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Research Paper

The quasi-polynomiality of mod q permutation representations for a linear finite group action on a lattice

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For given linear action of a finite group on a lattice and a positive integer q, we prove that the mod q permutation representation is a quasi-polynomial in q. Additionally, we establish several results that can be considered as mod q-analogues of results by Stapledon for equivariant Ehrhart quasi-polynomials. We also prove a reciprocity-type result for multiplicities of irreducible decompositions.

1. Introduction

1.1. Quasi-polynomials

Let R be a commutative ring. A function $f : \mathbb{Z}_{(>0)} \to R$ is called a *quasi-polynomial* if there exist a positive integer $\tilde{n} \in \mathbb{Z}_{>0}$ and polynomials $g_1(t), \dots, g_{\tilde{n}}(t) \in R[t]$ such that

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$$f(q) = g_r(q), \quad \text{if } q \equiv r \mod \tilde{n} \qquad (1 \le r \le \tilde{n}).$$

The positive integer $\tilde{n}$ is called a *period* and each polynomial g_r is called the *constituent* of f. The quasi-polynomial f has degree d if all the constituents have degree d. Moreover, the quasi-polynomial f has the gcd-*property* if the polynomial g_r depends on r only through $\gcd\{\tilde{n}, r\}$. In other words, $g_{r_1} = g_{r_2}$ if $\gcd\{\tilde{n}, r_1\} = \gcd\{\tilde{n}, r_2\}$.

Quasi-polynomials play important roles in many areas of mathematics. They appear frequently as counting functions (in this case, $R = \mathbb{Z}$). In particular, the following two notions have been actively studied.

Example 1.1 *(The Ehrhart quasi-polynomial).* Let $\mathcal{P}$ be a rational polytope in $\mathbb{R}^\ell$. For $q \in \mathbb{Z}_{\ge 0}$, define

$$\mathrm{L}_{\mathcal{P}}(q) := \#(q\mathcal{P} \cap \mathbb{Z}^\ell).$$

Then $\mathrm{L}_{\mathcal{P}}(q)$ is a quasi-polynomial ([1, Theorem 3.23]), known as the Ehrhart quasi-polynomial.

Example 1.2 *(The characteristic quasi-polynomial).* Let $L \simeq \mathbb{Z}^\ell$ be a lattice and $L^\vee := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ be the dual lattice. Given $\alpha_1, \dots, \alpha_n \in L^\vee \setminus \{0\}$, we can associate a hyperplane arrangement $\mathcal{A} := \{ H_1, \dots, H_n \}$ in $\mathbb{R}^\ell \simeq L \otimes \mathbb{R}$, where

$$H_i := \{ x \in L \otimes \mathbb{R} \mid \alpha_i(x) = 0 \}.$$

For a positive integer $q \in \mathbb{Z}_{>0}$, define the mod q complement of the arrangement by

$$\begin{aligned} M(\mathcal{A};\, q) &:= (L/qL) \setminus \bigcup_{i=1}^{n} \bar{H}_i \\ &= \{ \bar{x} \in L/qL \mid \alpha_i(x) \not\equiv 0 \pmod{q} \quad \text{for all } i \in \{1, \dots, n\} \}. \end{aligned}$$

It is known ([4, Theorem 2.4] and [5, Theorem 3.1] for non-central case) that

$$\chi_{\mathrm{quasi}}(\mathcal{A};\, q) := \# M(\mathcal{A};\, q)$$

is a quasi-polynomial. It is called a characteristic quasi-polynomial.

Roughly speaking, the notion of the characteristic quasi-polynomial is a mod q-version of the Ehrhart quasi polynomial. However, the characteristic quasi-polynomials possess some additional properties. First, the constituents $g_r(t)$ ($r \in \{1, \dots, \tilde{n}\}$) of the characteristic quasi-polynomial $\chi_{\mathrm{quasi}}(\mathcal{A};\, q)$ satisfy the gcd-property. Second, the first constituent $g_1(t)$ (and equivalently, $g_r(t)$ for r coprime to $\tilde{n}$) is known to be equal to the characteristic polynomial $\chi(\mathcal{A}, t)$ of the arrangement $\mathcal{A}$ (see [7]). Furthermore, $g_{\tilde{n}}(t)$ is the characteristic polynomial of the associated toric arrangement [6,11]. The characteristic quasi-polynomial is an important concept, not only in the context of enumerative problems but also in its connections with arithmetic matroids and toric arrangements [2,6].

1.2. Equivariant Ehrhart theory

In [10], Stapledon proposed an equivariant version of Ehrhart theory. Let $L \simeq \mathbb{Z}^\ell$ be a lattice and let Γ be a finite group acting linearly on L via $\rho : \Gamma \to \mathrm{GL}(L)$. Let $\mathcal{P}$ be a Γ-invariant lattice polytope. For a positive integer $q \in \mathbb{Z}_{>0}$, the group Γ acts on the lattice points $q\mathcal{P} \cap L$. Let $\chi_{q\mathcal{P}}$ denote the character of this permutation representation. Stapledon proved representation-theoretic analogues of several classical results in Ehrhart theory. For example, the map

$$F : \mathbb{Z}_{>0} \longrightarrow R(\Gamma), \quad q \longmapsto F(q) = \chi_{q\mathcal{P}}$$

is a quasi-polynomial of degree $\dim \mathcal{P}$ with the leading coefficient $\frac{\mathrm{vol}\,\mathcal{P}}{\#\Gamma}\chi_{\mathrm{R}}$, where χ_{R} is the regular (standard) character of Γ ([10, Theorem 5.7 and Corollary 5.9]). It was also proved that the number of Γ-orbits in $q\mathcal{P} \cap L$ is a quasi-polynomial in q.

Stapledon also proved the following reciprocity. Let $F^*(q) = \chi_{q\mathcal{P}^*}$ be the permutation representation of the lattice points in the interior of $q\mathcal{P}$. Then, from the Ehrhart reciprocity, the relation

$$F^*(q) = (-1)^{\dim \mathcal{P}} \det(\rho) F(-q) \tag{1}$$

holds. Note that when Γ is the trivial group, these results recover the classical results in Ehrhart theory.

1.3. Towards an equivariant version of characteristic quasi-polynomials

It is natural to consider $\mathrm{mod}\, q$-version of the equivariant Ehrhart quasi-polynomial, namely, the equivariant characteristic quasi-polynomials for an arrangement invariant under a group action. In this paper, as a stepping stone to the general case, we do not consider hyperplanes, and instead focus solely on the $\mathrm{mod}\, q$ permutation representation. For general arrangement cases, refer to the research [13] by the first author.

Let Γ be a finite group and let $L \simeq \mathbb{Z}^\ell$ be a lattice. Suppose that Γ acts linearly on L. Then the action of Γ on $L_q := L/qL$ is naturally induced for each $q \in \mathbb{Z}_{>0}$. Note that, in the case where Γ is the Weyl group and L is a lattice associated with a root system, there are several known results about L_q (e.g. [3,8]), especially for $q \equiv 1 \mod h$, where h is the Coxeter number.

One of our problems is how the permutation character χ_{L_q} of L_q depends on q. The main result of this paper is the following.

Theorem 1.3 *(Theorem 2.6 below). Consider the function* $F : \mathbb{Z}_{>0} \longrightarrow R(\Gamma)$ *defined by* $q \longmapsto \chi_{L_q}$*. Then* F *is a quasi-polynomial of degree* ℓ*. Furthermore,* F *has the* gcd-*property, the minimum period* $\tilde{n}$*, and the leading coefficient of the quasi-polynomial* χ_{L_q} *is* $\frac{\chi_{\mathrm{R}}}{\#\Gamma}$.

Note that this result corresponds to [10, Theorem 5.7 and Corollary 5.9].

Let $\chi_1, \dots, \chi_k$ be irreducible characters of Γ. Then χ_{L_q} can be expressed as

$$\chi_{L_q} = m(\chi_1; q) \cdot \chi_1 + \cdots + m(\chi_k; q) \cdot \chi_k, \tag{2}$$

with $m(\chi_k; q) \in \mathbb{Z}$. Theorem 1.3 is equivalent to that each $m(\chi_i; q)$ is a $\mathbb{Z}$-valued quasi-polynomial in q (Corollary 2.2).

For a character λ of a 1-dimensional representation of Γ and $q \in \mathbb{Z}_{>0}$, let $f_{L/\Gamma}(\lambda; q)$ denote the number of Γ-orbits on L_q whose isotropy subgroup is contained in the subgroup $\lambda^{-1}(1)$ of Γ. Then since $f_{L/\Gamma}(\lambda; q) = m(\lambda; q)$, we obtain the following results corresponding to [10, Corollary 5.8] as the corollary of the above theorem.

Corollary 1.4 *(Corollary 2.9 below). The function* $f_{L/\Gamma}(\lambda; -) : \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}$ *is a quasi-polynomial of degree* ℓ *and it has the* gcd-*property.*

There are several relations among these quasi-polynomials. In particular, there is a reciprocity-type relation between $m(\chi_i; q)$ and $m(\chi_i \otimes \delta_\rho; q)$, where $\delta_\rho = \det \rho$ is a 1-dimensional representation. More precisely, we obtain the following reciprocity theorem.

Theorem 1.5 *(Theorem 2.11 below). The following formula holds for an irreducible character* χ_i *of* Γ*:*

$$m(\chi_i \otimes \delta_\rho; q) = (-1)^\ell m(\chi_i; -q).$$

This implies the following relation.

Corollary 1.6 *(Corollary 2.12 below). The quasi-polynomial* $F : \mathbb{Z} \longrightarrow R(\Gamma)$ *satisfies*

$$F(q) = (-1)^\ell \delta_\rho F(-q). \tag{3}$$

Although the formula (3) appears similar to (1), they are different in nature. It is important to note that (1) represents a reciprocity between $F(q)$ and $F^*(q)$, whereas (3) is a self-duality of $F(q)$.

In section 2.6, we will also provide several explicit examples.

2. Quasi-polynomiality

2.1. Group action and representation

We recall several notions and basic facts about representations of finite groups [9].

Let Γ be a finite group. Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $\mathrm{GL}(V)$ denote the group of linear isomorphisms of V onto itself. A *(linear) representation* of Γ on V is a homomorphism $\rho : \Gamma \longrightarrow \mathrm{GL}(V)$. In this paper, we assume that ρ is injective. The space V is called the *representation space* of ρ.

The *character* $\chi_\rho : \Gamma \longrightarrow \mathbb{C}$ of the representation ρ is the function defined by $\gamma \longmapsto \operatorname{tr}\rho(\gamma)$, where tr denotes the trace function. The character χ_ρ is constant on each conjugacy class. A function $\phi : \Gamma \longrightarrow \mathbb{C}$ is called a class function if ϕ is constant on each conjugacy class. For functions $\phi, \psi : \Gamma \longrightarrow \mathbb{C}$, define the inner product (ϕ, ψ) by

$$(\phi, \psi) = \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \phi(\gamma)\overline{\psi(\gamma)},$$

where $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. Let $\chi_1, \ldots, \chi_k$ be the set of all irreducible characters of Γ. Then $\chi_1, \ldots, \chi_k$ form an orthonormal basis of the space of class functions. In particular, $(\chi_i, \chi_j) = \delta_{ij}$. Thus, if a class function χ is expressed as a linear combination of irreducible characters $\chi = m_1\chi_1 + \cdots + m_k\chi_k$, then we have $m_i = (\chi, \chi_i)$.

Let Γ' be a subgroup of Γ. The restriction of a class function $\chi : \Gamma \longrightarrow \mathbb{C}$ to Γ' is clearly a class function on Γ', which is denoted by $\operatorname{Res}^{\Gamma}_{\Gamma'} \chi : \Gamma' \longrightarrow \mathbb{C}$. Conversely, for a class function $\varphi : \Gamma' \longrightarrow \mathbb{C}$, define the *induced function* $\operatorname{Ind}^{\Gamma}_{\Gamma'} \varphi : \Gamma \longrightarrow \mathbb{C}$ by

$$\left(\operatorname{Ind}^{\Gamma}_{\Gamma'} \varphi\right)(\gamma) = \frac{1}{\#\Gamma'} \sum_{\substack{\eta \in \Gamma \\ \eta^{-1}\gamma\eta \in \Gamma'}} \varphi(\eta^{-1}\gamma\eta). \tag{4}$$

These two operators are related by the following Frobenius reciprocity:

$$\left(\chi, \operatorname{Ind}^{\Gamma}_{\Gamma'} \varphi\right) = \left(\operatorname{Res}^{\Gamma}_{\Gamma'} \chi, \varphi\right). \tag{5}$$

Recall that the *representation ring* $R(\Gamma)$ of Γ is $\bigoplus_V \mathbb{Z}[V]/\sim$, where V runs over all finite-dimensional representations of Γ, and $\sim$ is an equivalence relation generated by $[V] \sim [V']$ for isomorphic representations $V \simeq V'$ and $[V_1 \oplus V_2] \sim [V_1] + [V_2]$. The multiplication is defined by $[V_1] \cdot [V_2] = [V_1 \otimes V_2]$. The character gives a natural isomorphism of abelian groups

$$R(\Gamma) \simeq \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_k.$$

The trivial representation $\rho_{\mathbf{1}}$ is the unit element in $R(\Gamma)$. The character of $\rho_{\mathbf{1}}$ is denoted by $\mathbf{1}$.

Suppose that Γ acts on a finite set X. Let $\mathbb{C}X$ denote the vector space based on X, that is, $\mathbb{C}X = \bigoplus_{x\in X} \mathbb{C}x$. This gives rise to a natural representation $\rho_X : \Gamma \longrightarrow \mathrm{GL}(\mathbb{C}X)$, which is called the *permutation representation* of X. In the case of $X = \Gamma$ with action defined by the left multiplication, it is called the *regular (standard) representation*, denoted by ρ_{R}. Note that the character χ_{R} of the regular representation satisfies the following.

$$\chi_{\mathrm{R}} = \sum_{i=1}^{k} \chi_i(1)\chi_i, \qquad \chi_{\mathrm{R}}(\gamma) = \begin{cases} \#\Gamma & \text{if } \gamma = 1; \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in X$, the *Γ-orbit* $\Gamma(x)$ and the *isotropy subgroup* Γ_x are defined as follows:

$$\Gamma(x) = \{\, \gamma x \in X \mid \gamma \in \Gamma \,\},$$
$$\Gamma_x = \{\, \gamma \in \Gamma \mid \gamma x = x \,\}.$$

2.2. Multiplicities of irreducible decompositions

Let L be a lattice, and $\{\beta_1, \ldots, \beta_\ell\}$ be a $\mathbb{Z}$-basis of L, that is, $L = \mathbb{Z}\beta_1 \oplus \cdots \oplus \mathbb{Z}\beta_\ell \simeq \mathbb{Z}^\ell$. We identify an element $x = x_1\beta_1 + \cdots + x_\ell\beta_\ell$ of L with the row vector $x = (x_1, \ldots, x_\ell)$ of $\mathbb{Z}^\ell$.

Let Γ be a finite group. Let $\rho : \Gamma \longrightarrow \mathrm{GL}(L)$ be a group homomorphism. Let us denote the representation matrix of $\rho(\gamma)$ by R_γ, and we consider the right multiplication, namely,

$$\rho(\gamma) : L \longrightarrow L, \quad x \longmapsto xR_\gamma.$$

For $q \in \mathbb{Z}_{>0}$, define $\mathbb{Z}_q \coloneqq \mathbb{Z}/q\mathbb{Z}$. We will consider the following *q-reduction* of $x = (x_1, \ldots, x_\ell) \in \mathbb{Z}^\ell$:

$$[x]_q \coloneqq ([x_1]_q, \ldots, [x_\ell]_q) \in \mathbb{Z}_q^\ell,$$

where $[x_i]_q = x_i + q\mathbb{Z} \in \mathbb{Z}_q$. We similarly consider the q-reduction of an integer matrix $A = (a_{ij})_{ij}$:

$$[A]_q \coloneqq ([a_{ij}]_q)_{ij}\,.$$

Let $\varphi : \mathbb{Z}^\ell \longrightarrow \mathbb{Z}^\ell$ be a $\mathbb{Z}$-homomorphism represented by an $\ell \times \ell$ integer matrix A. We can define the induced morphism $\varphi_q : \mathbb{Z}_q^\ell \longrightarrow \mathbb{Z}_q^\ell$ by

$$x \longmapsto x[A]_q.$$

Let $L_q \coloneqq L/qL \simeq (\mathbb{Z}/q\mathbb{Z})^\ell$. The action of Γ on L_q is induced by $\rho(\gamma)_q : L_q \longrightarrow L_q$. Let χ_{L_q} denote the character of the permutation representation of L_q, and consider its irreducible decomposition:

$$\chi_{L_q} = m(\chi_1;\, q) \cdot \chi_1 + \cdots + m(\chi_k;\, q) \cdot \chi_k,$$

where $m(\chi_i;\, q)$ denotes the multiplicity of χ_i in χ_{L_q}. Since $\chi_{L_q}(\gamma)$ is equal to the number of elements in L_q fixed by $\gamma \in \mathit{\Gamma}$, we have

$$m(\chi_i;\, q) = (\chi_i, \chi_{L_q}) = \frac{1}{\#\mathit{\Gamma}} \sum_{\gamma \in \mathit{\Gamma}} \chi_i(\gamma)\overline{\chi_{L_q}(\gamma)} = \frac{1}{\#\mathit{\Gamma}} \sum_{\gamma \in \mathit{\Gamma}} \chi_i(\gamma) \cdot \#L_q^{\gamma}, \tag{6}$$

where $L_q^{\gamma} \coloneqq \{\, x \in L_q \mid \gamma x = x \,\}$. Thus, by studying the properties of $\#L_q^{\gamma}$, we can determine how $m(\chi_i;\, q)$ depends on q. Note that for the trivial character $\mathbf{1}$, the multiplicity $m(\mathbf{1};\, q)$ represents the number of $\mathit{\Gamma}$-orbits of L_q, according to Burnside's lemma.

The fixed point set L_q^{γ} is expressed as

$$\begin{aligned} L_q^{\gamma} &= \{\, x \in L_q \mid \gamma x = x \,\} \\ &= \{\, x \in L_q \mid x[R_\gamma]_q = x \,\} \\ &= \{\, x \in L_q \mid x[R_\gamma - I_\ell]_q = 0 \,\}, \end{aligned}$$

where I_ℓ is the identity matrix of size ℓ. Therefore, L_q^{γ} is equal to the kernel of the induced morphism $(\rho(\gamma) - \mathrm{id})_q$. The cardinality of the kernel of this morphism is known to be quasi-monomial, as shown in [4]:

Lemma 2.1 *([4, Lemma 2.1]). Let $\varphi : \mathbb{Z}^\ell \longrightarrow \mathbb{Z}^\ell$ be a $\mathbb{Z}$-homomorphism. Then the cardinality of the kernel of the induced morphism $\varphi_q : \mathbb{Z}_q^\ell \longrightarrow \mathbb{Z}_q^\ell$ is a quasi-monomial in q. Furthermore, suppose φ is represented by a matrix A. Then the quasi-monomial $\#\ker\varphi_q$ can be expressed as*

$$\#\ker\varphi_q = \left(\prod_{j=1}^{r} \gcd\{e_j,\, q\} \right) q^{\ell - r}, \tag{7}$$

where $r \coloneqq \operatorname{rank} A$ and $e_1, \ldots, e_r \in \mathbb{Z}_{>0}$, with $e_1 \mid e_2 \mid \cdots \mid e_r$, are the elementary divisors of A. Hence, the quasi-monomial $\#\ker\varphi_q$ has the gcd*-property and the minimum period e_r. If $r = 0$, we consider e_0 to be 1.*

Proof. Here, we only review quasi-monomiality. For further details, see [4, Lemma 2.1].

Since $\#\ker\varphi_q = q^\ell / \#\operatorname{im}\varphi_q$, we will study $\#\operatorname{im}\varphi_q$. Consider the Smith normal form

$$SAT = \begin{pmatrix} e_1 & & & \\ & \ddots & & \\ & & e_r & \\ & & & O \end{pmatrix}, \qquad r = \operatorname{rank} A, \quad e_1, \ldots, e_r, \in \mathbb{Z}_{>0}, \quad e_1 \mid e_2 \mid \cdots \mid e_r,$$

where S and T are $\ell \times \ell$ unimodular matrices. Since unimodularity is preserved under q-reductions, we may assume that A is a diagonal matrix $\operatorname{diag}(e_1, \dots, e_r, 0, \dots, 0)$ from the outset. Then, for $x = (x_1, \dots, x_\ell) \in \mathbb{Z}_q^\ell$, we have

$$\varphi_q(x) = ([e_1]_q x_1,\, \dots,\, [e_r]_q x_r,\, 0,\, \dots,\, 0)$$

and hence $\operatorname{im} \varphi_q = [e_1]_q \mathbb{Z}_q \times \cdots \times [e_r]_q \mathbb{Z}_q$. Therefore,

$$\# \operatorname{im} \varphi_q = \frac{q^r}{\prod_{j=1}^r \gcd\{e_j,\, q\}},$$

and we obtain (7). □

Corollary 2.2. *The multiplicity $m(\chi_i;\, q)$ of χ_i in χ_{L_q} is a quasi-polynomial in q. More explicitly,*

$$m(\chi_i;\, q) = \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \chi_i(\gamma) \cdot \left(\prod_{j=1}^{r(\gamma)} \gcd\{e_{\gamma,j},\, q\} \right) q^{\ell - r(\gamma)}, \tag{8}$$

where $r(\gamma) := \operatorname{rank}(R_\gamma - I_\ell)$ and $e_{\gamma,1}, \dots, e_{\gamma,r(\gamma)} \in \mathbb{Z}_{>0}$ with $e_{\gamma,1} \mid e_{\gamma,2} \mid \cdots \mid e_{\gamma,r(\gamma)}$, are the elementary divisors of $R_\gamma - I_\ell$.

Proof. The equation (8) is given by (6) and (7). □

Next, we present some properties of $m(\chi_i;\, q)$.

Proposition 2.3. *The quasi-polynomial $m(\chi_i;\, q)$ has the* gcd*-property with a period*

$$\tilde{n} := \operatorname{lcm}\{\, e_{\gamma,r(\gamma)} \mid \gamma \in \Gamma \,\}.$$

Furthermore, the minimum period of the quasi-polynomial $m(\mathbf{1};\, q)$ is equal to $\tilde{n}$.

Remark 2.4. If χ_i is not trivial, we do not know whether $\tilde{n}$ is the minimum period.

Proof. Let $\gamma \in \Gamma$ be an element that is not the identity, and let $e_{\gamma,1}, \dots, e_{\gamma,r(\gamma)}$ be the elementary divisors of $R_\gamma - I_\ell$. Since $e_{\gamma,j}$ divides $\tilde{n}$ for $j \in \{1, \dots, r(\gamma)\}$, we have

$$\prod_{j=1}^{r(\gamma)} \gcd\{e_{\gamma,j},\, q\} = \prod_{j=1}^{r(\gamma)} \gcd\{e_{\gamma,j},\, \tilde{n},\, q\} = \prod_{j=1}^{r(\gamma)} \gcd\{e_{\gamma,j},\, \gcd\{\tilde{n},\, q\}\}.$$

Hence $m(\chi_i;\, q)$ depends on q only through $\gcd\{\tilde{n},\, q\}$, which means that $\tilde{n}$ is a period of $m(\chi_i;\, q)$.

Let $g_1(t), \dots, g_{\tilde{n}}(t) \in \mathbb{Z}[t]$ denote the constituents of the quasi-polynomial $m(\mathbf{1};\, q)$. Since $\tilde{n}$ is divisible by $e_{\gamma,r(\gamma)}$ for all $\gamma \in \Gamma$, we have

$$g_{\tilde{n}}(t) = \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \left(\prod_{j=1}^{r(\gamma)} e_{\gamma,j} \right) t^{\ell - r(\gamma)} \tag{9}$$

from equation (8). Suppose that $s < \tilde{n} = \operatorname{lcm}\{\, e_{\gamma,r(\gamma)} \mid \gamma \in \Gamma \,\}$. Then there exists $\gamma \in \Gamma$ such that $\gcd\{e_{\gamma,r(\gamma)}, s\} \neq e_{\gamma,r(\gamma)}$. Since $\gcd\{e_{\gamma,j}, s\} \leq e_{\gamma,j}$ for any $\gamma \in \Gamma$ and $j \in \{1, \dots, r(\gamma)\}$, we conclude that

$$g_s(t) = \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \left(\prod_{j=1}^{r(\gamma)} \gcd\{e_{\gamma,j}, s\} \right) t^{\ell - r(\gamma)} \neq g_{\tilde{n}}(t)$$

by the equations (8) and (9), and hence s is not a period. Therefore, $\tilde{n}$ is the minimum period of $m(\mathbf{1};\, q)$. □

Proposition 2.5. *The leading term of the quasi-polynomial $m(\chi_i;\, q)$ is $\frac{\chi_i(1)}{\#\Gamma} q^\ell$.*

Proof. Since ρ is injective, $r(\gamma) = 0$ holds if and only if γ is the identity. Therefore, by Corollary 2.2, the leading term of $m(\chi_i;\, q)$ is q^ℓ with the coefficient $\frac{\chi_i(1)}{\#\Gamma}$. □

2.3. Permutation representations

Since each multiplicity $m(\chi_i;\, q)$ is a quasi-polynomial, the quasi-polynomiality of the function $F : q \longmapsto \chi_{L_q}$ follows immediately. The following theorem is the main result of this paper.

Theorem 2.6 *(Restatement of Theorem 1.3). Consider the function $F : \mathbb{Z}_{>0} \longrightarrow R(\Gamma)$ defined by $q \longmapsto \chi_{L_q}$. Then F is a quasi-polynomial of degree ℓ. Furthermore, F has the* gcd*-property, the minimum period $\tilde{n}$, and the leading coefficient of the quasi-polynomial χ_{L_q} is $\frac{\chi_{\mathrm{R}}}{\#\Gamma}$.*

Proof. By equation (8), we have

$$\begin{aligned} F(q) &= \chi_{L_q} \\ &= \sum_{i=1}^{k} m(\chi_i;\, q) \cdot \chi_i \\ &= \frac{1}{\#\Gamma} \sum_{i=1}^{k} \sum_{\gamma \in \Gamma} \chi_i(\gamma) \cdot \left(\prod_{j=1}^{r(\gamma)} \gcd\{e_{\gamma,j},\, q\} \right) \cdot \chi_i \cdot q^{\ell - r(\gamma)} \in R(\Gamma)[q], \end{aligned}$$

hence F is a quasi-polynomial with the gcd-property. Since $m(\chi_i;\, q)$ has a period $\tilde{n}$ for any $i \in \{1, \dots, k\}$, and especially $\tilde{n}$ is the minimum period of $m(\mathbf{1};\, q)$, the quasi-polynomial F has the minimum period $\tilde{n}$.

By Proposition 2.5, the leading term of each multiplicity $m(\chi_i;\, q)$ is $\frac{\chi_i(1)}{\#\Gamma} q^\ell$. Thus, we have

$$\sum_{i=1}^{k} \frac{\chi_i(1)}{\#\Gamma} \cdot \chi_i \cdot q^\ell = \frac{\chi_{\mathrm{R}}}{\#\Gamma} q^\ell$$

as the leading term of F. □

2.4. Number of orbits

In this section, we prove the quasi-polynomiality of the number of Γ-orbits. First, we describe the permutation character $\chi_{\Gamma(x)}$ on the Γ-orbit $\Gamma(x)$ of $x \in L_q$.

Lemma 2.7. *Let $\Gamma(x)$ denote the Γ-orbit of $x \in L_q$. Then we have*

$$\chi_{\Gamma(x)}(\gamma) = \#\Gamma(x)^\gamma = \left(\mathrm{Ind}_{\Gamma_x}^{\Gamma} \mathbf{1}\right)(\gamma).$$

Proof. An element ηx of $\Gamma(x)$ is fixed by γ if and only if $\eta^{-1}\gamma\eta$ fixes x. Thus, the cardinality of $\Gamma(x)^\gamma$ is

$$\#\Gamma(x)^\gamma = \frac{\#\{\, \eta \in \Gamma \mid \eta^{-1}\gamma\eta \in \Gamma_x \,\}}{\#\Gamma_x}.$$

On the other hand, it follows directly that the above expression is equal to $\left(\mathrm{Ind}_{\Gamma_x}^{\Gamma} \mathbf{1}\right)(\gamma)$:

$$\left(\mathrm{Ind}_{\Gamma_x}^{\Gamma} \mathbf{1}\right)(\gamma) = \frac{1}{\#\Gamma_x} \sum_{\substack{\eta \in \Gamma \\ \eta^{-1}\gamma\eta \in \Gamma_x}} \mathbf{1}(\eta^{-1}\gamma\eta) = \frac{\#\{\, \eta \in \Gamma \mid \eta^{-1}\gamma\eta \in \Gamma_x \,\}}{\#\Gamma_x}. \quad \square$$

For a character λ of a 1-dimensional representation of Γ and $q \in \mathbb{Z}_{>0}$, let $f_{L/\Gamma}(\lambda;\, q)$ denote the number of Γ-orbits on L_q whose isotropy subgroup is contained in the subgroup $\lambda^{-1}(1)$ of Γ. Using the Frobenius reciprocity (5), we obtain the following lemma.

Lemma 2.8. *Let λ be a character of a 1-dimensional representation of Γ. For $q \in \mathbb{Z}_{>0}$, we have*

$$f_{L/\Gamma}(\lambda;\, q) = (\lambda, \chi_{L_q}) = m(\lambda;\, q).$$

Proof. Note that the second equality is the definition of $m(\lambda;\, q)$.

Note that the permutation character χ_{L_q} can be decomposed into a sum of all permutation characters of Γ-orbit on L_q:

$$\chi_{L_q} = \sum_{\Gamma(x)\,:\,\Gamma\text{-orbit}} \chi_{\Gamma(x)}.$$

By Lemma 2.7 and Frobenius reciprocity (5), we have

$$\begin{aligned}(\lambda, \chi_{L_q}) &= \sum_{\Gamma(x)\,:\,\Gamma\text{-orbit}} (\lambda, \chi_{\Gamma(x)}) \\ &= \sum_{\Gamma(x)\,:\,\Gamma\text{-orbit}} \left(\lambda,\, \mathrm{Ind}_{\Gamma_x}^{\Gamma} \mathbf{1}\right) \\ &= \sum_{\Gamma(x)\,:\,\Gamma\text{-orbit}} \left(\mathrm{Res}_{\Gamma_x}^{\Gamma} \lambda,\, \mathbf{1}\right).\end{aligned}$$

Since $\mathrm{Res}_{\Gamma_x}^{\Gamma} \lambda$ is a character of a 1-dimensional representation of Γ_x, the orthogonality of irreducible characters implies that

$$\left(\mathrm{Res}_{\Gamma_x}^{\Gamma} \lambda,\, \mathbf{1}\right) = \begin{cases} 1 & \Gamma_x \subseteq \lambda^{-1}(1); \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have $(\lambda, \chi_{L_q}) = f_{L/\Gamma}(\lambda; q)$. □

Corollary 2.9 *(Restatement of Corollary 1.4). The function* $f_{L/\Gamma}(\lambda; -) : \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}$ *is a quasi-polynomial of degree* ℓ *and it has the* gcd*-property.*

Proof. This follows from Corollary 2.2 and Lemma 2.8. □

2.5. Reciprocity for the multiplicities

Let $\rho : \Gamma \longrightarrow \mathrm{GL}(L)$ be a representation and R_γ the representation matrix of $\rho(\gamma)$. Define the function $\delta_\rho : \Gamma \longrightarrow \mathbb{C}$ by

$$\delta_\rho(\gamma) := (-1)^{r(\gamma)},$$

where $r(\gamma) = \mathrm{rank}(R_\gamma - I_\ell)$. The following lemma shows that $\delta_\rho(\gamma) = \det R_\gamma$ and that δ_ρ is an irreducible character of Γ.

Lemma 2.10 *([10, Lemma 5.5]). Let* $R \in \mathrm{GL}_n(\mathbb{R})$ *be a real matrix of finite order. Let* $r := \mathrm{rank}(R - I_n)$*. Then* $\det R = (-1)^r$*.*

Proof. Since R is finite order, it is diagonalizable (in $\mathbb{C}$), and we can write $R = PDP^{-1}$, where $P, D \in \mathrm{GL}_n(\mathbb{C})$ with D diagonal. Clearly, $\mathrm{rank}(R - I_n) = \mathrm{rank}(D - I_n)$. Thus, r

is the number of diagonal entries of D that are not equal to 1. Since R is a real matrix, the set of eigenvalues is closed under complex conjugation. The finiteness of the order implies that all the eigenvalues have absolute value 1. Therefore, the diagonal entries of D are as follows (with multiplicities):

$$1^{p_1},\ (-1)^{p_2},\ \alpha_1^{q_1},\ \overline{\alpha}_1^{q_1},\ \alpha_2^{q_2},\ \overline{\alpha}_2^{q_2},\ \ldots,\ \alpha_m^{q_m},\ \overline{\alpha}_m^{q_m},$$

with $p_i, q_j \in \mathbb{Z}$ and $|\alpha_j| = 1$. Hence, we have

$$r = p_2 + 2(q_1 + q_2 + \cdots + q_m),$$

and $\det D = (-1)^{p_2}$. Thus, $\det R = (-1)^r$. □

The quasi-polynomials $m(\chi_i \otimes \delta_\rho;\, q)$ and $m(\chi_i;\, q)$ are connected by the following formula.

Theorem 2.11 *(Reciprocity theorem, restatement of Theorem 1.5). The following formula holds for an irreducible character χ_i of Γ:*

$$m(\chi_i \otimes \delta_\rho;\, q) = (-1)^\ell m(\chi_i;\, -q). \tag{10}$$

Proof. Using (8), we have

$$\begin{aligned} m(\chi_i \otimes \delta_\rho;\, q) &= \frac{1}{\#\Gamma} \sum_{\gamma\in\Gamma} (\chi_i \otimes \delta_\rho)(\gamma) \cdot \left(\prod_{j=1}^{r(\gamma)} \gcd\{e_{\gamma,j}, q\} \right) q^{\ell - r(\gamma)} \\ &= \frac{1}{\#\Gamma} \sum_{\gamma\in\Gamma} \chi_i(\gamma)(-1)^{r(\gamma)} \cdot \left(\prod_{j=1}^{r(\gamma)} \gcd\{e_{\gamma,j}, q\} \right) q^{\ell - r(\gamma)} \\ &= (-1)^\ell \cdot \frac{1}{\#\Gamma} \sum_{\gamma\in\Gamma} \chi_i(\gamma) \cdot \left(\prod_{j=1}^{r(\gamma)} \gcd\{e_{\gamma,j}, -q\} \right) (-q)^{\ell - r(\gamma)} \\ &= (-1)^\ell m(\chi_i;\, -q). \quad \square \end{aligned}$$

Note that the map $F(q) = \chi_{L_q}$ defined in Theorem 2.6 can be extended to $F : \mathbb{Z} \longrightarrow R(\Gamma)$ as a quasi-polynomial.

Corollary 2.12 *(Restatement of Corollary 1.6). The quasi-polynomial $F : \mathbb{Z} \longrightarrow R(\Gamma)$ satisfies*

$$F(q) = (-1)^\ell \delta_\rho F(-q).$$

Proof. By Theorem 2.11, it follows that

$$\begin{aligned} F(q) = \chi_{L_q} &= \sum_{i=1}^{k} m(\chi_i;\, q) \cdot \chi_i \\ &= \sum_{i=1}^{k} (-1)^{\ell} m(\chi_i \otimes \delta_\rho;\, -q) \cdot \chi_i \\ &= (-1)^{\ell} \sum_{i=1}^{k} m(\chi_i;\, -q)(\chi_i \otimes \delta_\rho) \\ &= (-1)^{\ell} \delta_\rho F(-q). \quad \square \end{aligned}$$

2.6. Examples

We present some simple examples involving cyclic groups and symmetric groups.

Example 2.13. Let $\Gamma \coloneqq \mathbb{Z}/6\mathbb{Z}$ be a cyclic group of order 6 generated by σ. Let $\chi : \Gamma \longrightarrow \mathbb{C}$ be the function that sends σ to $\zeta_6 \coloneqq e^{\frac{2\pi\sqrt{-1}}{6}}$. Then the irreducible characters of Γ are $\{\chi, \dots, \chi^5, \chi^6 = \mathbf{1}\}$, where $\mathbf{1}$ is the character of the trivial representation of Γ. Consider the action of Γ on $L \coloneqq \mathbb{Z}^2$ given by

$$\sigma \longmapsto R_\sigma \coloneqq \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Note that this setting is same as in [10, Example 6.11].

To compute χ_{L_q}, we need to compute the rank and the elementary divisors of $R_{\sigma^i} - I_\ell$ for each $i \in \{1, \dots, 5\}$. They are as follows:

$$\begin{aligned} r(\sigma^i) = 2 \ \text{ for all } i \in \{1, \dots, 5\}, \qquad (e_{\sigma^1,1}, e_{\sigma^1,2}) = (e_{\sigma^5,1}, e_{\sigma^5,2}) = (1, 1), \\ (e_{\sigma^2,1}, e_{\sigma^2,2}) = (e_{\sigma^4,1}, e_{\sigma^4,2}) = (1, 3), \qquad (e_{\sigma^3,1}, e_{\sigma^3,2}) = (2, 2). \end{aligned}$$

Hence, we obtain the multiplicity $m(\chi^j; q)$ as follows:

$$m(\chi^1;\, q) = m(\chi^5;\, q) = \begin{cases} \frac{1}{6}(q^2 - 1) & \gcd\{6, q\} = 1; \\ \frac{1}{6}(q^2 - 4) & \gcd\{6, q\} = 2; \\ \frac{1}{6}(q^2 - 3) & \gcd\{6, q\} = 3; \\ \frac{1}{6}(q^2 - 6) & \gcd\{6, q\} = 6, \end{cases}$$

$$m(\chi^2;\, q) = m(\chi^4;\, q) = \begin{cases} \frac{1}{6}(q^2-1) & \gcd\{6,q\}=1; \\ \frac{1}{6}(q^2+2) & \gcd\{6,q\}=2; \\ \frac{1}{6}(q^2-3) & \gcd\{6,q\}=3; \\ \frac{1}{6}q^2 & \gcd\{6,q\}=6, \end{cases}$$

$$m(\chi^3;\, q) = \begin{cases} \frac{1}{6}(q^2-1) & \gcd\{6,q\}=1; \\ \frac{1}{6}(q^2-4) & \gcd\{6,q\}=2; \\ \frac{1}{6}(q^2+3) & \gcd\{6,q\}=3; \\ \frac{1}{6}q^2 & \gcd\{6,q\}=6, \end{cases}$$

$$m(\mathbf{1};\, q) = \begin{cases} \frac{1}{6}(q^2+5) & \gcd\{6,q\}=1; \\ \frac{1}{6}(q^2+8) & \gcd\{6,q\}=2; \\ \frac{1}{6}(q^2+9) & \gcd\{6,q\}=3; \\ \frac{1}{6}(q^2+12) & \gcd\{6,q\}=6. \end{cases}$$

In this case, since $\delta_\rho = \mathbf{1}$, it follows that $m(\chi^j;\, q) = m(\chi^j;\, -q)$ for $j \in \{1, \dots, 6\}$.

We also obtain χ_{L_q} as

$$\chi_{L_q} = \begin{cases} \frac{1}{6}\Big(\chi_{\mathrm{R}} q^2 + 6(\mathbf{1}) - \chi_{\mathrm{R}}\Big) & \gcd\{6,q\}=1; \\ \frac{1}{6}\Big(\chi_{\mathrm{R}} q^2 + 12(\mathbf{1}) + 6(\chi^2+\chi^4) - 4\chi_{\mathrm{R}}\Big) & \gcd\{6,q\}=2; \\ \frac{1}{6}\Big(\chi_{\mathrm{R}} q^2 + 12(\mathbf{1}) + 6\chi^3 - 3\chi_{\mathrm{R}}\Big) & \gcd\{6,q\}=3; \\ \frac{1}{6}\Big(\chi_{\mathrm{R}} q^2 + 18(\mathbf{1}) + 6(\chi^2+\chi^3+\chi^4) - 6\chi_{\mathrm{R}}\Big) & \gcd\{6,q\}=6, \end{cases}$$

where $\chi_{\mathrm{R}} = \chi + \cdots + \chi^6$ is the regular character of Γ.

Example 2.14. As in the previous example, we consider the cyclic group $\Gamma = \mathbb{Z}/6\mathbb{Z}$. The action of Γ on $L := \mathbb{Z}^3$ is given by

$$\sigma \longmapsto R_\sigma := \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

By computing in the same way, we obtain the following:

$$r(\sigma^1) = r(\sigma^5) = 3, \qquad r(\sigma^2) = r(\sigma^4) = 2, \qquad r(\sigma^3) = 1,$$
$$(e_{\sigma^1,1}, e_{\sigma^1,2}, e_{\sigma^1,3}) = (e_{\sigma^5,1}, e_{\sigma^5,2}, e_{\sigma^5,3}) = (1,1,6),$$
$$(e_{\sigma^2,1}, e_{\sigma^2,2}) = (e_{\sigma^4,1}, e_{\sigma^4,2}) = (1,3), \qquad e_{\sigma^3,1} = 2,$$

and

$$m(\chi^1; q) = m(\chi^5; q) = \begin{cases} \frac{1}{6}(q^3 - q^2 - q + 1) & \gcd\{6,q\} = 1; \\ \frac{1}{6}(q^3 - 2q^2 - q + 2) & \gcd\{6,q\} = 2; \\ \frac{1}{6}(q^3 - q^2 - 3q + 3) & \gcd\{6,q\} = 3; \\ \frac{1}{6}(q^3 - 2q^2 - 3q + 6) & \gcd\{6,q\} = 6, \end{cases}$$

$$m(\chi^2; q) = m(\chi^4; q) = \begin{cases} \frac{1}{6}(q^3 + q^2 - q - 1) & \gcd\{6,q\} = 1; \\ \frac{1}{6}(q^3 + 2q^2 - q - 2) & \gcd\{6,q\} = 2; \\ \frac{1}{6}(q^3 + q^2 - 3q - 3) & \gcd\{6,q\} = 3; \\ \frac{1}{6}(q^3 + 2q^2 - 3q - 6) & \gcd\{6,q\} = 6, \end{cases}$$

$$m(\chi^3; q) = \begin{cases} \frac{1}{6}(q^3 - q^2 + 2q - 2) & \gcd\{6,q\} = 1; \\ \frac{1}{6}(q^3 - 2q^2 + 2q - 4) & \gcd\{6,q\} = 2; \\ \frac{1}{6}(q^3 - q^2 + 6q - 6) & \gcd\{6,q\} = 3; \\ \frac{1}{6}(q^3 - 2q^2 + 6q - 12) & \gcd\{6,q\} = 6, \end{cases}$$

$$m(\mathbf{1}; q) = \begin{cases} \frac{1}{6}(q^3 + q^2 + 2q + 2) & \gcd\{6,q\} = 1; \\ \frac{1}{6}(q^3 + 2q^2 + 2q + 4) & \gcd\{6,q\} = 2; \\ \frac{1}{6}(q^3 + q^2 + 6q + 6) & \gcd\{6,q\} = 3; \\ \frac{1}{6}(q^3 + 2q^2 + 6q + 12) & \gcd\{6,q\} = 6. \end{cases}$$

In this case, $\delta_\rho = \chi^3$. Then, we have $m(\chi^1; q) = -m(\chi^4; -q)$ and $m(\chi^3; q) = -m(\mathbf{1}; -q)$.

We also obtain χ_{L_q} as follows:

$$\chi_{L_q}=\begin{cases}\frac{1}{6}\Big(\chi_{\mathrm{R}}q^3+\big((\mathbf{1}-\chi^3)-(\chi_{15}-\chi_{24})\big)q^2+\big(2(\mathbf{1}+\chi^3)-(\chi_{15}+\chi_{24})\big)q \\ \qquad -\big(2(\mathbf{1}-\chi^3)+(\chi_{15}-\chi_{24})\big)\Big) & \gcd\{6,q\}=1; \\ \frac{1}{6}\Big(\chi_{\mathrm{R}}q^3+2\big((\mathbf{1}-\chi^3)-(\chi_{15}-\chi_{24})\big)q^2+\big(2(\mathbf{1}+\chi^3)-(\chi_{15}+\chi_{24})\big)q \\ \qquad -2\big(2(\mathbf{1}-\chi^3)+(\chi_{15}-\chi_{24})\big)\Big) & \gcd\{6,q\}=2; \\ \frac{1}{6}\Big(\chi_{\mathrm{R}}q^3+\big((\mathbf{1}-\chi^3)-(\chi_{15}-\chi_{24})\big)q^2+3\big(2(\mathbf{1}+\chi^3)-(\chi_{15}+\chi_{24})\big)q \\ \qquad -3\big(2(\mathbf{1}-\chi^3)+(\chi_{15}-\chi_{24})\big)\Big) & \gcd\{6,q\}=3; \\ \frac{1}{6}\Big(\chi_{\mathrm{R}}q^3+2\big((\mathbf{1}-\chi^3)-(\chi_{15}-\chi_{24})\big)q^2+3\big(2(\mathbf{1}+\chi^3)-(\chi_{15}+\chi_{24})\big)q \\ \qquad -6\big(2(\mathbf{1}-\chi^3)+(\chi_{15}-\chi_{24})\big)\Big) & \gcd\{6,q\}=6,\end{cases}$$

where $\chi_{15} \coloneqq \chi^1 + \chi^5$ and $\chi_{24} \coloneqq \chi^2 + \chi^4$. Since

$$\delta_\rho(\mathbf{1} \pm \chi^3) = (\chi^3 \pm \mathbf{1}), \quad \delta_\rho(\chi_{15} \pm \chi_{24}) = \chi_{24} \pm \chi_{15},$$

one can easily verify that Corollary 2.12 holds.

Example 2.15. Let $\varGamma \coloneqq \mathfrak{S}_3$ be the symmetric group of degree 3, which is also the Weyl group of type A_2. The group $\varGamma$ has three irreducible characters: the trivial character $\mathbf{1}$, the determinant character δ and the character χ of the 2-dimensional irreducible representation. Consider the (co)root lattice $L \coloneqq \mathbb{Z}(e_1 - e_2) \oplus \mathbb{Z}(e_2 - e_3)$. The group $\varGamma$ acts on L as a permutation of $\{ e_1, e_2, e_3 \}$.

Note that we only need to calculate the rank and the elementary divisors for the representative of each conjugacy class. Choose the representatives $\tau \coloneqq (1\ 2)$ and $\sigma \coloneqq (1\ 2\ 3)$. The representation matrices are given by

$$R_\tau = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad R_\sigma = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

Thus, we have

$$r(\tau) = 1, \quad r(\sigma) = 2, \qquad e_{\tau,1} = 1, \quad (e_{\sigma,1}, e_{\sigma,2}) = (1, 3).$$

Therefore, we obtain

$$m(\mathbf{1};\, q) = \begin{cases} \frac{1}{6}(q^2 + 3q + 2) & \gcd\{3, q\} = 1; \\ \frac{1}{6}(q^2 + 3q + 6) & \gcd\{3, q\} = 3, \end{cases}$$

$$m(\delta;\, q) = \begin{cases} \dfrac{1}{6}(q^2 - 3q + 2) & \gcd\{3, q\} = 1; \\ \dfrac{1}{6}(q^2 - 3q + 6) & \gcd\{3, q\} = 3, \end{cases}$$

$$m(\chi;\, q) = \begin{cases} \dfrac{1}{6}(2q^2 - 2) & \gcd\{3, q\} = 1; \\ \dfrac{1}{6}(2q^2 - 6) & \gcd\{3, q\} = 3. \end{cases}$$

In this case, $\delta_\rho = \delta$. Hence, we have $m(\mathbf{1};\, q) = m(\delta;\, -q)$ and $m(\chi;\, q) = m(\chi;\, -q)$.

We also obtain χ_{L_q} as

$$\chi_{L_q} = \begin{cases} \dfrac{1}{6}\Big(\chi_{\mathrm{R}} q^2 + 3(\mathbf{1} - \delta)q + 2(\mathbf{1} + \delta - \chi)\Big) & \gcd\{3, q\} = 1; \\ \dfrac{1}{6}\Big(\chi_{\mathrm{R}} q^2 + 3(\mathbf{1} - \delta)q + 6(\mathbf{1} + \delta - \chi)\Big) & \gcd\{3, q\} = 3, \end{cases}$$

where $\chi_{\mathrm{R}} = \mathbf{1} + \delta + 2\chi$. As Haiman mentions in [3, §7.4], the multiplicity $m(\mathbf{1};\, q)$ is equal to the Ehrhart quasi-polynomial $\mathrm{L}_{\overline{A_\circ}}(q) = \#(q\overline{A_\circ} \cap L)$ of the fundamental alcove $\overline{A_\circ}$ of type A_2.

Note that the first author, in [12, §3], computed χ_{L_q} in the setting where the Weyl group acts on the coroot lattice L for general classical root systems.

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