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ON CERTAIN HARDY SUMS AND THEIR $2m$ -TH POWER MEAN

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1. Introduction

For a positive integer k and an arbitrary integer h , the classical Dedekind sums $s(h, k)$ is defined by

$$s(h, k) = \sum_{a=1}^k \left(\left(\frac{a}{k} \right) \right) \left(\left(\frac{ah}{k} \right) \right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

The sum $s(h, k)$ plays an important role in the transformation theory of the Dedekind η function; See the Chapter 3 of [1]. There is an extensive literature about the Dedekind sums. H. Rademacher [8] wrote an introductory book on the subject.

Perhaps the most famous property of the Dedekind sums is the reciprocity formula

$$s(h, k) + s(k, h) = \frac{h^2 + k^2 + 1}{12hk} - \frac{1}{4}$$

for positive coprime integers h and k . Some three term versions of this formula were discovered by H. Rademacher [8], R.R. Hall, M.N. Huxley [5] and J. Pommersheim [7].

J.B. Conrey, E. Fransen, R. Klein and C. Scott [4] studied the mean value of Dedekind sums and proved the following proposition.

Proposition 1. *Suppose that m is a given positive integer and k is any sufficiently large integer. Then*

$$\sum_{h=1}^k s^{2m}(h, k) = f_m(k) \left(\frac{k}{12} \right)^{2m} + O \left(\left(k^{9/5} + k^{2m-1+1/(m+1)} \right) \log^3 k \right),$$

where \sum'_h denotes the summation over all h such that $(h, k) = 1$, and $f_m(k)$ is defined

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by the Dirichlet series

$$\sum_{k=1}^{\infty} \frac{f_m(k)}{k^s} = 2 \cdot \frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \cdot \zeta(s),$$

where $\zeta(s)$ is the Riemann zeta-function.

In [3], J. Chao-hua improved the error terms in Proposition 1. H. Walum [10] showed that for prime k ,

$$\sum_{\substack{\chi \pmod k \\ \chi(-1)=-1}} |L(1, \chi)|^4 = \frac{\pi^4(k-1)}{k^2} \sum_{h=1}^k |s(h, k)|^2.$$

In the spirit of [4] and [10], the second author [11] used an estimate for character sums to prove the following:

Proposition 2. *Suppose that p is any sufficiently large prime number and n is any positive integer. Then for $k = p^n$, we have*

$$\sum_{h=1}^k |s(h, k)|^2 = \frac{5}{144} \cdot \frac{(p^2 - 1)^2}{p(p^3 - 1)} \cdot k^2 + O\left(k \exp\left(\frac{3 \log k}{\log \log k}\right)\right),$$

where $\exp(y) = e^y$ and the constant implied in the O -symbol is absolute.

Also some interesting relations between Dedekind sums and Hurwitz zeta-function were established (see references [12], [13], [14] and [16]).

B.C. Berndt [2] gave an analogous transformation formula for the logarithm of the classical theta function

$$\theta(z) = \sum_{n=-\infty}^{+\infty} \exp(\pi i n^2 z), \quad \text{Im } z > 0,$$

and showed that for $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the theta group

$$\log \theta(Vz) = \log \theta(z) + \frac{1}{2} \log(cz + d) - \frac{1}{4} \pi i + \frac{1}{4} \pi i S(d, c),$$

where

$$S(d, c) = \sum_{j=1}^{c-1} (-1)^{j+1+\lfloor dj/c \rfloor}.$$

The sums $S(d, c)$ (and certain related ones) are sometimes called Hardy sums. They are closely connected with Dedekind sums [9]. Some arithmetical properties of $S(d, c)$

can be found in B.C. Berndt [2] and R. Sitaramachandra Rao [9]. In [15], the second author studied the $2m$ -th power mean of $S(d, c)$, and proved the following:

Proposition 3. *Let p be an odd prime and m be a positive integer, then*

$$\sum_{h=1}^{p-1} |S(h, p)|^{2m} = p^{2m} \frac{\zeta^2(2m) (1 - 1/4^m)}{\zeta(4m) (1 + 1/4^m)} + O\left(p^{2m-1} \exp\left(\frac{6 \ln p}{\ln \ln p}\right)\right).$$

In this paper, we use the important works of J.B. Conrey et al. [4] and J. Chaohua [3] to study the $2m$ -th power mean of $S(h, k)$, and give a sharp asymptotic formula for $\sum_{h=1}^k S^{2m}(h, k)$. That is, we shall prove the following theorem.

Theorem. *For any fixed integer $m \geq 2$ and any sufficiently large integer k , we have the asymptotic formula*

$$\sum_{h=1}^k S^{2m}(h, k) = g_m(k)k^{2m} + O(k^{2m-1}),$$

where $g_m(k)$ is defined by the Dirichlet series

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{g_m(k)}{k^s} &= \frac{2^s (2^{s+4m} - 2) (2^{2m} - 1)}{(2^{s+2m} - 1)^2 (2^{2m} + 1)} \cdot \frac{\zeta^2(2m) \zeta(s + 4m - 1)}{\zeta(4m) \zeta^2(s + 2m)} \zeta(s) \\ &+ \frac{2^s (2^{2m} - 1)}{(2^{s+2m} - 1)} \cdot \frac{\zeta(s)\zeta(2m)}{\zeta(s + 2m)}. \end{aligned}$$

2. Some lemmas

To prove the Theorem, we need following lemmas. First we have

Lemma 1. *For any given positive integer k and any integer h with $(h, k) = 1$ and any $P > 1$, there exist a positive integer $q \leq P$ and an integer a with $(a, q) = 1$ such that*

$$\left| \frac{h}{k} - \frac{a}{q} \right| < \frac{1}{qP}.$$

Proof. This is a well-known result; See Theorem 36 of [6]. □

Lemma 2. *Let a, b, c, d, h and k be positive integers with $ad - bc = 1$ and $(h, k) = 1$. Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

then we have

$$s(a, c) + s(h, k) - s(x, y) = \frac{c^2 + k^2 + y^2}{12cky} - \frac{1}{4}.$$

Proof. This is equation (26) of [5]. \square

Lemma 3. Let h and k denote relatively prime integers with $k > 0$, then

$$S(h, k) = \begin{cases} 4s(h, k) - 8s(h+k, 2k), & \text{if } h+k \text{ is odd;} \\ 0, & \text{if } h+k \text{ is even.} \end{cases}$$

Proof. This formula is an immediate consequence of (5.9) and (5.10) in [9]. \square

Lemma 4. For any positive integer q , we have

$$\sum_{a=1}^q |s(a, q)| \ll q \log^2 q.$$

Proof. This is Lemma 6 of [4]. \square

Lemma 5. Let k, h, a and q be positive integers with $(h, k) = (a, q) = 1$, and set $z = qh - ak$. If $h+k$ is odd and $1 \leq |z| \leq k/q$, then we have

$$S(h, k) = \begin{cases} -\frac{k}{qz} + O\left(|s(a, q)| + \left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right), & \text{if } a+q \text{ is an even number;} \\ O(q + |z|), & \text{if } a+q \text{ is an odd number.} \end{cases}$$

Proof. Suppose that $a+q$ is even. Since $(a, q) = 1$, a and q must be odd numbers.

First We consider the case that $z < 0$. Since $(a, q) = 1$, there exist positive integers b and d such that

$$ad - bq = 1, \quad 1 \leq d < q.$$

Let $f = 2dh - 2bk$. Then we have

$$\begin{pmatrix} 2d & -2b \\ -q & a \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} f \\ -z \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{a+q}{2} & b+d \\ q & 2d \end{pmatrix} \begin{pmatrix} f \\ -z \end{pmatrix} = \begin{pmatrix} h+k \\ 2k \end{pmatrix}.$$

The fact that $d < q$ and $z \geq -k/q$ yields

$$f = 2kd \left(\frac{h}{k} - \frac{b}{d} \right) = 2kd \left(\frac{h}{k} - \frac{a}{q} + \frac{1}{qd} \right) = 2kd \left(\frac{z}{qk} + \frac{1}{qd} \right) = \frac{2kd}{q} \left(\frac{z}{k} + \frac{1}{d} \right) > 0.$$

On the other hand, since $(h, k) = 1$ and z is odd, we get $(f, -z) = 1$. Then by Lemma 2,

$$s \left(\frac{a+q}{2}, q \right) + s(f, -z) - s(h+k, 2k) = -\frac{4k^2 + q^2 + z^2}{24kqz} - \frac{1}{4}.$$

That is,

$$s(h+k, 2k) = \frac{k}{6qz} + O \left(\left| s \left(\frac{a+q}{2}, q \right) \right| + |z| \right).$$

From Lemma 8 of [4] we also have

$$s(h, k) = \frac{k}{12qz} + O \left(|s(a, q)| + |z| \right).$$

Therefore by Lemma 3 we immediately have

$$S(h, k) = -\frac{k}{qz} + O \left(|s(a, q)| + \left| s \left(\frac{a+q}{2}, q \right) \right| + |z| \right), \quad \text{if } z < 0.$$

For $z > 0$, we can find positive integers b and d satisfying

$$ad - bq = -1, \quad 1 \leq d < q.$$

Let $f = 2bk - 2dh$. Then we have

$$\begin{pmatrix} 2b & -2d \\ -a & q \end{pmatrix} \begin{pmatrix} k \\ h \end{pmatrix} = \begin{pmatrix} f \\ z \end{pmatrix}$$

and

$$\begin{pmatrix} q & 2d \\ \frac{a+q}{2} & b+d \end{pmatrix} \begin{pmatrix} f \\ z \end{pmatrix} = \begin{pmatrix} 2k \\ h+k \end{pmatrix}.$$

Similarly we can get $(f, z) = 1$ and

$$f = 2kd \left(\frac{b}{d} - \frac{h}{k} \right) = 2kd \left(\frac{1}{qd} + \frac{a}{q} - \frac{h}{k} \right) = 2kd \left(\frac{1}{qd} - \frac{z}{qk} \right) = \frac{2kd}{q} \left(\frac{1}{d} - \frac{z}{k} \right) > 0.$$

Then by Lemma 2,

$$s\left(q, \frac{a+q}{2}\right) + s(f, z) - s(2k, h+k) = \frac{\left(\frac{(a+q)}{2}\right)^2 + z^2 + (h+k)^2}{12\left(\frac{(a+q)}{2}\right) \cdot z \cdot (h+k)} - \frac{1}{4}.$$

Noting that

$$s\left(q, \frac{a+q}{2}\right) + s\left(\frac{a+q}{2}, q\right) = \frac{\left(\frac{(a+q)}{2}\right)^2 + q^2 + 1}{12\left(\frac{(a+q)}{2}\right) \cdot q} - \frac{1}{4}$$

and

$$s(2k, h+k) + s(h+k, 2k) = \frac{(h+k)^2 + (2k)^2 + 1}{12(h+k) \cdot 2k} - \frac{1}{4},$$

we have

$$s(h+k, 2k) = \frac{k}{6qz} + O\left(\left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right).$$

So from Lemma 8 of [4] and Lemma 3 we immediately have

$$S(h, k) = -\frac{k}{qz} + O\left(\left|s(a, q)\right| + \left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right), \quad \text{for } z > 0.$$

This proves that

$$S(h, k) = -\frac{k}{qz} + O\left(\left|s(a, q)\right| + \left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right),$$

if $a+q$ is an even number.

On the other hand, if $a+q$ is an odd number, using the similar methods we can get

$$s(h+k, 2k) = \frac{k}{24qz} + O(q + |z|),$$

so we have

$$S(h, k) = 4s(h, k) - 8s(h+k, 2k) = O(q + |z|).$$

This completes the proof of Lemma 5. □

Lemma 6. For any real $s > 1$, we have the identities

$$\sum_{\substack{d=1 \\ 2 \nmid d}}^{\infty} \frac{\mu(d)}{d^s} = \frac{1}{(1-2^s)\zeta(s)}, \quad \sum_{\substack{d=1 \\ 2 \mid d}}^{\infty} \frac{\mu(d)}{d^s} = \frac{2^s}{(2^s-1)\zeta(s)};$$

$$\sum_{\substack{d=1 \\ 2|d}}^{\infty} \sum_{e|d} \frac{\mu(e)(d/e)^{1-2m}}{d^{s+2m}} = \frac{2^{s+2m} - 2^{1-2m}}{2^{s+2m} - 1} \cdot \frac{\zeta(s+4m-1)}{\zeta(s+2m)}$$

and

$$\sum_{\substack{d=1 \\ 2|d}}^{\infty} \sum_{e|d} \frac{\mu(e)(d/e)^{1-2m}}{d^{s+2m}} = \frac{2^{1-2m} - 1}{2^{s+2m} - 1} \cdot \frac{\zeta(s+4m-1)}{\zeta(s+2m)}.$$

Proof. Using elementary methods we can easily deduce these identities. □

3. Proof of Theorem

We suppose that $m \geq 2$ and a sufficiently large number k are given. We set

$$Q = \left[k^{1/2} \right], \quad P = 2Q.$$

For integers a and q with $1 \leq q \leq Q$, let $I(a, q)$ be an open interval given by

$$I(a, q) = \left(\frac{a}{q} - \frac{1}{qP}, \frac{a}{q} + \frac{1}{qP} \right).$$

When $a/q \neq \hat{a}/\hat{q}$ and $q, \hat{q} \leq Q$, one has

$$\left| \frac{a}{q} - \frac{\hat{a}}{\hat{q}} \right| \geq \frac{1}{q\hat{q}} \geq \left(\frac{1}{qP} + \frac{1}{\hat{q}P} \right).$$

Thus the intervals $I(a, q)$ are pairwise disjoint.

If $1 \leq h \leq k$, $(h, k) = 1$ and $h + k$ is odd, then by Lemma 1, h/k falls into an interval $I(a, q)$ with $1 \leq q \leq P$, $0 \leq a \leq q$ and $(a, q) = 1$.

Let $z = qh - ak$. It is easy to see that $z \neq 0$ and

$$|z| = qk \left| \frac{h}{k} - \frac{a}{q} \right| \leq \frac{k}{P} \leq \frac{k}{q}.$$

If h/k falls into an interval $I(a, q)$ with $1 \leq q \leq P$, $0 \leq a \leq q$, $(a, q) = 1$ and $a + q$ is an odd number, then by Lemma 5, we have

$$S(h, k) = O(q + |z|) \ll P + \frac{k}{P} \ll k^{1/2}.$$

Thus,

$$\sum^* S^{2m}(h, k) \ll k^{m+1} \ll k^{2m-1},$$

where the asterisk indicates summation over those integers h , $1 \leq h \leq k$, $(h, k) = 1$ and $h + k$ is odd, for which h/k falls into an interval $I(a, q)$ with $1 \leq q \leq P$, $0 \leq a \leq q$, $(a, q) = 1$ and $a + q$ is an odd number.

If h/k falls into an interval $I(a, q)$ with $Q \leq q \leq P$, $0 \leq a \leq q$, $(a, q) = 1$ and $a + q$ is an even number, then by Lemma 5, we have

$$S(h, k) = -\frac{k}{qz} + O\left(|s(a, q)| + \left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right) \\ \ll \frac{k}{q} + q + \frac{k}{P} \ll \frac{k}{Q} + P + \frac{k}{P} \ll k^{1/2}.$$

Thus,

$$\sum^* S^{2m}(h, k) \ll k^{m+1} \ll k^{2m-1},$$

where the asterisk indicates summation over those integers h , $1 \leq h \leq k$, $(h, k) = 1$ and $h + k$ is odd, for which h/k falls into an interval $I(a, q)$ with $Q \leq q \leq P$, $0 \leq a \leq q$, $(a, q) = 1$ and $a + q$ is an even number.

Therefore

$$\sum_{h=1}^k S^{2m}(h, k) = \sum_{\substack{h=1 \\ 2 \nmid h+k}}^k S^{2m}(h, k) = \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \sum_{\substack{a=1 \\ 2 \nmid a}}^q \sum_{h/k \in I(a, q)}^* S^{2m}(h, k) + O(k^{2m-1}),$$

where the asterisk means that $1 \leq h \leq k$, $(h, k) = 1$ and $h + k$ is odd.

Lemma 5 produces

$$S(h, k) = -\frac{k}{qz} + O\left(|s(a, q)| + \left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right), \\ \text{if } a \text{ and } q \text{ are odd numbers.}$$

Using the estimate

$$(A + B + C)^{2m} = A^{2m} + O\left(|A|^{2m-1}(|B| + |C|)\right) + O(B^{2m} + C^{2m}),$$

we obtain

$$S^{2m}(h, k) = \left(\frac{k}{qz}\right)^{2m} + O\left(\left(\frac{k}{q|z|}\right)^{2m-1} \left(|s(a, q)| + \left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right)\right) \\ + O\left(\left(|s(a, q)| + \left|s\left(\frac{a+q}{2}, q\right)\right|\right)^{2m} + z^{2m}\right).$$

Therefore

$$\sum_{\substack{q=1 \\ 2 \nmid q}}^Q \sum'_{\substack{a=1 \\ 2 \nmid a}}^q \sum^*_{h/k \in I(a,q)} S^{2m}(h, k) \equiv \Omega_1 + O(\Omega_2) + O(\Omega_3),$$

where

$$\begin{aligned} \Omega_1 &= \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \sum'_{\substack{a=1 \\ 2 \nmid a}}^q \sum^*_{h/k \in I(a,q)} \left(\frac{k}{qz}\right)^{2m}, \\ \Omega_2 &= \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \sum'_{\substack{a=1 \\ 2 \nmid a}}^q \sum^*_{h/k \in I(a,q)} \left(\frac{k}{q|z|}\right)^{2m-1} \left(|s(a, q)| + \left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right), \\ \Omega_3 &= \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \sum'_{\substack{a=1 \\ 2 \nmid a}}^q \sum^*_{h/k \in I(a,q)} \left(\left(|s(a, q)| + \left|s\left(\frac{a+q}{2}, q\right)\right|\right)^{2m} + z^{2m}\right). \end{aligned}$$

Noting that for the fixed a, q, k and z , the equation $z = qh - ak$ has at most one solution h . By Lemma 4, we have

$$\begin{aligned} \Omega_2 &\ll k^{2m-1} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \sum'_{\substack{a=1 \\ 2 \nmid a}}^q \sum^*_{h/k \in I(a,q)} \frac{1}{q^{2m-1}} \cdot \frac{1}{z^{2m-2}} \left(|s(a, q)| + \left|s\left(\frac{a+q}{2}, q\right)\right| + 1\right) \\ &\ll k^{2m-1} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{1}{q^{2m-1}} \sum'_{a=1}^q \left(|s(a, q)| + \left|s\left(\frac{a+q}{2}, q\right)\right| + 1\right) \sum_{z \neq 0} \frac{1}{z^2} \\ &\ll k^{2m-1} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{1}{q^{2m-1}} \cdot q \cdot \log^2(q+1) \ll k^{2m-1} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{\log^2(q+1)}{q^2} \ll k^{2m-1}. \end{aligned}$$

Moreover,

$$\Omega_3 \ll \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \sum'_{\substack{a=1 \\ 2 \nmid a}}^q \sum^*_{h/k \in I(a,q)} \left(q^{2m} + \left(\frac{k}{P}\right)^{2m}\right) \ll k^m \sum'_{h=1}^k 1 \ll k^{m+1} \ll k^{2m-1}.$$

Combining these estimates, we obtain

$$\sum'_{h=1}^k S^{2m}(h, k) = \Omega_1 + O(k^{2m-1}),$$

where

$$\Omega_1 = k^{2m} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \sum_{h/k \in I(a,q)}^* \frac{1}{z^{2m}}.$$

It remains to obtain an asymptotic formula for Ω_1 . Noting that if $1 \leq h \leq k$, then $h/k \notin I(a, q)$ if and only if $|z| \geq k/P$. Hence

$$\begin{aligned} k^{2m} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \sum_{h/k \notin I(a,q)}^* \frac{1}{z^{2m}} &\leq k^{2m} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \sum_{|z| \geq k/P} \frac{1}{z^{2m}} \\ &\ll k^{2m} \left(\frac{P}{k}\right)^{2m-1} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{1}{q^{2m-1}} \\ &\ll kP^{2m-1} \ll k^{m+1/2} \ll k^{2m-1}. \end{aligned}$$

Thus

$$\Omega_1 = k^{2m} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \sum_{\substack{h=1 \\ 2 \nmid h+k}}^k \frac{1}{(qh - ak)^{2m}} + O(k^{2m-1}).$$

Using the estimate

$$\sum_{h \geq k+1} \frac{1}{(qh - ak)^{2m}} \leq \int_k^\infty \frac{dx}{(qx - ak)^{2m}} = \int_{(q-a)k}^\infty \frac{dy}{qy^{2m}} \ll \frac{1}{qk^{2m-1}},$$

we get

$$k^{2m} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \sum_{h \geq k+1} \frac{1}{(qh - ak)^{2m}} \ll k^{2m} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \frac{1}{qk^{2m-1}} \ll k.$$

Since

$$\begin{aligned} \sum_{h \leq 0} \frac{1}{(qh - ak)^{2m}} &\leq \frac{1}{k^{2m}} + \sum_{r \geq 1} \frac{1}{(qr + ak)^{2m}} \leq \frac{1}{k^{2m}} + \int_0^\infty \frac{dx}{(qx + ak)^{2m}} \\ &= \frac{1}{k^{2m}} + \int_{ak}^\infty \frac{dy}{qy^{2m}} \ll \frac{1}{k^{2m}} + \frac{1}{qk^{2m-1}} \ll \frac{1}{k^{2m-1}}, \end{aligned}$$

we have

$$k^{2m} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \sum_{h \leq 0} \frac{1}{(qh - ak)^{2m}} \ll k^{2m} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \frac{1}{k^{2m-1}} \ll k.$$

Therefore

$$\Omega_1 = k^{2m} \sum_{\substack{q=1 \\ 2 \nmid q}}^Q \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \sum_{\substack{h=-\infty \\ (h,k)=1 \\ 2 \nmid h+k}}^{\infty} \frac{1}{(qh - ak)^{2m}} + O(k^{2m-1}).$$

Since

$$\begin{aligned} k^{2m} \sum_{\substack{q>Q \\ 2 \nmid q}} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \sum_{\substack{h=-\infty \\ (h,k)=1 \\ 2 \nmid h+k}}^{\infty} \frac{1}{(qh - ak)^{2m}} &\ll k^{2m} \sum_{\substack{q>Q \\ 2 \nmid q}} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \sum_{z=-\infty}^{\infty} \frac{1}{z^{2m}} \\ &\ll k^{2m} \sum_{\substack{q>Q \\ 2 \nmid q}} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q 1 \ll k^{2m} \sum_{\substack{q>Q \\ 2 \nmid q}} \frac{1}{q^{2m-1}} \\ &\ll \frac{k^{2m}}{Q^{2m-2}} \ll k^{m+1} \ll k^{2m-1}, \end{aligned}$$

we have

$$\Omega_1 = k^{2m} \sum_{\substack{q=1 \\ 2 \nmid q}}^{\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \sum_{\substack{h=-\infty \\ (h,k)=1 \\ 2 \nmid h+k}}^{\infty} \frac{1}{(qh - ak)^{2m}} + O(k^{2m-1}).$$

Therefore

$$\sum_{h=1}^k S^{2m}(h, k) = g_m(k)k^{2m} + O(k^{2m-1}),$$

where

$$g_m(k) = \sum_{\substack{q=1 \\ 2 \nmid q}}^{\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ 2 \nmid a}}^q \sum_{\substack{h=-\infty \\ (h,k)=1 \\ 2 \nmid h+k}}^{\infty} \frac{1}{(qh - ak)^{2m}}.$$

Let

$$\begin{aligned}
 U(s) &= \sum_{k=1}^{\infty} \frac{g_m(k)}{k^s} = \sum_{q=1}^{\infty} \frac{1}{q^{2m}} \sum_{a=1}^q \sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{\substack{h=-\infty \\ (h,k)=1 \\ 2|h+k}}^{\infty} \frac{1}{(qh - ak)^{2m}} \\
 &= \sum_{q=1}^{\infty} \frac{1}{q^{2m}} \sum_{a=1}^q \sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{\substack{h=-\infty \\ (h,k)=1 \\ 2|h}}^{\infty} \frac{1}{(qh - ak)^{2m}} + \sum_{q=1}^{\infty} \frac{1}{q^{2m}} \sum_{a=1}^q \sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{\substack{h=-\infty \\ (h,k)=1 \\ 2|h}}^{\infty} \frac{1}{(qh - ak)^{2m}} \\
 &\equiv U_1(s) + U_2(s).
 \end{aligned}$$

We proceed to find an expression for $U_1(s)$. We remove the coprimality conditions by use of the Möbius relation

$$\sum_{d|n} \mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n \neq 1. \end{cases}$$

After rearranging the sums, we have

$$U_1(s) = \sum_{d=1}^{\infty} \sum_{\substack{e=1 \\ 2|e}}^{\infty} \frac{\mu(d)}{d^{s+2m}} \frac{\mu(e)}{e^{4m}} \sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{q=1}^{\infty} \frac{1}{q^{2m}} \sum_{a=1}^q \sum_{h=-\infty}^{\infty} \frac{1}{(qh - ak)^{2m}}.$$

Let $g = (k, q)$, then the inner double sum is

$$\begin{aligned}
 &= \frac{1}{g^{2m}} \sum_{\substack{a=1 \\ 2|a}}^q \sum_{\substack{n=-\infty \\ 2|n}}^{\infty} \frac{1}{n^{2m}} \\
 &= \frac{1}{g^{2m}} \left[\sum_{\substack{a=1 \\ 2|a}}^q \sum_{\substack{n=1 \\ 2|n}}^{\infty} \frac{1}{n^{2m}} + \sum_{\substack{a=1 \\ 2|a}}^q \sum_{\substack{n=1 \\ 2|n}}^{\infty} \frac{1}{n^{2m}} \right] \\
 &= \frac{1}{g^{2m}} \sum_{a=0}^q \sum_{\substack{n=1 \\ 2|n}}^{\infty} \frac{1}{n^{2m}} = \frac{1}{g^{2m}} \left[\sum_{a=1}^q \sum_{\substack{n=1 \\ 2|n}}^{\infty} \frac{1}{n^{2m}} + \sum_{\substack{n=1 \\ 2|n}}^{\infty} \frac{1}{n^{2m}} \right] \\
 &= \frac{1}{g^{2m}} \left[g \frac{(2^{2m} - 1) \zeta(2m)}{2^{2m}} + \frac{g^{2m} (2^{2m} - 1) \zeta(2m)}{q^{2m}} \right] \\
 &= \frac{(2^{2m} - 1) \zeta(2m)}{2^{2m}} \left[g^{1-2m} + \frac{1}{q^{2m}} \right].
 \end{aligned}$$

Thus by Lemma 6,

$$U_1(s) = \frac{2^{s+4m}}{(2^{s+2m} - 1)(2^{2m} + 1)} \frac{\zeta(2m)}{\zeta(s + 2m)\zeta(4m)} \left[\sum_{\substack{k=1 \\ 2|k}}^{\infty} \sum_{\substack{q=1 \\ 2|q}}^{\infty} \frac{g^{1-2m}}{k^s q^{2m}} + \sum_{\substack{k=1 \\ 2|k}}^{\infty} \sum_{\substack{q=1 \\ 2|q}}^{\infty} \frac{1}{k^s q^{4m}} \right].$$

By Lemma 6 we have

$$\begin{aligned} \sum_{\substack{k=1 \\ 2|k}}^{\infty} \sum_{\substack{q=1 \\ 2|q}}^{\infty} \frac{g^{1-2m}}{k^s q^{2m}} &= \sum_{\substack{k=1 \\ 2|k}}^{\infty} \sum_{\substack{q=1 \\ 2|q}}^{\infty} \frac{1}{k^s q^{2m}} \sum_{d|(k,q)} \sum_{e|d} \mu(e) \left(\frac{d}{e}\right)^{1-2m} \\ &= \sum_{\substack{d=1 \\ 2|d}}^{\infty} \sum_{e|d} \frac{\mu(e)(d/e)^{1-2m}}{d^{s+2m}} \sum_{\substack{k=1 \\ 2|k}}^{\infty} \sum_{\substack{q=1 \\ 2|q}}^{\infty} \frac{1}{k^s q^{2m}} \\ &= \frac{(2^{s+2m} - 2^{1-2m})(2^{2m} - 1)}{2^{s+2m}(2^{s+2m} - 1)} \cdot \frac{\zeta(s + 4m - 1)\zeta(s)\zeta(2m)}{\zeta(s + 2m)}. \end{aligned}$$

Therefore

$$\begin{aligned} U_1(s) &= \frac{(2^{s+4m} - 2)(2^{2m} - 1)}{(2^{s+2m} - 1)^2(2^{2m} + 1)} \cdot \frac{\zeta^2(2m)}{\zeta(4m)} \frac{\zeta(s + 4m - 1)}{\zeta^2(s + 2m)} \zeta(s) \\ &\quad + \frac{(2^{2m} - 1)}{(2^{s+2m} - 1)} \cdot \frac{\zeta(s)\zeta(2m)}{\zeta(s + 2m)}. \end{aligned}$$

Using the same methods we can prove

$$\begin{aligned} U_2(s) &= \frac{(2^s - 1)(2^{s+4m} - 2)(2^{2m} - 1)}{(2^{s+2m} - 1)^2(2^{2m} + 1)} \cdot \frac{\zeta^2(2m)}{\zeta(4m)} \frac{\zeta(s + 4m - 1)}{\zeta^2(s + 2m)} \zeta(s) \\ &\quad + \frac{(2^s - 1)(2^{2m} - 1)}{(2^{s+2m} - 1)} \cdot \frac{\zeta(s)\zeta(2m)}{\zeta(s + 2m)}. \end{aligned}$$

So we have

$$\begin{aligned} U(s) &= \frac{2^s(2^{s+4m} - 2)(2^{2m} - 1)}{(2^{s+2m} - 1)^2(2^{2m} + 1)} \cdot \frac{\zeta^2(2m)}{\zeta(4m)} \frac{\zeta(s + 4m - 1)}{\zeta^2(s + 2m)} \zeta(s) \\ &\quad + \frac{2^s(2^{2m} - 1)}{(2^{s+2m} - 1)} \cdot \frac{\zeta(s)\zeta(2m)}{\zeta(s + 2m)}. \end{aligned}$$

This completes the proof of Theorem.

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