} \in C^{p-1}(\Gamma, H)$ such that

$$d_q h^{(p-1)} = \Omega^{(p-1)},$$

namely

$$\frac{\partial h^{(p-1)}}{\partial z_i} = h_i^{(p-1)} \quad (1 \le i \le q).$$

Hence

$$\frac{\partial}{\partial z_i}\delta h^{(p-1)} = \varphi_i^{(p)} = \frac{\partial \varphi^{(p)}}{\partial z_i}.$$

Put

$$a^{(p)} := \varphi^{(p)} - \delta h^{(p-1)} \in Z^p(\Gamma, H).$$

Then

$$\frac{\partial a^{(p)}}{\partial z_i} = 0 \quad (1 \le i \le q),$$

and we complete the proof of Theorem 2.2. □

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