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PERIODIC QUATERNION EXPANSION IN PISOT BASES

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Abstract

In this paper, we consider a positional numeration system in $\mathbb{R}^n$ called the rotational beta expansion. The expansion of an element $z \in \mathbb{R}^n$ is a sum of the form

$$z = (\beta M)^{-1}d_1 + (\beta M)^{-2}d_2 + \cdots,$$

where the radix is βM for some fixed real number $\beta > 1$ and matrix $M \in O(n)$. We reformulate the rotational beta expansion where $M \in SO(4)$ into the so-called q-expansion on the set $\mathbb{H}$ of real quaternions. In particular, we obtain necessary and sufficient conditions for the q-expansion of a quaternion to be periodic when the base q is a Pisot quaternion.

1. Introduction

Let $m \in \mathbb{N}$. Let $\eta := \{\eta_1, \ldots, \eta_m\} \subseteq \mathbb{R}^m$ be a set of linearly independent vectors over $\mathbb{R}$. Let $\mathcal{L}$ be the lattice of $\mathbb{R}^m$ with the fundamental domain

$$\mathcal{X} = \left\{ \sum_{i=1}^{m} x_i \eta_i \;\middle|\; x_i \in [0, 1) \right\}$$

generated by the vectors η_i. Let $1 < \beta \in \mathbb{R}$ and let M be an isometry in the orthogonal group $O(m)$ of dimension m. We define the rotational beta transformation map $T : \mathcal{X} \longrightarrow \mathcal{X}$ with parameters $[\beta, M, \eta]$ as the map given by

$$T(z) = \beta M z - d(z),$$

where $d(z)$ is the unique element in $\mathcal{L}$ satisfying $\beta M z \in \mathcal{X} + d(z)$. The *rotational beta expansion* of $z \in \mathcal{X}$ (with respect to T) is the expansion $D(z) := d_1 d_2 \ldots$, where $d_i := d(T^{i-1}z)$ for $i \in \mathbb{N}$. We have

$$z = \sum_{i=1}^{\infty} \frac{M^{-i} d_i}{\beta^i}.$$

The rotational beta expansion generalizes the beta expansion to higher dimensions (see [2, 1, 14, 12, 13]). The beta expansion of a real number $x \in [0, 1)$ is obtained by setting $m = 1$, $M = 1$ and $\eta = \{1\}$. Thus, $x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}$ where $d_i = \lfloor \beta T^{i-1}(x) \rfloor$. In 1980, Schmidt [16] showed that if β is a Pisot number, then the set $\mathrm{Per}(\beta)$ of real numbers $x \in [0, 1)$ with eventually periodic beta expansion coincides with $\mathbb{Q}(\beta) \cap [0, 1)$. On the other hand, if all the elements of $\mathbb{Q}(\beta) \cap [0, 1)$ have an eventually periodic beta expansion, then the base β is either a Pisot or Salem number. It is an open problem whether $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0, 1)$ when

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β is a Salem number.

In this article, our goal is to provide an analog of the periodicity result of Schmidt for a class of rotational beta expansions in dimension 4. We do this by reformulating the rotational beta expansion in the setting of the ring $\mathbb{H}$ of quaternions, thereby, introducing quaternion expansions in $\mathbb{H}$.

2. Preliminaries

Define distinct elements $\hat{\imath}, \hat{\jmath}, \hat{k}$ such that

$$\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = \hat{\imath}\hat{\jmath}\hat{k} = -1.$$

The set $\mathbb{H}$ of real quaternions is the 4-dimensional vector space over $\mathbb{R}$ given by

$$\mathbb{H} = \{a + b\hat{\imath} + c\hat{\jmath} + d\hat{k} : a, b, c, d \in \mathbb{R}\}.$$

As vector spaces, $\mathbb{H} \cong \mathbb{R}^4$. For $a, b, c, d \in \mathbb{R}$, the quaternion $x = a + b\hat{\imath} + c\hat{\jmath} + d\hat{k}$ is identified with the vector $\begin{bmatrix} a & b & c & d \end{bmatrix}^T$. Here, T denotes the transpose operator. We call a, b, c, d the coordinates of x. We define the real part of x by $\mathrm{Re}(x) := a$ and the imaginary part of x by $\mathrm{Im}(x) := b\hat{\imath} + c\hat{\jmath} + d\hat{k}$. The modulus $|x|$ of x is given by the 2-norm $\|x\| = \sqrt{a^2 + b^2 + c^2 + d^2}$. The (quaternion) conjugate of $x \in \mathbb{H}$ is

$$\overline{x} = \mathrm{Re}(x) - \mathrm{Im}(x).$$

Note that $\mathbb{H}$ is a noncommutative division ring. In particular, every nonzero quaternion has a unique multiplicative inverse.

Our goal is to define a numeration system on $\mathbb{H}$ corresponding to the rotational beta expansion on $\mathbb{R}^4$ under some mild assumptions.

2.1. Matrix Representation of Elements of $\mathbb{H}$. Note that $\mathbb{H}$ can be viewed as a 2-dimensional vector space over $\mathbb{C}$ since, for any $x \in \mathbb{H}$, we can write $x = p + q\hat{\jmath}$ for some unique $p, q \in \mathbb{C}$. We associate the complex number $z = a + b\hat{\imath}$ to its matrix form $C_z := \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

The multiplication on $\mathbb{H}$ is noncommutative. For $x \in \mathbb{H}$, we distinguish between the multiplication by x on the right and on the left. Define the maps $\cdot_L, \cdot_R : \mathbb{H} \longrightarrow M_4(\mathbb{R})$ by

$$x_L = \begin{bmatrix} C_p & -JC_q^T \\ C_qJ & JC_p^TJ \end{bmatrix} \text{ and } x_R = \begin{bmatrix} C_p & -JC_qJ \\ JC_q^TJ & C_p^T \end{bmatrix}$$

for $x = p + q\hat{\jmath}$ where $p, q \in \mathbb{C}$ and $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Note that, if $x = a + b\hat{\imath} + c\hat{\jmath} + d\hat{k}$, where $a, b, c, d \in \mathbb{R}$, then we can compute for x_L and x_R explicitly as

$$x_L = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \text{ and } x_R = \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}.$$

The following result follows directly from the definition of x_L and x_R.

Proposition 2.1. *For any $x, y \in \mathbb{H}$, we have*

$$xy = x_L y \qquad \text{and} \qquad yx = x_R y.$$

Moreover, x_L and x_R are the only matrices with the above properties.

For $x \in \mathbb{H}$, we call x_L and x_R the left and the right matrix representations of x, respectively.

Proposition 2.2. *Let $x, y \in \mathbb{H}$. Then*

1. $(x + y)_L = x_L + y_L$ *and* $(x + y)_R = x_R + y_R$
2. $x_L = y_R$ *if and only if* $x = y$ *and* $x \in \mathbb{R}$.

Proof. The first part is clear. Suppose $x, y \in \mathbb{H}$ such that $x_L = y_R$. If $y = 0$, then x_L and y_R are the zero matrix. So, $x = y = 0$. Suppose $y \neq 0$. Then

$$xy^{-1} = x_L y^{-1} = y_R y^{-1} = 1.$$

Hence, $x = y$. Moreover, for any $z \in \mathbb{H}$,

$$xz = x_L z = x_R z = zx.$$

This means that x is an element of the center $Z(\mathbb{H}) = \mathbb{R}$. The proof of the converse is straightforward. □

2.2. Isoclinic Matrices. An $M \in M_4(\mathbb{R})$ is called a left (right) pseudoskew matrix if $M = x_L$ (if $M = x_R$) for some $x \in \mathbb{H}$ (see [7]). In such a case, $M^T M = |x|^2 I_4$, where I_4 is the 4-by-4 identity matrix. We say that x is the quaternion representation of M. So, $\det(M) \in \{\pm|x|^4\}$.

A matrix $M \in M_4(\mathbb{R})$ is said to be isoclinic if M is a rotation about 2 orthogonal planes such that the rotation angles are equal, up to sign (see [15]). Suppose M is a rotation matrix about the planes P_1 and P_2 of angles α and β, respectively. If $\alpha = \beta$, then M is said to be left isoclinic. If $\alpha = -\beta$, then M is said to be right isoclinic.

Note that if $M \in M_4(\mathbb{R})$ is a left or right pseudoskew matrix and $\det(M) = 1$, then M is an isoclinic matrix [9, 7]. In what follows, we provide the details of this fact.

Note that if $x \in \mathbb{H}$ is nonzero, then $(x/|x|)_L^T (x/|x|)_L = I_4$ and $\det((x/|x|)_L) \in \{\pm 1\}$. In the next result, we determine the eigenvalues of a pseudoskew matrix.

Proposition 2.3. *Let* $0 \neq x \in \mathbb{H}$. *The eigenvalues of M are* $\mathrm{Re}(x) \pm \|\mathrm{Im}(x)\|\hat{\imath}$ *if* $M = x_L$ *or* x_R.

Proof. Let λ be an eigenvalue of x_L. Then $\det(x_L - \lambda I_4) = 0$. We have $(x_L)^T x_L = |x|^2 I_4$. Hence,

$$\begin{aligned}(x_L - \lambda I_4)^T (x_L - \lambda I_4) &= (|x|^2 + \lambda^2 - 2\lambda \mathrm{Re}(x)) I_4 \\ &= ((\mathrm{Re}(x) - \lambda)^2 + \|\mathrm{Im}(x)\|^2) I_4 \in M_{2^n}(\mathbb{C}).\end{aligned}$$

Thus,

$$\det(x_L - \lambda I_4) = \pm([\mathrm{Re}(x) - \lambda]^2 + \|\mathrm{Im}(x)\|^2)^{2^{n-1}}.$$

Hence, $\lambda = \mathrm{Re}(x) \pm \|\mathrm{Im}(x)\|\hat{\imath}$. □

If $x \in \mathbb{R}$, then $x_L = x_R = xI_4$. Suppose $x \notin \mathbb{R}$. Since x_L is normal, it is diagonalizable. Then $\lambda_\pm = \text{Re}(x) \pm \|\text{Im}(x)\|\hat{\imath}$ have algebraic and geometric multiplicities both equal to 2.

We now discuss the geometric properties of the matrices x_L and x_R. Let $x \in \mathbb{H} \setminus \mathbb{R}$. Let $\{v_1, v_2\} \subseteq \mathbb{C}^4$ be an orthogonal basis for the eigenspace $E(\lambda_+)$. Then $\{v_1^*, v_2^*\}$ is a corresponding basis for the eigenspace $E(\lambda_-)$, where * denotes the complex conjugation. For $j = 1, 2$, consider the plane

$$P_j := \{zv_j + z^* v_j^* \mid z \in \mathbb{C}\} \subseteq \mathbb{R}^4.$$

Observe that, for any $z \in \mathbb{C}$,

$$x_L(zv_j + z^* v_j^*) = z\lambda_+ v_j + z^* \lambda_- v_j^* = z\lambda_+ v_j + (z\lambda_+)^* v_j^* \in P_j.$$

Thus, x_L fixes the orthogonal planes P_1 and P_2. We now show that x_L and x_R are rotation matrices.

Proposition 2.4. *Let $x \in \mathbb{H} \setminus \mathbb{R}$ such that $|x| = 1$. Let $u \in \mathbb{H}$ and let $\theta_N \in [0, \pi)$ be the angle measure between u and $x_N u$ for $N \in \{L, R\}$. Then,* $\cos(\theta_N) = \text{Re}(x)$.

Proof. Note that $u^T x_L u = u^T x_R u = Re(x) u^T u$. Thus,

$$\begin{aligned}\text{Re}(x)|u|^2 &= \text{Re}(x) u^T u = u^T x_L u \\ &= |u||x_L u| \cos(\theta_L) = |u||xu| \cos(\theta_L) = |u|^2 \cos(\theta_L).\end{aligned}$$

The same argument applies for θ_R. □

It follows that x_L (and likewise, x_R) is similar to $\rho(\theta_1) \oplus \rho(\theta_2)$, where

$$\rho(\theta_i) = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{bmatrix}$$

for $\theta_i \in \{\pm \cos^{-1} \text{Re}(x)\}$ and each $\rho(\theta_i)$ fixes the orthogonal planes P_j.

Let $x \in \mathbb{H}$ such that $|x| = 1$. Hence, x_L is left-isoclinic and is similar to the direct sum $A \oplus A$ where $A = \begin{bmatrix} \text{Re}(x) & -|\text{Im}(x)| \\ |\text{Im}(x)| & \text{Re}(x) \end{bmatrix}$. Meanwhile, x_R is right-isoclinic and is similar to $A \oplus A^T$.

2.3. Quaternion Expansions. We now introduce the notion of quaternion expansions. Consider the rotational beta expansion in $\mathbb{R}^4$ with parameters $[\beta, M, \eta]$. Suppose M is left (or right) isoclinic. We can associate the expansion

$$z = \sum_{j=1}^{\infty} \beta^{-j} M^{-j} d_j$$

with an expansion in $\mathbb{H}$. Let $q := \beta M e_1 \in \mathbb{H}$, where $e_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$ is the vector form of $1 \in \mathbb{H}$. Note that if $x \in \mathbb{H}$ such that $M = x_L$ and $y \in \mathbb{H}$, then

$$q = \beta x \text{ and } \beta M y = \beta(x_L y) = \beta x y = q y.$$

The basis elements of η can be viewed as elements of $\mathbb{H}$. Consequently, we can view $\mathcal{L}$ as a lattice in $\mathbb{H}$ with corresponding fundamental domain $\mathcal{X}$. By a lattice in $\mathbb{H}$, we mean an additive abelian group $\mathcal{L} \subseteq \mathbb{H}$ such that $\inf\{|x - y| : x, y \in \mathcal{L} \text{ and } x \neq y\} > 0$. Moreover, given a lattice $\mathcal{L}$ in $\mathbb{H}$, a fundamental domain for $\mathcal{L}$ is a subset $\mathcal{X} \subseteq \mathbb{H}$ such that $\mathbb{H}$ can be partitioned as

$$\mathbb{H} = \bigcup_{d\in\mathcal{L}} (\mathcal{X} + d).$$

Define the transformation $T : \mathcal{X} \longrightarrow \mathcal{X}$ by $T(z) = qz - d(z)$ where $d(z) \in \mathcal{L}$ is the unique digit such that $qz \in \mathcal{X} + d(z)$. Then $D(z) = (d_1, d_2, \dots) \in \mathcal{L}^{\mathbb{N}}$ is the expansion of the quaternion $z \in \mathcal{X}$ and

$$z = \sum_{j=1}^{\infty} q^{-j} d_j,$$

where $d_j = d(T^{j-1}(z))$. We call $D(z)$ the *quaternion expansion* of z with respect to the base q (or simply, the q-expansion of z). Similarly, starting with an expansion of the above form with a base $q \in \mathbb{H}$, we can obtain a related rotational beta expansion such that $\beta M = q_L$.

To proceed, let us first introduce the ring $\mathbb{H}_L$ of Lipschitz quaternions and the ring $\mathbb{H}_H$ of Hurwitz quaternions. The rings $\mathbb{H}_L$ and $\mathbb{H}_H$ are lattices in $\mathbb{H}$.

1. $\mathbb{H}_L := \{a + b\hat{\imath} + c\hat{\jmath} + d\hat{k} \mid a, b, c, d \in \mathbb{Z}\}$.
2. $\mathbb{H}_H := \left\{\frac{a+b\hat{\imath}+c\hat{\jmath}+d\hat{k}}{2} \mid a, b, c, d \in \mathbb{Z} \text{ and } a \equiv b \equiv c \equiv d \mod 2\right\}$.

We provide some examples of q-expansions.

EXAMPLE 2.5. Let $\eta = \{1, \hat{\imath}, \hat{\jmath}, \hat{k}\}$. Then $\mathcal{L} = \mathbb{H}_L$ and $\mathcal{X} = \{a+b\hat{\imath}+c\hat{\jmath}+d\hat{k} \mid a, b, c, d \in [0, 1)\}$. Let $q = \left(1 + \sqrt{5}\right)\hat{\imath}/2$. The q-expansion of $z = (1 + \hat{\jmath})/2$ is the purely periodic expansion

$$D(z) = (\overline{0, -2 - 2\hat{\jmath}, \hat{\imath} + \hat{k}, -1 - \hat{\jmath}, \hat{\imath} + \hat{k}, -1 - \hat{\jmath}}).$$

Correspondingly, with respect to the parameters $\beta = (1 + \sqrt{5})/2$ and $M = \left(\frac{q_1}{|q_1|}\right)_L = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, we have the following rotational beta expansion:

$$\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} = \left(\overline{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}\right).$$

EXAMPLE 2.6. Let $\eta = \{1, \hat{\imath}, \hat{\jmath}, \hat{k}\}$. Then $\mathcal{L} = \mathbb{H}_L$ and $\mathcal{X} = \{a+b\hat{\imath}+c\hat{\jmath}+d\hat{k} \mid a, b, c, d \in [0, 1)\}$. Let $q = (\hat{\imath} - \hat{\jmath} + (2 + \sqrt{2})\hat{k})/2$. The q-expansion of $(1 + \hat{\jmath})/2$ is eventually periodic. Indeed,

$$\frac{1 + \hat{\jmath}}{2} = q^{-1}a_1 + q^{-2}a_2 + q^{-3}a_3 + q^{-4}a_4 + q^{-5}a_5 + \sum_{j=1}^{\infty} q^{-(j+5)} b_j,$$

where

$$a_1 = -\hat{\imath} - \hat{\jmath} + \hat{k}, \qquad a_2 = -2 + \hat{k}, \qquad a_3 = -1 - \hat{\imath} + \hat{\jmath},$$
$$a_4 = -2 - 2\hat{\imath} - \hat{\jmath} + \hat{k}, \qquad a_5 = -2 - \hat{\imath} - \hat{\jmath} + 2\hat{k}$$

and

$$b_k = \begin{cases} -2\hat{\imath} + 2\hat{k}, & \text{if } k \equiv 2 \mod 4 \\ -1 + \hat{\jmath}, & \text{if } k \equiv 3 \mod 4 \\ -1 - \hat{\jmath} + \hat{k}, & \text{if } k \equiv 0 \mod 4 \\ -1 - \hat{\imath} - \hat{\jmath} + \hat{k}, & \text{if } k \equiv 1 \mod 4. \end{cases}$$

The corresponding rotational beta expansion in $\mathbb{R}^4$ has the parameters $\beta = |q| = \sqrt{2+\sqrt{2}}$ and

$$M = (q/|q|)_L = \frac{1}{2\beta}\begin{bmatrix} 0 & -1 & 1 & -\beta^2 \\ 1 & 0 & -\beta^2 & -1 \\ -1 & \beta^2 & 0 & -1 \\ \beta^2 & 1 & 1 & 0 \end{bmatrix}.$$

Note that β is not a Pisot number.

Example 2.7. Let $\eta = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ where $(\eta_1, \eta_2, \eta_3, \eta_4) = (1, \sqrt{2}\hat{\imath}, \sqrt{2}\hat{\jmath}, \hat{k})$. Then the lattice $\mathcal{L} = \{a + b\sqrt{2}\hat{\imath} + c\sqrt{2}\hat{\jmath} + dk : a, b, c, d \in \mathbb{Z}\}$ is a ring distinct from $\mathbb{H}_L$ and $\mathbb{H}_H$. Let $q = -(1+\sqrt{2})\hat{\imath}$. The first few digits of the q-expansion of $z = (1+\sqrt{3}\hat{\jmath})/2 \in \mathcal{X}$ are

$$\begin{aligned} a_1 &= -\eta_2 - 3\eta_4, & a_2 &= \eta_3, & a_3 &= -\eta_2 - 2\eta_4, \\ a_4 &= 0, & a_5 &= -\eta_2 - \eta_4, & a_6 &= 0, \\ a_7 &= -\eta_2 - 2\eta_4, & a_8 &= 0, & a_9 &= -\eta_2 - \eta_4. \end{aligned}$$

This expansion is, in fact, not eventually periodic (see Example 3.22).

3. Periodicity

In this section, we extend the notion of Pisot numbers to quaternions. We then consider q-expansions where q is a Pisot quaternion and provide necessary and sufficient conditions for the q-expansion of a quaternion to be eventually periodic.

3.1. Pisot Quaternions. Recall that a real number $\beta > 1$ is Pisot if it is an algebraic integer and all of its nontrivial conjugates β' over $\mathbb{Z}$ satisfy $|\beta'| < 1$. In this section, we define Pisot numbers over a quaternion subring. To this end, we first study polynomials over a skew field. Note that the set $\mathcal{R}[X]$ of polynomials with coefficients in a (possibly noncommutative) ring $\mathcal{R}$ forms a ring under the usual addition and multiplication of polynomials assuming that the indeterminate X commutes with the elements of $\mathcal{R}$.

We now mention several useful results.

Proposition 3.1 ([6, Theorem 1])**.** *Let $\mathcal{D}$ be a skew field. Let $f(X) = \sum_{j=0}^{n} X^j a_j$ be a polynomial of degree n such that $a_j \in \mathcal{D}$ for each j. If $\alpha \in \mathcal{D}$ such that*

$$f(\alpha) = \sum_{j=0}^{n} \alpha^j a_j = 0,$$

then $f(X) = (X - \alpha)g(X)$ for some $g(X) \in \mathcal{D}[X]$.

Given a skew field $\mathcal{D}$, we can define an equivalence relation $\sim$ as follows: for $a, b \in \mathcal{D}$, we have $a \sim b$ if and only if $a = cbc^{-1}$ for some $c \in \mathcal{D}$.

Theorem 3.2 ([6, Theorem 2])**.** *Let $\mathcal{D}$ be a skew field. Let $f \in \mathcal{D}[X]$ be a polynomial of degree n. Then $|\{\alpha \in \mathcal{D} : f(\alpha) = 0\}/\sim| \leq n$. Moreover, if*

$$f(X) = (X - \alpha_1)(X - \alpha_2)\cdots(X - \alpha_n),$$

where $\alpha_j \in \mathcal{D}$ and $\alpha \in \mathcal{D}$ such that $f(\alpha) = 0$, then $\alpha \sim \alpha_k$ for some k.

Theorem 3.3 (Fundamental Theorem of Algebra for $\mathbb{H}$, [10])**.** *Let $f(X) \in \mathbb{H}[X]$ be nonzero. Then*

$$|\{\alpha \in \mathbb{H} \mid f(\alpha) = 0\}/\sim| = \deg f.$$

Now, let $\mathcal{R}$ be a subring of $\mathbb{H}$ with unity. Let $q \in \mathbb{H}$. Suppose that $f(q) = 0$ for some monic polynomial $f(X) \in \mathcal{R}[X]$. Assume that f is the minimal polynomial of q over $\mathcal{R}$, i.e., the degree of f is minimal. By Proposition 3.1, $f(X) = (X - q)g(X)$ for some $g(X) \in \mathbb{H}[X]$. We say that q is Pisot over $\mathcal{R}$ if $|q| > 1$ and whenever $g(\alpha) = 0$ for $\alpha \in \mathbb{H}$, we have $|\alpha| < 1$.

We provide some examples.

Example 3.4. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be subrings of $\mathbb{H}$ with unity. If $\mathcal{R}_1 \subseteq \mathcal{R}_2$ and $q \in \mathbb{H}$ is Pisot over $\mathcal{R}_1$, then q is Pisot over $\mathcal{R}_2$. Hence, we have the following.

1. If $\mathcal{R}$ is a subring of $\mathbb{H}$ with $\mathbb{Z} \subseteq \mathcal{R}$ and $\beta \in \mathbb{R}$ is a Pisot number (in the usual sense, i.e., over $\mathbb{Z}$), then β is also Pisot over $\mathcal{R}$.

   Let $\mathcal{R}$ be a subring of $\mathbb{H}$ with $\mathbb{Z} \subseteq \mathcal{R}$ and $\beta \in \mathbb{R}$ be a Pisot number with minimal polynomial $f(X) = \sum_{j=0}^{n} X^j a_j \in \mathbb{Z}[X]$. Let $\beta = \beta_1, \beta_2, \ldots, \beta_n$ be the roots of f in $\mathbb{R} + \mathbb{R}\hat{\imath}$. Let $\theta \in \mathcal{R}$ with $\theta^2 = -1$. Then

   $$f(X) = (X - \gamma_1)(X - \gamma_2)\cdots(X - \gamma_n)$$

   where $\gamma_j := \mathrm{Re}(\beta_j) + \mathrm{Im}(\beta_j)\theta$. Note that γ_j commutes with θ for all j. Consider the monic polynomial

   $$g(X) = f(X\theta^{-1})\theta^n = (X - \gamma_1\theta)(X - \gamma_2\theta)\cdots(X - \gamma_n\theta) \in \mathcal{R}[X].$$

   So, $\gamma_1\theta = \beta\theta$ is integral over $\mathcal{R}$ being a root of the g. Moreover, the (possible) nontrivial Galois conjugates of $\beta\theta$ over $\mathcal{R}$ have the form $\gamma_j\theta$ and observe that $|\gamma_j\theta| = |\beta_j| < 1$ for $j \geq 2$. In other words, $\theta\beta$ is Pisot over $\mathcal{R}$. For example, $\varphi\theta$ is Pisot over $\mathbb{H}_L$ with minimal polynomial $X^2 - X\theta + 1$ where φ is the Pisot number $(1 + \sqrt{2})/5$ and $\theta \in \{\pm\hat{\imath}, \pm\hat{\jmath}, \pm\hat{k}\}$.
2. Let $q \in \mathbb{C} \setminus \mathbb{R}$ be a (nonreal) complex Pisot number, i.e., q is an algebraic integer with $|q| > 1$ and the Galois conjugates of q over $\mathbb{Q}$ distinct from q and $\overline{q}$ have moduli less than 1. Let $\mathcal{R}$ be a subring of $\mathbb{H}$ with $\mathbb{Z} \subseteq \mathcal{R}$. Then q is integral over $\mathcal{R}$, say with the minimal polynomial $\mu(X) = (X - q)g(X) \in \mathcal{R}[X]$. Note that $\mu(X)$ divides the minimal polynomial of q over $\mathbb{Q}$. If $g(\overline{q}) \neq 0$, then q is Pisot over $\mathcal{R}$.

Example 3.5. Let $q = (a + b\hat{\imath} + c\hat{\jmath} + d\hat{k})/2$ where $a, b, c, d \in \mathbb{Z}$ are odd such that $3a^2 = b^2 + c^2 + d^2$. Then $q \in \mathbb{H}_H \setminus \mathbb{H}_L$ and q is Pisot over $\mathbb{H}_H$ (of degree 1). Now, $q^3 = -a$, i.e., q is a root of $f(X) = X^3 + a$. Thus, q is integral over $\mathbb{H}_L$ of degree 2 or 3. Suppose $f(X) = (X - q)g(X)$ and $\alpha \in \mathbb{H}$ is a root of g. By Theorem 3.2, $\alpha \sim \beta$ for some root β of

f. Thus, $1 \leq |a| = |\beta|^3 = |\alpha|^3$. This means that the roots of g have moduli greater than 1, that is, q is not Pisot over $\mathbb{H}_L$. This example is rather interesting since $\mathbb{H}_L$ and $\mathbb{H}_H$ have the same skew field of fractions (see Section 3.2) and yet q has different degrees over the two subrings.

Proposition 3.6. *Let $\mathcal{R}$ be a subring of $\mathbb{H}$ containing $\mathbb{Z} \subseteq \mathcal{R}$. Let $b, c, d \in \mathbb{R}$ such that $b^2 + c^2 + d^2$ is a Pisot number (over $\mathbb{R}$). Let $q = b\hat{\imath} + c\hat{\jmath} + d\hat{k}$. Then q is integral over $\mathcal{R}$. Moreover, if $\mu(X) = (X - q)g(X)$ is the minimal polynomial of q over $\mathcal{R}$ and $\overline{q}$ is not a root of $g(X)$, then q is Pisot over $\mathcal{R}$.*

Proof. Let $\gamma = b^2 + c^2 + d^2$. Then $q^2 = -|q|^2 = -\gamma$. Let $f(X) \in \mathbb{Z}[X]$ be the minimal polynomial of γ over $\mathbb{Z}$. Then $f(-q^2) = 0$. Since the coefficients of f are real and the powers of q commute, then q is a root of the polynomial $f(-X^2) \in \mathcal{R}[X]$ whose leading coefficient is either 1 or -1. This implies that q is integral over $\mathcal{R}$. Let $\gamma_n, \gamma_{n-1}, \ldots, \gamma_2, \gamma_1 = \gamma \in \mathbb{C}$ be the Galois conjugates of γ over $\mathbb{Z}$. Observe that $f(X) = (X - \gamma)h(X)$ where $h(X) \in \mathbb{C}[X]$. Then

$$0 = f(-X^2) = (-X^2 - \gamma)h(-X^2) = -(X - q)(X - \overline{q})h(-X^2).$$

Let $\mu(X) = (X - q)g(X)$ be the minimal polynomial of q over $\mathcal{R}$. So, $\mu(X)$ divides $f(-X^2)$. Assume that $g(\overline{q}) \neq 0$. If $\alpha \in \mathbb{H}$ is a root of g, then $\alpha \sim \beta$ for some $\beta \in \mathbb{H}$ such that β is a root of $h(-X^2)$. But, $h(X) = \prod_{j=2}^{n}(X - \gamma_j)$. Hence, β is a solution to $X^2 = -\gamma_j$ for some $j \geq 2$. So $|\alpha| = |\beta| = \sqrt{|\gamma_j|} < 1$ and therefore, q is Pisot over $\mathcal{R}$. □

Example 3.7. Let $q = (\hat{\imath} - \hat{\jmath} + (2 + \sqrt{2})\hat{k})/2$. Then q is integral over $\mathbb{H}_L$ with minimal polynomial

$$\mu(X) = X^2 - X(2\hat{k}) + (\hat{\imath} + \hat{\jmath}).$$

Note that $|q|^2 = 2 + \sqrt{2}$ is a Pisot number and $\mu(\overline{q}) = \mu(-q) \neq 0$. Therefore, q is Pisot over $\mathbb{H}_L$.

3.2. Ore Domains and Polynomial Skew Field Extensions. Recall that if $\mathcal{R}$ is a commutative ring with unity, then we can define its field $\mathcal{K}$ of fractions ab^{-1} where $a, b \in \mathcal{R}$ and $b \neq 0$. First studied by Ore in [11], Ore domains allow the formulation of field of fractions for noncommutative rings. A ring $\mathcal{R}$ (possibly noncommutative) with unity is an Ore domain if for any $(r, s) \in \mathcal{R} \times (\mathcal{R} \setminus \{0\})$, there exist $(r_1, s_1), (r_2, s_2) \in \mathcal{R} \times (\mathcal{R} \setminus \{0\})$ such that $rs_1 = sr_1$ and $s_2r = r_2s$. To an Ore domain $\mathcal{R}$, we associate a skew field $\mathcal{K}$ of elements ab^{-1} where $a, b \in \mathcal{R}$ and $b \neq 0$.

Proposition 3.8. *Let $\mathcal{R}$ be a subring of $\mathbb{H}$ with unity. Then $\mathcal{R}$ is an Ore domain.*

Proof. Given $r, s \in \mathcal{R} \setminus \{0\}$, take $(r_1, s_1) = (rsr^2 - r^2sr, c + r^2s^2 - (sr)^2)$ and $(r_2, s_2) = (r^2sr - rsr^2, c + r^2s^2 - (rs)^2)$ where $c = (rs - sr)^2$. □

Let $\mathcal{R}$ be a subring of $\mathbb{H}$ with unity. By Proposition 3.8, the set $\mathcal{K} = \{ab^{-1} \mid a, b \in \mathcal{R}, b \neq 0\}$ is the skew field of fractions of $\mathcal{R}$, that is, $\mathcal{K}$ is the smallest subskew field of $\mathbb{H}$ that contains $\mathcal{R}$. Note that $\mathcal{K}$ is unique up to isomorphism. The skew fields of fractions of $\mathbb{H}_L$ and $\mathbb{H}_H$ are both equal to the set

$$\{a + b\hat{\imath} + c\hat{\jmath} + d\hat{k} \mid a, b, c, d \in \mathbb{Q}\}.$$

Let $q \in \mathbb{H}$. Suppose $f(q) = 0$ where $f(X) \in \mathcal{R}[X]$ is the minimal polynomial of q over $\mathcal{R}$. Let $\mathcal{K}(q) \subseteq \mathbb{H}$ be the smallest skew field that contains both $\mathcal{K}$ and q. If $\mathcal{R}$ (and consequently, $\mathcal{K}$) is commutative, then $\mathcal{K}(q)$ is a field which is also a vector space over $\mathcal{K}$ of dimension $\deg f$. However, this is not always the case when $\mathcal{K}$ is not commutative. We say that $\mathcal{K}(q)$ is a *polynomial skew field extension* (PSFE) of $\mathcal{K}$ if

$$\mathcal{K}(q) = \left\{ \sum_{j=0}^{\deg f - 1} q^j a_j \;\middle|\; a_j \in \mathcal{K} \right\}.$$

In other words, $\{1, q, \ldots, q^{\deg f - 1}\}$ is a right basis for $\mathcal{K}(q)$ over $\mathcal{K}$. Note that $\mathcal{K}(q)$ is both a left and right $\mathcal{K}$-module.

In general, the problem of determining whether $\mathcal{K}(q)$ is a PSFE of $\mathcal{K}$ is a difficult problem. Interested readers may refer to [3, 8, 4, 5, 18]. When q is integral over a skew field $\mathcal{K} \subseteq \mathbb{H}$ of degree 2, we have the following result.

Theorem 3.9. *Let q be integral over a skew field $\mathcal{K} \subseteq \mathbb{H}$ of degree 2. Then $\mathcal{K}(q)$ is a PSFE of $\mathcal{K}$ if and only if there exist additive homomorphisms $S_0, S_1 : \mathcal{K} \to \mathcal{K}$ such that, for any $a \in \mathcal{K}$, the following "commutation rule" holds:*

$$aq = S_0(a) + qS_1(a).$$

Proof. The forward direction is Lemma 2.1 (b) of [18]. For the backward direction, observe that $q^2 = qA + B$ for some $A, B \in \mathcal{K}$ where $B \neq 0$ since q is algebraic over $\mathcal{K}$ of degree 2. Using the commutation rule, we have

$$qaqb = BS_1(a)b + q[S_1(a)b + S_0(a)b]$$

for all $a, b \in \mathcal{K}$. It follows that $\{a + qb : a, b \in \mathcal{K}\}$ is closed under multiplication. Moreover, a nonzero element $a + qb$ is invertible. Indeed, consider the case where $a, b \in \mathcal{K} \setminus \{0\}$. We have

$$1 = (a + qb)(x + qy),$$

where

$$x = -b^{-1}[S_1(a) + S_0(b) + AS_1(b)]y$$

and

$$\begin{aligned} y &= [S_0(a) + BS_1(b) - ab^{-1}(S_1(a) + S_0(b) + AS_1(b))]^{-1} \\ &= [qS_1(b) - S_1(a) - AS_1(b)]^{-1}[b^{-1} + a^{-1}q]^{-1}a^{-1}. \end{aligned}$$

Note that one of $b^{-1} + a^{-1}q$ and $qS_1(b) - S_1(a) - AS_1(b)$ being 0 implies that $q \in \mathcal{K}$, which is a contradiction. Hence, $\{a + qb : a, b \in \mathcal{K}\}$ is a skew field and it is equal to $\mathcal{K}(q)$. The other cases are easy. □

The "commutation rule" $aq = S_0(a) + qS_1(a)$ is relatively easy to verify when $\mathcal{K}$ and q are given. We provide some examples of PSFE. We also give an example where $\mathcal{K}(q)$ is not a PSFE of $\mathcal{K}$.

EXAMPLE 3.10. By Example 3.7, $q = (\hat{\imath} - \hat{\jmath} + (2+\sqrt{2})\hat{k})/2$ is integral over $\mathbb{H}_L$ of degree 2. Let $\mathcal{K}$ be the skew field of fractions of $\mathbb{H}_L$. Let $w, x, y, z \in \mathbb{Q}$. Then

$$(w + x\hat{\imath} + y\hat{\jmath} + z\hat{k})q = a + qb,$$

where $a := 2y - 2x + 2w\hat{\imath} + 2w\hat{\jmath} \in \mathcal{K}$ and $b := w - x\hat{\imath} - y\hat{\jmath} + z\hat{k} \in \mathcal{K}$. Thus, the set $\{a + qb \mid a, b \in \mathcal{K}\}$ is a skew field and is equal to $\mathcal{K}(q)$. Hence, $\mathcal{K}(q)$ is a PSFE of $\mathcal{K}$.

EXAMPLE 3.11. The quaternion $q = (1+\sqrt{5})\hat{\imath}/2$ is integral over $\mathbb{H}_L$ with minimal polynomial $f(X) = X^2 - X\hat{\imath} + 1 = (X - q)(X - q^{-1})$. So, q is Pisot over $\mathbb{H}_L$. Let $\mathcal{K}$ be the skew field of fraction of $\mathbb{H}_L$. It is easy to show that $\{a + qb \mid a, b \in \mathcal{K}\}$ is a skew field. Thus, $\mathcal{K}(q) = \{a + qb \mid a, b \in \mathcal{K}\}$ is a PSFE of $\mathcal{K}$.

EXAMPLE 3.12. The quaternion $q = \sqrt{2}(\hat{\imath} + \hat{\jmath})/2$ is integral over the (skew) field $\mathcal{K} = \{r + s\hat{\imath} \mid r, s \in \mathbb{Q}\}$ with minimal polynomial $f(X) = X^2 + 1$. Now, $(\hat{\imath}q)^2 = -\hat{k} \in \mathcal{K}(q)$ but $-\hat{k}$ cannot be written in the form $a + bq$ where $a, b \in \mathcal{K}$. Hence, $\mathcal{K}(q)$ is not a PSFE of $\mathcal{K}$.

3.3. Main Result. If X is a normed space with norm $|\cdot|$, we say that X has the property (BF) if for every $A \subseteq X$, we have that A is finite whenever $\sup_{a\in A} |a| < \infty$. In other words, every bounded subset of X is finite. The following are examples of subrings of $\mathbb{H}$ with the property (BF): $\mathbb{H}_L$, $\mathbb{H}_H$ and $\{a + \sqrt{2}b\hat{\imath} + \sqrt{2}c\hat{\jmath} + d\hat{k} \mid a, b, c, d \in \mathbb{Z}\}$.

We fix the following parameters: a linearly independent set $\eta = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ of quaternions over $\mathbb{R}$, the lattice $\mathcal{L} = \bigoplus_{j=1}^4 \mathbb{Z}\eta_j$ and its associated fundamental domain

$$\mathcal{X} = \left\{ \sum_{j=1}^{4} a_i\eta_j \;\middle|\; a_j \in [0,1) \right\}.$$

From hereon, we let $\mathcal{R}$ be the ring $\langle 1, \mathcal{L}\rangle$ generated by the lattice $\mathcal{L}$, together with 1. Then $\mathcal{R}$ is an Ore domain containing the digits of the numeration system under consideration. Let $\mathcal{K}$ be the skew field of fractions of $\mathcal{R}$. Let q be Pisot over $\mathcal{R}$ with minimal polynomial $P(X) \in \mathcal{R}[X]$ of degree d such that $\mathcal{K}(q)$ is a PSFE of $\mathcal{K}$.

We follow the exposition of Schmidt in [16].

Proposition 3.13. *Suppose $\mathcal{K}(q)$ is a PSFE of $\mathcal{K}$. If $\alpha \in \mathcal{K}(q)$, then $\alpha S = \sum_{j=0}^{d-1} q^j p_j$ for some $S, p_0, p_1, \ldots, p_{d-1} \in \mathcal{R}$.*

Proof. Since $\alpha \in \mathcal{K}(q)$, there exist $r_{0j}, s_{0j} \in \mathcal{R}$ such that $s_{0j} \neq 0$ and

$$\alpha = \sum_{j=0}^{d-1} q^j r_{0j} s_{0j}^{-1} = r_{00}s_{00}^{-1} + \sum_{j=1}^{d-1} q^j r_{0j} s_{0j}^{-1}.$$

Let $\mathcal{S} := \mathcal{R}\backslash\{0\}$. Recall that $\mathcal{R}$ is an Ore domain. Since $s_{00} \in \mathcal{R}$ and $s_{0j} \in \mathcal{S}$ for $1 \leq j \leq d-1$, then there exist $a_{0j} \in \mathcal{R}$ and $b_{0j} \in \mathcal{S}$ such that $s_{00}b_{0j} = s_{0j}a_{0j}$. Then $s_{0j}^{-1} = a_{0j}b_{0j}^{-1}s_{00}^{-1}$. Hence,

$$\alpha = r_{00}s_{00}^{-1} + \sum_{j=1}^{d-1} q^j r_{0j} a_{0j} b_{0j}^{-1} s_{00}^{-1}.$$

So,

$$\alpha s_{00} = r_{00} + \sum_{j=1}^{d-1} q^j r_{1j} s_{1j}^{-1}$$

where $r_{1j} := r_{0j}a_{0j}$ and $s_{1j} := b_{0j}$. Applying the same process,

$$\begin{aligned}\alpha s_{00} &= r_{00} + qr_{11}s_{11}^{-1} + \sum_{j=2}^{d-1} q^j r_{1j} s_{1j}^{-1} \\ &= r_{00} + qr_{11}s_{11}^{-1} + \sum_{j=2}^{d-1} q^j r_{1j} a_{1j} b_{1j}^{-1} s_{11}^{-1},\end{aligned}$$

for some $a_{1j} \in \mathcal{R}$ and $b_{1j} \in \mathcal{S}$. So,

$$\alpha s_{00} s_{11} = r_{00} s_{11} + qr_{11} + \sum_{j=2}^{d-1} q^j r_{1j} a_{1j} b_{1j}^{-1}.$$

Continuing this process yields

$$\begin{aligned}\alpha S_0 &= r_{00}S_1 + qr_{11}S_2 + \cdots + q^{d-2}r_{d-2,d-2}S_{d-2} + q^{d-1}r_{d-1,d-1}s_{d-1,d-1}^{-1} \\ &= \sum_{j=0}^{d-2} q^j r_{jj} S_{j+1} + q^{d-1} r_{d-1,d-1} s_{d-1,d-1}^{-1},\end{aligned}$$

where $S_j = \prod_{i=1}^{j} s_{ii}$ and each $s_{ii} \in \mathcal{S}$. Then $\alpha S = \sum_{j=0}^{d-1} q^j p_j$ where $S = \prod_{j=0}^{d-1} s_{jj}$ and $p_j = r_{jj}S_{j+1}$. □

Once S is fixed, the tuple $(p_0, \ldots, p_{d-1}) \in \mathcal{R}^d$ that satisfies

$$\alpha S = \sum_{j=0}^{d-1} q^j p_j$$

is uniquely determined since $\{1, q, \ldots, q^{d-1}\}$ is a right basis of $\mathcal{K}(q)$ over $\mathcal{K}$.

From hereon, we assume that $\alpha \in \mathcal{K}(q)$ has the form

$$\alpha = \sum_{j=0}^{d-1} q^j p_j S^{-1},$$

where $S, p_0, \ldots, p_{d-1} \in \mathcal{R}$ and $S \neq 0$. For each $n \in \mathbb{N}$, set $d_j := d(T^{j-1}(\alpha))$ and

$$\rho^{(n)}(\alpha) := T^n(\alpha) = q^n \left(\alpha - \sum_{j=0}^{n} q^{-j} d_j \right).$$

Lemma 3.14. *Let $\alpha \in \mathcal{K}(q) \cap \mathcal{X}$ and $n \in \mathbb{N}$. Then there is a unique tuple*

$$(r_1^{(n)}, \ldots, r_d^{(n)}) \in \mathcal{R}^d$$

such that

$$\rho^{(n)}(\alpha) = \sum_{k=1}^{d} q^{-k} r_k^{(n)} S^{-1}.$$

Proof. This follows from the fact that $\{1, q, \ldots, q^{d-1}\}$ is a right basis of $\mathcal{K}(q)$ over $\mathcal{K}$. □

Lemma 3.15. *Let $n \in \mathbb{N}$ and γ be a root of the minimal polynomial of q over $\mathcal{R}$. Then*

$$\gamma^n \left[\sum_{j=0}^{d-1} \gamma^j p_j S^{-1} - \sum_{j=0}^{n} \gamma^{-j} d_j \right] = \sum_{k=1}^{d} \gamma^{-k} r_k^{(n)} S^{-1}.$$

Moreover, if $|\gamma| > 1$ and α has a periodic q-expansion, then

$$\sum_{j=0}^{d-1} \gamma^j p_j S^{-1} = \sum_{j=0}^{\infty} \gamma^{-j} d_j.$$

Proof. For the first part, we replace γ by the indeterminate X and multiply S on the right to obtain a polynomial in X over $\mathcal{R}$. Then q and γ are roots of this polynomial.

Now, if α has a periodic q-expansion, then

$$c := \sup_{n \in \mathbb{N}} \max_{1 \le k \le d} |r_k^{(n)}| < \infty.$$

So,

$$\begin{aligned} \left| \sum_{j=0}^{d-1} \gamma^j p_j S^{-1} - \sum_{j=0}^{n} \gamma^{-j} d_j \right| &= \left| \sum_{k=1}^{d} \gamma^{-n-k} r_k^{(n)} S^{-1} \right| \\ &\le \frac{1}{|S|} \sum_{k=1}^{d} |\gamma|^{-(n+k)} |r_k^{(n)}| \\ &\le \frac{cd|\gamma|^{-n}}{|S|} \to 0 \end{aligned}$$

as $n \to \infty$. □

Theorem 3.16. *Let $\mathcal{R}$ be a subring of $\mathbb{H}$ with unity and the property (BF). Let $\mathcal{K}$ be its skew field of fractions. Suppose $\mathcal{K}(q)$ is a PSFE of $\mathcal{K}$. Let $q \in \mathbb{H}$ with $|q| > 1$ be integral over $\mathcal{R}$ with minimal polynomial $P(X) = g(X)(X - q)$. If every $z \in \mathcal{K}(q) \cap \mathcal{X}$ has a periodic q-expansion with digits in $\mathcal{R}$, then $|\alpha| \le 1$ whenever $g(\alpha) = 0$. In other words, q is Pisot or Salem over $\mathcal{R}$.*

Proof. Suppose $\alpha \in \mathbb{H}$ with $g(\alpha) = 0$ and $|\alpha| > 1$. Then $\alpha \sim \beta$ for some $\beta \in \mathbb{H}$ with $P(\beta) = 0$. Also, $|\beta| = |\alpha| > 1$. Note that $\beta \neq q$.

Let $\xi := \max\{|q|^{-1}, |\beta|^{-1}\} < 1$ and $D := \max_{z \in \mathcal{X}} |d(z)|$. Set $0 < \delta < |q^{-1} - \beta^{-1}|$. Choose $m \in \mathbb{N} \ge 2$ such that $\frac{D\xi^m}{1-\xi} < \frac{\delta}{3}$.

Now, take $z \in \mathcal{K} \cap \mathcal{X}$ such that $qz \notin \mathcal{X}$ but $qT^j(z) \in \mathcal{X}$ when $2 \le j \le m-2$. Then $d_1(z) \neq 0$ while $d_2(z) = d_3(z) = \cdots = d_{m-1}(z) = 0$ where $d_j(z)$ is the jth digit of the q-expansion of z. Then z has periodic q-expansion by assumption. By Lemma 3.15,

$$z = q^{-1} d_1(z) + \sum_{j=m}^{\infty} q^{-j} d_j(z) = \beta^{-1} d_1(z) + \sum_{j=m}^{\infty} \beta^{-j} d_j(z).$$

Since $d_1(z) \in \mathcal{R}$ and $\mathcal{R}$ has the property (BF), then $|d_1(z)| \ge 1$. Otherwise, the sequence $\{(d_1(z))^n \mid n \in \mathbb{N}\}$ is finite but has a strictly decreasing modulus. Observe that

$$\begin{aligned}\delta &< |d_1(z)||q^{-1}-\beta^{-1}| \\ &\leq \left|\sum_{j=m}^{\infty}(q^{-j}-\beta^{-j})d_j(z)\right| \\ &\leq 2D\sum_{j=m}^{\infty}\xi^j = \frac{2D\xi^m}{1-\xi} < \frac{2\delta}{3}.\end{aligned}$$

We have a contradiction. So, $|\alpha| \leq 1$. □

Let $C_1, C_2, \dots, C_d$ be the distinct equivalence classes with respect to $\sim$ containing the roots of the minimal polynomial $P(x)$ of q such that $q_1 = q \in C_1$. For $2 \leq j \leq d$, choose $q^{(j)} \in C_j$ to be a root of $P(x)$.

For $1 \leq i \leq d$ and $n \in \mathbb{N}$, set

$$\rho_i^{(n)}(\alpha) := \sum_{k=1}^{d}(q^{(i)})^{-k} r_k^{(n)} S^{-1}.$$

Lemma 3.17. *Suppose $\mathcal{R}$ has the property (BF). Then the following are equivalent:*

(1) *α has periodic q-expansion;*
(2) $\max_{1\leq i\leq d}\sup_{n\in\mathbb{N}}|\rho_i^{(n)}(\alpha)| < \infty$;
(3) $\sup_{n\in\mathbb{N}}\max_{1\leq k\leq d}|r_k^{(n)}| < \infty$.

Proof. Note that (1) $\Longrightarrow$ (3) follows from the previous lemma. Meanwhile, (3) $\Longrightarrow$ (2) is clear. We show (2) $\Longrightarrow$ (1).

Assume (2). Set

$$v^{(n)} = \begin{bmatrix}\rho_1^{(n)}(\alpha)\\ \rho_2^{(n)}(\alpha)\\ \vdots \\ \rho_d^{(n)}(\alpha)\end{bmatrix} S = \begin{bmatrix} q_1^{-1} & q_1^{-2} & \cdots & q_1^{-d}\\ q_2^{-1} & q_2^{-2} & \cdots & q_2^{-d}\\ \vdots & \vdots & \ddots & \vdots \\ q_d^{-1} & q_d^{-2} & \cdots & q_d^{-d}\end{bmatrix}\begin{bmatrix} r_1^{(n)}\\ r_2^{(n)}\\ \vdots \\ r_d^{(n)}\end{bmatrix}.$$

By (2), the set $\{v^{(n)} \mid n \in \mathbb{N}\}$ is bounded. So, the set $\{(r_1^{(n)}, r_2^{(n)}, \dots, r_d^{(n)} \mid n \in \mathbb{N}\}$ is also bounded. Thus, $\{r_k^{(n)}\}$ is bounded. By the property (BF) of $\mathcal{R}$, the set $\{r_k^{(n)}\}$ is finite. Thus, $\{T^n(\alpha)\}_{n\in\mathbb{N}}$ is finite. Therefore, α has a periodic q-expansion. □

Finally, we prove the main result.

Theorem 3.18. *Let $\mathcal{R} = \langle 1, \mathcal{L}\rangle$. Let $\mathcal{K}$ be the skew field of fractions of $\mathcal{R}$. Suppose $\mathcal{R}$ has the property (BF). If $q \in \mathbb{H}$ is Pisot over $\mathcal{R}$ and $\mathcal{K}(q)$ is a PSFE of $\mathcal{K}$, then $\alpha \in \mathcal{X}$ has periodic q-expansion if and only if $\alpha \in \mathcal{K}(q)$.*

Proof. The forward direction is clear. Now, let $\alpha \in \mathcal{X} \cap \mathcal{K}(q)$. We show that (2) in the previous lemma is satisfied.

Since $\mathcal{X}$ is bounded and $T^n(\alpha) \in \mathcal{X}$, then $\sup_{n\in\mathbb{N}}|\rho_1^{(n)}(\alpha)| = \sup_{n\in\mathbb{N}}|T^n(\alpha)| < \infty$. Let $2 \leq i \leq d$. Since q is Pisot over $\mathcal{R}$, then $|q_i| < 1$. Let $\lambda := \max_{2\leq i\leq d}|q_i| < 1$. By Lemma 3.15,

$$|\rho_i^{(n)}(\alpha)| = |q^{(i)}|^n\left|\sum_{j=0}^{d-1}(q^{(i)})^j p_j S^{-1} - \sum_{k=0}^{n}(q^{(i)})^{-k} d_k\right|$$

$$\le \frac{1}{|S|}\sum_{j=0}^{d-1}|q^{(i)}|^{n+j}|p_j| + B\sum_{k=0}^{n}|q^{(i)}|^{n-k}$$
$$\le \frac{1}{|S|}\sum_{j=0}^{d-1}|p_j| + B\sum_{k=0}^{n}|q^{(j)}|^{k}$$
$$\le \frac{1}{|S|}\sum_{j=0}^{d-1}|p_j| + B\sum_{k=0}^{\infty}|q^{(j)}|^{k}$$
$$= \frac{1}{|S|}\sum_{j=0}^{d-1}|p_j| + B\sum_{k=0}^{n}\lambda^{k}$$
$$= \frac{1}{|S|}\sum_{j=0}^{d-1}|p_j| + \frac{B}{1-\lambda}$$

where $B = \max_{z\in\mathcal{X}}|d(z)|$. Thus, $\sup_{n\in\mathbb{N}}|\rho_i^{(n)}(\alpha)| < \infty$. By the previous lemma, the q-expansion of α is periodic. □

This theorem can be translated into a rotational beta expansion version where M is left (right) isoclinic.

Corollary 3.19. *Let $\beta > 1$ and M be left (right) isoclinic of size 4. Let $\mathcal{R} = \langle 1, \mathcal{L}\rangle$. Suppose $\mathcal{R}$ has the property (BF). Let q be the vector representation of βM and $\mathcal{K}$ be the field of fractions of $\mathcal{R}$. If q is Pisot over $\mathcal{R}$ and $\mathcal{K}(q)$ is a PSFE of $\mathcal{K}$, then the following are equivalent:*

1. *z has a periodic rotational beta expansion with respect to the parameter $[\beta, M, \eta]$;*
2. *$z \in \mathcal{K}(q)$.*

We illustrate the previous theorem through the following examples.

EXAMPLE 3.20. We revisit Examples 2.5 and 3.11. Note that $\mathcal{R} = \langle 1, \mathcal{L}\rangle = \mathbb{H}_L$ has the property (BF). The quaternion base $q = (1+\sqrt{5})\hat{\imath}/2$ is Pisot over $\mathbb{H}_L$. The set $\mathcal{K} = \{a + b\hat{\imath} + c\hat{\jmath} + d\hat{k} \mid a,b,c,d \in \mathbb{Q}\}$ is the skew field of fractions of $\mathbb{H}_L$. The set $\mathcal{K}(q) = \{qa + b \mid a, b \in \mathcal{K}\}$ is a PSFE of $\mathcal{K}$. Clearly, $(1+\hat{\jmath})/2 \in \mathcal{K}(q) \cap \mathcal{X}$ and it is expected that its q-expansion is periodic.

EXAMPLE 3.21. We revisit Examples 2.6, 3.7 and 3.10. The quaternion base $q = (\hat{\imath} - \hat{\jmath} + (2+\sqrt{2})\hat{k})/2$ is Pisot over $\mathbb{H}_L$. Moreover, $\mathcal{K}(q)$ is a PSFE of $\mathcal{K} = \{a + b\hat{\imath} + c\hat{\jmath} + d\hat{k} \mid a,b,c,d \in \mathbb{Q}\}$. Then $(1+\hat{\jmath})/2 \in \mathcal{K}(q) \cap \mathcal{X}$ and its q-expansion is expected to be periodic.

EXAMPLE 3.22. We revisit Example 2.7. The quaternion base $q = -(1+\sqrt{2})\hat{\imath}$ is Pisot over the ring (lattice) $\mathcal{R} = \{a + b\sqrt{2}\hat{\imath} + c\sqrt{2}\hat{\jmath} + d\hat{k} : a,b,c,d \in \mathbb{Z}\}$ since it is a root of

$$\mu(X) = X^2 + X(2\sqrt{2}\hat{\imath}) - 1 = (X-q)(X-q')$$

where $q' = (1-\sqrt{2})\hat{\imath}$ and $|q'| < 1$. Note that $\mathcal{R}$ has property (BF) and the skew field of fractions of $\mathcal{R}$ is

$$\mathcal{K} = \{a + b\sqrt{2}\hat{\imath} + c\sqrt{2}\hat{\jmath} + d\hat{k} : a,b,c,d \in \mathbb{Q}\}.$$

Moreover, observe that $\mathcal{K}(q) = \{a + b\hat{\imath} + c\hat{\jmath} + d\hat{k} : a, b, c, d \in \mathbb{Q}(\sqrt{2})\} = \{a + qb : a, b \in \mathcal{K}\}$. Hence, $\mathcal{K}(q)$ is a PSFE of $\mathcal{K}$. Therefore, $D((1 + \sqrt{3}\hat{\jmath})/2)$ is not periodic since $(1 + \sqrt{3}\hat{\jmath})/2 \notin \mathcal{K}(q)$.

3.4. Quasi-Pisot Base. Let $\mathcal{R}$ be a subring of $\mathbb{H}$ with unity. We say that $q \in \mathbb{H}$ with $|q| > 1$ is quasi-Pisot over $\mathcal{R}$ if q is integral over $\mathcal{R}$ with minimal polynomial $\mu(X)$ such that

$$\mu(X) = (X - q)(X - \overline{q})g(X)$$

and $|\gamma| < 1$ whenever $\gamma \in \mathbb{H}$ with $g(\gamma) = 0$.

Example 3.23. Let $\mathcal{R}$ be a subring of $\mathbb{H}$ with unity such that $\mathbb{Z} \subseteq \mathcal{R}$. Let $q \in \mathbb{C} \setminus \mathbb{R}$ be a complex Pisot number. Then q is integral over $\mathcal{R}$. Let $\mu(X) = (X - q)g(X) \in \mathcal{R}[X]$ be the minimal polynomial over q. If $g(\overline{q}) = 0$, then q is quasi-Pisot over $\mathcal{R}$. Otherwise, q is Pisot over $\mathcal{R}$.

Example 3.24. Let $\mathcal{R}$ be a subring of $\mathbb{H}$ with unity. Observe that $q \in \mathbb{H}$ is quasi-Pisot over $\mathcal{R}$ of degree 2 if and only if its minimal polynomial over $\mathcal{R}$ is $\mu(X) = (X - q)(X - \overline{q})$. Hence, $q \in \mathbb{H}$ is quasi-Pisot over $\mathcal{R}$ if and only if $q \notin \mathcal{R}$ and $2\mathrm{Re}(q), |q|^2 \in \mathcal{R}$. For example, let $q \in \mathbb{H}_H \setminus \mathbb{H}_L$. Then $q = (a + b\hat{\imath} + c\hat{\jmath} + d\hat{k})/2$ for some odd integers a, b, c, d. Then q is quasi-Pisot over $\mathbb{H}_L$ of degree 2.

Now, let β, M and $\mathcal{R}$ be the same as in Corollary 3.19 and suppose $q \in \mathbb{H}$ is integral over $\mathcal{R}$ of degree d. Then for $\alpha \in \mathcal{K}(q) \cap \mathcal{X}$ and for each $n \in \mathbb{N}$, we have

$$T^n(\alpha) = \sum_{k=1}^{d} q^{-k} r_k^{(n)} S^{-1}$$

for some tuple $(r_1^{(n)}, \ldots, r_d^{(n)}) \in \mathcal{R}^d$ and $S \in \mathcal{R} \setminus \{0\}$. If q is quasi-Pisot over $\mathcal{R}$ and the real dimension of the span

$$\mathrm{Span}\{r_k^{(n)} \mid 1 \leq k \leq d, n \in \mathbb{N}\}$$

is at most 1, then α has a periodic rotational beta expansion with parameter $[\beta, M, \eta]$. In particular, this is the case when $\mu(X) \in \mathbb{R}[X]$ and the digits of the q-expansion of α are all real.

4. Quaternion Zeta Expansions

Let $\theta \in \mathbb{H}$ such that $\theta^2 = -1$. Let $q \in C(\theta) := \mathbb{R} + \mathbb{R}\theta \cong \mathbb{C}$. Fix $\varepsilon \in [0, 1)$. The zeta expansion [17] on the fundamental domain

$$\mathcal{D}(\varepsilon) := \{a_1 + a_2(-\overline{q}) : a_1, a_2 \in [-\varepsilon, 1 - \varepsilon)\} \subseteq C(\theta)$$

is the rotational beta expansion with parameter $[|q|, M, \{1, -\overline{q}\}]$ where M is the 2×2 rotation matrix form of $q/|q|$ (as an element of $C(\theta)$). For $z \in \mathcal{D}(\varepsilon)$, the digits of the zeta expansion of z are all integers. We drop the arguments θ and ε from $C(\theta)$ and $D(\varepsilon)$, respectively, whenever the context is clear.

EXAMPLE 4.1. Let $\theta = (\hat{\imath} + \hat{\jmath} + \hat{k})/\sqrt{3}$ and $\varepsilon = 1/2$. Let $q = 1 + \theta$. Then the zeta expansion of $(3 + \theta)/10 \in \mathcal{D}(\varepsilon)$ in base q is $\{1, \overline{-2, 2}\}$.

We say that $q \in \mathcal{C} \setminus \mathbb{R}$ is a $\mathcal{C}$-Pisot number if q is an algebraic integer with minimal polynomial $\mu(X) \in \mathbb{Z}[X]$ such that the roots of μ distinct from q and $\overline{q}$ have moduli less than 1. On the other hand, we say that q is a $\mathcal{C}$-Salem number if q is an algebraic integer with minimal polynomial $\mu(X) \in \mathbb{Z}[X]$ such that the roots of μ aside from q and $\overline{q}$ moduli less than 1 with at least one root γ with $|\gamma| = 1$. Clearly, $q \in \mathcal{C}$ is $\mathcal{C}$-Pisot ($\mathcal{C}$-Salem) if and only if $\overline{q}$ is $\mathcal{C}$-Pisot ($\mathcal{C}$-Salem).

Proposition 4.2. *Let $q \in \mathcal{C}(\theta)$ and $0 \neq c \in \mathbb{H}$ with $|c| = 1$. Then*

1. *q is $\mathcal{C}(\theta)$-Pisot ($\mathcal{C}(\theta)$-Salem) if and only if cqc^{-1} is $\mathcal{C}(c\theta c^{-1})$-Pisot ($\mathcal{C}(c\theta c^{-1})$-Salem);*
2. *in particular, q is $\mathcal{C}$-Pisot ($\mathcal{C}$-Salem) if and only if $\mathrm{Re}(q) + \mathrm{Im}(q)\hat{\imath}$ is complex Pisot (complex Salem).*

Proof. For (1), it is enough to show one direction. Let $\mu(X) \in \mathbb{Z}[X]$ be the minimal polynomial of q over $\mathbb{Z}$. Then $\mu(X) = g(X)h(X)$ where $h(X) = X^2 - 2\,\mathrm{Re}(q)X + |q|^2$ for some $g(X) \in \mathbb{R}[X]$. Then cqc^{-1} and $\overline{cqc^{-1}}$ are roots of h and μ. This implies that $cqc^{-1} \in \mathcal{C}(c\theta c^{-1})$ is an algebraic integer whose possible Galois conjugates distinct from cqc^{-1} and $\overline{cqc^{-1}}$ are roots of g. By Theorem 3.2, if $\gamma \in \mathcal{C}(c\theta c^{-1})$ is a root of g, then $\gamma \sim \gamma'$ for a root $\gamma' \in \mathcal{C}(\theta)$ of g. Hence, the moduli of the roots of g in $\mathcal{C}(\theta)$ and the moduli of the roots of g in $\mathcal{C}(c\theta c^{-1})$ are the same. Thus, if $\gamma \in \mathcal{C}(c\theta c^{-1})$ is a root of g, then $|\gamma| < 1$ ($|\gamma| \leq 1$) because q is $\mathcal{C}(\theta)$-Pisot ($\mathcal{C}(\theta)$-Salem).

For the second part, suppose $\theta = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ for some $x, y, z \in \mathbb{R}$. Then $|\theta|^2 = x^2 + y^2 + z^2 = 1$. If $\theta = \pm\hat{\imath}$, then we are done since $-\hat{\imath} = \hat{\jmath}\hat{\imath}\hat{\jmath}^{-1}$. Suppose $|x| < 1$. Then $x + 1 > 0$. We have $\hat{\imath} = c\theta c^{-1}$ where

$$c = \frac{(x+1)\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{2(x+1)}}. \qquad \square$$

We have the following result.

Theorem 4.3 ([17])**.** *Let $q \in \mathcal{C}$ and $\varepsilon \in [0, 1)$.*

1. *If q is $\mathcal{C}$-Pisot, then for any $z \in \mathcal{D}(\varepsilon)$, the zeta expansion (with base q) of z is periodic if and only if $z \in \mathbb{Q}(q)$.*
2. *If z has periodic zeta expansion (with base q) for any $z \in \mathbb{Q}(q)$, then z is either $\mathcal{C}$-Pisot or $\mathcal{C}$-Salem.*

We now consider the extension of the zeta expansion on $\mathbb{H}$. Let $\phi \in \mathbb{H}$ such that $\mathrm{Re}(\phi) = 0, |\phi| = 1$. Suppose $\phi \cdot \theta = 0$, that is, ϕ and θ are perpendicular when viewed as elements of $\mathbb{R}^3$.

We have $\mathbb{H} \cong \mathcal{C} + \mathcal{C}\phi$. Let $(\eta_1, \eta_2, \eta_3, \eta_4) = (1, -\overline{q}, \phi, -\overline{q}\phi)$. Then $\eta = \{\eta_j \mid 1 \leq j \leq 4\}$ is a basis for $\mathbb{H}$ over $\mathbb{R}$. Fix $\varepsilon \in [0, 1)$. Consider the fundamental domain generated by η:

$$\mathscr{D} = \mathscr{D}(\varepsilon) := \left\{ \sum_{j=1}^{4} a_j \eta_j \;\middle|\; a_j \in [\varepsilon, 1 - \varepsilon) \right\}.$$

Note that $\mathscr{D} = \mathcal{D} + \mathcal{D}\phi$ where $\mathcal{D}$ is the fundamental domain of the zeta expansion on $\mathcal{C}(\theta)$

with base q. For $z \in \mathscr{D}$, define $d(z) \in \mathscr{L} = \bigoplus_{j=1}^{4} \mathbb{Z}\eta_j$ to be the unique element of $\mathscr{L}$ such that $qz - d(z) \in \mathscr{D}$. Let $T(z) := qz - d(z)$. The quaternion zeta expansion of $z \in \mathscr{D}$ is given by

$$z = \sum_{j=1}^{\infty} q^{-j} d_j$$

where $d_j = d(T^{j-1}(z))$ for $j \in \mathbb{N}$. If $z = z_1 + z_2\phi$ where $z_1, z_2 \in \mathcal{D}$, then the jth digit of the quaternion zeta expansion of z is

$$d_j = d_{1,j} + d_{2,j}\phi$$

where $d_{k,j}$ is the jth digit of the zeta expansion (on $C(\theta)$) of z_k with respect to the base q for $k = 1, 2$.

EXAMPLE 4.4. Let $\theta = (\hat{\imath} + \hat{\jmath} + \hat{k})/\sqrt{3}$. Then $\theta^2 = -1$. Let $\phi = (\hat{\imath} - \hat{\jmath})/\sqrt{2}$. Then ϕ is perpendicular to θ in $\mathbb{R}^3$. Moreover, $\mathbb{H} = C(\theta) + [C(\theta)]\phi$. Let $q = 1 + \theta \in C(\theta)$. Consider $z_1 = (3 + \theta)/10$ and $z_2 = (2 - 2\theta)/5$. Then zeta expansion of z_1 in base q is $\{1, \overline{-2, 2}\}$ while the zeta expansion of z_2 in base q is $\{\overline{1, 0, 0, 0}\}$. Therefore, the quaternion zeta expansion of $z = z_1 + z_2\phi$ in base q is $\{1 + \phi, \overline{-2, 2, -2, 2 + \phi}\}$.

We have the following results.

Proposition 4.5. *Let $z = z_1 + z_2\phi \in \mathscr{D}$ where $z_1, z_2 \in \mathcal{D}$. The quaternion zeta expansion with base q of z is periodic if and only if the zeta expansions on C with base q of z_1 and z_2 are both periodic.*

Theorem 4.6. *If $q \in C$ is C-Pisot, then the following are equivalent:*

1. *$z \in \mathscr{D}$ has periodic quaternion zeta expansion with base q;*
2. *$z \in (\mathcal{D} \cap \mathbb{Q}(q)) + (\mathcal{D} \cap \mathbb{Q}(q))\phi$.*

Theorem 4.7. *If every $z \in (\mathcal{D} \cap \mathbb{Q}(q)) + (\mathcal{D} \cap \mathbb{Q}(q))\phi$ has periodic quaternion zeta expansion with base q, then q is either C-Pisot or C-Salem.*

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