OUKA
The University of Osaka
Institutional Knowledge Archive

Title	Examples of foliations with non trivial exotic characteristic classes
Author(s)	Yamato, Kenzi
Citation	Osaka Journal of Mathematics. 1975, 12(2), p. 401-417
Version Type	VoR
URL	https://doi.org/10.18910/10380
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

The University of Osaka

Yamato, K.
Osaka J. Math.
12 (1975), 401-417

EXAMPLES OF FOLIATIONS WITH NON TRIVIAL EXOTIC CHARACTERISTIC CLASSES

KENZI YAMATO

(Received August 12, 1974)

Introduction

An example of foliation of codimension one with non trivial Godbillon-Vey invariant ([3]) was constructed by R. Roussarie (see Bott [1]). Generalizing the Godbillon-Vey invariant, R. Bott [1] has defined exotic characteristic classes for foliations. In this paper, we shall construct examples of foliations with non trivial exotic characteristic classes.

Roussarie's example was constructed on a compact quotient space of $SL(2;\boldsymbol{R})$ by a discrete subgroup. This example may be regarded as an Anosov foliation arising from the geodesic flow on the unit tangent sphere bundle of a surface with constant negative curvature. This suggests us to consider such a foliation on the unit tangent sphere bundle of a closed $(q+1)$-manifold $(q\geqq 1)$ with constant negative curvature. In fact, our example is constructed as follows. Let G denote the identity component of the Lie group

$$O(q+1,\ 1) = \{X\in GL(q+2;\boldsymbol{R});\ {}^tXBX= B\}\ ,$$

where

$$B = \begin{pmatrix} I_{q+1} & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider a compact subgroup

$$H = \left\{\begin{pmatrix} X & 0 \\ 0 & I_2 \end{pmatrix};\ X\in SO(q)\right\}$$

of G, and a closed subgroup K consisting of $X=(x_{ij})\in G$such that

$$\det\begin{pmatrix} x_{1\,q+1} & \cdots & x_{q+1\,q+2} \\ x_{q+2\,q+1} & & x_{q+2\,q+2} \end{pmatrix} - 1$$

and

$$x_{i\,q+1}+x_{i\,q+2}=0\ (i= 1,\ \cdots,q)\ .$$

By a theorem of A. Borel [2], there exists a discrete subgroup D of G such that $D\backslash\bar{M}$ is a closed manifold, where $\bar{M}=G/H$. Foliate $\bar{M}$ into the fibres of the fibre bundle $\bar{M}=G/H\to G/K$, and consider the foliation on $M=D\backslash\bar{M}$ induced naturally from the G-invariant foliation on $\bar{M}$. Then it is proved that the foliation on M has non trivial exotic characteristic classes.

In §1, we review differential geometry which will be needed. In §2, we define exotic characteristic classes following R. Bott [1] and state our result precisely. §3 is devoted to construct examples of foliations with non trivial exotic characteristic classes. The proof of our result will be given in §4.

The author is grateful to Professor H. Ozeki and Mr. A. Ikeda for their advices.

1. Preliminaries

1.1. First, we shall fix some notations:

For a smooth manifold N, we put

$\mathfrak{X}(N)=\{$smooth vector fields on $N\}$,

$C^\infty(N)=\{$smooth real valued functions on $N\}$,

$A_C^*(N)=$the space of complex smooth forms on N,

$A^*(N)=$the space of (real) smooth forms on N,

$A^p(N)=\{\omega\in A^*(N);\omega$ is p-form$\}$,

$A_0^p(N)=\{\omega\in A^p(N);\omega$ has compact support$\}$,

$\Gamma(\xi)=$the set of smooth cross-sections of a smooth vector bundle ξ over N.

For a C^∞-smooth codimension q foliation $\mathcal{F}$ on N, we denote by $\tau(\mathcal{F})$ (resp. $\nu(\mathcal{F})$) the subbundle of $\tau(N)$ tangent (resp. normal with respect to a Riemannian metric on N) to the foliation, where $\tau(N)$ denotes the tangent bundle of N.

1.2. Connections

Let N be a smooth manifold and ξ a smooth q-dimensional vector bundle over N.

(1.2.1) (1) A connection on ξ is an $\boldsymbol{R}$-bilinear map

$$\mathrm{V}:\ \mathfrak{X}(N)\times\Gamma(\xi)\to\Gamma(\xi)$$

such that

(i) $\nabla_X(fs)=X(f)s+f\nabla_X(s)$

(ii) $\nabla_{fX}(s)=f\nabla_X(s)$

for all $X\in\mathfrak{X}(N), s\in\Gamma(\xi), f\in C^\infty(N)$, where $\nabla_X(s)=\nabla(X,s)$.

(2) Let $S=\{s_1,\cdots,s_q\}$ be a smooth frame of ξ defined on some open set U in N. The connection form of V relative to the frame S is a $q\times q$ matrix $\theta=(\theta_{ij})$ of 1-forms on U such that

$$\nabla_X(s_i)=\sum_{j=1}^{q}\theta_{ij}(X)s_j$$

for $i=1, \cdots, q$.

The curvature matrix of V associated to the frame S is a $qx\ q$ matrix $k=(k_{ij})$ of 2-forms on U such that

$$k_{ij} = d\theta_{ij} - \sum_{h=1}^{q} \theta_{ih} \wedge \theta_{hj}$$

for $i, j=1, \cdots, q$.

Let $(N, \mathcal{F})$ be a C^∞-smooth codimension q foliation on N, and $(\ ,\ )$ be a Riemannian metric on $\nu(\mathcal{F})$ not necessarily induced from a Riemannian metric on N.

(1.2.2) (1) A metric connection on $\nu(\mathcal{F})$ is a connection V° on $\tau(F)$ such that

$$d(s_1, s_2)(X) = (\nabla^0_X(s_1), s_2) + (s_1, \nabla^0_X(s_2))$$

for $X \in \mathfrak{X}(N)$, $s_1, s_2 \in \Gamma(\nu(\mathcal{F}))$.

(2) A basic connection on $\nu(\mathcal{F})$ is a connection ∇^1 on $\nu(\mathcal{F})$ such that

$$\nabla^1_X(s) = \pi[X, \tilde{s}]$$

for $X \in \Gamma(\tau(\mathcal{F}))$, $s \in \Gamma(\nu(\mathcal{F}))$, where $\pi: \tau(N) \to \nu(\mathcal{F})$ is the natural projection, and $\tilde{s} \in \mathfrak{X}(N)$ is such that $\pi(\tilde{s})=s$.

(3) Let $\nu(\mathcal{F}) \times \boldsymbol{R}$ denote the vector bundle over $N \times \boldsymbol{R}$ with the same fibre dimension as $\nu(\mathcal{F})$. Given a metric (resp. basic) connection V° (resp. ∇^1) on $\nu(\mathcal{F})$, a unique connection ∇^{01} on $\nu(\mathcal{F}) \times R$ is defined by requiring

(i) On sections s which are constant in $\boldsymbol{R}$-direction, let $\nabla^{01}_{\partial/\partial t}(s)=0$;

(ii) If $X \in T_{(x,t)}(N \times \{t\})$, define

$$\nabla^{01}_X(s) = (1-t)\nabla^0_X(s) + t\nabla^1_X(s)$$

for $s \in \Gamma(\nu(\mathcal{F}) \times \boldsymbol{R})$.

Clearly, for a smooth frame $S = \{s_1, \cdots, s_q\}$ of $\tau(F)$ defined on some open set U in N, a smooth frame $S' = \{s'_1, \cdots, s'_q\}$ of $\nu(\mathcal{F}) \times \boldsymbol{R}$ on $U \times \boldsymbol{R}$ is defined by

$$s'_i(x, t) = (s_i(x), t), \quad i = 1, \cdots, q,$$

then connection form θ^{01} of ∇^{01} relative to the frame S' is represented as follows:

$$\theta^{01} = (1-t)\theta^0 + t\theta^1,$$

where θ^0 (resp. θ^1) is the connection form of ∇^0 (resp. ∇^1) relative to the frame 5.

Let N be a Riemannian manifold, $(\ ,\ )$ be the Riemannian metric on N. The following is well known.

(1.2.3) *There exists a unique connection* V *on* $\tau(N)$ *statisfying the following conditions:*

(i) $\nabla_X(Y) - \nabla_Y(X) = [XY]$

for $X, Y \in \mathfrak{X}(N)$;

(ii) $d(Y_1, Y_2)(X) = (\nabla_X(Y_1), Y_2) + (Y_1, \nabla_X(Y_2))$

for $X, Y_1, Y_2 \in \mathfrak{X}(N)$.

This connection is called the Riemannian connection on N. Clearly, let V be the Riemannian connection on N, and f be an isometry on N, then

$$f_*(\nabla_X(Y)) = \nabla_{f_*X}(f_*Y)$$

for $X, Y \in \mathfrak{X}(N)$.

1.3. Integration along the fibre

(1.3.1) (Bott [1]) *Let N be an oriented smooth manifold and π: NX [0, 1]$\to$ N be the natural* ***projection,*** *then there* ***exists a unique homomorphism*** *of $C^\infty(N)$-modules*

$$\pi_*: A^p(N \times [0,1]) \to A^{p-1}(N), \qquad \text{for } p \geqq 1 ,$$

satisfying *the equation*

$$\int_{N \times [0,1]} \phi \wedge \pi^* \psi = \int_N \pi_* \phi \wedge \psi$$

for all $\phi \in A^p(N \times [0, 1])$, $\psi \in A_0^r(N)$, *where* $r = \dim N - p + 1$.

This homomorphism π_* is called integration along the fibre. Then it is easy to see the following.

(1.3.2) *Let $\bar{N}$, N be oriented smooth manifolds of dimension n, and $\bar{\pi}$: $\bar{N}X$ [0, 1]$\to \bar{N}$, π: $N \times$ [0, 1]$\to N$ be the natural* ***projections,*** *then for any* ***immersion*** *f: $\bar{N} \to$ N, the following diagram is* ***commutative***:

$$\begin{array}{ccc} A^p(\bar{N} \times [0,1]) & \xrightarrow{\bar{\pi}_*} & A^{p-1}(\bar{N}) \\ \uparrow (f \times id)^* & & \uparrow f^* \\ A^p(N \times [0,1]) & \xrightarrow{\pi_*} & A^{p-1}(N) \end{array} \qquad \text{for } p \geqq 1 .$$

2. Exotic characteristic classes and Theorem

In R. Bott [1], exotic characteristic classes for foliations have been defined as follows.

Let $q \geqq 1$ be an integer.

First, a cochain complex (WO_q, d) is defined. Let $R[c_1, \cdots, c_q]$ denote the graded polynomial aglebra over R generated by the elements c_i with degree $2i$. Set

$$R_q[c_1, \cdots, c_q] = R[c_1, \cdots, c_q] / \{\phi;\ \deg(\phi) > 2q\} .$$

Let $E(h_1, h_3, \cdots, h_r)$ denote the exterior algebra over R generated by the elements

h_i with degree $2i-1$, where r is the largest odd integer $\leqq q$. Then as a graded algebra over R

$$WO_q = \boldsymbol{R}_q[c_1, \cdots, c_q] \otimes E(h_1 h_3, \cdots, h_r),$$

and a unique antiderivation of degree 1

$$d\colon WO_q \to WO_q$$

is defined by requiring

$$\begin{aligned} d(c_i) &= 0, \qquad i = 1, \cdots, q \\ d(h_i) &= c_i, \qquad i = 1, 3, \cdots, r . \end{aligned}$$

Let ξ be a smooth q-dimensional vector bundle over a manifold N and V a connection on ξ. For a curvature matrix k of V, local $2i$-forms $c_i(k)$ on N are defined by the following formula

$$\det(I_q + tk) = 1 + \sum_{i=1}^{q} t^i c_i(k) .$$

Since $c_i(k)$ do not depend on the choice of the local frame of ξ, $c_i(k)$ define global $2i$-forms on N. Then a homomorphism of graded $\boldsymbol{R}$-algebras

$$\lambda(\nabla)\colon \boldsymbol{R}[c_1, \cdots, c_q] \to A_C^*(N)$$

is defined by requiring

$$\lambda(\nabla)(c_i) = (\sqrt{-1}/2\pi)^i c_i(k), \qquad \text{for } i = 1, \cdots, q .$$

Let N be an oriented smooth manifold without boundary and $(N, \mathcal{F})$ a C^∞-smooth codimension q foliation on N. Let V° (resp. ∇^1) be a metric (resp. basic) connection on $\nu(\mathcal{F})$ and ∇^{01} be as in (1.2.2) (3). Then the followings hold.

(2.1) (1) $\lambda(\nabla^1)(\phi) \in A_C^*(N)$ *is a closed form for any* $\phi \in \boldsymbol{R}[c_1, \cdots, c_q]$, *and if* $\deg(\phi) > 2q$ *then* $\lambda(\nabla^1)(\phi) = 0$.

(2) $\lambda(\mathrm{V}^\circ)(\phi) = 0$ *for* $\phi \in \boldsymbol{R}[c_1, \cdots, c_q]$ *such that* $\deg(\phi)/2$ *is an odd integer.*

(3) *Let* $\pi\colon NX\,[0, 1] \to N$ *be the natural projection and* $i\colon NX\,[0, 1] \to N \times \boldsymbol{R}$ *the inclusion mapping, then*

$$d(\pi_* i^* \lambda(\nabla^{01})(\phi)) = \lambda(\nabla^1)(\phi) - \lambda(\nabla^0)(\phi)$$

for $\phi \in \boldsymbol{R}[c_1, \cdots, c_q]$, *especially*

$$d(\pi_* i^* \lambda(\nabla^{01})(c_{2j-1})) = \lambda(\nabla^1)(c_{2j-1}) ,$$

where π_* *is the integration along the fibre.*

In view of (2.1), given a C^∞-smooth codimension q foliation $(N, \mathcal{F})$ on an oriented smooth manifold N without boundary, a homomorphism of cohain

complexes

$$\lambda_{(N,\mathcal{F})}: WO_q \to A_C^*(N)$$

is defined by requiring

$$\lambda_{(N,\mathcal{F})}(c_i) = \lambda(\nabla^1)(c_i),$$
$$\lambda_{(N,\mathcal{F})}(h_j) = \pi_* i^* \lambda(\nabla^{01})(c_j).$$

We used the notation $\lambda_{(N,30}$ in place of λ_E of Bott [1]. Here the homomorphism $\lambda_{(N,30}$ depends only on the choics of two connections ∇^0 and ∇^1 on $\nu(\mathcal{F})$. In cohomology, $\lambda_{(N,\mathcal{F}}$ induces a homomorphism of graded $\boldsymbol{R}$-algebras

$$\lambda^*_{(N,\mathcal{F})}: H^*(WO_q) \to H^*(N; C)$$

which depends only on the foliation *(N, F)*.

The elements of $\lambda^*_{(N,\mathcal{F})}(H^*(WO_q))$ are called the exotic characteristic classes for the foliation $(N, \mathcal{F})$.

It is easy to see the foillowing lemma.

Lemma *2.2. Each canonical generator of* $H^{2q+1}(WO_q)$ *is represented by some* $\phi\ h_j \in WO_q$, *where* $\phi \in \boldsymbol{R}_q[c_1, \cdots, c_q]$ *is a monomial with degree* $2(q-j+1)$.

Then we have

Theorem. *For any integer* $q \geqq 1$, *there exists a* C^∞*-smooth codimensíon q foliation* $(M, \mathcal{F})$ *on a closed* $(2q+1)$*-manifold such that all the exotic characteristic classes for the foliation which correspond to the canonical generators* $[\phi\ h_j]$ *of* $H^{2q+1}(WO_q)$ *are non zero in* $H^{2q+1}(M; C)$.

REMARK. When $q=1$, the generator fo-AJ of $H^3(WO_1) \cong \boldsymbol{R}$ is the Godbillon-Vey invariant, and our foliation of codimension one is diffeomorphic to the foliation constructed by R. Roussarie (cf. [1]).

3. Construction of the foliation (M, $\mathcal{F}$)

Throughout this paper, integer $q \geqq 1$ is to be fixed, and all foliations are to be C^∞-smooth codimension q foliations. Let

$$O(q+1, 1) = \{X \in GL(q+2\boldsymbol{R});\ {}^tXBX = B\}, \text{where } B = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

We can define subgroups $H \subset K \subset G$ of $O(q+1, 1)$ as follows:

(3.1) *Let G be the identity component of* $O(q+1, 1)$. *Then* $H = \left\{ \begin{pmatrix} X & 0 \\ 0 & I_2 \end{pmatrix};\ X \in SO(q) \right\}$ *is a compact subgroup of* G, *and* G/H *is an open* $(2q+1)$*-manifold.*

(3.2) *Let K be a subspace of G consisting of* $X = (x_{ij}) \in G$ *such that*

$$\det \begin{pmatrix} x_{q+1\,q+1} & x_{q+1\,q+2} \\ x_{q+2\,q+1} & x_{q+2\,q+2} \end{pmatrix} = 1$$

and

$$x_{i\,q+1} + x_{i\,q+2} = 0 \quad (i = 1, \cdots, q)$$

then K is a subgroup of G, *and G/Kis a q-maniford.*

Proof. The proof of (3.1) is trivial. We shall prove (3.2). Let $X=(x_{ij}) \in GL(q+2; \boldsymbol{R})$ such that

$$x_{i\,q+1} + x_{i\,q+2} = 0, \quad \text{for} \quad i = 1, \cdots, q .$$

If $X \in G \subset O(q+1, 1)$, then the followings hold.

1) $\det \begin{pmatrix} x_{q+1\,q+1} & x_{q+1\,q+2} \\ x_{q+2\,q+1} & x_{q+2\,q+2} \end{pmatrix} = \pm 1$,

and $\det \begin{pmatrix} x_{q+1\,q+1} & x_{q+1\,q+2} \\ x_{q+2\,q+1} & x_{q+2\,q+2} \end{pmatrix} = 1$, if and only if

$$x_{q+1\,q+1} + x_{q+1\,q+2} = x_{q+2\,q+1} + x_{q+2\,q+2};$$

2) $x_{q+1\,i} - x_{q+2\,i} = 0, \quad \text{for} \quad i = 1, \cdots, q .$

If the above equality holds, (3.2) folows from 1) and 2) (q.e.d.)

Set $\bar{M}=G/H$,then $\bar{M}$ is an open $(2q+1)$-manifold and $\bar{M}$ is foliated into the fibres of the fibre bundle $\bar{M}=G/H \to G/K$. We denote this foliation by $(\bar{M}, \bar{\mathcal{F}})$. Clearly, the foliation $(\bar{M}, \bar{\mathcal{F}})$ is a G-invariant foliation of codimension q on $\bar{M}$.

By A. Borel [2], the connected semi-simple Lie subgroup G of $GL(q+2; C)$ has the discrete subgroup Γ of G which contains a normal torsion free subgroup D of finite index. Since H is compact subgroup of G, the subgroup D acts freely on $\bar{M}=G/H$ and $D\backslash\bar{M}$ is compact. Therefore we have.

(3.3) *There exists a discrete subgroup DofG such that* $D\backslash\bar{M}$ *is a closed* $(2q+1)$-*manifold.*

Set $M=D\backslash\bar{M}$. Since the foliation $(\bar{M}, \bar{\mathcal{F}})$ is G-invariant, the closed $(2q+1)$-manifold M has a codimension q foliation $(M, \mathcal{F})$ induced naturally from $(\bar{M}, \bar{\mathcal{F}})$. This foliation $(M, \mathcal{F})$ is the example of foliation with non trivial exotic characteristic classes.

4. Proof of Theorem

4.1. Naturality of the homomorphism $\lambda(N, \mathcal{F})$

Let N be an oriented manifold without boundary and (,) be a Riemannian metric on N. Let V be the Riemannian connection on N. Given a foliation $(N, \mathcal{F})$ a metric connection V^0 and a basic connection ∇^1 on $\nu(\mathcal{F})$ are

defined as follows:

$$\nabla^0_X(Y) = \pi\nabla_X(Y)$$
$$\nabla^1_X(Y) = \pi[X_{\tau(\mathcal{F})}, Y] + \nabla^0_{X_{\nu(\mathcal{F})}}(Y)$$

for any $X \in \mathfrak{X}(N)$, $Y \in \Gamma(\nu(\mathcal{F}))$, where $\pi\colon \tau(N) \to \nu(\mathcal{F})$ is the natural projection, and $X_{\tau(\mathcal{F})} \in \Gamma(\tau(\mathcal{F}))$ $X_{\nu(\mathcal{F})} \in \Gamma(\nu(\mathcal{F}))$ are such that $X = X_{\tau(\mathcal{F})} + X_{\nu(\mathcal{F})}$. Here, of course, we consider the Riemannian metric on $\nu(\mathcal{F})$ induced naturally from the Riemannian metric $(\ ,\ )$ on $\tau(N)$.

Then the homomorphism of cochin complexes

$$\lambda_{(N,\mathcal{F})}\colon \quad WO_q \to A^*_C(N)$$

is uniquely determined from the above connections V° and ∇^1, hence from the foliation $(N, \mathcal{F})$ and the Riemannian metric $(\ ,\ )$. Thus we denote this $\lambda_{(N,30}(\omega)$ by $\omega((N, \mathcal{F}), (\ ,\ ))$ for $\omega \in WO_q$.

Now, let $O(q+1, 1)$, G, H, K, $(\bar{M}, \bar{\mathcal{F}})$ and (M, ff) be as in Section 3. Let $\mathfrak{o}(q+1, 1)$ (resp. $\mathfrak{gl}_{q+2}$) denote the Lie algebra of $O(q+1, 1)$ (resp. $GL(q+2; \boldsymbol{R})$), then clearly

$$\mathfrak{o}(q+1, 1) = \{X \in \mathfrak{gl}_{q+2};\ {}^tXB + BX = 0\}\ ,$$

and a basis of $\mathfrak{o}(q+1, 1)$ is given by the following elements:

$$Z = \left(\begin{array}{c|cc} O & O & \\ \hline & 0 & -2 \\ O & -2 & 0 \end{array}\right)$$

$$H_{ij} = \left(\begin{array}{ccc} \vdots & & \vdots \\ \cdots\ 0 & \cdots & 1\ \cdots \\ \vdots & & \vdots \\ \cdots -1 & \cdots & 0 \\ \vdots & & \vdots \\ \hat{i} & & \hat{j} \end{array}\right) \begin{array}{l} \\ \\ <i \\ \\ <j \\ \\ \\ \end{array}, \qquad 1 \leqq i < j \leqq q\ ,$$

$$X_i = \left(\begin{array}{c|cc} O & 1 & -1 \\ \hline -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right) \begin{array}{l} \\ <i \\ \\ \\ \end{array}, \qquad i = 1, \cdots, q\ ,$$
$$\hat{1}$$

$$Y_i = \left(\begin{array}{c|cc} O & 1 & 1 \\ \hline -1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right) \begin{array}{l} \\ <i \\ \\ \\ \end{array}, \qquad i = 1, \cdots, q\ .$$
$$\hat{i}$$

It is known that $\{H_{ij}\}_{1\leqq i<j\leqq q}$ is a basis of the Lie algebra of H. Then $T_0(\bar{M})$ is identified naturally with the subspace of $T_e(G)$ spanned by $\{X_1, \cdots, X_q, Y_1, \cdots, Y_q, Z\}$, where $o=H\in G/H=\bar{M}$.

In this time, we have

$$(Ad(g)(X_1), \cdots, Ad(g)(X_q), Ad(g)(Y_1), \ldots, Ad(g)(Y_q), Ad(g)(Z))$$
$$= (X_1, \cdots, X_q, Y_1, \cdots, Y_q, Z)\cdot\begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $g=\begin{pmatrix} A & 0 \\ 0 & I_2 \end{pmatrix}\in H$ $(A\in SO(q))$. Therefore, $\overline{\langle\ ,\ \rangle}_o=\sum_{i=1}^{q} X_i^*\otimes X_i^*+\sum_{i=1}^{q} Y_i^*\otimes Y_i^*+Z^*\otimes Z^*$ is an $Ad(H)$-invariant innerproduct on $T_o(\bar{M})$ where $\{X_1^*, \cdots, X_q^*, Y_1^*, \cdots, Y_q^*, Z^*\}$ denote the dual basis of $\{X_1, \cdots, X_q, Y_1, \cdots, Y_q, Z\}$. Hence, for any $u=g\ o\in\bar{M}(g\in G)$, an innerproduct $\overline{\langle\ ,\ \rangle}_u$ on $T_u(\bar{M})$ is defined by $\overline{\langle\ ,\ \rangle}_u=(g^{-1})^*\overline{\langle\ ,\ \rangle}_o$. Therefore we have

(4.1.1) *$\overline{\langle\ ,\ \rangle}$ is a G-invariant* ***Riemannian metric*** *on $\bar{M}$ and $\bar{M}$ is* ***orientable.***

Then we have the followings.

Lemma 4.1.2. *For any $\omega\in WO_q$, $\omega((\bar{M}, \bar{\mathcal{F}}), \overline{\langle\ ,\ \rangle})$ is a G-invariant differentiaform on $\bar{M}$.*

Lemma 4.1.3. *Let $\langle\ ,\ \rangle$ denots the* ***Riemannian metric*** *on M induced naturally from the* ***Riemannian*** *metric $\overline{\langle\ ,\ \rangle}$ on $\bar{M}$, then*

$$p^*\omega((M, \mathcal{F}), \langle\ ,\ \rangle) = \omega((\bar{M}, \bar{\mathcal{F}}), \overline{\langle\ ,\ \rangle})$$

for $\omega\in WO_q$, where p: $\bar{M}\to M$ is the natural projection.

Proof of (4.1.2). Let V be the Riemannian connection on the Riemannian manifold $(\bar{M}, \overline{\langle\ ,\ \rangle})$. Since the Riemannian metric $\overline{\langle\ ,\ \rangle}$ on $\bar{M}$ is G-invariant,

$$g_*(\bar{\nabla}_X(Y)) = \bar{\nabla}_{g_*X}(g_*Y)$$

for any $g\in G$ and $X,\ Y\in\mathfrak{X}(\bar{M})$.

Since the foliation $(\bar{M}, \bar{\mathcal{F}})$ is G-invariant, g_* maps $\Gamma(\tau(\bar{\mathcal{F}}))$(resp. $\Gamma(\nu(\bar{\mathcal{F}}))$) into $\Gamma(\tau(\bar{\mathcal{F}}))$ (resp. $\Gamma(\nu(\bar{\mathcal{F}}))$) and $g_*\pi=\pi g_*$ for $g\in G$, where π: $\tau(\bar{M})\to\nu(\bar{\mathcal{F}})$ is the natural projection.

Therefore we have

$$(*)\qquad \begin{aligned} g_*(\nabla^0_X(Y)) &= \nabla^0_{g_*X}(g_*Y), \\ g_*(\nabla^1_X(Y)) &= \nabla^1_{g_*X}(g_*Y), \qquad \text{for} \quad g\in G. \end{aligned}$$

Let k^1(resp. k^0) be the curvature matrix of ∇^1(resp. V°) associated to some local frame $S=\{s_1, \cdots, s_q\}$ of $\nu(\bar{\mathcal{F}})$. Then by (*), $(g^{-1})^*k^1$(resp. $(g^{-1})^*k^0$) is the curvature matrix of ∇^1(resp. V°) associated to the local frame $g_*S=\{g_*s_1, \cdots, g_*s_q\}$. But $c_i(k^1)\in A^{2i}(\bar{M})$ is independent of the choices of local frames, hence $c_i(k^1)$ is G-invariant. Therefore

$$c_i((\bar{M}, \bar{\mathcal{F}}), \overline{\langle\ ,\ \rangle}) = \lambda(\nabla^1)(c_i) = (\sqrt{-1}/2\pi)^i c_i(k^1)$$

is also G-invariant.

Similarly, $\lambda(\nabla^{01})(c_i)\in A_C^{2i}(\bar{M}\times \boldsymbol{R})$ is G-invariant. Hence it follows from (1.3.2) that $h_i((M, \mathcal{F}), \overline{\langle\ ,\ \rangle})$ is G-invariant. (q.e.d)

Proof of (4.1.3). Let V be the Riemannian connection on $(M, \langle\ ,\ \rangle)$. Since the natural projection p: $(\bar{M}, \overline{\langle\ ,\ \rangle})\to(M, \langle\ ,\ \rangle)$ is a local isometry, locally we have

$$p_*(\bar{\nabla}_X(Y)) = \nabla_{p_*X}(p_*Y)\,.$$

Therefore, the proof is similar to that of (4.1.2). (q.e.d.)

4.2. Local frame of $\nu(\bar{\mathcal{F}})$

Let $o=H\in\bar{M}=G/H$. To calculate the connection forms, we define local vector fields around $o\in\bar{M}$ as follows:

Define a prametrization ϕ around $o\in\bar{M}$ by

$$\begin{aligned}&\bar{\phi}(y_1, \cdots, y_q, x_1, \cdots, x_q, z)\\ &= \exp(\sum_{i=1}^{q} y_i Y_i)\exp(\sum_{i=1}^{q} x_i X)\exp(z\cdot Z)H\in\bar{M} = G/H\,.\end{aligned}$$

In the sequel, we use the vector notations such as $x=(x_1, \cdots, x_q), y=(y_1, \cdots, y_q)$

Set local vector fields $\bar{Z}, \bar{X}_1, \cdots, \bar{X}_q, \bar{Y}_1, \cdots, \bar{Y}_q$, around $o\in\bar{M}$, $\bar{Z}=\bar{\phi}_*(\partial/\partial z)$,

$$\bar{X}_i = \bar{\phi}_*(e^{-2z}\partial/\partial x_i), \qquad i = 1, \cdots, q\,,$$

$$\bar{Y}_j = \bar{\phi}_*(e^{2z}(\partial/\partial y_j + \sum_{k=1}^{q} x_k^2\partial/\partial x_j - 2x_j\sum_{k=1}^{q} x_k\partial/\partial x_k + x_j\partial/\partial z)), \qquad j = 1, \cdots, q\,,$$

at $u=\bar{\phi}(y, x, z)\in\bar{M}$.

Then we have

Lemma 4.2.

(1) *$\bar{X}_1, \cdots, \bar{X}_q, \bar{Z}$ are tangent to the foliation* $(\bar{M}, \bar{\mathcal{F}})$.

(2) *$\{\bar{X}_1, \cdots, X_q, Z, \bar{Y}_1, \cdots, \bar{Y}_q\}$ is a local orthonormal frame of $\tau(\bar{M})$ with respect to the Riemannian metric $\overline{\langle\ ,\ \rangle}$.*

(3) *(bracket relations)*

$$[\bar{X}_i, \bar{Z}] = 2X_i \qquad [\bar{X}_i, \bar{X}_j] = 0,$$
$$[\bar{Y}_j, \bar{Z}] = -2\bar{Y}_j,$$
$$[\bar{Y}_i, \bar{Y}_j]_u = 2e^{2z}(x_i(\bar{Y}_j)_u - x_j(\bar{Y}_i)_u)$$
$$[\bar{X}_i, \bar{Y}_j]_u = \begin{cases} (\bar{Z})_u - 2e^{2z}\sum_{\substack{k=1 \\ (k\neq i)}}^{q} x_k(\bar{X}_k)_u, & i = j \\ 2e^{2z}x_i(\bar{X}_j)_u, & i \neq j, \end{cases}$$

where $u=\bar{\phi}(y, x, z)\in\bar{M}$, *and* $i, j=1, \cdots, q$.

Proof. The bracket relations of (3) are calculated directly by the definitions of Z, X_i, $\bar{Y}_j$.

We shall prove (1) and (2).

First, we define a local parametrization ϕ around $e\in G$ and a local section σ around $o\in\bar{M}$ as follows:

Set

$$\phi(y, x, z, (h_{ij})_{1\leq i<j\leq q})$$
$$= \exp(\sum_{i=1}^{q} y_i Y_i)\exp(\sum_{i=1}^{q} x_i X_i)\exp(z\cdot Z)\exp(\sum_{1\leq i<j\leq q} h_{ij}H_{ij})\in G.$$

The local section σ is defined by requiring

$$\sigma\bar{\phi}(y,x, z) = \phi(y, x, z, (0)).$$

Then the next (4.2.1) and (4.2.2) follows from tedious calculations, which will be left to the reader.

(4.2.1) (1) *For* $g_1=\exp(\sum_{i=1}^{q} y_i^0 Y_i)$,

$$L_{g_1}(\phi(y, x, z, (h_{ij})) = \phi(y+y^0, x, z, (h_{ij}))$$

(2) *For* $g_2=\exp(\sum_{i=1}^{q} x_i^0 X_i)$,

$$R_{g_2}(\phi(y, x, z, (h_{ij})) = \phi(y, \bar{x}, z, (h_{ij})).$$

where $\bar{x}=x+e^{-2z}\,x^0\ {}^t(\exp(\sum h_{ij}H_{ij}))$.

(3) $\quad =\exp(z^0 Z+\sum_{1\leq i<j\leq q} h_{ij}^0 H_{ij})$

$$R_{g_3}(\phi(y, x, z, (h_{ij})) = \phi(y, x, z+z^0, (h_{ij}+h_{ij}^0)).$$

Here, of course, L_g*(resp.* R_g*) denote the left (resp. right) translation by* $g\in G$.

(4.2.2) *Let* $g=\phi(y, x, z, (h_{ij}))\in G$, *and* g_1, g_2, g_3 *be as in* (4.2.1), *and let* $X_i'=\phi_*(\partial/\partial x_i)$, $Y_i'=\phi_*(\partial/\partial y_i)$, $Z'=\phi_*(\partial/\partial z)$. *Then,*

(1) $(L_{g_1})_*((X_i')_g)=(X_i')_{g_1 g}$
$(L_{g_1})_*((Y_j')_g)=(Y_i')_{g_1 g}$
$(L_{g_1})_*(Z_g')=Z'_{g_1 g}$

for $i=1, \cdots, q$;

(2) $(R_{g_2})_*((X'_i)_g)=(X'_i)_{gg_2}$
$(R_{g_2})_*((Y'_i)_g)=(Y'_i)_{gg_2}$

for $i=1, \cdots, q$;

$$(R_{g_2})_*(Z'_g) = Z'_{gg_2}-2e^{-2z}\sum_{i,j=1}^{q} a_{ij}x_j^0(X'_i)_{gg_2}$$

where a_{ij} *is the* (i,j) *compaonent of* $\exp(\sum h_{ij}H_{ij})$;

(3) $(R_{g_3})_*((X'_j)_g)=(X'_i)_{gg_3}$
$(R_{g_3})_*((Y'_i)_g)=(Y'_i)_{gg_3}$
$(R_{g_3})_*(Z'_g)=Z'_{gg_3}$

for $i=1, \cdots, q$.

Then we have the following key lemma for the proof of Lemma 4.2.

(4.2.3) *The following* (a), (b), (c) *hold at* $g=\phi(y, x, z, (0))\in G$.

(a) $Z'_g=(L_g)_*Z$.

(b) $(X'_j)_g=e^{2z}(L_g)_*X_i$, $j=1, \cdots, q$.

(c) $$(Y'_i)_g=(L_g)_*(e^{-2z}Y_j+2e^{2z}x_j\sum_{k=1}^{q}x_kX_k-e^{2z}\sum_{k=1}^{q}x_k^2X_j-x_jZ+\sum_{\substack{k=1\\(k\neq j)}}^{q}2x_kH_{kj}),$$

for $j=1, \cdots, q$.

Here, of course, the elements $X_1, \cdots, X_q, Y_1, \cdots, Y_q, Z, H_{ij}$ *of* $\mathfrak{o}(q+1, 1)$ *are regarded as the elements of* $T_e(G)$, *and* $H_{ij}=-H_{ji}$ *foi*$>j$.

Proof. First, notice that

(*) $(X'_i)_e=X_i$, for $i=1, \cdots, q$,
$(Y'_i)_e=Y_i$, for $i=1, \cdots, q$,
$Z'_e=Z$.

For $g=(y, x, z, (0))$, set $g_1=\exp(\sum_{i=1}^{q}y_iY_i)$, $g_2=\exp(\sum_{i=1}^{q}x_iX_i)$, $g_3=\exp(z\cdot Z)$.

Then $g=g_1\cdot g_2\cdot g_3$.

We shall prove *(a)*. By (4.2.2),

$$\begin{aligned} Z'_g &= (L_{g_1})_*(Z'_{g_2g_3}) \\ &= (L_{g_1})_*(R_{g_3})_*Z'_{g_2} \\ &= (L_{g_1})_*(R_{g_3})_*(R_{g_2})_*(Z'_e+2\sum_{k=1}^{q}x_k(X'_k)_e)\,. \end{aligned}$$

Then, by (*),

$$\begin{aligned} Z'_g &= (L_{g_1})_*(R_{g_3})_*(R_{g_2})_*(Z+2\sum_{k=1}^{q}x_kX_k) \\ &= (L_g)_*(L_{(g_2g_3)^{-1}})_*(R_{g_2g_3})_*(Z+2\sum_{k=1}^{q}x_kX_k) \\ &= (L_g)_*Ad((g_2g_3)^{-1})\,(Z+2\sum_{k=1}^{q}x_kX_k) \\ &= (L_g)_*Ad(g_3^{-1})Ad(g_2^{-1})\,(Z+2\sum_{k=1}^{q}x_kX_k)\,. \end{aligned}$$

But, $Ad(g_2^{-1})(Z)=Z-2\sum_{k=1}^{q}x_kX_k$,

$Ad(g_2^{-1})(X_k)=X_k$, for $k=1, \cdots, q$,

and $Ad(g_3^{-1})(Z)=Z$.

Therfore, $Z'_g=(L_g)_*Z$.

(b) and (c) are proved similarly. (q.e.d.)

Now, let $\bar{P}$: $G\to\bar{M}=G/H$ be the natural projection, then clearly we have

$$\begin{aligned}
&\bar{P}_*(X_i)=(\bar{X}_i)_o \qquad i=1, \cdots, q,\\
&\bar{P}_*(Y_j)=(\bar{Y}_j)_o \qquad j=1, \cdots, q,\\
&\bar{P}_*(Z)=(\bar{Z})_o\\
&\bar{P}_*(H_{ij})=0, \qquad 1\leqq i<j\leqq q,
\end{aligned}$$

where $o=H\in\bar{M}=G/H$.

Hence, by the definition of the G-invariant Riemannian metric $\langle\overline{\ ,\ }\rangle$ on $\bar{M}$, the following lemma shows Lemma 4.2 (2).

(4.2.4) *Let σ be the local section defined as before, then*

(a) $(\bar{Z})_u=\sigma(u)_*(\bar{Z})_o$,

(b) $(\bar{X}_i)_u=\sigma(u)_*(\bar{X}_i)_o, i=1, \cdots, q$,

(c) $(\bar{Y}_j)_u=\sigma(u)_*(\bar{Y}_j)_o, j=1, \cdots, q$,

for any point u of some neighborhood of o in $\bar{M}$.

Proof. Clearly, $\bar{P}_*(L_{\sigma(u)})_*=\sigma(u)_*\bar{P}_*$, and $(\phi_*)_{\sigma(u)}=\sigma_*\circ(\bar{\phi}_*)_u$. Then, in view of the definitions of $\bar{X}_i$, $\bar{Y}_j$, $\bar{Z}$, we have (4.2.4) easily from (4.2.3). (q.e.d.)

By the definition of Lie algebras of Lie groups, we have the following easily.

(4.2.5) *The following elements $\mathfrak{o}(q+1, 1)$ form a basis of the Lie algebra of K,*

$$Z, X_1, \cdots, X_q, H_{ij}, 1\leqq i<j\leqq q,$$

where the subgroup K of $O(q+1, 1)$ is as in (3.2).

Since the foliation $(\bar{M}, \bar{\mathcal{F}})$ is G-invariant and each leaf of this foliation is a fibre of the fibre bundle $\bar{M}=G/H\to G/K$, we have Lemma 4.2 (1) from (4.2.4) and (4.2.5).

This completes the proof of Lemma 4.2.

REMARK. Consider Z, X_i, Y_j of $\mathfrak{o}(q+1,1)$ as left-invariant vector fields on G. It hold that $\bar{Z}=\bar{P}_*(Z)$ and $\bar{Z}$ is G-invariant, then we may define $\bar{Z}$ by $\bar{P}_*(Z)$. However it is impossible to define X_i (resp. $\bar{Y}_j$) by P_*X_i (resp. P_*Y_j), for X_i, Y_j are not $Ad(H)$-invariant.

4.3. Calculation of $c_i((\bar{M}, \bar{\text{ff}}), \langle\overline{\ ,\ }\rangle)$ and $h_j((\bar{M}, \bar{\mathcal{F}}), \langle\overline{\ ,\ }\rangle)$.

Let Z^* (resp. $\bar{X}_i^*$, $\bar{Y}_j^*$) denote the dual one form of Z (resp. X_i, $\bar{Y}_j$) with

respect to the G-invariant Riemannian metric $\overline{\langle\ ,\ \rangle}$ on $\bar{M}$.

Then we have

Lemma 4.3. *At* $o=H\in\bar{M}=G/H$,

(1) $c_i((M,\mathrm{ff}),\overline{\langle\ ,\ \rangle})=\alpha_i(\sqrt{-1}/2\pi)^i(d\bar{Z}^*)^i$, $\alpha_i>0$, *for* $i=1,\ \cdots,\ q$;

(2) $h_j((M,\mathcal{F}),\overline{\langle\ ,\ \rangle})=\beta_j(\sqrt{-1}/2\pi)^j\bar{Z}^*\wedge(d\bar{Z}^*)^{j-1}$, $\beta_j<0$, *for* $j=1,3,\ \cdots,r$.

We shall prove Lemma 4.3. As usual, dx_i and rfy_z are regarded as local 1-forms on $\bar{M}$ *by* the parametrization $\bar{\phi}$. It is easy to see the following.

(4.3.1) *Let* Z^*, $\bar{X}_i^*,\bar{Y}_j^*$ *be as above, then*

(1) $\bar{Y}_j^*=e^{-2z}dy_j$ *for* $j=1,\ \cdots,\ q$, *at* $u=\bar{\phi}(y_1,\ \cdots,\ y_q,\ x_1,\ \cdots,\ x_q,\ z)\in\bar{M}$;

(2) $d\bar{Z}=\sum_{i=1}^{q}dy_i\wedge dx_i=\sum_{i=1}^{q}\bar{Y}_i^*\wedge\bar{X}_i^*$.

Since the Riemannian metric $\overline{\langle\ ,\ \rangle}$ on $\bar{M}$ is given, connections V° and ∇^1 on $\nu(\bar{\mathcal{F}})$ are uniquely defined as in Section 4.1. Then $\{\bar{Y}_1,\ \cdots,\bar{Y}_q\}$ is a local orthonormal frame of $\nu(\bar{\mathcal{F}})$ by Lemma 4.2. Let $\theta^0=(\theta_{ij}^0)$ (resp. $\theta^1=(\theta_{ij}^1)$) be the connection form of ∇^0 (resp. ∇^1) relative to the frame $\{\bar{Y}_1,\cdots,\ \bar{Y}_q\}$, then we have

(4.3.2) *At* $u=\phi(y_1,\ \cdots,y_q,x_1,\ \ \cdots,\ x_q,\ z)\in\bar{M}$

(1) $\theta_{ij}^0=\begin{cases}0 & i=j\\ e^{2z}(x_j\bar{Y}_i^*-x_i\bar{Y}_j^*), & i\neq j\end{cases}$

(2) $\theta_{ij}^1=\begin{cases}2\bar{Z}^*, & i=j\\ \theta_{ij}^0, & i\neq j.\end{cases}$

Proof. Let V be the Riemannian connection on the Riemannian manifold $(\bar{M},\overline{\langle\ ,\ \rangle})$ and $\theta=(\theta_{ij})$ be the connection form of V relative to the frame $\{\bar{Y}_1,\ \cdots,\ \bar{Y}_q,\ \bar{X}_1,\ \cdots,\ \bar{X}_q,\ \bar{Z}\}$ of $\tau(\bar{M})$.

Set $s_1=\bar{Y}_1,\ \cdots,\ s_q=\bar{Y}_q$, $s_{q+1}=\bar{X}_1,\cdots,\ s_{2q}=\bar{X}_q, s_{2q+1}=\bar{Z}$. Then by the definition of V,

$$d\overline{\langle s_i,\ s_j\rangle}(X)=\overline{\langle\nabla_X s_i,\ s_j\rangle}+\overline{\langle s_i,\ \nabla_X s_j\rangle}$$

for $X\in\mathfrak{X}(\bar{M})$. Hence

(i) $\theta_{ij}=-\theta_{ij}$, for $i,j=1,\ \cdots,\ 2q+1$.

Moreover $\nabla_{s_i}(s_j)-\nabla_{s_j}(s_i)=[s_i,\ s_j]$, then we have the followings (ii) (iii) from Lemma 4.2 (3).

For $i,j=1,\ \cdots,\ q$ and $i\neq j$.

(ii) $\theta_{ij}(s_j)=-2e^{2z}x_i$, $\theta_{ij}(s_i)=2e^{2z}x_j$.

(iii) $\theta_{jk}(s_i)=\theta_{ik}(s_j)$, for $k=1,\cdots,2q+1$ and $k\neq i,j$.

$$\theta_{q+j\,k}(s_i)=\theta_{ik}(s_{q+j}),\text{for } k=1,\ \cdots,\ q.$$
$$\theta_{2q+1\ k}(s_i)=\theta_{ik}(s_{2q+1})\text{for } k=1,\ \cdots,\ 2q+1,$$

and $k\neq i,\ 2q+l.$

Let $i, j=1,\ \cdots, q.$ Now, let $k=1,\ \cdots, q,$ and $k\neq i, j,$ then by (i) and (iii),

$$\theta_{ij}(s_k)=\ -\theta_{ji}(s_k)=\ \text{-MO}=\text{MO}=\theta_{jk}(s_i)=\ -\theta_{kj}(s_i)=\ -\theta_{ij}(s_k)$$

Hence $\theta_{ij}(s_k)=0.$ It is shown similarly by making use of (i) and (iii) that $\theta_{ij}(s_k)=0$for $k=q+1,\ \cdots, 2q+1.$ Therefore we have

$$\theta_{ij}(s_k)=\begin{cases}-2e^{2z}x_i\,, & k=j\\ 2e^{2z}x_j\,, & k=i\\ 0\,, & k\neq i, j\end{cases}$$

for $i, j=1,\ \ ,\ q$ and $i\,\Phi j.$

But by the definition of V°,

$$\theta^0_{ij}=\theta_{ij},\quad \text{for}\quad i, j=1,\ \cdots, q\,.$$

Hence we have (4.3.2) (1).

(4.3.2) (2) is shown easily by the following.

By the definition of ∇^1 and Lemma 4.2 (3),

$$\nabla^1_X(\bar{Y}_i)=\begin{cases}0\,, & \text{if}\quad X=\bar{X}_1,\ \cdot\cdot\ ,\ \bar{X}q\\ 2\bar{Y}_i\,, & \text{if}\quad X=\bar{Z}\\ \nabla^0_X(\bar{Y}_i), & \text{if}\quad X=\ \bar{Y}_1,\ \cdot\cdot\quad \bar{Y}_q\end{cases}$$

for $i=1,\ \cdots, q.$ (q.e.d.)

Now, let $k^1=(k^1_{ij})$ be the curvature matrix of ∇^1 associated to the frame $\{\bar{Y}_1,\cdots,\ \bar{Y}_q\}.$ Let ∇^{01} be the connection on $\nu(\bar{\mathcal{F}})\times\boldsymbol{R}$ defined by V° and ∇^1 as in (1.2.2) (3), and $\theta^{01}=(\theta^{01}{}_{ij})$ be the connection form of $\nabla^{01},$ that is,

$$\theta^{01}=(1-t)\theta^0+t\theta^1\,,$$

and $k^{01}=(k^{01}{}_{ij})$ be the curvature matrix of $\nabla^{01}.$ Then we have

(4.3.3) (1) *At* $o=H\in\bar{M}=G/H,$

$$k^0_{ij}=\begin{cases}2d\bar{Z}^* & i=j\\ 2(dx_j\wedge dy_i-dx_i\wedge dy_j), & i\neq j\,.\end{cases}$$

(2) *At* $(o,\ t)\in o\times\boldsymbol{R}\subset\bar{M}\times\boldsymbol{R},$

$$k^{01}_{ij}=\begin{cases}2dt\wedge\bar{Z}^*+2td\bar{Z}^*, & i=j\\ 2\pi^*(dx_j\wedge dy_i-dx_i\wedge dy_j), & i\ \neq\end{cases}$$

where $\pi: \bar{M}\times \boldsymbol{R}\to\bar{M}$ *in the natnral projection.*

Proof. By (4.3.2) (1), proof of (1) is trivial. By the definition of θ^{01},

$$\theta^{01}_{ij}=\begin{cases}2t\bar{Z}_*, & i=j\\ 2e^{2z}(x_j\bar{Y}^*_i-x_i\bar{Y}^*_j), & i\neq j,\end{cases}$$

then by (4.3.1), we have (2). (q.e.d.)

By the definition of the determinant of matrices, we have the following.

(4.3.4) *Let* $K=(K_{ij})$ *be a qxq matrix of* 2*-forms. Assume that*

$$K_{ij}=\begin{cases}\omega, & i=j\\ \gamma_j\wedge\eta_i-\gamma_i\wedge\eta_j, & i\neq j,\end{cases}$$

where $\gamma_1, \cdots, \gamma_q, \eta_1, \cdots, \eta_q$ *are* 1*-forms. Then*

$$c_i(K)=\sum_{\substack{0\leqq n\leqq i\\ n \text{ are even}}} a_{in}\cdot\omega^{i-n}\wedge\Big(\sum_{k=1}^{q}\eta_k\wedge\gamma_k\Big)^n,$$

for $i=1, \cdots, q$ *and each* a_{in} *is a positive number.*

Now, $c_i((\bar{M},\bar{\mathcal{F}}),\langle\ ,\ \rangle)$ and $h_j((\bar{M},\ \text{ff}),\langle\ ,\ \rangle)$ are calculated as follows. In view of (4.3.1), we have Lemma 4.3 (1) from (4.3.3) (1) and (4.3.4). Similarly, by (4.3.3) (2),

$$c_j(k^{01})=\sum_{\substack{0\leqq n\leqq j\\ n \text{ are even}}} a_{jn}\cdot 2^n(j-n)t^{j-n-1}dt\wedge\bar{Z}^*\wedge(d\bar{Z}^*)^{j-1}$$
$$+\ (\text{terms which do not contain } dt),$$

for $j=1, \cdots, q$, at $(o,t)\in\bar{M}\times\boldsymbol{R}$.

Let $i: \bar{M}\times[0,\ 1]\hookrightarrow\bar{M}\times\boldsymbol{R}$ be the inclusion mapping, and π: $\bar{M}\times[0,\ 1]\to\bar{M}$ be the natural projection, then then by the definition of Integration along the fibre π_*,

$$h_j((\bar{M},\bar{\mathcal{F}}),\overline{\langle\ ,\ \rangle})=(\sqrt{-1}/2\pi)^j\pi_*i^*c_j(k^{01})$$
$$=\beta_j(\sqrt{-1}/2\pi)^j\cdot\bar{Z}^*\wedge(d\bar{Z}^*)^{j-1},\ \beta_j>0.$$

This completes the proof of Lemma 4.3.

4.4. Proof of Theorem

Let ω be an element of WO_q with degree $2q+1$ such that $\omega=\phi\ h_j$ for some monomial $\phi\in\boldsymbol{R}_q[c_1, \cdots, c_q]$ as in Lemma 2.2. By Lemma 4.2 (1), $\{\bar{X}_1, \cdots, \bar{X}_q, \bar{Y}_1, \ , \bar{Y}_q,$ is a local frame of $\tau(\bar{M})$. Then by (4.3.1) (2), $(2q+1)$-form $\bar{Z}^*\wedge(d\bar{Z}^*)^q=q!\cdot\bar{Z}^*\wedge(\bar{Y}^*_1\wedge\bar{X}^*_1)\wedge\cdots\wedge(\bar{Y}^*_q\wedge\bar{X}^*_q)$ is non zero around $o\in\bar{M}$. Hence $\omega((\bar{M},\bar{\mathcal{F}}),\overline{\langle\ ,\ \rangle})$ is non zero at $o\in\bar{M}$ by Lemma 4.3. On the other hand, $\omega((\bar{M},\bar{\mathcal{F}}),\overline{\langle\ ,\ \rangle})$ is a G-invariant form on $\bar{M}$ by Lemma 4.1.2.

Therefore $\omega((\bar{M}, \bar{\mathcal{F}}), \overline{< , >})$ is nowhere zero on $\bar{M}$. In view of Lemma 4.1.3, $\omega((M, \mathcal{F}), < , >)$ is a nowhere zero $(2q+1)$-form on the closed orientable $(2q+1)$-manifold M. Therefore $\omega((M, \mathcal{F}), < , >)$ represents a non zero cohomology class of $H^{2q+1}(M\ C)$.

OSAKA UNIVERSITY

References

[1] R. Bott: Lectures on Characteristic Classes and Foliations, Lecture Notes in Math. No. 279, Springer-Verlag, 1972.

[2] A. Borel: *Compact Clifford-Klein forms of symmetric spaces*, Topology 2 (1962), 111-123.

[3] C. Godbillon and J. Vey: *Un invariant des feuilletages de condimension* 1. C.R. Acad. Sci Paris Sér. A. 273 (1971) 92.