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A THREE-MANIFOLD INVARIANT VIA THE KONTSEVICH INTEGRAL

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We construct an invariant for closed, oriented three-manifolds from the Kontsevich integral of framed links, and show that it includes Lescop's generalization of the Casson-Walker invariant. Combining this result and a formula for computing the Kontsevich integral in [17], we can compute the Casson-Walker invariant combinatorially in terms of q-tangles (non-associative tangles in [3]).

Our invariant is obtained from the Kontsevich integral by imposing the three-term (3T) relation, orientation independence (OI) relation, 0-vanishing relation and 1-vanishing relation to the space of chord diagrams subjected to the four-term relation. The *3T relation* is given by

(3T relation) 

Here, dotted lines present chords and the three chord diagrams are identical except within the region where they are as above. The *OI relation* is given as follows. Let D be a chord diagram and let D' be a chord diagram obtained by changing the orientation of a string s of D. Then

(OI relation) $$D' = (-1)^{e(s)}\, D.$$

Here $e(s)$ denotes the number of end points of chords on s. The *0-vanishing relation* means that a chord diagram having a string with no end points of chords is equal to 0, and the *1-vanishing relation* means that a chord diagram having a string with only one end point of chords is equal to 0.

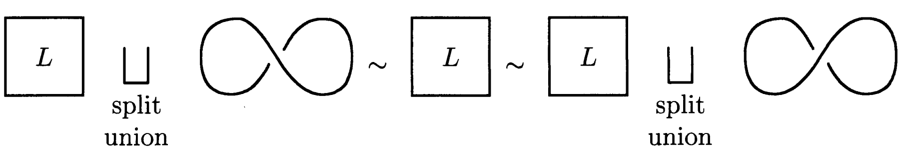
The Kontsevich integral $\hat{Z}_f$ of a framed link has values in the space of chord diagrams subject to the four-term relation [13, 2, 17]. Let $\nu = \hat{Z}_f(\bigcirc)$ for the trivial knot $\bigcirc$, which is equal to the factor introduced in [2, 17] to normalize the effect of maximal and minimal points. For an ℓ-component oriented framed link L, let

$$\check{Z}_f(L) = \hat{Z}_f(L)\#(\nu, \nu, \cdots, \nu).$$

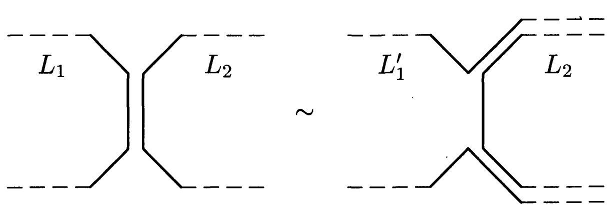
This means that we connect-sum ν to each string of $\hat{Z}_f(L)$. Let $\Lambda'(L)$ be the image of $\check{Z}_f(L)$ by the quotient of the space of chord diagrams by 3T, OI, 0-vanishing and 1-vanishing relations, and then normalize $\Lambda'(L)$ by using the signature of the linking matrix of L, we get an invariant of 3-manifolds $\Lambda(M_L)$ where M_L is the 3-manifold given by the surgery on the framed link L.

We first define $\Lambda(L)$ and prove that it is an invariant of 3-manifolds by showing the invariance of $\Lambda(L)$ under the Kirby moves in Figure 1 [12]. Any oriented 3-manifold is obtained by the surgery on a non-oriented framed link in S^3 [23], and the two oriented 3-manifolds obtained from two links are homeomorphic if and only if one of the links are obtained from the other one by a sequence of Kirby moves. In Figure 1, framings of the links are given by the blackboard framings and the part of L_1' in L' parallel to L_2 is actually parallel on the blackboard.

(KI)



(KII)



$$L = L_1 \bigcup L_2 \bigcup \cdots \bigcup L_k \sim L' = L_1' \bigcup L_2 \bigcup \cdots \bigcup L_k$$

FIGURE 1. Kirby Moves

In the latter half of this paper, we study $\Lambda(L)$ concretely, and show that it consists of the order of the first homology group and Lescop's generalization of the Casson-Walker invariant.

After finishing this work, the theory of the universal perturbative invariant of three-manifolds is developed in [20] with this paper as a starting point.

The main results (contents of Section 1) are announced in [14].

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1. Construction of a 3-manifold invariant

We use notations in [17, 19]. Let C be a chord diagram with a distinguished string s, and let k be the number of end points of chords on s. Let $\Delta(C)$ denote the sum of 2^k diagrams obtained by adding a string parallel to s and changing each point on s as in Figure 2.

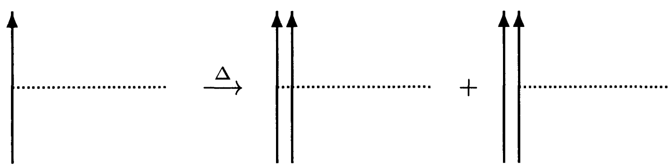


FIGURE 2. Parallel of a chord diagram

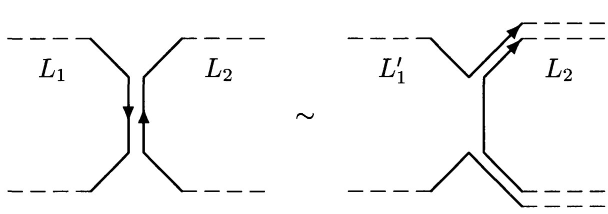
Proposition 1. *Let L and L' be two framed oriented links as in Figure 3, and let*

$$\check{Z}_f(L) = \sum_{X\ :\ \text{chord diagram}} \check{c}_X\, X. \tag{1.1}$$

Then $\check{Z}_f(L')$ is obtained from $\check{Z}_f(L)$ by

$$\check{Z}_f(L') = \sum_{X\ :\ \text{chord diagram}} \check{c}_X\, X', \tag{1.2}$$

where $\check{c}_X$ is the same as in (1,1) and X' is obtained from X as in Figure 4.



$$L = L_1 \bigcup L_2 \bigcup \cdots \bigcup L_k \sim L' = L_1' \bigcup L_2 \bigcup \cdots \bigcup L_k$$

FIGURE 3. KII move for oriented framed links

This is proved in Section 2.

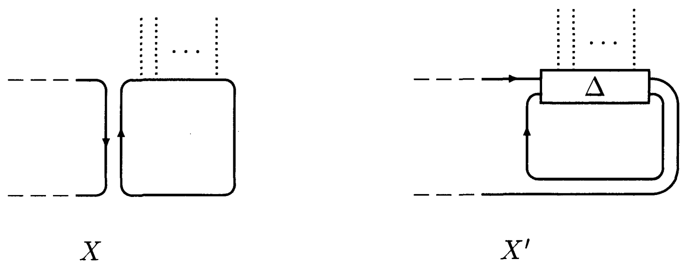


FIGURE 4. Difference of $\check{Z}_f$ by the second Kirby move

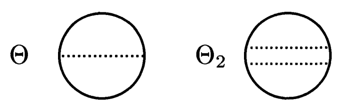


FIGURE 5. Θ and Θ_2

Let $\mathcal{A}^{(\ell)}$ denote the $\mathbf{C}$-linear space spanned by the chord diagrams on a disjoint union of ℓ copies of S^1 subject to the four-term relation. Let

$$\mathcal{A}_1^{(\ell)} = \mathcal{A}^{(\ell)}/(\text{3T, OI, 0-vanishing and 1-vanishing relations}).$$

Let $\Lambda'(L)$ be the image of $\check{Z}_f(L)$ in $\mathcal{A}_1^{(\ell)}$ for an ℓ-component framed oriented link L.

Proposition 2. *$\Lambda'(L)$ is invariant under the KII moves and the orientation change of each component.*

This is proved in Section 3.

The structure of $\mathcal{A}_1^{(\ell)}$ is given as follows.

Proposition 3. *$\mathcal{A}_1^{(\ell)}$ is a two-dimensional $\mathbf{C}$-vector space spanned by the two elements $\underbrace{\Theta \sqcup \Theta \sqcup \cdots \sqcup \Theta}_{\ell}$ and $\Theta_2 \sqcup \underbrace{\Theta \sqcup \cdots \sqcup \Theta}_{\ell-1}$, where Θ_2 denotes the chord diagram on a circle with two chords as in Figure 5.*

This is proved in Section 4.

We normalize $\Lambda'(L)$ for the KI moves. For $\ell_1, \ell_2 \geq 1$, $\mathcal{A}_1^{(\ell_1)}$ and $\mathcal{A}_1^{(\ell_2)}$ are isomorphic by identifying the corresponding basis. Let $\mathcal{A}_1$ be a two dimensional space spanned by e_0 and e_1, and we identify $\mathcal{A}_1^{(\ell)}$ with $\mathcal{A}_1$ by identifying $\Theta \sqcup \Theta \sqcup \cdots \sqcup \Theta$ with e_0 and $\Theta_2 \sqcup \Theta \sqcup \cdots \sqcup \Theta$ with e_1. The image of $\Lambda'(L)$ in $\mathcal{A}_1$ is also denoted by $\Lambda'(L)$. We give an algebra structure to $\mathcal{A}_1$ by

$$e_0\, e_0 = e_0, \qquad e_0\, e_1 = e_1\, e_0 = e_1, \qquad e_1\, e_1 = 0.$$

Then, for a split union $L_1 \sqcup L_2$ of two framed links L_1 and L_2, we have

$$\Lambda'(L_1 \sqcup L_2) = \Lambda'(L_1)\,\Lambda'(L_2) \in \mathcal{A}_1. \tag{1.3}$$

For trivial knots $\infty_{\pm 1}$ with ± 1 framings, we know (from (6.3) in Section 6) that

$$\Lambda'(\infty_{+1}) = \frac{1}{2}\,e_0 + \frac{3}{8}\,e_1, \qquad \Lambda'(\infty_{-1}) = -\frac{1}{2}\,e_0 + \frac{3}{8}\,e_1. \tag{1.4}$$

These elements are invertible in $\mathcal{A}_1$ and their inverses are

$$\Lambda'(\infty_{+1})^{-1} = 2\,e_0 - \frac{3}{2}\,e_1, \qquad \Lambda'(\infty_{-1})^{-1} = -2\,e_0 - \frac{3}{2}\,e_1.$$

So we can modify $\Lambda'(L)$ for the KI moves as in the case of the Jones-Witten invariant. Let $\sigma_+(L)$ (resp. $\sigma_-(L)$) denote the number of positive (resp. negative) eigenvalues of the linking matrix B_L of L, and let

$$\Lambda(L) = 2^{\ell-\sigma_+(L)-\sigma_-(L)}\,\Lambda'(\infty_{+1})^{-\sigma_+(L)}\,\Lambda'(\infty_{-1})^{-\sigma_-(L)}\,\Lambda'(L). \tag{1.5}$$

For a framed link L and the corresponding three-manifold M_L, we have the following.

Theorem 1. *$\Lambda(L)$ is a topological invariant of the three-manifold M_L.*

Let $|H_1(M_L)|$ denote the order of the first homology group of M_L and $b_1(M_L)$ the first Betti number of M_L. Let $\tilde{\lambda}(M_L)$ be Lescop's generalization [22] of the Casson-Walker invariant $\lambda(M_L)$. If $b_1(M_L) = 0$, it satisfies

$$\tilde{\lambda}(M_L) = |H_1(M_L)|\,\lambda(M_L).$$

Let $\Lambda_0(L)$ and $\Lambda_1(L)$ be the coefficients of e_0 and e_1 in $\Lambda(L)$, i.e.

$$\Lambda(L) = \Lambda_0(L)\,e_0 + \Lambda_1(L)\,e_1. \tag{1.6}$$

Then $\Lambda_0(L)$ and $\Lambda_1(L)$ satisfy the following.

Theorem 2.

(1) $$\Lambda_0(L) = \begin{cases} |H_1(M_L)| & \text{if } b_1(M_L) = 0, \\ 0 & \text{if } b_1(M_L) > 0. \end{cases}$$

(2) $$\Lambda_1(L) = -3\,\tilde{\lambda}(M_L).$$

Theorem 1 is a direct consequence of our construction of Λ_1. To prove Theorem 2, we use the fourth author's diagonalizing lemma given in [26, Corollary 2.5] and [25, Lemma 2.2]. According to the diagonalizing lemma, we can reduce the proof to the case of algebraically split links, for which we can prove (1). See Section 5 for detail. To prove (2), adding to the diagonalizing lemma, we use Dehn surgery formula obtained in [10] and [22] which expresses the Casson-Walker invariant in terms of linking numbers and coefficients of the Conway polynomial [22]. For algebraically split links, this formula is rather simple and we can prove similar formula for our invariant. Comparing these formulas, we get (2). For detail, see Sections 6 and 7.

2. Proof of Proposition 1

We prepare several lemmas. Suppose X is a one-dimensional oriented manifold whose components are numbered. A *chord diagram with support* X is a set consisting of a finite number of unordered pairs of distinct non-boundary points on X, regarded up to orientation and component preserving homeomorphisms. We view each pair of points as a chord on X and represent it as a dashed line connecting them. Let $\mathcal{A}(X)$ be the vector space over $\mathbf{C}$ spanned by all chord diagrams with support X, subject to the well-known 4-term relation (see, for example, [2, 17]). The vector space $\mathcal{A}(X)$ is graded by the number of chords, and, abusing notation, we use the same $\mathcal{A}(X)$ for the completion of this vector space with respect to the grading. When X is n numbered lines, $\mathcal{A}(X)$ is denoted by $\mathcal{P}_n$. All the $\mathcal{P}_n$ are algebras: the product of two chord diagrams D_1 and D_2 is obtained by placing D_1 on top of D_2. The algebra P_1 is commutative [2, 13].

We recall the associator $\Phi \in \mathcal{P}_3$ in [16, 17], which is equal to $Z_f(|\diagdown|)$, where $|\diagdown|$ presents the trivial q-tangle on three strings with brackets $((**)*)$ at the top and $(*(**))$ at the bottom. Let $\Phi_{321} = Z_f(|\diagup|)$, where $|\diagup|$ presents the trivial q-tangle on three strings with brackets $(*(**))$ at the top and $((**)*)$ at the bottom. These associators correspond to the associators of quasi-Hopf algebras in [5, 6] and are also studied in [3]. For $\mathbf{p} = (p_1, \cdots, p_g)$, $g(\mathbf{p}) = g$ is the length of $\mathbf{p}$, and $|\mathbf{p}| = p_1 + p_2 + \cdots + p_g$. For $\mathbf{p}$ and $\mathbf{r}$ with the same length g, $\mathbf{p} \geq \mathbf{r}$ means $p_i \geq r_i$, $\mathbf{p} > \mathbf{r}$ means $p_i > r_i$, $\mathbf{p} > \mathbf{0}$ means $p_i > 0$, and $\mathbf{p} \geq \mathbf{0}$ means $p_i \geq 0$ for $1 \leq i \leq g$. Let ζ be Zagier's multiple zeta function defined by

$$\zeta(s_1, \dots, s_k) = \sum_{m_1 < \cdots < m_k \in \mathbf{N}} \frac{1}{m_1^{s_1} \dots m_k^{s_k}}, \tag{2.1}$$

and let

$$\tau(p_1, q_1, \dots, p_n, q_n) = \zeta(\underbrace{1, \dots, 1}_{p_1 - 1}, q_1 + 1, \underbrace{1, \dots, 1}_{p_2 - 1}, q_2 + 1, \dots, q_n + 1). \tag{2.2}$$

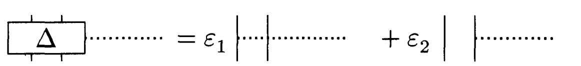
Then

$$\Phi = 1 + \sum_{k=2}^{\infty} \frac{1}{(2\pi\sqrt{-1})^k} \sum_{g\geq 1} \sum_{\substack{\mathbf{p}>\mathbf{0},\ \mathbf{q}>\mathbf{0},\\ |\mathbf{p}|+|\mathbf{q}|=\mathbf{k},\\ g(\mathbf{p})=g(\mathbf{q})=g}} (-1)^{|\mathbf{q}|}\tau(p_1,q_1,\cdots,p_g,q_g)\times \tag{2.3}$$

$$\sum_{\substack{g(\mathbf{r})=g(\mathbf{s})=g,\\ 0\leq\mathbf{r}\leq\mathbf{p},\ 0\leq\mathbf{s}\leq\mathbf{q}}} (-1)^{|\mathbf{r}|} \Big(\prod_{i=1}^{g} \binom{p_i}{r_i}\binom{q_i}{s_i}\Big) B^{|\mathbf{s}|}A^{p_1-r_1}B^{q_1-s_1}\ldots A^{p_g-r_g}B^{q_g-s_g}A^{|\mathbf{r}|},$$

where A (resp. B) denotes the chord connecting the first and second (resp. the second and third) strings. Another associator Φ_{321} is obtained from Φ by substituting B to A and A to B.

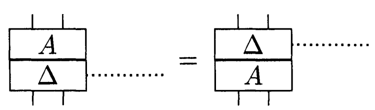
Let



where $\varepsilon_i = 1$ (resp. -1) if the i–th string is oriented downward (resp. upward).

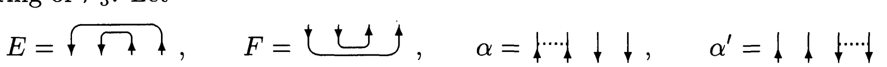
Lemma 1. *For any $A \in \mathcal{P}_2$, we have*

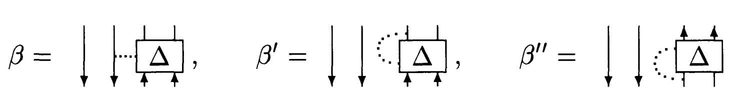
(2.4)



Proof. This is a special case of Lemma 2.1 in [19]. □

Let Δ_3 be a mapping from $\mathcal{P}_3$ to $\mathcal{P}_4$ applying Δ to the third (right most) string of $\mathcal{P}_3$. Let





Lemma 2. $E\,\Delta_3(\Phi) = E, \qquad \Delta_3(\Phi_{321})\,F = F.$

Proof. Note that

$$E\,\alpha = E\,\alpha', \qquad E\,\beta = E\,\beta', \qquad \alpha\,F = \alpha'\,F, \qquad \beta\,F = \beta''\,F.$$

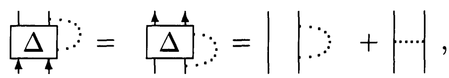
Let X be an element of $\mathcal{P}_3$. By applying Lemma 1 with $A = \alpha'$ or $A = \beta'$ to each end point on the third string of X, we get

$$E\,\alpha\,\Delta_3(X) = E\,\Delta_3(X)\,\alpha', \qquad E\,\beta\,\Delta_3(X) = E\,\Delta_3(X)\,\beta'. \tag{2.5a}$$

Similarly, we get

(2.5b) $$\Delta_3(X)\,\alpha\,F = \alpha'\,\Delta_3(X)\,F, \qquad \Delta_3(X)\,\beta\,F = \beta'\,\Delta_3(X)\,F.$$

Since



we have

(2.6) $$\alpha'\,\beta' = \beta'\,\alpha' = \alpha'\,\beta'' = \beta''\,\alpha'.$$

We use the above expression for Φ and Φ_{321}. In $\Delta_3(\Phi)$ and $\Delta_3(\Phi_{321})$, $\Delta_3(A) = \alpha$ and $\Delta_3(B) = -\beta$. By (2.4), (2.5) and (2.6), we have

$$E\,\beta^{|\mathbf{s}|}\,\alpha^{p_1-r_1}\,\beta^{q_1-s_1}\ldots\alpha^{p_g-r_g}\,\beta^{q_g-s_g}\,\alpha^{|\mathbf{r}|} = E\,\alpha'^{|\mathbf{p}|}\,\beta'^{|\mathbf{q}|},$$
$$\alpha^{|\mathbf{s}|}\,\beta^{p_1-r_1}\,\alpha^{q_1-s_1}\ldots\beta^{p_g-r_g}\,\alpha^{q_g-s_g}\,\beta^{|\mathbf{r}|}\,F = \alpha'^{|\mathbf{q}|}\,\beta''^{|\mathbf{p}|}\,F.$$

Hence

(2.7a) $$E\,\Delta_3(\Phi) = E + \sum_{k=2}^{\infty}\frac{1}{(2\pi\sqrt{-1})^k}\sum_{g\geq 1}\sum_{\substack{\mathbf{p}>\mathbf{0},\ \mathbf{q}>\mathbf{0},\\ |\mathbf{p}|+|\mathbf{q}|=\mathbf{k},\\ g(\mathbf{p})=g(\mathbf{q})=g}}\tau(p_1,q_1,\cdots,p_g,q_g)\times$$
$$\sum_{\substack{g(\mathbf{r})=g(\mathbf{s})=g,\\ \mathbf{0}\leq\mathbf{r}\leq\mathbf{p},\ \mathbf{0}\leq\mathbf{s}\leq\mathbf{q}}}(-1)^{|\mathbf{q}|}\,(-1)^{|\mathbf{r}|}\,(\prod_{i=1}^{g}\binom{p_i}{r_i}\binom{q_i}{s_i})\,E\,\alpha'^{|\mathbf{p}|}\,\beta'^{|\mathbf{q}|},$$

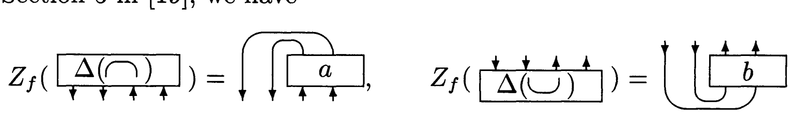
(2.7b) $$\Delta_3(\Phi)\,F = F + \sum_{k=2}^{\infty}\frac{1}{(2\pi\sqrt{-1})^k}\sum_{g\geq 1}\sum_{\substack{\mathbf{p}>\mathbf{0},\ \mathbf{q}>\mathbf{0},\\ |\mathbf{p}|+|\mathbf{q}|=\mathbf{k},\\ g(\mathbf{p})=g(\mathbf{q})=g}}\tau(p_1,q_1,\cdots,p_g,q_g)\times$$
$$\sum_{\substack{g(\mathbf{r})=g(\mathbf{s})=g,\\ \mathbf{0}\leq\mathbf{r}\leq\mathbf{p},\ \mathbf{0}\leq\mathbf{s}\leq\mathbf{q}}}(-1)^{|\mathbf{p}|}\,(-1)^{|\mathbf{r}|}\,(\prod_{i=1}^{g}\binom{p_i}{r_i}\binom{q_i}{s_i})\,\alpha'^{|\mathbf{q}|}\,\beta'^{|\mathbf{p}|}\,F.$$

However, we know that

$$\sum_{s_i=0}^{q_i}(-1)^{s_i}\binom{q_i}{s_i} = 0 \quad \text{if } q_i > 0, \qquad \sum_{r_i=0}^{p_i}(-1)^{r_i}\binom{p_i}{r_i} = 0 \quad \text{if } p_i > 0.$$

Therefore, all the terms except the first one of the right hand sides of (2.7a) and (2.7b) vanish. Hence we get $E\,\Delta_3(\Phi) = E$ and $\Delta_3(\Phi_{321})\,F = F$. $\square$

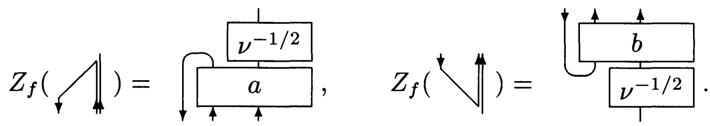
Let ν be an element in $\mathcal{P}_2$ such that $\nu^{-1} = Z_f($  $)$. By the remark at the end of Section 5 in [19], we have



for some a and b in $\mathcal{P}_2$ such that

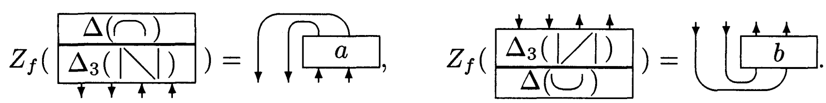
$$a\,b = \Delta(\nu)\,(\nu^{-1} \otimes \nu^{-1}). \tag{2.8}$$

With these a and b, we have the following.

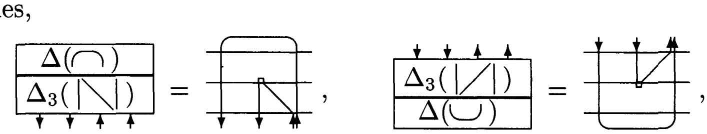
Lemma 3. 

Proof. By (2.7a) and (2.7b), we have

(2.9)



As q-tangles,



where the brackets aregiven as $(*(*(**)))$. On the other hand, $\hat{Z}_f$ is obtained from Z_f by adding $\nu^{1/2}$ at each maximal and minimal point. So, by multiplying $\nu^{-1/2}$ at the outer strings of (2.9), we get Lemma 3. $\square$

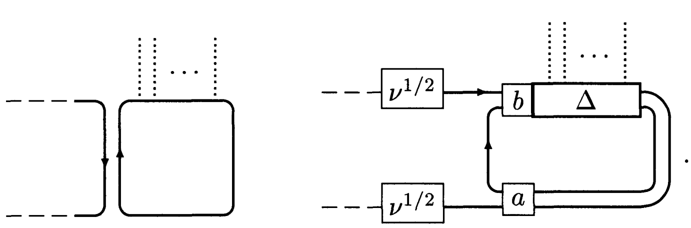
Proof of Proposition 1. Let

$$\hat{Z}_f(L) = \sum_{X\,:\ \text{chord diagram}} \hat{c}_X\, X. \tag{2.10}$$

Then, from Lemma 3, $\hat{Z}_f(L')$ is given by

$$\hat{Z}_f(L') = \sum_{X\,:\ \text{chord diagram}} \hat{c}_X\, X', \tag{2.11}$$

where $\hat{c}_X$ is given in (2.10) and X' is obtained from X as in Figure 6. Hence, $\check{Z}_f(L)$



X X'

FIGURE 6. Difference of $\hat{Z}_f$ by the second Kirby move

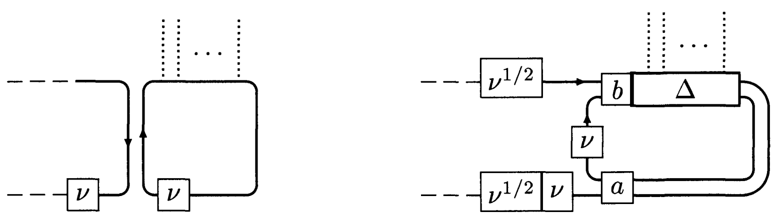
is given by

$$\check{Z}_f(L) = \sum_{X\ :\ \text{chord diagram}} \hat{c}_X\, X\#(\nu, \cdots, \nu). \tag{2.12}$$

and $\check{Z}_f(L')$ is given

$$\check{Z}_f(L') = \sum_{X\ :\ \text{chord diagram}} \hat{c}_X\, X'\#(\nu, \cdots, \nu). \tag{2.13}$$

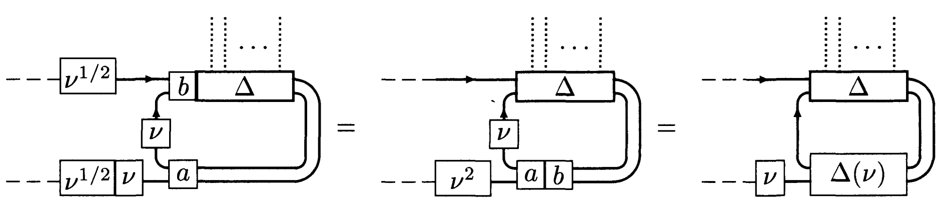
The difference of $X\#(\nu, \cdots, \nu)$ and $X'\#(\nu, \cdots, \nu)$ are given as in Figure 7. Ap-



$X\#(\nu, \cdots, \nu)$ $\qquad$ $X'\#(\nu, \cdots, \nu)$

FIGURE 7. Difference of $\check{Z}_f$ by the second Kirby move

plying Lemma 1 and (2.8) to $X'\#(\nu, \cdots, \nu)$ in Figure 7, we have

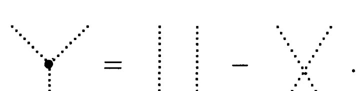


Comparing the above last term and $X\#(\nu, \cdots, \nu)$ in Figure 7, we get Proposition 1. □

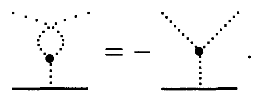
3. Proof of Proposition 2

In this and the next sections, we extend the notion of chord diagrams by introducing trivalent vertices of dashed lines (chords) satisfying the following STU relation as in [2].

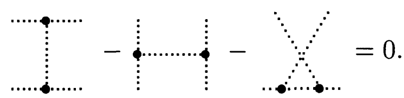
(STU relation)



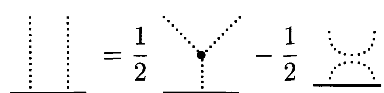
From this definition, the order of the edges around a trivalent vertex has meaning and we have

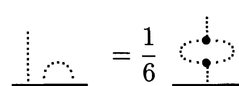
(anti-symmetry relation) 

According to the four-term relations, an element with a trivalent vertex of chords is well-defined in the space spanned by the chord diagrams subjected to the four-term relations. From the STU relation, we have the following relation, which is called the *IHX relation* in [2].

(IHX relation) 

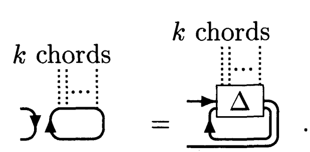
Moreover, from STU and 3T relations, we get following two relations.

(two legs reduction) 

(three legs reduction) 

Proof of Proposition 2. According to the OI relation and the construction of the Kontsevich integral, $\Lambda'(L)$ does not depend on the orientation of components of the link L. Hence Λ' is an invariant of non-oriented framed links.

Next, we show that Λ' is invariant by the KII move of non-oriented framed links. Since Λ' does not depend on the orientation, we may give orientations to the links L and L' in Figure 1 as in Figure 3. So, we prove that



For $k = 0, 1$, this formula is satisfied by the 0 and 1 vanishing formulas. For $k = 2$, the right hand side of the formula is a sum of four diagrams. However, three of these four diagrams vanish by the 0 and 1 vanishing formulas, and the remaining diagram is the same as the diagram of the left hand side of the formua. Therefore, the formula is true for $k = 2$ case. The $k > 2$ cases are reduced to the above cases by using the two and three legs reductions. So the above formula is true for all k. □

4. Proof of Proposition 3

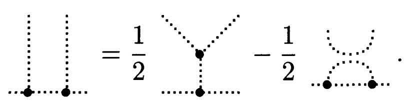
Let $\mathcal{A}^{(0)}$ be the $\mathbf{C}$-vector space spanned by trivalent graphs subjected to the IHX relation and the anti-symmetry relation. Then we first show the following lemma.

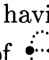
Lemma 5. *The space $\mathcal{A}_1^{(\ell)}$ is spanned by*

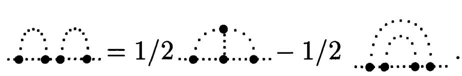
$$\{D \sqcup (\overset{\ell}{\sqcup}\Theta) \mid D : \text{connected chord diagram in } \mathcal{A}^{(0)} \text{ with less than three vertices}\}.$$

Proof. By using the 3T relation, the two and three legs reductions, 0-vanishing and 1-vanishing fomulas, and the anti-symmetry relation, we can reduce any chord diagram to a linear combination of diagrams of the form $D \sqcup (\overset{\ell}{\sqcup}\Theta)$ with $D \in \mathcal{A}^{(0)}$. Then, by using the 3T relation again, we reduce D to a linear combination of connected diagrams.

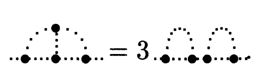
Assume that D has $2d$ vertices with $d > 1$ and we will show that D vanishes in $\mathcal{A}_1^{(\ell)}$. From the IHX and 3T relations, we have

(4.1) 

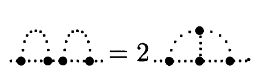
Focus on a cyclic path of D with n edges ($n \geq 3$). Replace an edge of the path by (4.1), D is equal to a linear combination of a diagram having a cyclic path with $n-1$ edges and a diagram which is a connected sum of  and a diagram with $2d-2$ vertices. Applying this process repeatedly, D is shown to be equal to a linear combination of diagrams having a part  Using (4.1) again, we have

(4.2) 

Since  =  by the IHX relation, (4.2) implies

(4.3) 

On the other hand, the IXH relation implies

(4.4) 

Hence, (4.3) and (4.4) implies that 5 $= 0$, and so  $= 0$. Hence $D = 0$. $\square$

Proof of Proposition 3. For a chord diagram $D \in \mathcal{A}^{(\ell)}$, let $\deg(D)$ be

$$\deg(D) = (\text{number of vertices and end points of } D)/2 - \ell,$$

and we call it the *degree of* D. All defining relations of $\mathcal{A}_1^\ell$ are homogeneous with respect to this degree. Lemma 2 shows that the degree 0 and 1 parts of $\mathcal{A}_1^\ell$ are both at most 1-dimensional and other parts are all 0-dimensional. Therefore, it is enough to show that the elements $\overset{\ell}{\sqcup}\Theta$ and $\Theta_2 \sqcup \left(\overset{\ell}{\sqcup}\Theta\right)$ do not vanish. To do this, we list up all the non-zero chord diagrams of degree 0 and 1 in $\mathcal{A}_1^{(\ell)}$ and show that they are reduced uniquely to scalar multiples of $\overset{\ell}{\sqcup}\Theta$ and $\Theta_2 \sqcup \left(\overset{\ell-1}{\sqcup}\Theta\right)$ by the IHX, 3T, orientation independence relations and 0 vanishing formula. We write down exactly all such non-zero diagrams as scalar multiples of $\overset{\ell}{\sqcup}\Theta$ and $\Theta_2 \sqcup \left(\overset{\ell-1}{\sqcup}\Theta\right)$, and check that all the relations are compatible with these elements.

Let D be a non-zero diagram in $\mathcal{A}_1^\ell$ of degree 0 without vertices of chords. Then, because of the 0 and 1 vanishing formulas, each string (real line) of D has just two end points of chords. Hence D is a disjoint union of chord diagrams with several components connected by chords as a chain as in Figure 8. For these diagrams, we have

$$D^{(k)} = 1/(-2)^{k-1} \left(\overset{k}{\sqcup}\Theta\right). \tag{4.2}$$

Hence,

$$D = (-2)^{-(\ell-j)} \left(\overset{\ell}{\sqcup}\Theta\right),$$

where j is the number of connected components of D.

Let D be a non-zero diagram in $\mathcal{A}_1^\ell$ of degree 1 without vertices of chords. Then the total number of end points of chords is equal to $\ell + 2$ and the number of end points of chords on each string is equal to 2, 3 or 4. Hence, D is connected, or a disjoint union of one degree 1 component and other degree 0 components. Non-zero connected diagrams of degree 1 are $D_=^{(k_1,k_2)}$, $D_\mathrm{x}^{(k_1,k_2)}$ and $D_{\pm 1}^{(k_1,k_2,k_3)}$ in Figure 8. For these diagrams, we have

$$\begin{aligned} D_=^{(k_1,k_2)} &= 1/(-2)^{k_1+k_2}\, \Theta_2 \sqcup \left(\overset{k_1+k_2}{\sqcup}\Theta\right), \\ D_\mathrm{x}^{(k_1,k_2)} &= 1/(-2)^{k_1+k_2-1}\, \Theta_2 \sqcup \left(\overset{k_1+k_2}{\sqcup}\Theta\right), \\ D_{\pm 1}^{(k_1,k_2,k_3)} &= \pm 3\, (-2)^{-(k_1+k_2+k_3+2)}\, \Theta_2 \sqcup \left(\overset{k_1+k_2+k_3+1}{\sqcup}\Theta\right). \end{aligned} \tag{4.3}$$

Therfore, combining (4.2) and (4.3), we get expressions for all the non-vanishing diagrams.

Now we can check all the relation for the above non-vanishing elements. Computation is rather elementally and we omit the detail. $\square$

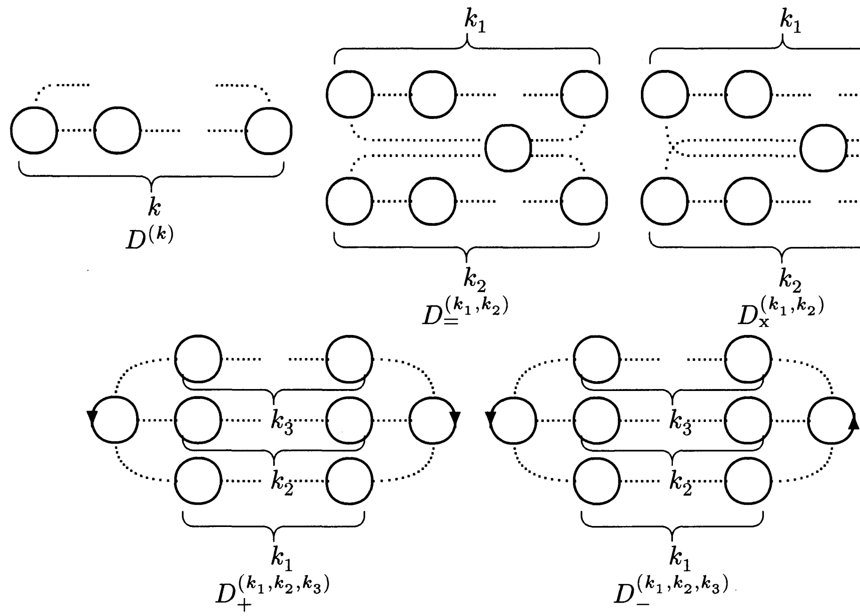


FIGURE 8. Non-vanishing diagrams

5. Order of the first homology group

In this section, we prove the first part of Theorem 2. We first recall the fourth author's diagonalizing Lemma, by which we can reduce our work to simple cases. For a framed link L, let B_L denote the linking matrix of L. A framed link L is called an *algebraically split link* if B_L is a diagonal matrix.

Lemma 6 (diagonalizing Lemma, [26, Corollary 2.5] and [25, Lemma 2.2]). *Let L be a framed link. There is an algebraically split link L' with a non-degenerate linking matrix such that $L \sqcup L'$ is equivalent to an algebraically split link by the Kirby moves.*

Proof of Theorem 2 (1). Since L_0 is additive for disjoint union of links, it will be enough to prove for algebraically split links. Let L be an algebraically split framed link. To prove (1), we compute the integral for L corresponding to the configuration of disjoint union of $D^{(k)}$. We give a coordinate in $\mathbf{R}^3$ by $(z,t) \in \mathbf{C}\times\mathbf{R} \cong \mathbf{R}^3$, and call the *t-coordinate level.* Let L be an ℓ-component Morse link in $\mathbf{C}\times\mathbf{R}$ such that L is in the plane $\mathbf{R}\times\mathbf{R}$ except a small neighborhood of the crossing points of L. Then L has a framing given by the normal vectors orthogonal to the plane $\mathbf{R}\times\mathbf{R}$, and we regard L as a framed link with this framing. Note that every framed link can be given by this way.

A set of k chords (dashed line segments) with end points on L, parallel to $\mathbf{C} \times \{0\}$ on different levels each other is called a *horizontal configuration with k chords.* Two horizontal configurations P_1 and P_2 are considered as the same one if we can get P_2 from P_1 by sliding the chords of P_1 keeping them parallel to the plane $\mathbf{C} \times \{0\}$ and keeping their order with respect to the level. Then the Kontsevich integral $Z_f(L)$ in [17] of L is written as follows.

$$Z_f(L) = \sum_{k=0}^{\infty} \sum_{\substack{P:\\ \text{horizontal configuration}\\ \text{with } k \text{ chords}}} c_P(L)\, D_P, \tag{5.1}$$

where D_P denote the chord diagram describing the configuration of the end points of chords of P, and the coefficient c_P is given by the modified iterated integral corresponding to P as in [17]. Let

$$\Lambda'(L) = \Lambda'_0(L)\, e_0 + \Lambda'_1(L)\, e_1. \tag{5.2}$$

Then $\Lambda'_0(L)$ is defined from $\check{Z}_f(L)$, which is given by adding several ν's to $Z_f(L)$. Since $\Lambda'_0(L)$ corresponds to the degree 0 part of $\check{Z}_f(L)$, and, by the argument for the degree 0 part in the previous section, only chord diagrams with just two end points of chords on each component contribute to $\Lambda'_0(L)$. Since, the non-trivial terms of ν have more than three end points, ν does not contribute to $\Lambda'_0(L)$. More precisely, $\Lambda'_0(L)$ is the image of

$$\sum_{\substack{P\,:\, \text{horizontal configuration with}\\ \text{two end points on each component of } L}} c_P(L)\, D_P,$$

which is equal to

$$\sum_{\substack{\text{D:}\\ \text{chord diagram with two end points}\\ \text{on each component}}} \left(\sum_{P\,:\, D_P = D} c_P(L) \right) D.$$

We first assume that L is a knot. In this case, we need the coefficient of $D^{(1)}(= \Theta)$, which is equal to $\sum_{P\,:\, D_P = D^{(1)}} c_P(L)$. From the description of Z_f by using a tangle decomposition in [17, 18], we know that the coefficient of Θ in $Z_f(L)$ is a half of the writhe $w(L)$ of L. Hence, we have

$$\Lambda'_0(L) = \frac{w(L)}{2}. \tag{5.3}$$

Next, we assume that L is a two component algebraically split link with components L_1 and L_2. In this case, we need the coefficients of $D^{(2)}$ and $D^{(1)} \sqcup D^{(1)}$, which are given by the following integrals: The coefficient of $D^{(2)}$ is

$$\frac{1}{(2\pi\sqrt{-1})^2}\int_{t_1<t_2}\sum_{\substack{z_1\in L_1\cap\{t=t_1\},z_1'\in L_2\cap\{t=t_1\}\\ z_2\in L_1\cap\{t=t_2\},z_2'\in L_2\cap\{t=t_2\}}}\varepsilon_1\,\varepsilon_1'\,\varepsilon_2\,\varepsilon_2'\,d\log(z_1-z_1')\wedge d\log(z_2-z_2'),$$

where ε_i, (resp. ε_i') is 1 if the link L is oriented upward at the point (t_i, z_i) (resp. (t_i, z_i')), and is equal to -1 if L is oriented downward at this point. This integral is equal to

$$\frac{1}{2}\left(\frac{1}{2\pi\sqrt{-1}}\int_{t_1}\sum_{\substack{z_1\in L_1\cap\{t=t_1\}\\ z_1'\in L_2\cap\{t=t_1\}}}\varepsilon_1\,\varepsilon_1'\,d\log(z_1-z_1')\right)$$
$$\times\left(\frac{1}{2\pi\sqrt{-1}}\int_{t_2}\sum_{\substack{z_2\in L_1\cap\{t=t_2\}\\ z_2'\in L_2\cap\{t=t_2\}}}\varepsilon_2\,\varepsilon_2'\,d\log(z_2-z_2')\right).$$

As in [13], we know that

$$\frac{1}{2\pi\sqrt{-1}}\int_{t_1}\sum_{z\in L_1\cap\{t=t_1\},z'\in L_2\cap\{t=t_1\}}\varepsilon_1\,\varepsilon_1'\,d\log(z-z')=\mathrm{lk}(L_1,L_2), \tag{5.4}$$

and so the coefficient is equal to

$$\frac{1}{2}\,\mathrm{lk}(L_1,L_2)^2=0,$$

since L is an algebraically split link. As in the case of knot, the coefficient of $D^{(1)}\sqcup D^{(1)}$ is equal to

$$\frac{w(L_1)\,w(L_2)}{4}.$$

Hence we have

$$\Lambda_0'(L)=\frac{w(L_1)\,w(L_2)}{4}.$$

Now consider the case for ℓ-component algebraically split link L with components L_1, L_2, $\cdots$, L_ℓ for $\ell\geq 3$. We first compute the coefficient of $D^{(\ell)}$ whose components corresponding to L_1, $\cdots$, L_ℓ are connected by chords in that order. It is given by the following integral:

$$\frac{1}{(2\pi\sqrt{-1})^\ell}\int_{t_1,t_2,\cdots,t_\ell}\sum_{\substack{z_1,\in L_1\cap\{t=t_1\},z_1',\in L_2\cap\{t=t_1\}\\ z_2,\in L_2\cap\{t=t_2\},z_2'\in L_3\cap\{t=t_2\}\\ \cdots\\ z_\ell,\in L_\ell\cap\{t=t_\ell\},z_\ell',\in L_1\cap\{t=t_\ell\}}}\bigwedge_{k=1}^{\ell}d\log(z_k-z_k').$$

Let $L_{\ell+1}$ denote L_1, then the above integral is equal to

$$\prod_{k=1}^{\ell} \frac{1}{2\pi\sqrt{-1}} \int_{t_k} \sum_{\substack{z\in L_k\cap\{t=t_k\} \\ z'\in L_{k+1}\cap\{t=t_k\}}} d\log(z-z').$$

Hence, the coefficient is equal to

$$\prod_{k=1}^{\ell} \operatorname{lk}(L_k, L_{k+1}) = 0,$$

since L is an algebraically split link. Next we compute the coefficient of a disjoint union of several $D^{(k)}$'s. In this case, the result is a product of coefficients corresponding to every $D^{(k)}$ given above, and so only the coefficient of $\sqcup^{\ell} D^{(1)}$ does not vanish. Hence we get

$$\Lambda_0'(L) = \prod_{i=1}^{\ell} \frac{w(L_i)}{2}.$$

Therefore, from (1.1) and (1.2), we have

$$\begin{aligned}\Lambda_0(L) = 2^{\ell-\sigma_+(L)-\sigma_-(L)} \left(\frac{1}{2}\right)^{-\sigma_+(L)} \left(-\frac{1}{2}\right)^{-\sigma_-(L)} \prod_{i=1}^{\ell} \frac{w(L_i)}{2} &= |\prod_{i=1}^{\ell} w(L_i)| \\ &= |\det B_L|,\end{aligned}$$

since $\sigma_+(L)$ and $\sigma_-(L)$ are equal to the number of components of L with positive writhes and negative writhes respectively. This implies Theorem 2 (1) because $|\det B_L|$ is equal to the order of the first homology group of M_L if $b_1(M_L) = 0$, and is equal to 0 if $b_1(M_L) > 0$. $\square$

6. Coincidence with the Casson-Walker invariant

To show the equivalence of Λ_1 and $\tilde{\lambda}$, it will be enough to show for knots and algebraically split links since we can apply Lemma 6 (diagonalization Lemma) in Section 4. For knots and algebraically split links, we derive skein relations of Λ_1, and compare them with the Lescop's formula for the Casson-Walker invariant.

Let L_1 and L_2 be two framed links. Assume that the linking matrix of L_2 is non-degenerate. Let $L_1 \sqcup L_2$ denote the split union of these two links. Then we have

$$\Lambda_1(L_1 \sqcup L_2) = \Lambda_0(L_2)\,\Lambda_1(L_1) + \Lambda_0(L_1)\,\Lambda_1(L_2). \tag{6.1}$$

On the other hand, [22] shows that, for a connected sum $M_{L_1}\#M_{L_2}$, $\tilde{\lambda}$ satisfies
(6.2)

$$\tilde{\lambda}(M_{L_1}\#M_{L_2}) = \begin{cases} |H_1(M_{L_2})|\,\tilde{\lambda}(M_{L_1}) + |H_1(M_{L_1})|\tilde{\lambda}(M_{L_2}) & \text{if } b_1(M_{L_1}) = 0, \\ |H_1(M_{L_2})|\,\tilde{\lambda}(M_{L_1}) & \text{if } b_1(M_{L_1}) > 0. \end{cases}$$

We already know that $\Lambda_0(L) = |H_1(M_L)|$ if $b_1(M_L) = 0$, and $\Lambda_0(L) = 0$ if $b_1(M_L) > 0$. Therefore, if Theorem 2 (2) is true for certain link L_2 with non-degenerate linking matrix, and for the split union $L_1 \sqcup L_2$, then it is also true for L_1. Hence, it is good enough to prove Theorem 2 (2) only for algebraically split links.

Next, we introduce skein relations for Λ_1.

Lemma 7. *Let L_+ and L_- be two framed knots which are identical except in a small ball B where $L_+ \cap B$ is a positive crossing and $L_- \cap B$ is a negative crossing. Let $\tilde{L}_-$ be the knot obtained from L_- by adding a positive full twist as in Figure 9. Let $K^{(1)}$ and $K^{(2)}$ be the components of the two component link obtained from L_+ by the smoothing at the crossing in the ball B. Then $\Lambda_1(L_+)$ and $\Lambda_1(\tilde{L}_-)$ satisfy the following.*

(skein I) $$\Lambda_1(L_+) - \Lambda_1(\tilde{L}_-) = -6\,\mathrm{sign}(w(L_+))\,\mathrm{lk}(K^{(1)}, K^{(2)}).$$

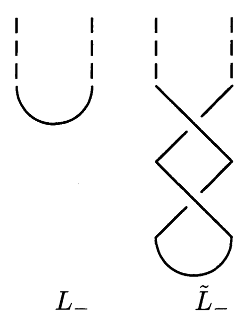


FIGURE 9. Adding full twist to L_-.

Lemma 8. *Let $L_{+-} = L_{+-}^{(1)} \cup L_{+-}^{(2)} \cup L^{(3)} \cup \cdots L^{(\ell)}$ be an ℓ-component algebraically split link with a positive crossing (resp. a negative crossing) consisting of the strings $L_{+-}^{(1)}$ and $L_{+-}^{(2)}$ in a small ball B_1 (resp. B_2). Let $L_{-+} = L_{-+}^{(1)} \cup L_{-+}^{(2)} \cup L^{(3)} \cup \cdots L^{(\ell)}$ be a link obtained from L_{+-} by crossing changes in B_1 and B_2. Let L_1 be the link obtained from L_{+-} by smoothing at the crossing in the ball B_1, and* *let L_2 be the link obtained from L_{-+} by smoothing at the crossing in the ball B_2. Then $\Lambda_1(L_{+-})$ and $\Lambda_1(L_{-+})$ satisfy the following.*

(skein II)

$$\Lambda_1(L_{+-}) - \Lambda_1(L_{-+}) = -\mathrm{sign}(w(L^{(1)}_{+-})\, w(L^{(2)}_{+-})\, w(L_1))\ (\Lambda_1(L_1) - \Lambda_1(L_2)).$$

Proofs of Lemmas 7 and 8 are given in the next section.

For an ℓ-component algebraically split link $L = L_1 \cup L_2 \cup \cdots \cup L_\ell$, a Dehn surgery formula for the Casson invariant is obtained in [10, 21, 22], which is given by

(Dehn surgery formula)

$$\tilde{\lambda}(M_L) = -\sum_{i=1}^{\ell} \mathrm{sign}(w(L_i)) \left(\prod_{j\neq i} |w(L_j)|\right) \frac{(|w(L_i)|-1)\,(|w(L_i)|-2)}{12} +$$
$$\sum_{I\subset\{1,2,\cdots,\ell\}} 2 \left(\prod_{i\in I} \mathrm{sign}(w(L_i))\right) \left(\prod_{i\notin I} |w(L_i)|\right) a_{|I|+1}(\cup_{i\in I} L_i).$$

Here $|I|$ denotes the number of elements in I, and $a_k(L)$ is the coefficient of t^k in the Conway polynomial $\nabla_L(t)$, which is defined by the following skein relation:

$$\nabla_{L_+}(t) - \nabla_{L_-}(t) = -t\,\nabla_{L_0}(t),$$

where L_+, L_-, L_0 are links identical except within a ball at which they are a positive crossing, negative crossing and their smoothing as usual. Note that there is a minus at the right hand side of the relation. Recall that, if M_L is a rational homology sphere,

$$\tilde{\lambda}(M_L) = |H_1(M_L)|\,\lambda(M_L) = \prod_{i=1}^{\ell} |w(L_i)|\,\lambda(M_L).$$

Before proving Theorem 2 (2), we show some properties of $\tilde{\lambda}$ for three simple cases.

Lemma 9. *Lescop's generalization of the Casson-Walker invariant $\tilde{\lambda}$ satisfies the following.*

(i) *For any trivial framed knot L, $-3\,\tilde{\lambda}(M_L) = \Lambda_1(L)$.*

(ii) *For any knots L_+, $\tilde{L}_-$, $K^{(1)}$ and $K^{(2)}$ as in Lemma 7,*

$$\tilde{\lambda}(M_{L_+}) - \tilde{\lambda}(M_{\tilde{L}_-}) = 2\,\mathrm{sign}(w(L_+))\,\mathrm{lk}(K^{(1)}, K^{(2)}).$$

(iii) *For any links* $L_{+-} = L^{(1)}_{+-} \cup L^{(2)}_{+-} \cup L^{(3)} \cup \cdots L^{(\ell)}$, $L_{-+} = L^{(1)}_{-+} \cup L^{(2)}_{-+} \cup L^{(3)} \cup \cdots L^{(\ell)}$, L_1 *and* L_2 *as in Lemma 8,*

$$\tilde{\lambda}_1(M_{L_{+-}}) - \tilde{\lambda}_1(M_{L_{-+}}) = -\mathrm{sign}(w(L^{(1)}_{+-})\, w(L^{(2)}_{+-})\, w(L_1)) \left(\tilde{\lambda}_1(M_{L_1}) - \tilde{\lambda}_1(M_{L_2})\right).$$

This lemma is proved after proving Theorem 2 (2).

Proof of Theorem 2 (2). For an ℓ-component algebraically split link L with non-degenerate linking matrix, we will prove that $-3\,\tilde{\lambda}(M_L) = \Lambda_1(L)$. The computation of Λ_1 is reduced to those for split links by the relation (skein II). By using (6.1) and (6.2), the proof for a split link is reduced to the proof for each component. Moreover, by using (skein I), Λ_1 of a knot is reduced to Λ_1 of trivial knots with framings. Therefore, Lemmas 7, 8 and 9 show that Λ_1 and $-3\,\tilde{\lambda}$ satisfy the same recursive relations and the same initial conditions. Hence they are identical. □

Proof of Lemma 9. We first show (i). For a trivial framed knot L given by vertical twist as in Figure 10,

$$Z_f(L) = \bigcirc + \frac{w(L)}{2}\,\Theta + \frac{w(L)^2}{8}\,\Theta_2 + \text{(terms with more than two chords)},$$

and

$$\nu = \bigcirc + \frac{1}{24}\,(\Theta_2 - \oplus\,) + \text{ (terms with more than two chords)}.$$

Hence we have

$$\begin{aligned}\check{Z}_f(L) &= Z_f(L)\#\nu\#\nu \\ &= \bigcirc + \frac{w(L)}{2}\,\Theta + \frac{(3\,w(L)^2+2)\,\Theta_2 - 2\,\oplus}{24} \\ &\qquad + \text{(terms with more than two chords)}.\end{aligned}$$

In $\mathcal{A}_1$, we have

$$\bigcirc = 0, \qquad \oplus = -2\,\Theta_2,$$

by the 0-vanishing formula and the 3T relation. Therfore, in $\mathcal{A}_1$, we have

$$\Lambda'(L) = \frac{w(L)}{2}\,e_0 + \frac{w(L)^2+2}{8}\,e_1, \tag{6.3}$$

and so

$$\begin{aligned}\Lambda(L) &= \left(\frac{w(L)}{2}\,e_0 + \frac{w(L)^2+2}{8}\,e_1\right)\left(\mathrm{sign}(w(L))\,\frac{1}{2}\,e_0 + \frac{3}{8}\,e_1\right)^{-1} \\ &= \left(\frac{w(L)}{2}\,e_0 + \frac{w(L)^2+2}{8}\,e_1\right)\left(2\,\mathrm{sign}(w(L))\,e_0 - \frac{3}{2}\,e_1\right) \\ &= |w(L)|\,e_0 + \mathrm{sign}(w(L))\,\frac{|w(L)|^2 - 3\,|w(L)| + 2}{4}\,e_1.\end{aligned}$$

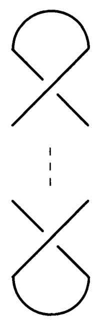


FIGURE 10. trivial framed knot L with vertical twist

Hence

$$\Lambda_1(L) = \operatorname{sign}(w(L)) \frac{|w(L)|^2 - 3\,|w(L)| + 2}{4}.$$

On the other hand, by the Dehn surgery formula of $\tilde{\lambda}$ described before, we have

$$\tilde{\lambda}(M_L) = -\operatorname{sign}(w(L)) \frac{|w(L)|^2 - 3\,|w(L)| + 2}{12}.$$

Hence (i) is true.

Now we show (ii). Since $w(L_+) = w(\tilde{L}_-)$,

$$\tilde{\lambda}(M_{L_+}) - \tilde{\lambda}(M_{\tilde{L}_-}) = 2 \operatorname{sign}(w(L_+)) \left(a_2(L_+) - a_2(\tilde{L}_-)\right).$$

From (4.7) in [7] and Theorem 2 in [8], we have

$$a_2(L_+) - a_2(\tilde{L}_-) = a_2(L_+) - a_2(L_-) = -a_1(K^{(1)} \cup K^{(2)}) = \operatorname{lk}(K^{(1)}, K^{(2)}).$$

These two formulas imply (ii).

It remains to prove (iii). We use the notations in (skein II). Then, we have

$$\text{(6.4)} \quad \tilde{\lambda}(M_{L_{+-}}) - \tilde{\lambda}(M_{L_{-+}}) = 2 \sum_{J \subset \{3,4,\cdots,\ell\}} \left(\prod_{i \in \{3,4,\cdots,\ell\} \setminus J} |w(L^{(i)})| \right) \times$$
$$\left(a_{|J|+3}(L_{+-}^{(1)} \cup L_{+-}^{(2)} \cup (\cup_{j\in J} L^{(j)})) - a_{|J|+3}(L_{-+}^{(1)} \cup L_{--}^{(2)} \cup (\cup_{j\in J} L^{(j)}))\right),$$

since the other terms of $\tilde{\lambda}(M_{L_{+-}})$ are identical to the correspoinding terms of $\tilde{\lambda}(M_{L_{-+}})$. Similarly, we have

$$\text{(6.5)} \quad \tilde{\lambda}(M_{L_1}) - \tilde{\lambda}(M_{L_2}) = 2 \sum_{J \subset \{3,4,\cdots,\ell\}} \left(\prod_{i \in \{3,4,\cdots,\ell\} \setminus J} |w(L^{(i)})| \right) \times$$
$$\left(a_{|J|+2}(L_1^{(1)} \cup (\cup_{j\in J} L^{(j)})) - a_{|J|+2}(L_2^{(1)} \cup (\cup_{j\in J} L^{(j)}))\right),$$

where $L_1^{(1)}$ and $L_2^{(1)}$ are components of L_1 and L_2 such that $L_1 = L_1^{(1)} \cup L^{(3)} \cup \cdots \cup L^{(\ell)}$ and $L_2 = L_2^{(1)} \cup L^{(3)} \cup \cdots \cup L^{(\ell)}$. From the definition of ∇, we have

(6.6)

$$a_{|J|+3}(L_{+-}^{(1)} \cup L_{+-}^{(2)} \cup (\underset{j\in J}{\cup} L^{(j)})) - a_{|J|+3}(L_{--}^{(1)} \cup L_{--}^{(2)} \cup (\underset{j\in J}{\cup} L^{(j)}))$$
$$= -a_{|J|+2}(L_1^{(1)} \cup (\underset{j\in J}{\cup} L^{(j)})),$$

(6.7)

$$a_{|J|+3}(L_{-+}^{(1)} \cup L_{-+}^{(2)} \cup (\underset{j\in J}{\cup} L^{(j)})) - a_{|J|+3}(L_{--}^{(1)} \cup L_{--}^{(2)} \cup (\underset{j\in J}{\cup} L^{(j)}))$$
$$= -a_{|J|+2}(L_2^{(1)} \cup (\underset{j\in J}{\cup} L^{(j)})).$$

We also know that

(6.8)

$$w(L_1^{(1)}) = w(L_2^{(1)}), \qquad w(L_{+-}^{(1)}) = w(L_{-+}^{(1)}), \qquad w(L_{+-}^{(2)}) = w(L_{-+}^{(2)}).$$

Hence, normalizing by the signatures of the writhes of $L_1^{(1)}$, $L_1^{(2)}$, $L_{+-}^{(1)}$, $L_{-+}^{(1)}$, $L_{+-}^{(2)}$ and $L_{-+}^{(2)}$, we get (iii) from (6.4) $\sim$ (6.8). □

7. Skein relation of Λ_1

We prove the skein relations Lemmas 7 and 8. These two lemmas are obtained from the following skein relations for Λ_1' by using the relation between Λ and Λ'.

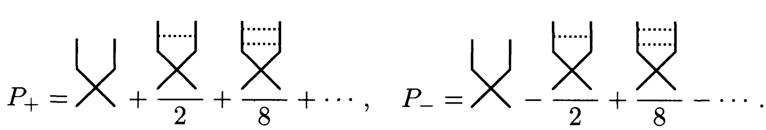
Lemma 10. *Let L_+, $\tilde{L}_-$, $K^{(1)}$ and $K^{(2)}$ be as in Lemma 7. Then*

(skein I') $$\Lambda_1'(L_+) - \Lambda_1'(L_-) = -3\mathrm{lk}(K^{(1)}, K^{(2)}).$$

Lemma 11. *Let L_{+-}, $\tilde{L}_{-+}$, L_1 and L_2 be as in Lemma 8. Then*

(skein II') $$\Lambda_1'(L_{+-}) - \Lambda_1'(L_{-+}) = -\frac{1}{2}\left(\Lambda_1'(L_1) - \Lambda_1'(L_2)\right).$$

Proof of Lemma 10. We separate $\Lambda_1(L_+)$ and $\Lambda_1(\tilde{L}_-)$ into the q-tangles corresponding to the crossing points in B and the added full twist, and the contribution from the other parts. Let $P_\pm$ be the invariant $\hat{Z}_f$ from the crossing, T be that from the full twist, and Q be that from the other part. Then, P_+, P_- are given in [17] and



The contribution of T is given by the connected sum of $\bigcirc + \Theta + \Theta_2 + \cdots$ to the integral of L_-. For a sum X of chord diagrams, let $X^{(k)}$ denote the part of X consisting of the terms with diagrams having k chords. Since $\check{Z}_f(K) = \hat{Z}_f(K)\#\nu$ for a knot K, we have

$$\begin{aligned}\Lambda_1'(L_+) &= P_+^{(0)}\,(Q\#\nu)^{(2)} + P_+^{(1)}\,(Q\#\nu)^{(1)} + P_+^{(2)}\,(Q\#\nu)^{(0)}\\ &= P_+^{(0)}\,(Q\#\nu)^{(2)} + P_+^{(1)}\,Q^{(1)} + P_+^{(2)}\,Q^{(0)},\end{aligned}$$

$$\begin{aligned}\Lambda_1'(L_-) &= P_-^{(0)}\,(Q\#\nu)^{(2)} + P_-^{(1)}\,(Q\#\nu)^{(1)} + P_-^{(2)}\,(Q\#\nu)^{(0)}\\ &= P_-^{(0)}\,(Q\#\nu)^{(2)} + P_-^{(1)}\,Q^{(1)} + P_-^{(2)}\,Q^{(0)},\end{aligned}$$

$$\begin{aligned}\Lambda_1'(\tilde{L}_-) &= P_-^{(0)}\,(Q\#\nu)^{(2)}\,T^{(0)} + P_-^{(1)}\,Q^{(1)}\,T^{(0)} + P_-^{(2)}\,Q^{(0)}\,T^{(0)}\\ &\qquad + P_-^{(0)}\,Q^{(1)}\,T^{(1)} + P_-^{(1)}\,Q^{(0)}\,T^{(1)} + P_-^{(0)}\,Q^{(0)}\,T^{(2)}\\ &= P_-^{(0)}\,(Q\#\nu)^{(2)} + P_-^{(1)}\,Q^{(1)} + P_-^{(2)}\,Q^{(0)}\\ &\qquad + (P_-^{(0)}\,Q^{(1)})\#\Theta + (P_-^{(1)}\,Q^{(0)})\#\Theta + (P_-^{(0)}\,Q^{(0)})\#\Theta_2/2.\end{aligned}$$

Therefore, we have

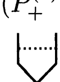
$$\begin{aligned}\Lambda_1'(L_+) - \Lambda_1'(\tilde{L}_-) &= (P_+^{(0)} - P_-^{(0)})\,(Q\#\nu)^{(2)} + (P_+^{(1)} - P_-^{(1)})\,Q^{(1)} - (P_-^{(0)}\,Q^{(1)})\#\Theta\\ &\qquad + (P_+^{(2)} - P_-^{(2)})\,Q^{(0)} - (P_-^{(1)}\,Q^{(0)})\#\Theta - (P_-^{(0)}\,Q^{(0)})\#\Theta_2/2.\end{aligned}$$

Since

$$\begin{aligned}(P_-^{(1)}\,Q^{(0)})\#\Theta &= -\,\Theta_2/2,\\ (P_-^{(0)}\,Q^{(0)})\#\Theta_2 &= \Theta_2/2,\\ (P_+^{(0)} - P_-^{(0)}) &= (P_+^{(2)} - P_-^{(2)}) = 0,\end{aligned}$$

we have

$$\Lambda_1'(L_+) - \Lambda_1'(\tilde{L}_-) = (P_+^{(1)} - P_-^{(1)})\,Q^{(1)} - (P_-^{(0)}\,Q^{(1)})\#\Theta$$

$=$  $\cup\, Q^{(1)} - ($  $\cup\, Q^{(1)})\#\Theta.$

We compute $Q^{(1)}$ exactly. There are three kinds of chord diagrams in $Q^{(1)}$ given in Figure 11. In this figure, $s^{(1)}$ and $s^{(2)}$ denote the two components of $Q^{(1)}$ corresponding to $K^{(1)}$ and $K^{(2)}$ respectively. Let α_1, α_2, α_3 be the coefficient of the invariant $\hat{Z}_f$ of the configurations to $Q_1^{(1)}$, $Q_2^{(1)}$, $Q_3^{(1)}$. Then, from (5.3) and (5.4), we have

$$\alpha_1 = w(K^{(1)})/2, \qquad \alpha_2 = \mathrm{lk}(K^{(1)}, K^{(2)}), \qquad \alpha_3 = w(K^{(2)})/2.$$

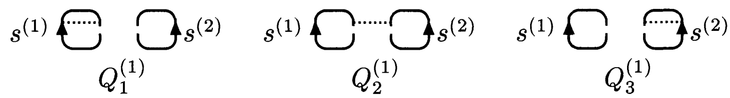
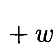


FIGURE 11. Chord diagrams in $Q^{(1)}$, where $s^{(i)}$ denote the string corresponding to $K^{(i)}$.

Hence $(\Lambda_1'(L_+) - \Lambda_1'(\tilde{L}_-))\,g_1$ is equal to the image in $\mathcal{A}_1$ of

$$\frac{w(K^{(1)}) + w(K^{(2)})}{2}\,\Theta_2 + \mathrm{lk}(K^{(1)}, K^{(2)})$$  $-$

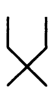
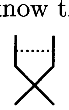
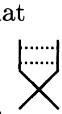
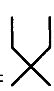
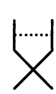
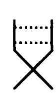
$$\frac{w(K^{(1)}) + 2\,\mathrm{lk}(K^{(1)}, K^{(2)}) + w(K^{(2)})}{2}\,\Theta_2 = -3\mathrm{lk}(K^{(1)}, K^{(2)})\,e_1.$$

Here we use the relation  $= -2\Theta_2$. So we get Lemma 10. $\square$

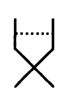
Proof of Lemma 11.

Let $L_{+-} = L_{+-}^{(1)} \cup L_{+-}^{(2)} \cup L^{(3)} \cup \cdots \cup L^{(\ell)}$, $L_{-+} = L_{-+}^{(1)} \cup L_{-+}^{(2)} \cup L^{(3)} \cup \cdots \cup L^{(\ell)}$, L_1, L_2, B_1 and B_2 are as in Lemma 8 and let $L_{--} = L_{--}^{(1)} \cup L_{--}^{(2)} \cup L^{(3)} \cup \cdots \cup L^{(\ell)}$ be a link obtained from L_{+-} by a crossing change in B_1. Note that L_{-+} is an algebraically split link since so is L_{+-}.

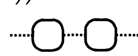
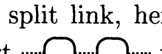
We first compute $\Lambda_1'(L_{+-}) - \Lambda_1'(L_{--})$. We express $\Lambda_1'(L_{+-}) - \Lambda_1'(L_{--})$ in terms of $\Lambda_1'(L_1)$. Let P_+, P_-, P_1 denote $\hat{Z}_f$ of the q-tangles corresponding to the parts of L_+, L_-, L_1 in B_1, and Q be $\hat{Z}_f$ form the other part. Note that Q is the same one for L_+, L_-, L_1 because L_+, L_- and L_1 are identical in the complement of B_1. We know that

$P_+ =$  $+$  $/\,2 +$  $/\,8 + \cdots$, $P_- =$  $-$  $/\,2 +$  $/\,8 - \cdots$, $P_1 = \mid\ \mid.$

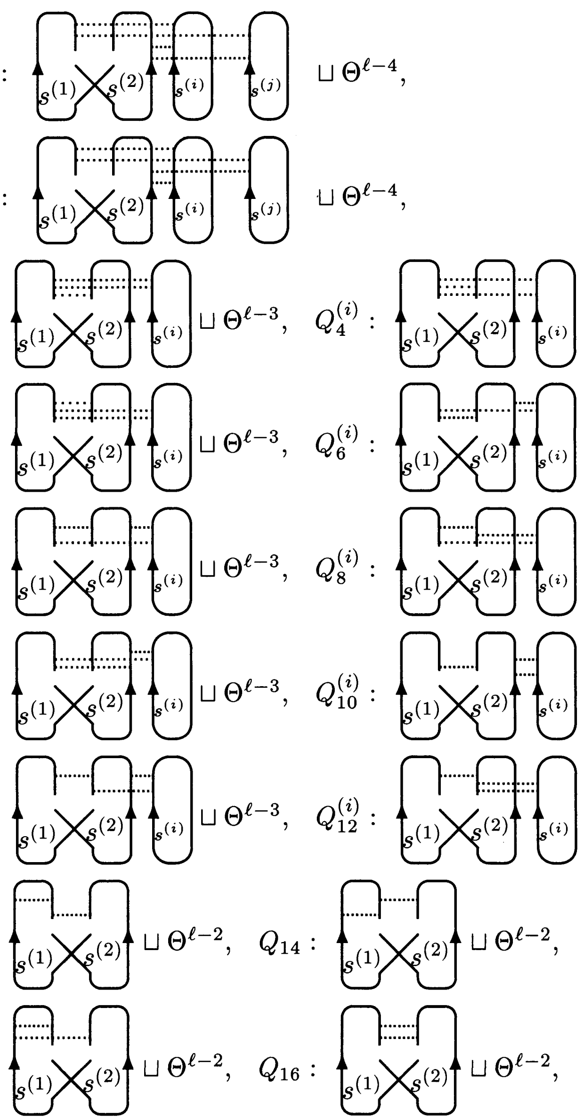
Let $X^{(k)}$ denote the part consisting of terms with diagrams having k chords as before. Since, for any framed link L, $\check{Z}_f(L)$ is obtained from $\hat{Z}_f(L)$ by adding ν to each component by the connect sum, we have

$\Lambda_1'(L_{+-}) - \Lambda_1'(L_{--}) =$  $\cup\,(Q\#(\nu, \cdots, \nu))^{(\ell)}$,

$$\Lambda_1'(L_1) = \mid\ \ \mid \cup\,(Q\#(\nu, \cdots, \nu))^{(\ell)}.$$

The integral corresponding to a middle chord of the part  is given by the linking number of two components of the link corresponding to the component containing the end points of the middle chord. We assumed that L_{+-} is an algebraically split link, hence the integral corresponding to a chord diagram containing a part  vanishes. By using this, we list up in Figure 12. all the chord diagrams of $(Q\#(\nu,\cdots,\nu))^{(\ell)}$ which do not vanish. In this figure, $s^{(1)}$, $s^{(2)}$, $s^{(i)}$ denote the compoents of $(Q\#(\nu,\cdots,\nu))^{(\ell)}$ corresponding to $L^{(1)}_{\pm-}$, $L^{(2)}_{\pm-}$ and $L^{(i)}$ respectively. Let $Q_1^{(i,j)}$, $\cdots$, $Q_{33}(D_1^{(\ell-2)})$ be the diagrams in Figure 12. In this figure, $\Theta^k = \sqcup^k\Theta$.

Case that the number of end points of chords on $s^{(1)}$ and $s^{(2)}$ are equal to 4,



$Q_1^{(i,j)}$: $\sqcup\, \Theta^{\ell-4}$,

$Q_2^{(i,j)}$: $\sqcup\, \Theta^{\ell-4}$,

$Q_3^{(i)}$: $\sqcup\, \Theta^{\ell-3}$, $Q_4^{(i)}$: $\sqcup\, \Theta^{\ell-3}$,

$Q_5^{(i)}$: $\sqcup\, \Theta^{\ell-3}$, $Q_6^{(i)}$: $\sqcup\, \Theta^{\ell-3}$,

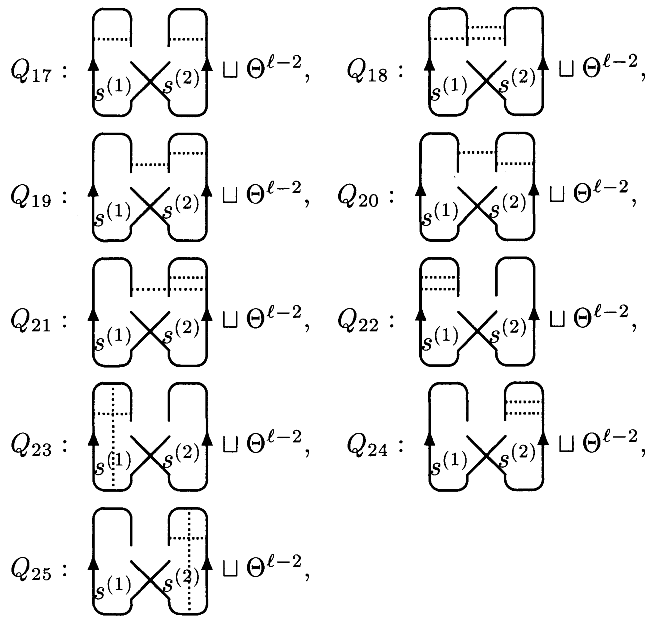
$Q_7^{(i)}$: $\sqcup\, \Theta^{\ell-3}$, $Q_8^{(i)}$: $\sqcup\, \Theta^{\ell-3}$,

$Q_9^{(i)}$: $\sqcup\, \Theta^{\ell-3}$, $Q_{10}^{(i)}$: $\sqcup\, \Theta^{\ell-3}$,

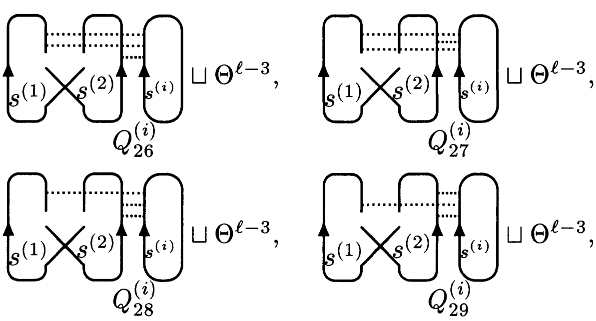
$Q_{11}^{(i)}$: $\sqcup\, \Theta^{\ell-3}$, $Q_{12}^{(i)}$: $\sqcup\, \Theta^{\ell-3}$,

Q_{13} : $\sqcup\, \Theta^{\ell-2}$, Q_{14} : $\sqcup\, \Theta^{\ell-2}$,

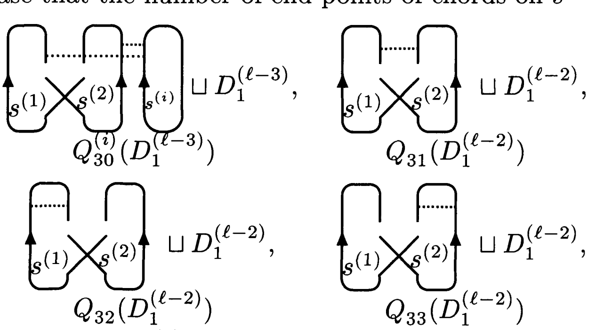
Q_{15} : $\sqcup\, \Theta^{\ell-2}$, Q_{16} : $\sqcup\, \Theta^{\ell-2}$,



Case that the number of end points of chords on $s^{(1)}$ and $s^{(2)}$ are equal to 3,

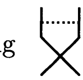


Case that the number of end points of chords on $s^{(1)}$ and $s^{(2)}$ are equal to 2,



where $D_1^{(k)}$ is a non-vanishing chord diagram of degree 1 with k components.

FIGURE 12. Non-vanishing chord diagrams in Q^ℓ for an algebraically split link

Let $E_1^{(i,j)}, \cdots, E_{33}(D_1^{(\ell-2)})$ be diagrams obtained by inserting  to $Q_1^{(i,j)}$, $\cdots, Q_{33}(D_1^{(\ell-2)})$, and let $F_1^{(i,j)}, \cdots, F_{33}(D_1^{(\ell-2)})$ be diagrams obtained by inserting $|\ |$ to $Q_1^{(i,j)}, \cdots, Q_{33}(D_1^{(\ell-2)})$. By using relations in $\mathcal{A}_1^{(\ell)}$, $E_1^{(i,j)}, \cdots, F_{33}(D_1^{(\ell-2)})$ are reduced as in Table 1. In the table, $\Theta_2^{(k)} = \left(\sqcup^{k-1}\Theta\right) \sqcup \Theta_2$ and $\Theta^k = \sqcup^k\Theta$.

$$
\begin{array}{llllllll}
E_1^{(i,j)} = & -\frac{3}{16}\Theta_2^{(\ell)}, & E_2^{(i,j)} = & \frac{3}{16}\Theta_2^{(\ell)}, & E_3^{(i)} = & \frac{1}{4}\Theta_2^{(\ell)}, & E_4^{(i)} = & -\frac{1}{2}\Theta_2^{(\ell)}, \\
E_5^{(i)} = & \frac{1}{4}\Theta_2^{(\ell)}, & E_6^{(i)} = & -\frac{3}{8}\Theta_2^{(\ell)}, & E_7^{(i)} = & \frac{3}{8}\Theta_2^{(\ell)}, & E_8^{(i)} = & -\frac{3}{8}\Theta_2^{(\ell)}, \\
E_9^{(i)} = & \frac{3}{8}\Theta_2^{(\ell)}, & E_{10}^{(i)} = & \frac{1}{4}\Theta_2^{(\ell)}, & E_{11}^{(i)} = & -\frac{1}{2}\Theta_2^{(\ell)}, & E_{12}^{(i)} = & \frac{1}{4}\Theta_2^{(\ell)}, \\
E_{13} = & -\frac{1}{2}\Theta_2^{(\ell)}, & E_{14} = & \Theta_2^{(\ell)}, & E_{15} = & -\frac{1}{2}\Theta_2^{(\ell)}, & E_{16} = & \frac{3}{4}\Theta_2^{(\ell)}, \\
E_{17} = & 0\ , & E_{18} = & -\frac{3}{4}\Theta_2^{(\ell)}, & E_{19} = & -\frac{1}{2}\Theta_2^{(\ell)}, & E_{20} = & \Theta_2^{(\ell)}, \\
E_{21} = & -\frac{1}{2}\Theta_2^{(\ell)}, & E_{22} = & 0\ , & E_{23} = & 0\ , & E_{24} = & 0\ , \\
E_{25} = & 0\ , & E_{26}^{(i)} = & \frac{3}{8}\Theta_2^{(\ell)}, & E_{27}^{(i)} = & -\frac{3}{8}\Theta_2^{(\ell)}, & E_{28}^{(i)} = & -\frac{3}{8}\Theta_2^{(\ell)},
\end{array}
$$

$$E_{29}^{(i)} = \frac{3}{8}\Theta_2^{(\ell)}, \qquad E_{30}^{(i)}(D_1^{(\ell-3)}) = \frac{1}{4}D_1^{(\ell-3)} \sqcup \Theta^3, \quad E_{31}(D_1^{(\ell-2)}) = -\frac{1}{2}D_1^{(\ell-2)} \sqcup \Theta^2,$$

$$E_{32}(D_1^{(\ell-2)}) = 0, \quad E_{33}(D_1^{(\ell-2)}) = 0,$$

$$
\begin{array}{llllllll}
F_1^{(i,j)} = & \frac{1}{4}\Theta_2^{(\ell-1)}, & F_2^{(i,j)} = & -\frac{1}{2}\Theta_2^{(\ell-1)}, & F_3^{(i)} = & -\frac{1}{2}\Theta_2^{(\ell-1)}, & F_4^{(i)} = & \Theta_2^{(\ell-1)}, \\
F_5^{(i)} = & -\frac{1}{2}\Theta_2^{(\ell-1)}, & F_6^{(i)} = & \Theta_2^{(\ell-1)}, & F_7^{(i)} = & -\frac{1}{2}\Theta_2^{(\ell-1)}, & F_8^{(i)} = & \Theta_2^{(\ell-1)}, \\
F_9^{(i)} = & -\frac{1}{2}\Theta_2^{(\ell-1)}, & F_{10}^{(i)} = & -\frac{1}{2}\Theta_2^{(\ell-1)}, & F_{11}^{(i)} = & \Theta_2^{(\ell-1)}, & F_{12}^{(i)} = & -\frac{1}{2}\Theta_2^{(\ell-1)}, \\
F_{13} = & \Theta_2^{(\ell-1)}, & F_{14} = & -2\,\Theta_2^{(\ell-1)}, & F_{15} = & \Theta_2^{(\ell-1)}, & F_{16} = & -2\,\Theta_2^{(\ell-1)}, \\
F_{17} = & \Theta_2^{(\ell-1)}, & F_{18} = & \Theta_2^{(\ell-1)}, & F_{19} = & \Theta_2^{(\ell-1)}, & F_{20} = & -2\,\Theta_2^{(\ell-1)}, \\
F_{21} = & \Theta_2^{(\ell-1)}, & F_{22} = & \Theta_2^{(\ell-1)}, & F_{23} = & -2\,\Theta_2^{(\ell-1)}, & F_{24} = & \Theta_2^{(\ell-1)}, \\
F_{25} = & -2\,\Theta_2^{(\ell-1)}, & F_{26}^{(i)} = & -\frac{3}{4}\Theta_2^{(\ell-1)}, & F_{27}^{(i)} = & \frac{3}{4}\Theta_2^{(\ell-1)}, & F_{28}^{(i)} = & \frac{3}{4}\Theta_2^{(\ell-1)},
\end{array}
$$

$$F_{29}^{(i)} = -\frac{3}{4}\Theta_2^{(\ell-1)}, \quad F_{30}^{(i)}(D_1^{(\ell-3)}) = -\frac{1}{2}D_1^{(\ell-3)} \sqcup \Theta^2, \quad F_{31}(D_1^{(\ell-2)}) = D_1^{(\ell-2)} \sqcup \Theta,$$

$$F_{32}(D_1^{(\ell-2)}) = D_1^{(\ell-2)} \sqcup \Theta, \quad F_{33}(D_1^{(\ell-2)}) = D_1^{(\ell-2)} \sqcup \Theta.$$

Table 1

Let $\alpha_k^{(i,j)}$ (resp. $\alpha_k^{(i)}$, α_k, $\alpha_k^{(i)}(D_1^{(\ell-3)})$, $\alpha_k(D_1^{(\ell-2)})$) denote the coefficient of $\check{Z}_f$ of the configuration $Q_k^{(i,j)}$ (resp. $Q_k^{(i)}$, Q_k, $Q_k^{(i)}(D_1^{(\ell-3)})$, $Q_k(D_1^{(\ell-2)})$), then we have

$$\begin{aligned}
&\Lambda_1'(L_{+-}) - \Lambda_1'(L_{--}) + \frac{1}{2}\Lambda_1'(L_1) = \\
&\quad \sum_{3\le i<j\le \ell}\left(-\frac{\alpha_1^{(i,j)}}{16} - \frac{\alpha_2^{(i,j)}}{16}\right) + \sum_{3\le i\le \ell}\left(\frac{\alpha_6^{(i)}}{8} + \frac{\alpha_7^{(i)}}{8} + \frac{\alpha_8^{(i)}}{8} + \frac{\alpha_9^{(i)}}{8}\right) \\
&\qquad - \frac{\alpha_{16}}{4} + \frac{\alpha_{17}}{2} - \frac{\alpha_{18}}{4} + \frac{\alpha_{22}}{2} - \alpha_{23} + \frac{\alpha_{24}}{2} - \alpha_{25} \\
&\qquad + \sum_{D_1^{(\ell-2)}}\left(\frac{\alpha_{32}(D_1^{(\ell-2)})}{2} + \frac{\alpha_{33}(D_1^{(\ell-2)})}{2}\right).
\end{aligned}$$

We know that

$$\alpha_1^{(i,j)} + \alpha_2^{(i,j)} = \mathrm{lk}(L_{+-}^{(1)}, L^{(i)})\,\mathrm{lk}(L_{+-}^{(1)}, L^{(j)})\,\mathrm{lk}(L_{+-}^{(2)}, L^{(i)})\,\mathrm{lk}(L_{+-}^{(2)}, L^{(j)}),$$

and so it is equal to 0 because L_{+-} is an algebraically split link. Similarly, we have

$$\begin{aligned}
&\alpha_6^{(i)} + \alpha_7^{(i)} + \alpha_8^{(i)} + \alpha_9^{(i)} \\
&\quad = \left(\mathrm{lk}(L_{+-}^{(1)}, L_{+-}^{(2)}) - \frac{1}{2}\right)\mathrm{lk}(L_{+-}^{(1)}, L^{(i)})\,\mathrm{lk}(L_{+-}^{(2)}, L^{(i)})\,\mathrm{lk}(L^{(i)}, L^{(j)}) = 0,
\end{aligned}$$

and

$$\alpha_{16} + \alpha_{18} = \left(\mathrm{lk}(L_{+-}^{(1)}, L_{+-}^{(2)}) - \frac{1}{2}\right)^2 = 0.$$

We also know that

$$\alpha_{17} = \left(\prod_{j=3}^{\ell}\frac{w(L^{(j)})}{2}\right)\frac{w(L_{+-}^{(1)})\,w(L_{+-}^{(2)})}{4}.$$

Moreover,

$$\alpha_{22} - 2\,\alpha_{23} = \Lambda_1'(L_{+-}^{(1)})\left(\prod_{j=3}^{\ell}\frac{w(L^{(j)})}{2}\right), \qquad \alpha_{24} - 2\,\alpha_{25} = \Lambda_1'(L_{+-}^{(2)})\left(\prod_{j=3}^{\ell}\frac{w(L^{(j)})}{2}\right),$$

and

$$\sum_{D_1^{(\ell-2)}}\alpha_{32}(D_1^{(\ell-2)}) = \Lambda_1'(L')\,\frac{w(L_{+-}^{(1)})}{2}, \qquad \sum_{D_1^{(\ell-2)}}\alpha_{33}(D_1^{(\ell-2)}) = \Lambda_1'(L')\,\frac{w(L_{+-}^{(2)})}{2},$$

where $L' = L^{(3)} \cup L^{(4)} \cup \cdots \cup L^{(\ell)}$. Hence, we have

$$\begin{aligned}
&\Lambda_1'(L_{+-}) - \Lambda_1'(L_{--}) + \frac{1}{2}\Lambda_1'(L_1) \\
&\quad = \frac{w(L_{+-}^{(1)})\, w(L_{+-}^{(2)}) + 2\,\Lambda_1'(L_{+-}^{(1)}) + 2\,\Lambda_1'(L_{+-}^{(2)})}{8} \left(\prod_{j=3}^{\ell} \frac{w(L^{(j)})}{2} \right) \\
&\qquad + \frac{\Lambda_1'(L')}{4} \left(w(L_{+-}^{(1)}) + w(L_{+-}^{(2)}) \right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\Lambda_1'(L_{-+}) - \Lambda_1'(L_{--}) + \frac{1}{2}\Lambda_1'(L_2) = \\
&\quad \frac{w(L_{-+}^{(1)})\, w(L_{-+}^{(2)}) + 2\,\Lambda_1'(L_{-+}^{(1)}) + 2\,\Lambda_1'(L_{-+}^{(2)})}{8} \left(\prod_{j=3}^{\ell} \frac{w(L^{(j)})}{2} \right) \\
&\qquad + \frac{\Lambda_1'(L')}{4} \left(w(L_{-+}^{(1)}) + w(L_{-+}^{(2)}) \right).
\end{aligned}$$

Since $L_{+-}^{(i)}$ and $L_{-+}^{(i)}$ are of the same knot type for $i = 1$ and 2, we get Lemma 11 by subtracting the above two formulas. □

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