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EXAMPLES OF FOLIATIONS WITH NON TRIVIAL EXOTIC CHARACTERISTIC CLASSES

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Introduction

An example of foliation of codimension one with non trivial Godbillon-Vey invariant ([3]) was constructed by R. Roussarie (see Bott [1]). Generalizing the Godbillon-Vey invariant, R. Bott [1] has defined exotic characteristic classes for foliations. In this paper, we shall construct examples of foliations with non trivial exotic characteristic classes.

Roussarie's example was constructed on a compact quotient space of $SL(2; \mathbf{R})$ by a discrete subgroup. This example may be regarded as an Anosov foliation arising from the geodesic flow on the unit tangent sphere bundle of a surface with constant negative curvature. This suggests us to consider such a foliation on the unit tangent sphere bundle of a closed $(q+1)$ -manifold ($q \geq 1$) with constant negative curvature. In fact, our example is constructed as follows. Let G denote the identity component of the Lie group

$$O(q+1, 1) = \{X \in GL(q+2; \mathbf{R}); {}^tXBX = B\},$$

where

$$B = \begin{pmatrix} I_{q+1} & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider a compact subgroup

$$H = \left\{ \begin{pmatrix} X & 0 \\ 0 & I_2 \end{pmatrix}; X \in SO(q) \right\}$$

of G , and a closed subgroup K consisting of $X = (x_{ij}) \in G$ such that

$$\det \begin{pmatrix} x_{11} & \cdots & x_{q+1, q+2} \\ \vdots & \ddots & \vdots \\ x_{q+2, q+1} & \cdots & x_{q+2, q+2} \end{pmatrix} = 1$$

and

$$x_{i, q+1} + x_{i, q+2} = 0 \quad (i = 1, \dots, q).$$

By a theorem of A. Borel [2], there exists a discrete subgroup D of G such that $D \backslash \bar{M}$ is a closed manifold, where $\bar{M} = G/H$. Foliate \bar{M} into the fibres of the fibre bundle $\bar{M} = G/H \rightarrow G/K$ and consider the foliation on $M = D \backslash \bar{M}$ induced naturally from the G -invariant foliation on \bar{M} . Then it is proved that the foliation on M has non trivial exotic characteristic classes.

In §1, we review differential geometry which will be needed. In §2, we define exotic characteristic classes following R. Bott [1] and state our result precisely. §3 is devoted to construct examples of foliations with non trivial exotic characteristic classes. The proof of our result will be given in §4.

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1. Preliminaries

1.1. First, we shall fix some notations:

For a smooth manifold N , we put

$\mathfrak{X}(N) = \{\text{smooth vector fields on } N\}$,

$C^\infty(N) = \{\text{smooth real valued functions on } N\}$,

$A_C^*(N) = \{\text{the space of complex smooth forms on } N\}$,

$A^*(N) = \{\text{the space of (real) smooth forms on } N\}$,

$A^p(N) = \{\omega \in A^*(N); \omega \text{ is } p\text{-form}\}$,

$A_o^p(N) = \{\omega \in A^p(N); \omega \text{ has compact support}\}$,

$\Gamma(\xi) = \{\text{the set of smooth cross-sections of a smooth vector bundle } \xi \text{ over } N\}$.

For a C^∞ -smooth codimension q foliation \mathcal{F} on N , we denote by $\tau(\mathcal{F})$ (resp. $\nu(\mathcal{F})$) the subbundle of $\tau(N)$ tangent (resp. normal with respect to a Riemannian metric on N) to the foliation, where $\tau(N)$ denotes the tangent bundle of N .

1.2. Connections

Let N be a smooth manifold and ξ a smooth q -dimensional vector bundle over N .

(1.2.1) (1) A connection on ξ is an \mathbf{R} -bilinear map

$$V: \mathfrak{X}(N) \times \Gamma(\xi) \rightarrow \Gamma(\xi)$$

such that

$$(i) \quad \nabla_X(f\xi) = X(f)\xi + f\nabla_X(\xi)$$

$$(ii) \quad \nabla_{fX}(\xi) = f\nabla_X(\xi)$$

for all $X \in \mathfrak{X}(N)$, $\xi \in \Gamma(\xi)$, $f \in C^\infty(N)$, where $\nabla_X(\xi) = \nabla(X\xi)$.

(2) Let $S = \{s_1, \dots, s_q\}$ be a smooth frame of ξ defined on some open set U in N . The connection form of V relative to the frame S is a $q \times q$ matrix $\theta = (\theta_{ij})$ of 1-forms on U such that

$$\nabla_X(s_i) = \sum_{j=1}^q \theta_{ij}(X) s_j$$

for $i=1, \dots, q$.

The curvature matrix of V associated to the frame S is a $q \times q$ matrix $k=(k_{ij})$ of 2-forms on U such that

$$k_{ij} = d\theta_{ij} - \sum_{h=1}^q \theta_{ih} \wedge \theta_{hj}$$

for $i, j=1, \dots, q$.

Let (N, \mathcal{F}) be a C^∞ -smooth codimension q foliation on N , and (\cdot, \cdot) be a Riemannian metric on $\nu(\mathcal{F})$ not necessarily induced from a Riemannian metric on N .

(1.2.2) (1) A metric connection on $\nu(\mathcal{F})$ is a connection V° on $\tau(F)$ such that

$$d(s_1, s_2)(X) = (\nabla_X^0(s_1), s_2) + (s_1, \nabla_X^0(s_2))$$

for $X \in \mathfrak{X}(N)$, $s_1, s_2 \in \Gamma(\nu(\mathcal{F}))$.

(2) A basic connection on $\nu(\mathcal{F})$ is a connection ∇^1 on $\nu(\mathcal{F})$ such that

$$\nabla_X^1(s) = \pi[X, \tilde{s}]$$

for $X \in \Gamma(\tau(\mathcal{F}))$, $s \in \Gamma(\nu(\mathcal{F}))$, where $\pi: \tau(N) \rightarrow \nu(\mathcal{F})$ is the natural projection, and $\tilde{s} \in \mathfrak{X}(N)$ is such that $\pi(\tilde{s})=s$.

(3) Let $\nu(\mathcal{F}) \times \mathbf{R}$ denote the vector bundle over $N \times \mathbf{R}$ with the same fibre dimension as $\nu(\mathcal{F})$. Given a metric (resp. basic) connection V° (resp. ∇^1) on $\nu(\mathcal{F})$, a unique connection ∇^{01} on $\nu(\mathcal{F}) \times \mathbf{R}$ is defined by requiring

- (i) On sections s which are constant in \mathbf{R} -direction, let $\nabla_{\partial/\partial t}^{01}(s)=0$;
- (ii) If $X \in T_{(x,t)}(N \times \{t\})$, define

$$\nabla_X^{01}(s) = (1-t)\nabla_X^0(s) + t\nabla_X^1(s)$$

for $s \in \Gamma(\nu(\mathcal{F}) \times \mathbf{R})$.

Clearly, for a smooth frame $S=\{s_1, \dots, s_q\}$ of $\tau(F)$ defined on some open set U in N , a smooth frame $S'=\{s'_1, \dots, s'_q\}$ of $\nu(\mathcal{F}) \times \mathbf{R}$ on $U \times \mathbf{R}$ is defined by

$$s'_i(x, t) = (s_i(x), t), \quad i = 1, \dots, q,$$

then connection form θ^{01} of ∇^{01} relative to the frame S' is represented as follows:

$$\theta^{01} = (1-t)\theta^0 + t\theta^1$$

where θ^0 (resp. θ^1) is the connection form of ∇^0 (resp. ∇^1) relative to the frame S .

Let N be a Riemannian manifold, (\cdot, \cdot) be the Riemannian metric on N . The following is well known.

(1.2.3) *There exists a unique connection V on $\tau(N)$ satisfying the following conditions:*

- (i) $\nabla_X(Y) - \nabla_Y(X) = [XY]$

for $X, Y \in \mathfrak{X}(N)$;

$$(ii) \quad d(Y_1, Y_2)(X) = (\nabla_X(Y_1), Y_2) + (Y_1, \nabla_X(Y_2))$$

for $X, Y_1, Y_2 \in \mathfrak{X}(N)$.

This connection is called the Riemannian connection on N . Clearly, let ∇ be the Riemannian connection on N , and $/$ be an isometry on N , then

$$f_*(\nabla_X(Y)) = \nabla_{f_*X}(f_*Y)$$

for $X, Y \in \mathfrak{X}(N)$.

1.3. Integration along the fibre

(1.3.1) (Bott [1]) Let N be an oriented smooth manifold and $\pi: NX[0, 1] \rightarrow N$ be the natural *projection*, then there exists a *unique homomorphism of $C^\infty(N)$ -modules*

$$\pi_*: A^p(N \times [0, 1]) \rightarrow A^{p-1}(N), \quad \text{for } p \geq 1,$$

satisfying the equation

$$\int_{N \times [0, 1]} \phi \wedge \pi^* \psi = \int_N \pi_* \phi \wedge \psi$$

for all $\phi \in A^p(N \times [0, 1])$, $\psi \in A^r(N)$, where $r = \dim N - p + 1$.

This homomorphism π_* is called integration along the fibre. Then it is easy to see the following.

(1.3.2) Let \bar{N}, N be oriented smooth manifolds of dimension n , and $\pi: \bar{N}X[0, 1] \rightarrow \bar{N}$, $\pi: N \times [0, 1] \rightarrow N$ be the natural *projections*, then for any *immersion* $f: \bar{N} \rightarrow N$, the following diagram is *commutative*:

$$\begin{array}{ccc} A^p(\bar{N} \times [0, 1]) & \xrightarrow{\pi_*} & A^{p-1}(\bar{N}) \\ \uparrow (f \times id)^* & & \uparrow f^* \\ A^p(N \times [0, 1]) & \xrightarrow{\pi_*} & A^{p-1}(N) \end{array} \quad \text{for } p \geq 1.$$

2. Exotic characteristic classes and Theorem

In R. Bott [1], exotic characteristic classes for foliations have been defined as follows.

Let $q \geq 1$ be an integer.

First, a cochain complex (WO_q, d) is defined. Let $R[c_1, \dots, c_q]$ denote the graded polynomial algebra over R generated by the elements c_i with degree $2i$. Set

$$R_q[c_1, \dots, c_q] = R[c_1, \dots, c_q] / \{\phi; \deg(\phi) > 2q\}.$$

Let $E(h_1, h_3, \dots, h_r)$ denote the exterior algebra over R generated by the elements

h_i with degree $2i-1$, where r is the largest odd integer $\leq q$. Then as a graded algebra over R

$$WO_q = \mathbf{R}_q[c_1, \dots, c_q] \otimes E(h_1, h_3, \dots, h_r),$$

and a unique antiderivation of degree 1

$$d: WO_q \rightarrow WO_q$$

is defined by requiring

$$\begin{aligned} d(c_i) &= 0, & i &= 1, \dots, q \\ d(h_i) &= c_i, & i &= 1, 3, \dots, r. \end{aligned}$$

Let ξ be a smooth q -dimensional vector bundle over a manifold N and V a connection on ξ . For a curvature matrix k of V , local $2i$ -forms $c_i(k)$ on N are defined by the following formula

$$\det(I_q + tk) = 1 + \sum_{i=1}^q t^i c_i(k).$$

Since $c_i(k)$ do not depend on the choice of the local frame of ξ , $c_i(k)$ define global $2i$ -forms on N . Then a homomorphism of graded \mathbf{R} -algebras

$$\lambda(\nabla): \mathbf{R}[c_1, \dots, c_q] \rightarrow A_C^*(N)$$

is defined by requiring

$$\lambda(\nabla)(c_i) = (\sqrt{-1}/2\pi)^i c_i(k), \quad \text{for } i = 1, \dots, q.$$

Let N be an oriented smooth manifold without boundary and (N, \mathcal{F}) a C^∞ -smooth codimension q foliation on N . Let V° (resp. ∇^1) be a metric (resp. basic) connection on $\nu(\mathcal{F})$ and ∇^{01} be as in (1.2.2) (3). Then the followings hold.

(2.1) (1) $\lambda(\nabla^1)(\phi) \in A_C^*(N)$ is a closed form for any $\phi \in \mathbf{R}[c_1, \dots, c_q]$, and if $\deg(\phi) > 2q$ then $\lambda(\nabla^1)(\phi) = 0$.

(2) $\lambda(V^\circ)(\phi) = 0$ for $\phi \in \mathbf{R}[c_1, \dots, c_q]$ such that $\deg(\phi)/2$ is an odd integer.

(3) Let $\pi: NX[0, 1] \rightarrow N$ be the natural projection and $i: NX[0, 1] \rightarrow NX$ the inclusion mapping, then

$$d(\pi_* i^* \lambda(\nabla^{01})(\phi)) = \lambda(\nabla^1)(\phi) - \lambda(\nabla^0)(\phi)$$

for $\phi \in \mathbf{R}[c_1, \dots, c_q]$, especially

$$d(\pi_* i^* \lambda(\nabla^{01})(c_{2j-1})) = \lambda(\nabla^1)(c_{2j-1}),$$

where π_* is the integration along the fibre.

In view of (2.1), given a C^∞ -smooth codimension q foliation (N, \mathcal{F}) on an oriented smooth manifold N without boundary, a homomorphism of cochain

complexes

$$\lambda(N, \mathcal{F}): WO_q \rightarrow A_c^*(N)$$

is defined by requiring

$$\begin{aligned}\lambda(N, \mathcal{F})(c_i) &= \lambda(\nabla^1)(c_i), \\ \lambda(N, \mathcal{F})(h_j) &= \pi_* i^* \lambda(\nabla^{01})(c_j).\end{aligned}$$

We used the notation $\lambda(N, 30$ in place of λ_E of Bott [1]. Here the homomorphism $\lambda(N, 30$ depends only on the choices of two connections ∇^0 and ∇^1 on $\nu(\mathcal{F})$. In cohomology, $\lambda(N, \mathcal{F})$ induces a homomorphism of graded \mathbf{R} -algebras

$$\lambda_{(N, \mathcal{F})}^*: H^*(WO_q) \rightarrow H^*(N; C)$$

which depends only on the foliation (N, F) .

The elements of $\lambda_{(N, \mathcal{F})}^*(H^*(WO_q))$ are called the exotic characteristic classes for the foliation (N, \mathcal{F}) .

It is easy to see the following lemma.

Lemma 2.2. *Each canonical generator of $H^{2q+1}(WO_q)$ is represented by some $\phi \cdot h_j \in WO_q$, where $\phi \in \mathbf{R}_q[c_1, \dots, c_q]$ is a monomial with degree $2(q-j+1)$.*

Then we have

Theorem. *For any integer $q \geq 1$, there exists a C^∞ -smooth codimension q foliation (M, \mathcal{F}) on a closed $(2q+1)$ -manifold such that all the exotic characteristic classes for the foliation which correspond to the canonical generators $[\phi \cdot h_j]$ of $H^{2q+1}(WO_q)$ are non zero in $H^{2q+1}(M; C)$.*

REMARK. When $q=1$, the generator fo-AJ of $H^3(WO_1) \cong \mathbf{R}$ is the Godbillon-Vey invariant, and our foliation of codimension one is diffeomorphic to the foliation constructed by R. Roussarie (cf. [1]).

3. Construction of the foliation (M, \mathcal{F})

Throughout this paper, integer $q \geq 1$ is to be fixed, and all foliations are to be C^∞ -smooth codimension q foliations. Let

$$O(q+1, 1) = \{X \in GL(q+2, \mathbf{R}); {}^t X B X = B\}, \text{ where } B = \begin{pmatrix} 1 & & \\ & \text{---} & \\ & & 1 \end{pmatrix}.$$

We can define subgroups $H \subset K \subset G$ of $O(q+1, 1)$ as follows:

(3.1) Let G be the identity component of $O(q+1, 1)$. Then $H = \left\{ \begin{pmatrix} X & 0 \\ 0 & I_2 \end{pmatrix}; X \in SO(q) \right\}$ is a compact subgroup of G , and G/H is an open $(2q+1)$ -manifold.

(3.2) Let K be a subspace of G consisting of $X = (x_{ij}) \in G$ such that

$$\det \begin{pmatrix} x_{q+1, q+1} & x_{q+1, q+2} \\ x_{q+2, q+1} & x_{q+2, q+2} \end{pmatrix} = 1$$

and

$$x_{i, q+1} + x_{i, q+2} = 0 \quad (i = 1, \dots, q)$$

then K is a subgroup of G , and G/K is a q -manifold.

Proof. The proof of (3.1) is trivial. We shall prove (3.2). Let $X = (x_{ij}) \in GL(q+2; \mathbf{R})$ such that

$$x_{i, q+1} + x_{i, q+2} = 0, \quad \text{for } i = 1, \dots, q.$$

If $X \in G \subset O(q+1, 1)$, then the followings hold.

$$1) \quad \det \begin{pmatrix} x_{q+1, q+1} & x_{q+1, q+2} \\ x_{q+2, q+1} & x_{q+2, q+2} \end{pmatrix} = \pm 1,$$

$$\text{and} \quad \det \begin{pmatrix} 1 & x_{q+1, q+2} \\ x_{q+2, q+1} & x_{q+2, q+2} \end{pmatrix} = 1, \quad \text{if and only if}$$

$$x_{q+1, q+1} + x_{q+1, q+2} = x_{q+2, q+1} + x_{q+2, q+2};$$

$$2) \quad x_{q+1, i} - x_{q+2, i} = 0, \quad \text{for } i = 1, \dots, q.$$

If the above equality holds, (3.2) follows from 1) and 2) (q.e.d.)

Set $\bar{M} = G/H$, then \bar{M} is an open $(2q+1)$ -manifold and \bar{M} is foliated into the fibres of the fibre bundle $\bar{M} = G/H \rightarrow G/K$. We denote this foliation by $(\bar{M}, \bar{\mathcal{F}})$. Clearly, the foliation $(\bar{M}, \bar{\mathcal{F}})$ is a G -invariant foliation of codimension q on \bar{M} .

By A. Borel [2], the connected semi-simple Lie subgroup G of $GL(q+2; \mathbf{C})$ has the discrete subgroup Γ of G which contains a normal torsion free subgroup D of finite index. Since H is compact subgroup of G , the subgroup D acts freely on $\bar{M} = G/H$ and $D \backslash \bar{M}$ is compact. Therefore we have.

(3.3) *There exists a discrete subgroup D of G such that $D \backslash \bar{M}$ is a closed $(2q+1)$ -manifold.*

Set $M = D \backslash \bar{M}$. Since the foliation $(\bar{M}, \bar{\mathcal{F}})$ is G -invariant, the closed $(2q+1)$ -manifold M has a codimension q foliation (M, \mathcal{F}) induced naturally from $(\bar{M}, \bar{\mathcal{F}})$. This foliation (M, \mathcal{F}) is the example of foliation with non trivial exotic characteristic classes.

4. Proof of Theorem

4.1. Naturality of the homomorphism $\lambda(N, \mathcal{F})$

Let N be an oriented manifold without boundary and (\cdot, \cdot) be a Riemannian metric on N . Let ∇ be the Riemannian connection on N . Given a foliation (N, \mathcal{F}) a metric connection ∇° and a basic connection ∇^1 on $\nu(\mathcal{F})$ are

defined as follows:

$$\begin{aligned}\nabla_X^0(Y) &= \pi \nabla_X(Y) \\ \nabla_X^1(Y) &= \pi[X_{\tau(\mathcal{F})}, Y] + \nabla_{X_{\nu(\mathcal{F})}}^0(Y)\end{aligned}$$

for any $X \in \mathfrak{X}(N)$, $Y \in \Gamma(\nu(\mathcal{F}))$, where $\pi: \tau(N) \rightarrow \nu(\mathcal{F})$ is the natural projection, and $X_{\tau(\mathcal{F})} \in \Gamma(\tau(\mathcal{F}))$, $X_{\nu(\mathcal{F})} \in \Gamma(\nu(\mathcal{F}))$ are such that $X = X_{\tau(\mathcal{F})} + X_{\nu(\mathcal{F})}$. Here, of course, we consider the Riemannian metric on $\nu(\mathcal{F})$ induced naturally from the Riemannian metric (\cdot, \cdot) on $\tau(N)$.

Then the homomorphism of cochain complexes

$$\lambda(N, \mathcal{F}): WO_q \rightarrow A_c^*(N)$$

is uniquely determined from the above connections ∇^0 and ∇^1 , hence from the foliation (N, \mathcal{F}) and the Riemannian metric (\cdot, \cdot) . Thus we denote this $\lambda(N, \mathfrak{so}(\omega))$ by $\omega((N, \mathcal{F}), (\cdot, \cdot))$ for $\omega \in WO_q$.

Now, let $O(q+1, 1)$, $G, H, K, (\bar{M}, \bar{\mathcal{F}})$ and (M, \mathfrak{ff}) be as in Section 3. Let $\mathfrak{o}(q+1, 1)$ (resp. \mathfrak{gl}_{q+2}) denote the Lie algebra of $O(q+1, 1)$ (resp. $GL(q+2; \mathbf{R})$), then clearly

$$\mathfrak{o}(q+1, 1) = \{X \in \mathfrak{gl}_{q+2}; {}^tXB + BX = 0\},$$

and a basis of $\mathfrak{o}(q+1, 1)$ is given by the following elements:

$$\begin{aligned}Z &= \left(\begin{array}{c|cc} O & O & \\ \hline O & 0 & -2 \\ & -2 & 0 \end{array} \right) \\ H_{ij} &= \left(\begin{array}{c|ccc} \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & \dots & 1 \dots \\ \vdots & \vdots & \vdots & \vdots \\ \dots & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \hat{i} & \hat{j} & & \end{array} \right) \begin{matrix} < i \\ < j \end{matrix}, \quad 1 \leq i < j \leq q, \\ X_i &= \left(\begin{array}{c|cc} O & 1 & -1 \\ \hline -1 & 0 & 0 \\ \vdots & 0 & 0 \end{array} \right) \begin{matrix} < i \\ \hat{i} \end{matrix}, \quad i = 1, \dots, q, \\ Y_i &= \left(\begin{array}{c|cc} O & 1 & 1 \\ \hline -1 & 0 & 0 \\ \vdots & 0 & 0 \end{array} \right) \begin{matrix} < i \\ \hat{i} \end{matrix}, \quad i = 1, \dots, q.\end{aligned}$$

It is known that $\{H_{ij}\}_{1 \leq i < j \leq q}$ is a basis of the Lie algebra of H . Then $T_0(\bar{M})$ is identified naturally with the subspace of $T_o(G)$ spanned by $\{X_1, \dots, X_q, Y_1, \dots, Y_q, Z\}$, where $o=H \in G/H=\bar{M}$.

In this time, we have

$$\begin{aligned} & (Ad(g)(X_1), \dots, Ad(g)(X_q), Ad(g)(Y_1), \dots, Ad(g)(Y_q), Ad(g)(Z)) \\ &= (X_1, \dots, X_q, Y_1, \dots, Y_q, Z) \cdot \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

for $g = \begin{pmatrix} A & 0 \\ 0 & I_2 \end{pmatrix} \in H$ ($A \in SO(q)$). Therefore, $\langle \cdot, \cdot \rangle_o = \sum_{i=1}^q X_i^* \otimes X_i^* + \sum_{i=1}^q Y_i^* \otimes Y_i^* + Z^* \otimes Z^*$ is an $Ad(H)$ -invariant innerproduct on $T_o(\bar{M})$ where $\{X_1^*, \dots, X_q^*, Y_1^*, \dots, Y_q^*, Z^*\}$ denote the dual basis of $\{X_1, \dots, X_q, Y_1, \dots, Y_q, Z\}$. Hence, for any $u=g \circ \bar{M}(g \in G)$, an innerproduct $\langle \cdot, \cdot \rangle_u$ on $T_u(\bar{M})$ is defined by $\langle \cdot, \cdot \rangle_u = (g^{-1})^* \langle \cdot, \cdot \rangle_o$. Therefore we have

(4.1.1) $\langle \cdot, \cdot \rangle$ is a G -invariant Riemannian metric on \bar{M} and \bar{M} is orientable.

Then we have the followings.

Lemma 4.1.2. For any $\omega \in WO_q, \omega((\bar{M}, \bar{\mathcal{F}}), \langle \cdot, \cdot \rangle)$ is a G -invariant differential form on \bar{M} .

Lemma 4.1.3. Let $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric on M induced naturally from the Riemannian metric $\langle \cdot, \cdot \rangle$ on \bar{M} , then

$$p^* \omega((M, \mathcal{F}), \langle \cdot, \cdot \rangle) = \omega((\bar{M}, \bar{\mathcal{F}}), \langle \cdot, \cdot \rangle)$$

for $\omega \in WO_q$, where $p: \bar{M} \rightarrow M$ is the natural projection.

Proof of (4.1.2). Let ∇ be the Riemannian connection on the Riemannian manifold $(\bar{M}, \langle \cdot, \cdot \rangle)$. Since the Riemannian metric $\langle \cdot, \cdot \rangle$ on \bar{M} is G -invariant,

$$g_*(\bar{\nabla}_X(Y)) = \bar{\nabla}_{g_*X}(g_*Y)$$

for any $g \in G$ and $X, Y \in \mathfrak{X}(\bar{M})$.

Since the foliation $(\bar{M}, \bar{\mathfrak{F}})$ is G -invariant, g_* maps $\Gamma(\tau(\bar{\mathcal{F}}))$ (resp. $\Gamma(\nu(\bar{\mathcal{F}}))$) into $\Gamma(\tau(\mathcal{F}))$ (resp. $\Gamma(\nu(\mathcal{F}))$) and $g_*\pi = \pi g_*$ for $g \in G$, where $\pi: \tau(\bar{M}) \rightarrow \nu(\mathcal{F})$ is the natural projection.

Therefore we have

$$\begin{aligned} (*) \quad & g_*(\nabla_X^0(Y)) = \nabla_{g_*X}^0(g_*Y), \\ & g_*(\nabla_X^1(Y)) = \nabla_{g_*X}^1(g_*Y), \quad \text{for } g \in G. \end{aligned}$$

Let k^1 (resp. k^0) be the curvature matrix of ∇^1 (resp. ∇^0) associated to some local frame $S = \{s_1, \dots, s_q\}$ of $\nu(\bar{\mathcal{F}})$. Then by (*), $(g^{-1})^*k^1$ (resp. $(g^{-1})^*k^0$) is the curvature matrix of ∇^1 (resp. ∇^0) associated to the local frame $g_*S = \{g_*s_1, \dots, g_*s_q\}$. But $c_i(k^1) \in A_c^{2i}(\bar{M})$ is independent of the choices of local frames, hence $c_i(k^1)$ is G-invariant. Therefore

$$c_i(\bar{M}, \bar{\mathcal{F}}), \langle \cdot, \cdot \rangle = \lambda(\nabla^1)(c_i) = (\sqrt{-1}/2\pi)^i c_i(k^1)$$

is also G-invariant.

Similarly, $\lambda(\nabla^{01})(c_i) \in A_c^{2i}(\bar{M} \times \mathbf{R})$ is G-invariant. Hence it follows from (1.3.2) that $h_i((\bar{M}, \bar{\mathcal{F}}), \langle \cdot, \cdot \rangle)$ is G-invariant. (q.e.d.)

Proof of (4.1.3). Let ∇ be the Riemannian connection on $(M, \langle \cdot, \cdot \rangle)$. Since the natural projection $p: (\bar{M}, \langle \cdot, \cdot \rangle) \rightarrow (M, \langle \cdot, \cdot \rangle)$ is a local isometry, locally we have

$$p_*(\bar{\nabla}_X(Y)) = \nabla_{p_*X}(p_*Y).$$

Therefore, the proof is similar to that of (4.1.2). (q.e.d.)

4.2. Local frame of $\nu(\bar{\mathcal{F}})$

Let $o = H \in \bar{M} = G/H$. To calculate the connection forms, we define local vector fields around $o \in \bar{M}$ as follows:

Define a parametrization ϕ around $o \in \bar{M}$ by

$$\begin{aligned} & \bar{\phi}(y_1, \dots, y_q, x_1, \dots, x_q, z) \\ &= \exp\left(\sum_{i=1}^q y_i Y_i\right) \exp\left(\sum_{i=1}^q x_i X_i\right) \exp(z \cdot Z) H \in \bar{M} = G/H. \end{aligned}$$

In the sequel, we use the vector notations such as $x = (x_1, \dots, x_q)$, $y = (y_1, \dots, y_q)$

Set local vector fields $Z, \bar{X}_1, \dots, \bar{X}_q, \bar{Y}_1, \dots, \bar{Y}_q$ around $o \in \bar{M}$, $Z = \bar{\phi}_*(\partial/\partial z)$,

$$\bar{X}_i = \bar{\phi}_*(e^{-2z} \partial/\partial x_i), \quad i = 1, \dots, q,$$

$$\bar{Y}_j = \bar{\phi}_*(e^{2z} (\partial/\partial y_j + \sum_{k=1}^q x_k^2 \partial/\partial x_j - 2x_j \sum_{k=1}^q x_k \partial/\partial x_k + x_j \partial/\partial z)), \quad j = 1, \dots, q,$$

at $u = \bar{\phi}(y, x, z) \in \bar{M}$.

Then we have

Lemma 4.2.

- (1) $\bar{X}_1, \dots, \bar{X}_q, Z$ are tangent to the foliation $(\bar{M}, \bar{\mathcal{F}})$.
- (2) $\{\bar{X}_1, \dots, \bar{X}_q, Z, \bar{Y}_1, \dots, \bar{Y}_q\}$ is a local orthonormal frame of $\tau(\bar{M})$ with respect to the Riemannian metric $\langle \cdot, \cdot \rangle$.
- (3) (bracket relations)

$$\begin{aligned}
[\bar{X}_i, \bar{Z}] &= 2X_i & [\bar{X}_i, \bar{X}_j] &= 0, \\
[\bar{Y}_j, \bar{Z}] &= -2\bar{Y}_j, \\
[\bar{Y}_i, \bar{Y}_j]_u &= 2e^{2z}(x_i(\bar{Y}_j)_u - x_j(\bar{Y}_i)_u) \\
[\bar{X}_i, \bar{Y}_j]_u &= \begin{cases} (\bar{Z})_u - 2e^{2z} \sum_{\substack{k=1 \\ k \neq i}}^q x_k(\bar{X}_k)_u, & i = j \\ 2e^{2z} x_i(\bar{X}_j)_u, & i \neq j, \end{cases}
\end{aligned}$$

where $u = \bar{\phi}(y, x, z) \in \bar{M}$, and $i, j = 1, \dots, q$.

Proof. The bracket relations of (3) are calculated directly by the definitions of Z, X_i, \bar{Y}_j .

We shall prove (1) and (2).

First, we define a local parametrization ϕ around $e \in G$ and a local section σ around $o \in \bar{M}$ as follows:

Set

$$\begin{aligned}
&\phi(y, x, z, (h_{ij})_{1 \leq i < j \leq q}) \\
&= \exp\left(\sum_{i=1}^q y_i Y_i\right) \exp\left(\sum_{i=1}^q x_i X_i\right) \exp(z \cdot Z) \exp\left(\sum_{1 \leq i < j \leq q} h_{ij} H_{ij}\right) \in G.
\end{aligned}$$

The local section σ is defined by requiring

$$\sigma \bar{\phi}(y, x, z) = \phi(y, x, z, (0)).$$

Then the next (4.2.1) and (4.2.2) follows from tedious calculations, which will be left to the reader.

$$(4.2.1) \quad (1) \quad \text{For } g_1 = \exp\left(\sum_{i=1}^q y_i^0 Y_i\right),$$

$$L_{g_1}(\phi(y, x, z, (h_{ij}))) = \phi(y + y^0, x, z, (h_{ij}))$$

$$(2) \quad \text{For } g_2 = \exp\left(\sum_{i=1}^q x_i^0 X_i\right),$$

$$R_{g_2}(\phi(y, x, z, (h_{ij}))) = \phi(y, \bar{x}, z, (h_{ij})).$$

where $\bar{x} = x + e^{-2z} x^0 \cdot (\exp(\sum h_{ij} H_{ij}))$.

$$(3) \quad = \exp(z^0 Z + \sum_{1 \leq i < j \leq q} h_{ij}^0 H_{ij})$$

$$R_{g_3}(\phi(y, x, z, (h_{ij}))) = \phi(y, x, z + z^0, (h_{ij} + h_{ij}^0)).$$

Here, of course, L_g (resp. R_g) denote the left (resp. right) translation by $g \in G$.

(4.2.2) Let $g = \phi(y, x, z, (h_{ij})) \in G$, and g_1, g_2, g_3 be as in (4.2.1), and let $X'_i = \phi_*(\partial/\partial x_i)$, $Y'_i = \phi_*(\partial/\partial y_i)$, $Z' = \phi_*(\partial/\partial z)$. Then,

$$\begin{aligned}
(1) \quad &(L_{g_1})_*((X'_i)_g) = (X'_i)_{g_1 g} \\
&(L_{g_1})_*((Y'_i)_g) = (Y'_i)_{g_1 g} \\
&(L_{g_1})_*(Z'_g) = Z'_{g_1 g}
\end{aligned}$$

for $i=1, \dots, q$;

$$(2) \quad (R_{g_2})*((X'_i)_g) = (X'_i)_{gg_2} \\ (R_{g_2})*((Y'_i)_g) = (Y'_i)_{gg_2}$$

for $i=1, \dots, q$;

$$(R_{g_2})*(Z'_g) = Z'_{gg_2} - 2e^{-2z} \sum_{i,j=1}^q a_{ij} x_j^0 (X'_i)_{gg_2}$$

where a_{ij} is the (i, j) component of $\exp(\sum h_{ij} H_{ij})$;

$$(3) \quad (R_{g_3})*((X'_i)_g) = (X'_i)_{gg_3} \\ (R_{g_3})*((Y'_i)_g) = (Y'_i)_{gg_3} \\ (R_{g_3})*(Z'_g) = Z'_{gg_3}$$

for $i=1, \dots, q$.

Then we have the following key lemma for the proof of Lemma 4.2.

(4.2.3) The following (a), (b), (c) hold at $g = \phi(y, x, z, (0)) \in G$.

$$(a) \quad Z'_g = (L_g)*Z.$$

$$(b) \quad (X'_j)_g = e^{2z}(L_g)*X_j, j=1, \dots, q.$$

$$(c) \quad (Y'_i)_g = (L_g)*(e^{-2z}Y_j + 2e^{2z}x_j \sum_{k=1}^q x_k X_k - e^{2z} \sum_{k=1}^q x_k^2 X_j - x_j Z + \sum_{\substack{k=1 \\ (k \neq j)}}^q 2x_k H_{kj}),$$

for $j=1, \dots, q$.

Here, of course, the elements $X_1, \dots, X_q, Y_1, \dots, Y_q, Z, H_{ij}$ of $\mathfrak{o}(q+1, 1)$ are regarded as the elements of $T_e(G)$, and $H_{ij} = -H_{ji}$ for $i > j$.

Proof. First, notice that

$$(*) \quad (X'_i)_e = X_i, \quad \text{for } i=1, \dots, q, \\ (Y'_i)_e = Y_i, \quad \text{for } i=1, \dots, q, \\ Z'_e = Z.$$

For $g = (y, x, z, (0))$, set $g_1 = \exp(\sum_{i=1}^q y_i Y_i)$, $g_2 = \exp(\sum_{i=1}^q x_i X_i)$, $g_3 = \exp(z \cdot Z)$.

Then $g = g_1 \cdot g_2 \cdot g_3$.

We shall prove (a). By (4.2.2),

$$Z'_g = (L_{g_1})*(Z'_{g_2 g_3}) \\ = (L_{g_1})*(R_{g_3})*Z'_{g_2} \\ = (L_{g_1})*(R_{g_3})*(R_{g_2})*(Z'_e + 2 \sum_{k=1}^q x_k (X'_k)_e).$$

Then, by (*),

$$Z'_g = (L_{g_1})*(R_{g_3})*(R_{g_2})*(Z + 2 \sum_{k=1}^q x_k X_k) \\ = (L_g)*(L_{(g_2 g_3)^{-1}})*(R_{g_2 g_3})*(Z + 2 \sum_{k=1}^q x_k X_k) \\ = (L_g)*Ad((g_2 g_3)^{-1})(Z + 2 \sum_{k=1}^q x_k X_k) \\ = (L_g)*Ad(g_3^{-1})Ad(g_2^{-1})(Z + 2 \sum_{k=1}^q x_k X_k).$$

But, $Ad(g_2^{-1})(Z) = Z - 2 \sum_{k=1}^q x_k X_k$,

$$Ad(g_2^{-1})(X_k) = X_k, \text{ for } k=1, \dots, q,$$

and $Ad(g_3^{-1})(Z) = Z$.

Therefore, $Z'_g = (L_g)_* Z$.

(b) and (c) are proved similarly. (q.e.d.)

Now, let $\bar{P}: G \rightarrow \bar{M} = G/H$ be the natural projection, then clearly we have

$$\bar{P}_*(X_i) = (\bar{X}_i)_o \quad i = 1, \dots, q,$$

$$\bar{P}_*(Y_j) = (\bar{Y}_j)_o \quad j = 1, \dots, q,$$

$$\bar{P}_*(Z) = (\bar{Z})_o$$

$$\bar{P}_*(H_{ij}) = 0, \quad 1 \leq i < j \leq q,$$

where $o = H \in \bar{M} = G/H$.

Hence, by the definition of the G -invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on \bar{M} , the following lemma shows Lemma 4.2 (2).

(4.2.4) *Let σ be the local section defined as before, then*

$$(a) \quad (\bar{Z})_u = \sigma(u)_*(\bar{Z})_o,$$

$$(b) \quad (\bar{X}_i)_u = \sigma(u)_*(\bar{X}_i)_o, i = 1, \dots, q,$$

$$(c) \quad (\bar{Y}_j)_u = \sigma(u)_*(\bar{Y}_j)_o, j = 1, \dots, q,$$

for any point u of some neighborhood of o in \bar{M} .

Proof. Clearly, $\bar{P}_*(L_{\sigma(u)})_* = \sigma(u)_* \bar{P}_*$, and $(\phi_*)_{\sigma(u)} = \sigma_* \circ (\phi_*)_u$. Then, in view of the definitions of $\bar{X}_i, \bar{Y}_j, \bar{Z}$, we have (4.2.4) easily from (4.2.3).

(q.e.d.)

By the definition of Lie algebras of Lie groups, we have the following easily.

(4.2.5) *The following elements $\mathfrak{o}(q+1, 1)$ form a basis of the Lie algebra of K ,*

$$Z, X_1, \dots, X_q, H_{ij}, 1 \leq i < j \leq q,$$

where the subgroup K of $O(q+1, 1)$ is as in (3.2).

Since the foliation $(\bar{M}, \bar{\mathcal{F}})$ is G -invariant and each leaf of this foliation is a fibre of the fibre bundle $\bar{M} = G/H \rightarrow G/K$, we have Lemma 4.2 (1) from (4.2.4) and (4.2.5).

This completes the proof of Lemma 4.2.

REMARK. Consider Z, X_i, Y_j of $\mathfrak{o}(q+1, 1)$ as left-invariant vector fields on G . It holds that $\bar{Z} = \bar{P}_*(Z)$ and \bar{Z} is G -invariant, then we may define \bar{Z} by $\bar{P}_*(Z)$. However it is impossible to define X_i (resp. \bar{Y}_j) by $\bar{P}_* X_i$ (resp. $\bar{P}_* Y_j$), for X_i, Y_j are not $Ad(H)$ -invariant.

4.3. Calculation of $c_i((\bar{M}, \bar{\mathfrak{H}}), \langle \cdot, \cdot \rangle)$ and $h_j((\bar{M}, \bar{\mathcal{F}}), \langle \cdot, \cdot \rangle)$.

Let Z^* (resp. \bar{X}_i^*, \bar{Y}_j^*) denote the dual one form of Z (resp. X_i, Y_j) with

respect to the G -invariant Riemannian metric $\langle \overline{}, \overline{} \rangle$ on \bar{M} .

Then we have

Lemma 4.3. At $o=H \in \bar{M}=G/H$,

- (1) $c_i((M, \mathfrak{F}), \langle \overline{}, \overline{} \rangle) = \alpha_i (\sqrt{-1}/2\pi)^i (d\bar{Z}^*)^i, \alpha_i > 0$,
for $i=1, \dots, q$;
(2) $h_j((M, \mathfrak{F}), \langle \overline{}, \overline{} \rangle) = \beta_j (\sqrt{-1}/2\pi)^j \bar{Z}^* \wedge (d\bar{Z}^*)^{j-1}, \beta_j < 0$,
for $j=1, 3, \dots, r$.

We shall prove Lemma 4.3. As usual, dx_i and rfy_z are regarded as local 1-forms on \bar{M} by the parametrization $\bar{\phi}$. It is easy to see the following.

(4.3.1) Let $Z^*, \bar{X}_i^*, \bar{Y}_j^*$ be as above, then

(1) $\bar{Y}_j^* = e^{-2z} dy_j$ for $j=1, \dots, q$, at $u = \bar{\phi}(y_1, \dots, y_q, x_1, \dots, x_q, z) \in \bar{M}$;

(2) $d\bar{Z} = \sum_{i=1}^q dy_i \wedge dx_i = \sum_{i=1}^q \bar{Y}_i^* \wedge \bar{X}_i^*$.

Since the Riemannian metric $\langle \overline{}, \overline{} \rangle$ on \bar{M} is given, connections ∇^0 and ∇^1 on $\nu(\bar{\mathfrak{F}})$ are uniquely defined as in Section 4.1. Then $\{\bar{Y}_1, \dots, \bar{Y}_q\}$ is a local orthonormal frame of $\nu(\bar{\mathfrak{F}})$ by Lemma 4.2. Let $\theta^0 = (\theta_{ij}^0)$ (resp. $\theta^1 = (\theta_{ij}^1)$) be the connection form of ∇^0 (resp. ∇^1) relative to the frame $\{\bar{Y}_1, \dots, \bar{Y}_q\}$, then we have

(4.3.2) At $u = \phi(y_1, \dots, y_q, x_1, \dots, x_q, z) \in \bar{M}$

(1) $\theta_{ij}^0 = \begin{cases} 0 & i=j \\ -2e^{2z}(x_j \bar{Y}_i^* - x_i \bar{Y}_j^*), & i \neq j \end{cases}$

(2) $\theta_{ij}^1 = \begin{cases} 2\bar{Z}^*, & i=j \\ \theta_{ij}^0, & i \neq j. \end{cases}$

Proof. Let V be the Riemannian connection on the Riemannian manifold $(\bar{M}, \langle \overline{}, \overline{} \rangle)$ and $\theta = (\theta_{ij})$ be the connection form of V relative to the frame $\{\bar{Y}_1, \dots, \bar{Y}_q, \bar{X}_1, \dots, \bar{X}_q, \bar{Z}\}$ of $\tau(\bar{M})$.

Set $s_1 = \bar{Y}_1, \dots, s_q = \bar{Y}_q, s_{q+1} = \bar{X}_1, \dots, s_{2q} = \bar{X}_q, s_{2q+1} = \bar{Z}$. Then by the definition of V ,

$$d\langle \overline{s_i}, \overline{s_j} \rangle(X) = \langle \overline{\nabla_X s_i}, \overline{s_j} \rangle + \langle \overline{s_i}, \overline{\nabla_X s_j} \rangle$$

for $X \in \mathfrak{X}(\bar{M})$. Hence

(i) $\theta_{ij} = -\theta_{ji}$, for $i, j=1, \dots, 2q+1$.

Moreover $\nabla_{s_i}(s_j) - \nabla_{s_j}(s_i) = [s_i, s_j]$, then we have the followings (ii) (iii) from Lemma 4.2 (3).

For $i, j=1, \dots, q$ and $i \neq j$.

(ii) $\theta_{ij}(s_j) = -2e^{2z}x_i, \theta_{ij}(s_i) = 2e^{2z}x_j$.

(iii) $\theta_{jk}(s_i) = \theta_{ik}(s_j)$, for $k=1, \dots, 2q+1$ and $k \neq i, j$.

$$\theta_{q+j}k(s_i) = \theta_{ik}(s_{q+j}), \text{ for } k=1, \dots, q.$$

$$\theta_{2q+1}k(s_i) = \theta_{ik}(s_{2q+1}) \text{ for } k=1, \dots, 2q+1,$$

and $k \neq i, 2q+1$.

Let $i, j=1, \dots, q$. Now, let $k=1, \dots, q$, and $k \neq i, j$, then by (i) and (iii),

$$\theta_{ij}(s_k) = -\theta_{ji}(s_k) = -\mathbf{MO} = \mathbf{MO} = \theta_{jk}(s_i) = -\theta_{kj}(s_i) = -\theta_{ij}(s_k)$$

Hence $\theta_{ij}(s_k)=0$. It is shown similarly by making use of (i) and (iii) that $\theta_{ij}(s_k)=0$ for $k=q+1, \dots, 2q+1$. Therefore we have

$$\theta_{ij}(s_k) = \begin{cases} -2e^{2z}x_i, & k=j \\ 2e^{2z}x_j, & k=i \\ 0, & k \neq i, j \end{cases}$$

for $i, j=1, \dots, q$ and $i \neq j$.

But by the definition of V° ,

$$\theta_{ij}^0 = \theta_{ij}, \text{ for } i, j = 1, \dots, q.$$

Hence we have (4.3.2) (1).

(4.3.2) (2) is shown easily by the following.

By the definition of ∇^1 and Lemma 4.2 (3),

$$\nabla_X^1(\bar{Y}_i) = \begin{cases} 0, & \text{if } X = \bar{X}_1, \dots, \bar{X}_q \\ 2\bar{Y}_i, & \text{if } X = \bar{Z} \\ \nabla_X^0(\bar{Y}_i), & \text{if } X = \bar{Y}_1, \dots, \bar{Y}_q \end{cases}$$

for $i=1, \dots, q$. (q.e.d.)

Now, let $k^1=(k_{ij}^1)$ be the curvature matrix of ∇^1 associated to the frame $\{\bar{Y}_1, \dots, \bar{Y}_q\}$. Let ∇^{01} be the connection on $\nu(\bar{\mathcal{F}}) \times \mathbf{R}$ defined by V° and ∇^1 as in (1.2.2) (3), and $\theta^{01}=(\theta_{ij}^{01})$ be the connection form of ∇^{01} , that is,

$$\theta^{01} = (1-t)\theta^0 + t\theta^1,$$

and $k^{01}=(k_{ij}^{01})$ be the curvature matrix of ∇^{01} . Then we have

$$(4.3.3) \quad (1) \quad At \ o=H \in \bar{M}=G/H,$$

$$k_{ij}^{01} = \begin{cases} 2d\bar{Z}^* & i=j \\ 2(dx_j \wedge dy_i - dx_i \wedge dy_j), & i \neq j. \end{cases}$$

$$(2) \quad At \ (o, t) \in o \times \mathbf{R} \subset \bar{M} \times \mathbf{R},$$

$$k_{ij}^{01} = \begin{cases} 2dt \wedge \bar{Z}^* + 2td\bar{Z}^*, & i=j \\ 2\pi^*(dx_j \wedge dy_i - dx_i \wedge dy_j), & i \neq j. \end{cases}$$

where $\pi: \bar{M} \times \mathbf{R} \rightarrow \bar{M}$ in the natural projection.

Proof. By (4.3.2) (1), proof of (1) is trivial. By the definition of θ^{01} ,

$$\theta_{ij}^{01} = \begin{cases} 2t\bar{Z}^*, & i = j \\ 2e^{2z}(x_j\bar{Y}_i^* - x_i\bar{Y}_j^*), & i \neq j, \end{cases}$$

then by (4.3.1), we have (2). (q.e.d.)

By the definition of the determinant of matrices, we have the following.

(4.3.4) Let $K = (K_{ij})$ be a $q \times q$ matrix of 2-forms. Assume that

$$K_{ij} = \begin{cases} \omega, & i = j \\ \gamma_j \wedge \eta_i - \gamma_i \wedge \eta_j, & i \neq j, \end{cases}$$

where $\gamma_1, \dots, \gamma_q, \eta_1, \dots, \eta_q$ are 1-forms. Then

$$c_i(K) = \sum_{\substack{0 \leq n \leq i \\ n \text{ are even}}} a_{in} \cdot \omega^{i-n} \wedge \left(\sum_{k=1}^q \eta_k \wedge \gamma_k \right)^n,$$

for $i=1, \dots, q$ and each a_{in} is a positive number.

Now, $c_i((\bar{M}, \mathcal{F}), \langle \bar{\cdot}, \bar{\cdot} \rangle)$ and $h_j((\bar{M}, \text{ff}), \langle \bar{\cdot}, \bar{\cdot} \rangle)$ are calculated as follows. In view of (4.3.1), we have Lemma 4.3 (1) from (4.3.3) (1) and (4.3.4). Similarly, by (4.3.3) (2),

$$c_j(k^{01}) = \sum_{\substack{0 \leq n \leq j \\ n \text{ are even}}} a_{jn} \cdot 2^n (j-n) t^{j-n-1} dt \wedge \bar{Z}^* \wedge (d\bar{Z}^*)^{j-1} \\ + (\text{terms which do not contain } dt),$$

for $j=1, \dots, q$, at $(o, t) \in \bar{M} \times \mathbf{R}$.

Let $i: \bar{M} \times [0, 1] \hookrightarrow \bar{M} \times \mathbf{R}$ be the inclusion mapping, and $\pi: \bar{M} \times [0, 1] \rightarrow \bar{M}$ be the natural projection, then then by the definition of Integration along the fibre π_* ,

$$h_j((\bar{M}, \mathcal{F}), \langle \bar{\cdot}, \bar{\cdot} \rangle) = (\sqrt{-1}/2\pi)^j \pi_* i^* c_j(k^{01}) \\ = \beta_j (\sqrt{-1}/2\pi)^j \cdot \bar{Z}^* \wedge (d\bar{Z}^*)^{j-1}, \beta_j > 0.$$

This completes the proof of Lemma 4.3.

4.4. Proof of Theorem

Let ω be an element of WO_q with degree $2q+1$ such that $\omega = \phi h_j$ for some monomial $\phi \in \mathbf{R}_q[c_1, \dots, c_q]$ as in Lemma 2.2. By Lemma 4.2 (1), $\{\bar{X}_1, \dots, \bar{X}_q, \bar{Y}_1, \dots, \bar{Y}_q\}$ is a local frame of $\tau(\bar{M})$. Then by (4.3.1) (2), $(2q+1)$ -form $\bar{Z}^* \wedge (d\bar{Z}^*)^q = q! \cdot \bar{Z}^* \wedge (\bar{Y}_1^* \wedge \bar{X}_1^*) \wedge \dots \wedge (\bar{Y}_q^* \wedge \bar{X}_q^*)$ is non zero around $o \in \bar{M}$. Hence $\omega((\bar{M}, \mathcal{F}), \langle \bar{\cdot}, \bar{\cdot} \rangle)$ is non zero at $o \in \bar{M}$ by Lemma 4.3. On the other hand, $\omega((\bar{M}, \mathcal{F}), \langle \bar{\cdot}, \bar{\cdot} \rangle)$ is a G-invariant form on \bar{M} by Lemma 4.1.2.

Therefore $\omega((\bar{M}, \bar{\mathcal{F}}), \overline{\langle \ , \ \rangle})$ is nowhere zero on \bar{M} . In view of Lemma 4.1.3, $\omega((M, \mathcal{F}), \langle \ , \ \rangle)$ is a nowhere zero $(2q+1)$ -form on the closed orientable $(2q+1)$ -manifold M . Therefore $\omega((M, \mathcal{F}), \langle \ , \ \rangle)$ represents a non zero cohomology class of $H^{2q+1}(M; \mathbb{C})$.

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