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Some Generalizations of Quasi-Frobenius Rings

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Let A be a ring satisfying the minimum condition for left and right ideals. (We shall understand by a ring always such one.) Let S be a set of elements of A. We shall denote the set $\{x \mid xS=0\ x\in A\}$ by $l(S)$ and the set $\{y \mid Sy=0\ y\in A\}$ by $r(S)$. Let N be the radical of A, $A/N=\bar{A}=\bar{A}_1+\ldots+\bar{A}_n$ be the direct decomposition of $\bar{A}$ into simple two-sided ideals $\bar{A}_\kappa$ and let $f(\kappa)$, $e_{\kappa,i}$, $e_\kappa=e_{\kappa,1}$, $c_{\kappa,i,j}$, and $E_\kappa=\sum_{i=1}^{f(\kappa)} e_{\kappa,i}$ $(\kappa=1,\ldots,n)$ have the same meaning as in Fr. I § 1 or S. I[1]. Namely $e_{\kappa,i}$, $(\kappa=1,\ldots,n;\ i=1,\ldots,f(\kappa))$ are mutually orthogonal primitive idempotents whose sum is a principal idempotent E of A, whence $A=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)} Ae_{\kappa,i}+l(E)$ $(=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)} e_{\kappa,i}A+r(E))$ is the direct decomposition of A into directly indecomposable left ideals $Ae_{\kappa,i}$ (right ideals $e_{\kappa,i}A$) and a left ideal $l(E)$ (right ideal $r(E)$), here $Ae_{\kappa,i}(e_{\kappa,i}A)$ and $Ae_{\lambda,j}(e_{\lambda,j}A)$ are operator isomorphic if and only if $\kappa=\lambda$. And $c_{\kappa,i,j}$ $(\kappa=1,\ldots,n;\ i=1,\ldots,f(\kappa),\ j=1,\ldots,f(\kappa))$ are matric units, $c_{\kappa,i,j}c_{\lambda,h,k}=\delta_{\kappa,\lambda}\delta_{j,h}c_{\kappa,i,k}$ for any κ, λ, i, j, h, k and $c_{\kappa,i,i}=e_{\kappa,i}$ for each κ, i. The residue class $\bar{E}_\kappa$ of E_κ mod. N is the identity element of $\bar{A}_\kappa$ for each κ. For the sake of brevity we shall call a ring A a *D_l-ring* (*D_r-ring*) if in A the duality relation $l(r(\mathfrak{l}))=\mathfrak{l}$ $(r(l(\mathfrak{r}))=\mathfrak{r})$ holds for every left ideal $\mathfrak{l}$ (right ideal $\mathfrak{r}$) of A. And if in A the duality relation $l(r(\mathfrak{l}))=\mathfrak{l}$ $(r(l(\mathfrak{r}))=\mathfrak{r})$ holds for every nilpotent simple left ideal and zero (every nilpotent simple right ideal and zero), then we shall call A an *S. D_l-ring* (*S. D_r-ring*).

Recently T. Nakayama studied the structure of quasi-Frobenius rings[2] and, in the previous note S. II[3], T. Nakayama and the writer proved some properties of *D_l-rings*. In this note we shall consider the structure of *S. D_l-rings* and refine Theorem 3 in Fr. I and Theorem 2 in S. II. Finally we shall consider some special *D_l-rings*, and give some results about them.

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1) T. Nakayama: On Frobeniusean algebras I, II, Annals of Math. 42 (referred to Fr. I, II,) T. Nakayama: Supplementary remarks on Frobenius algebras I, Proc. Japan Acad. (1949) (referred to S. I)

2) See Footnote 1).

3) T. Nakayama and M. Ikeda: Supplementary remarks on Frobenius algebras II. Osaka Math. Journ. 2 (1950) (referred to S. II)

1. *S. D_l-rings.*

Theorem 1. *Any S. D_l-ring has the properties;*

(I) A has an identity element.

(II) There exists a permutation π of $(1, 2, \dots, n)$ such that for each κ, a) Ae_κ has a unique simple left subideal which is isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$, and b) the largest completely reducible right subideal of $e_{\pi(\kappa)}A$ is a direct sum of simple right subideals of the form $\xi\mathfrak{m}$, where $\mathfrak{m}$ is an arbitrary simple right subideal of $e_{\pi(\kappa)}A$ isomorphic to $e_\kappa A/e_\kappa N$ and ξ's are suitable units[4] of $e_{\pi(\kappa)}Ae_{\pi(\kappa)}$.

(III) $f(\kappa)=1$ if the largest completely reducible right subideal of $e_{\pi(\kappa)}A$ is not simple.

Proof. We can prove (I) and (II) by a slight modification of the proof of Theorem 1 in S. II.

For the proof of (III), we shall use

Lemma 1. *Let A be an S. D_l-ring. Then eAe is also an S. D_l-ring, where $e=\sum_{\kappa=1}^{n} e_{\kappa, i(\kappa)}$ and $e_{\kappa, i(\kappa)}$ is one of $e_{\kappa, i}$ $(i=1, \dots, f(\kappa))$ for each κ.*

Proof. We shall denote the right annihilator (in eAe) of a set $*$ by $r^\circ(*)$ and left one by $l^\circ(*)$. Let $\mathfrak{l}^\circ$ be a left ideal of eAe, then $A\mathfrak{l}^\circ=\mathfrak{l}$ is a left ideal of A and $e\mathfrak{l}=\mathfrak{l}^\circ$. $\mathfrak{l}^\circ$ is contained in $\mathfrak{l}$ so $\mathfrak{l}^\circ x=\mathfrak{l}^\circ ex=0$ for any element x of $r(\mathfrak{l})$. This shows $er(l)\frown eAe=er(\mathfrak{l})\,e\subseteqq r^\circ(\mathfrak{l}^\circ)$. Conversely, if x is an element of $r^\circ(\mathfrak{l}^\circ)$, then $\mathfrak{l}x=A\mathfrak{l}^\circ x=0$. Therefore, x belongs to $r(\mathfrak{l})$, and consequently $r^\circ(\mathfrak{l}^\circ)=er^\circ(\mathfrak{l}^\circ)\,e\subseteqq er(\mathfrak{l})\,e$. Thus we have $r^\circ(\mathfrak{l}^\circ)=er(\mathfrak{l})\,e=er(A\mathfrak{l}^\circ)\,e$. A similar relation holds obviously for any right ideal $\mathfrak{r}^\circ$ of eAe: $l^\circ(\mathfrak{r}^\circ)=el(\mathfrak{r}^\circ A)\,e$. Since $r^\circ(\mathfrak{l}^\circ)=er(\mathfrak{l})\,e$, we have $r^\circ(\mathfrak{l}^\circ)\,A\subseteqq er(\mathfrak{l})$. Conversely if x is an element of $er(\mathfrak{l})$, then $er(\mathfrak{l})$ contains $xc_{\kappa, i, j(\kappa)}$ and $er(\mathfrak{l})\,eA$ contains $xc_{\kappa, i, j(\kappa)}\,ec_{, j(\kappa)\, i}=x\cdot e_{\kappa, i}$ for arbitrary κ and i. Since A has an identity element, $r^\circ(\mathfrak{l}^\circ)\,A=er(\mathfrak{l})\,eA$ contains $x=\sum_{\kappa, i} x\cdot e_{\kappa, i}$. This shows $r^\circ(\mathfrak{l}^\circ)\,A=er(\mathfrak{l})$. So we have $l^\circ(r^\circ(\mathfrak{l}^\circ))=el(r^\circ(\mathfrak{l}^\circ)\,A)\,e=el(er(\mathfrak{l}))\,e$. It is easily seen that $l(er(\mathfrak{l}))=A(1-e)+(Ae\frown l(r(\mathfrak{l})))$. Hence $l^\circ(r^\circ(\mathfrak{l}^\circ))=e(A(1-e)+(Ae\frown l(r(\mathfrak{l}))))\,e=e(Ae\frown l(r(\mathfrak{l})))$. Now let $\mathfrak{l}^\circ$ be a simple left ideal of eAe, and assume that $\mathfrak{l}=A\mathfrak{l}^\circ$ is not simple and $\mathfrak{l}'$ is its proper subideal. Then, since $\mathfrak{l}^\circ$ is simple and $e\mathfrak{l}'\subseteqq e\mathfrak{l}=\mathfrak{l}^\circ$, $e\mathfrak{l}'$ is either equal to $\mathfrak{l}^\circ$ or zero. If $e\mathfrak{l}'=\mathfrak{l}^\circ$, then $\mathfrak{l}'\supseteqq Ae\mathfrak{l}'=A\mathfrak{l}^\circ=\mathfrak{l}$ and this is a contradiction. If $e\mathfrak{l}'=0$, then $e_{\kappa, i(\kappa)}e\mathfrak{l}'=e_{\kappa, i(\kappa)}\mathfrak{l}'=e_{\kappa, i(\kappa)}\mathfrak{l}'=0$, and $0=c_{\kappa, j, i(\kappa)}e_{\kappa, i(\kappa)}\mathfrak{l}'\supseteqq c_{\kappa, j, i(\kappa)}e_{\kappa, i(\kappa)}c_{\kappa, i(\kappa), j}\mathfrak{l}'=e_{\kappa, j}\mathfrak{l}'$ for each κ, j. Since A has an identity element and $\mathfrak{l}'=\bigcup_{\kappa, j} e_{\kappa, j}\mathfrak{l}'$, $\mathfrak{l}'$ must be zero. This is a contradic-

4) A unit is an element which has its inverse.

tion. Therefore $\mathfrak{l} = A\mathfrak{l}^{\circ}$ is a simple left ideal of A. By the assumption, $l(r(\mathfrak{l})) = \mathfrak{l}$ in A. Hence $l^{\circ}(r^{\circ}(\mathfrak{l}^{\circ})) = e(Ae \cap l(r(\mathfrak{l}))) = e(Ae \cap \mathfrak{l})$. But since $\mathfrak{l} = A\mathfrak{l}^{\circ}$, $\mathfrak{l} = \mathfrak{l}e$ and $Ae \cap \mathfrak{l} = \mathfrak{l}$. So we have $l^{\circ}(r^{\circ}(\mathfrak{l}^{\circ})) = e\mathfrak{l} = \mathfrak{l}^{\circ}$.

Now we shall prove (III). By the above lemma, eAe is also an $S.\ D_l$-*ring*, where we take $\sum_{\kappa=1}^{n} e_{\kappa}$ as e. We shall denote eAe by A°, and the radical eNe of eAe by N°. Let $\bar{A}^{\circ} = A^{\circ}/N^{\circ} = \bar{A}_1^{\circ} + \dots + \bar{A}_n^{\circ}$ be the direct decomposition of $\bar{A}^{\circ}$ into two-sided ideals $\bar{A}_{\kappa}^{\circ}$, then since each $e_{\kappa} = e_{\kappa, 1}$ is a primitive idempotent, we have that $\bar{A}_{\kappa}^{\circ}$ is a quasi-field having $\bar{e}_{\kappa}$ as its unit element for each κ. Since A° is an $S.\ D_l$-*ring*, $r^{\circ}(N^{\circ}) = l^{\circ}(N^{\circ}) = M^{\circ}$ has the unique direct decomposition into simple two-sided ideals; $M^{\circ} = M^{\circ}e_1 + \dots + M^{\circ}e_n = e_{\pi(1)} M^{\circ} + \dots + e_{\pi(n)} M^{\circ}$, and here $M^{\circ}e_{\kappa} = e_{\pi(\kappa)} M^{\circ}$. Moreover $M^{\circ}e_{\kappa}$ is the unique simple left ideal of $A^{\circ}e_{\kappa}$, and is isomorphic to $A^{\circ}e_{\pi(\kappa)}/N^{\circ}e_{\pi(\kappa)}$, therefore $M^{\circ}e_{\kappa} = e_{\pi(\kappa)} Ae_{\pi(\kappa)} m_{\kappa}$, where m_{κ} is an element of $M^{\circ}e_{\kappa}$. Now let $e_{\pi(s)}A$ be an indecomposable right ideal the largest completely reducible right subideal of which is not simple.

Since $e_{\pi(s)} M^{\circ}$ is not a simple right ideal of A°, $M^{\circ}e_s = e_{\pi(s)} M^{\circ} = e_{\pi(s)} Ae_{\pi(s)} m_s \supsetneqq m_s e_s A e_s$. Assume that $f(s) \geqq 1$ for such an s.

It can easily be seen that $Me_s = Am_s$, and $Me_{s, 2} = Am_s c_{s, 1, 2}$.[5] Now we shall take a left subideal $\mathfrak{l} = A(m_s + dm_s c_{s, 1, 2})$ of the completely reducible left ideal $Me_s + Me_{s, 2}$, where d is an element of $e_{\pi(s)} A e_{\pi(s)}$ such that there exists no element d' in $e_s A e_s$ which satisfies $dm_s = m_s d'$. Since $e_{\pi(s)} A e_{\pi(s)} m_s \supsetneqq m_s e_s A e_s$, we can take such an element. $\mathfrak{l}$ is a simple left ideal, for $\mathfrak{l} = Ae_{\pi(s)} (m_s + dm_s c_{s, 1, 2})$ is homomorphic to $Ae_{\pi(s)}$ and $m_s + dm_s c_{s, 1, 2}$ is annihilated by $Ne_{\pi(s)}$. Since A is an $S.\ D_l$-*ring* it should be $r(\mathfrak{l}) \supsetneqq r(Me_s + Me_{s, 2}) = r(Me_s) \cap r(Me_{s, 2}) = (1 - e_s - e_{s, 2})\, A \dot{+} e_s N + e_{s, 2} N$. The composition length of $A/r(Me_s + Me_{s, 2})$ is 2, and $A/r(\mathfrak{l}) \simeq (m_s + dm_s c_{s, 1, 2})\, A$, hence the composition length of $(m_s + dm_s c_{s, 1, 2})\, A$ should be 1.

On the other hand it can easily be seen that $(m_s + dm_s c_{s, 1, 2})\, A = m_s A \cup dm_s c_{s, 1, 2} A$. $dm_s c_{s, 1, 2}$ does not belong to $m_s A$, for if $dm_s c_{s, 1, 2} \in m_s A$, then there exists an element d' such that $dm_s c_{s, 1, 2} = m_s d'$, and this leads $dm_s c_{s, 1, 2}\, c_{s, 2, 1} = dm_s = m_s d' c_{s, 2, 1}$, and $d' c_{s, 2, 1} \in e_s A e_s$, but this is impossible. Therefore the composition length of $(m_s + dm_s c_{s, 1, 2})\, A$ is not 1, and thus the assumption that $f(s) \geqq 1$ leads to a contradiction. This completes our proof. As a special case of this theorem, we have

Corollary 1. *Let A be a primary $S.\ D_l$-ring and not a quasi-Frobenius ring. Then A is a completely primary $S.\ D_l$-ring.*

Corollary 2. *Let A be a primary decomposable ring. Then A is an $S.\ D_l$-ring if and only if every component of A is either a quasi-Frobenius*

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5) $M = r(N) = l(N)$.

ring or a completely primary S. D_l-ring and not a quasi-Frobenius ring.

Corollary 3. *Let A be a ring in which the duality relation $l(r(\mathfrak{l}))=\mathfrak{l}$ holds for nilpotent simple left ideals and zero, and the duality relation $r(l(\mathfrak{r}))=\mathfrak{r}$ holds for nilpotent simple right ideals. Then A is a quasi-Frobenius ring.*[6]

As the converse of Theorem I, we have the following

Theorem 2. *Let A be a ring which has the properties (I), (II) and (III). Then in A the duarity relation $l(r(\mathfrak{l}))=\mathfrak{l}$ holds for every completely reducible left ideal, every left ideal which contains the radical N of A, the radical N itself, and zero. Furthermore in A the duality relation $r(l(\mathfrak{r}))=\mathfrak{r}$ holds for every right ideal which contains N, and N itself. Therefore A is an S. D_l-ring.*

Proof.

(1) It can easily be seen that the duality relation holds for zero.

(2) Since the unique simple left subideal $r(N)\,e_{\kappa,i}$ of $Ae_{\kappa,i}$ is isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$, we have that $r(N)\,e_{\kappa,i}=E_{\pi(\kappa)}r(N)\,e_{\kappa,i}$ and consequently $r(N)\,E_\kappa=\sum_{i=1}^{f(\kappa)}r(N)\,e_{\kappa,i}=E_{\pi(\kappa)}r(N)\,E_\kappa$ for each κ. Since A has an identity element $E_{\pi(\kappa)}r(N)=\sum_{\lambda=1}^{n}E_{\pi(\kappa)}r(N)\,E_\lambda=r(N)\,E_\kappa$. This shows that $r(N)\,E_\kappa=E_{\pi(\kappa)}r(N)$ is a two-sided ideal for each κ.

(3) $r(N)\,E_\kappa$ is a simple two-sided ideal. For if $m\neq 0$ is an arbitrary element of $r(N)\,E_\kappa$, then $me_{\kappa,i}\neq 0$ for at least one i, and it can easily be seen that $r(N)\,e_{\kappa,j}=Am\,c_{\kappa,i,j}$, hence $AmA\supseteqq Am\,e_{\kappa,i}A\supseteqq\sum_{j=1}^{f(\kappa)}r(N)\,e_{\kappa,j}=r(N)\,E_\kappa$. This shows that $r(N)\,E_\kappa$ is a simple two-sided ideal. Therefore $r(N)\,E_\kappa\subset l(N)$, and consequently $r(N)=\sum_{\kappa=1}^{n}r(N)\,E_\kappa\subseteqq l(N)$.

(4) Since the largest completely reducible right subideal $e_{\pi(\kappa)}l(N)$ of $e_{\pi(\kappa)}A$ is a direct sum of simple subideals which are isomorphic to $e_\kappa A/e_\kappa N$, $e_{\pi(\kappa),i}\,l(N)=c_{\pi(\kappa),i,1}\,e_{\pi(\kappa)}\,l(N)$ is a direct sum of simple subideals which are isomorphic to $e_\kappa A/e_\kappa N$. Hence $e_{\pi(\kappa),i}\,l(N)=e_{\pi(\kappa),i}\,l(N)\,E_\kappa$. Therefore $E_{\pi(\kappa)}\,l(N)=\sum_{i=1}^{f(\pi(\kappa))}e_{\pi(\kappa),i}\,l(N)=E_{\pi(\kappa)}\,l(N)\,E_\kappa$. Then by the same way as above, we have that $E_{\pi(\kappa)}\,l(N)=l(N)\,E_\kappa$ is a two-sided ideal for each κ.

(5) Let $m\neq 0$ be an arbitrary element of $E_{\pi(\kappa)}\,l(N)$. Then $e_{\pi(\kappa),i}\,me_{\kappa,j}\neq 0$ for suitable i and j. It can easily be seen that $e_{\pi(\kappa),i}\,me_{\kappa,j}A$ is a simple right subideal of $e_{\pi(\kappa),i}\,l(N)$. Since $e_{\pi(\kappa)}\,l(N)$ is a direct

6) Fr. II Theorem 6.

sum of simple right subideals of the form $\xi\mathfrak{m}$ for an arbitrary simple right subideal $\mathfrak{m}$, $e_{\pi(\kappa),i}\, l(N)$ is a direct sum of simple right subideals of the form $\xi'\mathfrak{m}'$ for an arbitrary simple right subideal $\mathfrak{m}'$ and suitable units ξ' of $e_{\pi(\kappa),i}Ae_{\pi(\kappa),i}$, and we have that $e_{\pi(\kappa),i}\, l(N) \subseteqq e_{\pi(\kappa),i}Ae_{\pi(\kappa),i} \cdot e_{\pi(\kappa),i}\, me_{\kappa,j}A$. Since $e_{\pi(\kappa),j}\, l(N) = c_{\pi(\kappa),j,i}\, e_{\pi(\kappa),i}\, l(N)$, we have that $e_{\pi(\kappa),j} \cdot l(N) \subseteq c_{\pi(\kappa),j,i}\, e_{\pi(\kappa),i}\, A_{\pi(\kappa),i}\, me_{\kappa,j}A$. Then $AmA \supseteqq \sum_{j=1}^{f(\pi(\kappa))} e_{\pi(\kappa),j}\, l(N) = E_{\pi(\kappa)}\, l(N)$. This shows that $E_{\pi(\kappa)}\, l(N)$ is a simple two-sided ideal. Therefore $E_{\pi(\kappa)}\, l(N) \subseteqq r(N)$, and consequently $l(N) = \sum_{\kappa=1}^{n} E_{\pi(\kappa)}\, l(N) \subseteqq r(N)$. By (3) and this relation, we have that $l(N) = r(N)$. We shall denote this by M.

(6) Now let $\mathfrak{l}$ be a simple left ideal which is isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$. Then $E_{\pi(\kappa)}\mathfrak{l} = \mathfrak{l}$. Since $\mathfrak{l} \subseteqq M$, $\mathfrak{l} = E_{\pi(\kappa)}\mathfrak{l} \subseteqq E_{\pi(\kappa)}M = ME_{\kappa}$. Now if κ is an index such that the largest completely reducible right subideal of $e_{\pi(\kappa)}A$ is not simple, then $f(\kappa) = 1$ and consequently $E_{\kappa} = e_{\kappa}$ is a primitive idempotent. Since Me_{κ} is a simple left ideal, $\mathfrak{l} = Me_{\kappa}$. In this case we can easily show that the duality relation holds for $\mathfrak{l}$. Now assume that κ is not such an index as above. Since $\mathfrak{l}$ is isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$, $\mathfrak{l} = Ae_{\pi(\kappa)}m$ for a suitable element $m \neq 0$ of $\mathfrak{l}$. Since m is an element of M, $e_{\pi(\kappa)}mA$ is contained in $e_{\pi(\kappa)}M$. From the assumption of κ, $e_{\pi(\kappa)}mA$ is a simple right ideal. On the other hand $e_{\pi(\kappa)}mA \simeq A/r(e_{\pi(\kappa)}m)$. Therefore $r(e_{\pi(\kappa)}m)$ must be a maximal right ideal of A and there exists a primitive idempotent e such that $r(e_{\pi(\kappa)}m) = N \cup (1-e)A$. Therefore $e_{\pi(\kappa)}m = e_{\pi(\kappa)}me + e_{\pi(\kappa)}m(1-e) = e_{\pi(\kappa)}me$ and $\mathfrak{l} = Ae_{\pi(\kappa)}me = Me$. Then we can show easily that the duality relation holds for $\mathfrak{l}$. Thus we have that the duality relation holds for every simple left ideal.

(7) Let L be a completely reducible left ideal and $L = \sum_{i=1}^{s} \mathfrak{m}_i$ be a direct decomposition of L into simple left ideals $\mathfrak{m}_i$. $r(\mathfrak{m}_i)$ is a maximal right ideal of A for each i. Therefore the composition length of $A/r(L)$ is at most s. Let $\mathfrak{r} \supset \mathfrak{r}_0$ are two right ideals and let $\mathfrak{r}/\mathfrak{r}_0$ be irreducible. Then $l(\mathfrak{r}_0)/l(\mathfrak{r})$ is an irreducible left module or zero.[7] Therefore the composition length of $l(r(L))$ is at most s. Therefore we have that $l(r(L)) = L$ Thus the duality relation holds for every completely reducible left ideal.

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7) Let $\mathfrak{r}/\mathfrak{r}_0$ be isomorphic to $\bar{e}_{\kappa}\bar{A}$, then $\mathfrak{r} = \mathfrak{r}_0 \cup be_{\kappa}A$ for a suitable element b of $\mathfrak{r}$. We see readily that $l(\mathfrak{r}_0)\, be_{\kappa}$ is contained in Me_{κ}. Since Me_{κ} is a simple left ideal, $l(\mathfrak{r}_0)\, be_{\kappa} = Me_{\kappa}$ or zero. On the other hand $l(\mathfrak{r}_0)\, be_{\kappa} \cong l(\mathfrak{r}_0)/l(\mathfrak{r}_0) \cap l(be_{\kappa}) = l(\mathfrak{r}_0)/l(\mathfrak{r}_0 \cup Abe_{\kappa}) = l(\mathfrak{r}_0)/l(\mathfrak{r})$. Therefore $l(\mathfrak{r}_0)/l(\mathfrak{r})$ is irreducible or zero.

(8) Let $\mathfrak{l}$ be a maximal left ideal. Now if $l(r(\mathfrak{l}))\supsetneqq\mathfrak{l}$ then $l(r(\mathfrak{l}))$ must be equal to A, since $\mathfrak{l}$ is a maximal left ideal. Therefore $r(l(r(\mathfrak{l})))=r(\mathfrak{l})=r(A)=0$. But $\mathfrak{l}=N\cup A(1-e)$ for a suitable primitive idempotent e, and $r(\mathfrak{l})=r(N\cup A(1-e))=eM\neq 0$. This is a contradiction. Therefore the duality relation holds for every maximal ideal. It can easily be seen that if the duality relation holds for $\mathfrak{l}_1$ and $\mathfrak{l}_2$ then it holds for $\mathfrak{l}_1\cap\mathfrak{l}_2$.[8] Now let L be a left ideal which is either equal to the radical N or contains the radical N. Then it is expressed as a cross-cut of a finite number of maximal ideals. Therefore, as mentioned above, the duality relation holds for L.

(9) By the same way as in (8), we have that the duality relation $r(l(\mathfrak{r}))=\mathfrak{r}$ holds for every right ideal which contains N, and for N itself.

This completes our proof.

In the case of algebras, we have, by Theorem I, the following

Theorem 3. *An algebra A, with a finite rank over a field F, is a quasi-Frobenius algebra if (and only if) the duality relation $l(r(\mathfrak{l}))=\mathfrak{l}$ holds for every nilpotent simple left ideal and zero. A is further a Frobenius algebra if (and only if) the rank relation $(A:F)=(\mathfrak{l}:F)+(r(\mathfrak{l}):F)$ holds for every nilpotent simple left ideal, besides the duality relation for nilpotent simple left ideals and zero.*

Proof. Assume that A is an *S. D_l-algebra* and is not a quasi-Frobenius algebra. By **Lemma 1.** $eAe=A^{\circ}$ is also an *S. D_l-algebra* and is not a Frobenius algebra,[9] where we take $\sum_{\kappa=1}^{n} e_\kappa$ as e. Therefore there exists at least one κ such that the largest completely reducible right subideal $e_{\pi(\kappa)}M^{\circ}$ of $e_{\pi(\kappa)}A^{\circ}$ is not simple. As in the proof of (III) of Theorem 1, $M^{\circ}e_\kappa=e_{\pi(\kappa)}Ae_{\pi(\kappa)}m_\kappa\supsetneqq m_\kappa e_\kappa Ae_\kappa$. Since m_κ is an element of M°, αm_κ and $m_\kappa\beta(\alpha\in e_{\pi(\kappa)}Ae_{\pi(\kappa)},\ \beta\in e_\kappa Ae_\kappa)$ are determined uniquely by the residue classes $\bar\alpha$ and $\bar\beta$ (mod. N) to which α and β belongs respectively. Hence we may express $e_{\pi(\kappa)}Ae_{\pi(\kappa)}m_\kappa$ by $\bar e_{\pi(\kappa)}\bar A\bar e_{\pi(\kappa)}m_\kappa$ and $m_\kappa e_\kappa Ae_\kappa$ by $m_\kappa\bar e_\kappa\bar A\bar e_\kappa$. From $\bar e_{\pi(\kappa)}\bar A\bar e_{\pi(\kappa)}m_\kappa\supsetneqq m_\kappa\bar e_\kappa\bar A\bar e_\kappa$, we have a "properly into" isomorphism θ of $\bar e_\kappa\bar A\bar e_\kappa$ into $\bar e_{\pi(\kappa)}\bar A\bar e_{\pi(\kappa)}$. Similarly we have an isomorphism θ' of $\bar e_{\pi(\kappa)}\bar A\bar e_{\pi(\kappa)}$ onto or into $\bar e_{\pi(\pi(\kappa))}\bar A\bar e_{\pi(\pi(\kappa))}$, and so on. Finally we have a "properly into" isomorphism $\Theta(=\theta\cdot\theta'\dots)$ of $\bar e_\kappa\bar A\bar e_\kappa$ into $\bar e_\kappa\bar A\bar e_\kappa$, since π is a permutation. But since $\bar e_\kappa\bar A\bar e_\kappa$ is a division algebra with a finite rank over a field F, this is impossible. Therefore

8) $r(\mathfrak{l}_1\cap\mathfrak{l}_2)\supseteqq r(\mathfrak{l}_1)\cup r(\mathfrak{l}_2)$, hence $l(r(\mathfrak{l}_1)\cup r(\mathfrak{l}_2))=l(r(\mathfrak{l}_1))\cap l(r(\mathfrak{l}_2))=\mathfrak{l}_1\cap\mathfrak{l}_2\supseteqq l(r(\mathfrak{l}_1\cap\mathfrak{l}_2))\supseteqq\mathfrak{l}_1\cap\mathfrak{l}_2$. Therefore $l(r(\mathfrak{l}_1\cap\mathfrak{l}_2))=\mathfrak{l}_1\cap\mathfrak{l}_2$.

9) From the definition of quasi-Frobenius rings, we see readily that A is a quasi-Frobenius ring if and only if eAe is a Frobenius ring, where $e=\sum_{\kappa=1}^{n} e_{\kappa,\,i(\kappa)}$.

A must be a quasi-Frobenius algebra. The converse is trivial. The latter half is same as Theorem 6 in S. II.

2. *Some special D_l-rings.*

Now we shall consider D_l-rings. Of course a D_l-ring has the properties (I), (II) and (III).

Moreover we shall give a necessary condition for a ring to be a D_l-*ring*, other than (I), (II) and (III).

Theorem 4. *Let A be a D_l-ring. Then A satisfies the following condition.*

(*) *Every left ideal which has a unique simple left subideal, is contained in an indecomposable left ideal Ae which is generated by a suitable primitive idempotent e.*

Proof. Since A has an identity element, A satisfies the maximum condition also. Therefore A has a composition series. Now let $\mathfrak{l}$ be a left ideal which has a unique simple left ideal. Then a maximal subideal $\mathfrak{l}'$ of $\mathfrak{l}$ satisfies the same assumption as $\mathfrak{l}$. Therefore we shall apply induction.

Let $\mathfrak{t}$ be a simple left ideal. Then $r(\mathfrak{t})=(1-e)\ A\cup N$ for a suitable primitive idempotent e, since $r(\mathfrak{t})$ is a maximal right ideal. Therefore $l(r(\mathfrak{t}))=\mathfrak{t}=Me$, by the duality relation.

Now assume that the condition holds for left ideals which have shorter composition lengths than that of $\mathfrak{l}$. Thus a maximal left subideal $\mathfrak{l}'$ of $\mathfrak{l}$ is contained in Ae' for a suitable primitive idempotent e'. If $\mathfrak{l}=\mathfrak{l}e'$, then $\mathfrak{l}\subset Ae'$. This case in trivial. Therefore we assume that $\mathfrak{l}\neq\mathfrak{l}e'$. Since $\mathfrak{l}$ contains $\mathfrak{l}'=\mathfrak{l}'e'$, $\mathfrak{l}e'$ contains $\mathfrak{l}'$. If $\mathfrak{l}e'=\mathfrak{l}'$, then $\mathfrak{l}\supset\mathfrak{l}e'=\mathfrak{l}'$ and we can decompose $\mathfrak{l}$ into $\mathfrak{l}e'$ and $\mathfrak{l}(1-e')\neq 0$. But this contradicts the fact that $\mathfrak{l}$ has a unique simple left ideal. Thus $\mathfrak{l}e'\supsetneqq\mathfrak{l}'$. Since $\mathfrak{l}(1-e')\cong\mathfrak{l}/\mathfrak{l}\cap l(1-e')$ and $\mathfrak{l}\cap l(1-e')\supseteqq\mathfrak{l}'$, we have that $\mathfrak{l}(1-e')$ is either a simple left ideal or zero. If $\mathfrak{l}(1-e')=0$, then we have $\mathfrak{l}=\mathfrak{l}e'$. This contradicts the above assumption $\mathfrak{l}\neq\mathfrak{l}e'$. Thus $\mathfrak{l}(1-e')$ is a simple left ideal. It can easily be seen that $r(\mathfrak{l}')=r(\mathfrak{l}'e')=(1-e')\ A+e'r(\mathfrak{l}')$. Therefore $r(\mathfrak{l})\subsetneqq(1-e')\ A+e'r(\mathfrak{l}')$. Similarly $r(\mathfrak{l}(1-e'))=e'A+((1-e')\ A\cap r(\mathfrak{l}))$. Moreover, since $\mathfrak{l}(1-e')$ is a simple left ideal, $r(\mathfrak{l}(1-e'))$ is a maximal right ideal. Therefore $r(\mathfrak{l}(1-e'))=(1-e)\ A+eN$, where e is a primitive idempotent and $ee'=e'e=0$.[10] Therefore $(1-e')A$

10) Since $r(\mathfrak{l}(1-e'))$ is a maximal right ideal, $\overline{r(\mathfrak{l}(1-e'))}$, the residue class of $r(\mathfrak{l}(1-e'))$ mod. N, is a maximal right ideal of the residue class ring $\bar{A}$ of A mod. N. $\overline{r(\mathfrak{l}(1-e'))}$ is generated by an idempotent $\bar{E}$. Since $\overline{r(\mathfrak{l}(1-e'))}$ contains $\bar{e}'$ (the residue class of e' mod. N), $\bar{E}\bar{e}'=\bar{e}'$. We shall denote $\bar{E}-\bar{e}'\bar{E}$ by $\bar{E}'$. Then $\bar{E}'$ is an idempotents and $\bar{E}'\bar{e}'=\bar{e}'\bar{E}'=0$ and $\overline{r(\mathfrak{l}(1-e'))}=\bar{E}\bar{A}=\bar{e}'\bar{A}+\bar{E}'\bar{A}$. We shall decompose $\bar{E}'$ into primitive idempotents $\bar{e}_i(i=2,\cdots,n-1)$. Then $\bar{e}_1=\bar{e}'$, $\bar{e}_2,\cdots,\bar{e}_{n-1}$, $\bar{e}_n=1-\sum_{i=1}^{n-1}\bar{e}_i$ form a system of orthogonal idempotents. As is well known we can constract orthogonal primitive idempotents $e_i(i=1,\cdots,n)$ such that $e_i\in\bar{e}_i$, where we can take e' as e_1. We see readily that $r(\mathfrak{l}(1-e'))=e_1A+\cdots+e_{n-1}A+e_nN=(1-e_n)\ A+e_nN$, where $e'e_n=e_ne'=0$.

$\bigwedge r(\mathfrak{l})=(1-e-e')\ A+eN$ and consequently $r(\mathfrak{l})\supseteqq(1-e-e')\ A+eN$. From the above relations, we have $(1-e')\ A+e'r(\mathfrak{l}')\supset r(\mathfrak{l})\supseteqq(1-e-e')A+eN$. Here we may assume that $e'r(\mathfrak{l}')$ is a nilpotent ideal. For if $e'r(l')$ is not nilpotent, then $e'r(\mathfrak{l}')$ contains an idempotent and $r(\mathfrak{l}')=(1-e')\ A+e'r(\mathfrak{l}')$ is equal to A, hence $\mathfrak{l}'=l(r(\mathfrak{l}'))=l(A)=0$, but in this case $\mathfrak{l}$ is a simple left ideal and the condition holds for $\mathfrak{l}$. Since $(1-e')\ A+e'r(\mathfrak{l}')/(1-e-e')\ A+eN+e'r(\mathfrak{l}')$ is an irreducible right module, we have that either $r(\mathfrak{l})$ is contained in $(1-e-e')\ A+eN+e'r(\mathfrak{l}')$ or $r(\mathfrak{l})\bigvee e'r(\mathfrak{l}')=(1-e')\ A+e'r(\mathfrak{l}')$. If $r(\mathfrak{l})\subseteqq(1-e-e')\ A+eN+e'r(\mathfrak{l}')$, then $\mathfrak{l}\ =\ l(r(\mathfrak{l}))\supseteqq l((1-e-e')\ A+eN+e'r(\mathfrak{l}'))\ =\ A(e+e')\bigwedge(A(1-e)+Me)\bigwedge(A(1-e')+(Ae'\bigwedge l(r(\mathfrak{l}'))))=Me+\mathfrak{l}'$. This contradicts the assumption that $\mathfrak{l}$ has a unique simple left ideal. Therefore $r(\mathfrak{l})\bigvee e'r(\mathfrak{l}')=(1-e')\ A+e'r(\mathfrak{l}')$. Since $r(\mathfrak{l})$ contains $(1-e-e')\ A$, we can decompose $r(\mathfrak{l})$ into $(1-e-e')\ A$ and $(e+e')\ r(\mathfrak{l})$. Therefore $(e+e')\ r(\mathfrak{l})\bigvee e'r(\mathfrak{l}')=eA+e'r(\mathfrak{l}')$. This shows that $(e+e')\ r(\mathfrak{l})$ is not nilpotent. Therefore $(e+e')\ r(\mathfrak{l})$ contains an idempotent e_1. Since $(e+e')\ r(\mathfrak{l})\ni e_1$, we have that $(e+e')\ e_1=e_1$. If we denote $e_1(e+e')$ by e_2, then e_2 is not zero. For if e_2 is zero, then $e_1=e_1^2=((e+e')\ e_1)^2=0$, and this is a contradiction. It can easily be seen that e_2 is an idempotent and $(1-e-e')\ e_2=e_2(1-e-e')=0$. Now we set $e_3=(e+e')-e_2$. Then e_3 is an idempotent and is not zero. Because, if e_3 is zero, then $(e+e')\ r(\mathfrak{l})$ contains $eA+e'A$ and consequently $\mathfrak{l}=l(r(\mathfrak{l}))=l((1-e-e')\ A+eA+e'A)=l(A)=0$. Moreover, since e_2 and e_3 are mutually orthogonal idempotent, e_3 is a primitive idempotent. Since $(e+e')\ r(\mathfrak{l})\supset e_2A$, we have a direct decomposition of $(e+e')\ r(\mathfrak{l})$ into e_2A and $\mathfrak{r}_0$, where $\mathfrak{r}_0$ is a right ideal consisting of elements such that $e_2x=0$. From $e_2\mathfrak{r}_0=0$ and $(e+e')\ \mathfrak{r}_0=\mathfrak{r}_0$, we have that $e_3\mathfrak{r}_0=\mathfrak{r}_0$. Therefore $\mathfrak{l}=l(r(\mathfrak{l}))=l((1-e_3)\ A+e_3\mathfrak{r}_0)=Ae_3\bigwedge l(\mathfrak{r}_0)\subseteqq Ae_3$, and this shows that $\mathfrak{l}$ is contained in an indecomposable ideal Ae_3. This completes our proof.

Remark. Let A be a ring which satisfies (I), (II) and (*). Then $r(N)=l(N)=M$ and every simple left ideal is of the form Me for a suitable primitive idempotent e, by (*). Therefore the duality relation holds for every simple left ideal. Thus A is an *S. D_l-ring,* and consequently A satisfies (III). Actually the conditions (I), (II) and (*) are stronger than (I), (II) and (III).

Example 1. Let $K(x)$ be the rational function field over a field K. Then

$$A=K(x)\ e_1+K(x)\ e_2+K(x)\ u_1+K(x)\ u_2+K(x)\ u_1u_2+K(x)\ u_2u_1$$
$$e_1x=xe_1\quad e_2x=xe_2\quad u_1x=x^2u_1\quad u_2x=x^2u_2$$

	e_1	e_2	u_1	u_2	u_1u_2	u_2u_1
e_1	e_1	0	0	u_2	0	u_2u_1
e_2	0	e_2	u_1	0	u_1u_2	0
u_1	u_1	0	0	u_1u_2	0	0
u_2	0	u_2	u_2u_1	0	0	0
u_1u_2	0	u_1u_2	0	0	0	0
u_2u_1	u_2u_1	0	0	0	0	0

is an *S. D_l-ring*. But A does not satisfy $(*)$.

For example, the left ideal $A(u_1+xu_1u_2)$ is homomorphic to Ae_2 which has a unique composition series, hence $A(u_1+xu_1u_2)$ has a unique simple left subideal. Now assume that $A(u_1+xu_1u_2)$ is contained in an indecomposable left ideal Ae. Then e is congruent to e_1 or e_2 mod. N. Therefore e is either $e_1+f_1(x)\ u_1+f_2(x)\ u_2+f_3(x)\ u_1u_2+f_4(x)\ u_2u_1$ or $e_2+g_1(x)\ u_1+g_2(x)\ u_2+g_3(x)\ u_1u_2+g_4(x)\ u_2u_1$. If $e=e_1+f_1(x)\ u_1+f_2(x)\ u_2+f_3(x)\ u_1u_2+f_4(x)\ u_2u_1$, then $(u_1+xu_1u_2)\ e=u_1+f_2(x^2)\ u_1u_2\neq u_1+xx_1u_2$. If $e=e_2+g_1(x)\ u_1+g_2(x)\ u_2+g_3(x)\ u_1u_2+g_4(x)\ u_2u_1$, then $(u_1+xu_1u_2)\ e=(g_2(x^2)+x)\ u_1u_2\neq u_1+xu_1u_2$. These are contradictions. Therefore A does not satisfy $(*)$.

In general, the conditions (I), (II) and $(*)$ are not sufficient for a ring to be a *D_l-ring*.

Example 2. Let $K(x)$ be the rational function field over a field K. Then $A=K(x)+K(x)\ u_1+K(x)\ u_2+K(x)\ u_1u_2$ ($u_1^2=u_2^2=0$ $u_1u_2=u_2u_1$ $u_1x=x^2u_1$ $u_2x=x^2u_2$) is a completely primary ring and satisfies (I), (II) and $(*)$, but is not a *D_l-ring*.

Next we shall consider some special *D_l-rings*.

We shall call a ring A a *generalized left uni-serial ring*, if A has an identity element and every indecomposable left ideal Ae generated by a primitive idempotent e has a unique composition series. A left uniserial ring is a primary decomposable generalized left uni-serial ring.

Let A be a *generalized left uni-serial ring* which satisfies (II). Then, since an indecomposable left ideal Ae_κ has a unique composition series, $Ae_\kappa \supset Ne_\kappa \supset \dots N^{\sigma(\kappa)-2}\,e_\kappa \supset N^{\sigma(\kappa)-1}\,e_\kappa \supset 0$ is the unique composition series of Ae_κ and $N^{\sigma(\kappa)-1}\,e_\kappa \cong \bar{A}\bar{e}_{\pi(\kappa)}$.

Now we assume that $Ae_\kappa/Ne_\kappa \cong \bar{A}\bar{e}_\kappa$, $Ne_\kappa/N^2e_\kappa \cong \bar{A}\bar{e}_{\lambda_1} \dots N^ie_\kappa/N^{i+1}\,e_\kappa \cong \bar{A}\bar{e}_{\lambda_i}, \dots, N^{\sigma(\kappa)-1}\,e_\kappa \cong \bar{A}\bar{e}_{\pi(\kappa)}$. If we take an element d of Ne_κ which is not in N^2e_κ, then we have $Ne_\kappa=Ae_{\lambda_1}de_\kappa$, hence Ne_κ is homomorphic to Ae_{λ_1}.

If $\kappa=\lambda_1$, then we can easily show that $N^ie_\kappa/N^{i+1}e_\kappa$ is isomorphic to $\bar{A}\bar{e}_\kappa$ for every i, If $\kappa \neq \lambda_1$, then the composition factor groups of Ae_{λ_1} are as follows; $Ae_{\lambda_1}/Ne_{\lambda_1} \simeq \bar{A}\bar{e}_{\lambda_1}$, $Ne_{\lambda_1}/N^2e_{\lambda_1} \simeq \bar{A}\bar{e}_{\lambda_2}$, ..., $N^{\sigma(\kappa)-2}e_{\lambda_1}/N^{\sigma(\kappa)-1}e_{\lambda_1} \simeq \bar{A}\bar{e}_{\pi(\kappa)}$, $N^{\sigma(\kappa)-1}e_{\lambda_1}/N^{\sigma(\kappa)}e_{\lambda_1}, \dots, N^{\sigma(\lambda_1)-1}e_{\lambda_1} \simeq \bar{A}\bar{e}_{\pi(\lambda_1)}$, but since $\pi(\kappa) \neq \pi(\lambda_1)$ if $\kappa \neq \lambda_1$, we have $[Ae_{\lambda_1}]_l \geqq [Ae_\kappa]_l$. (For the sake of brevity we shall denote the composition length of a left module $\mathfrak{M}$ by $[\mathfrak{M}]_l$.) In this case $\lambda_1 \neq \lambda_2$, for if $\lambda_1 = \lambda_2$, then as above $N^ie_{\lambda_1}/N^{i+1}e_{\lambda_1} \simeq \bar{A}\bar{e}_{\lambda_1}$ for every i, hence $\bar{A}\bar{e}_{\pi(\kappa)} \simeq \bar{A}\bar{e}_{\lambda_1}$ and $\bar{A}\bar{e}_{\pi(\lambda_1)} \simeq \bar{A}\bar{e}_{\lambda_1}$, consequently $\pi(\kappa) = \lambda_1 = \pi(\lambda_1)$, but this is a contradiction. Thus we have $[Ae_{\lambda_2}]_l \geqq [Ae_{\lambda_1}]_l \geqq [Ae_\kappa]_l$, as above, and so on. Finally we have $[Ae_{\pi^i(\kappa)}]_l \geqq [Ae_{\pi^{i-1}(\lambda_{\sigma(\kappa)-2})}]_l \geqq \dots \geqq [Ae_\kappa]_l$, where $\pi^i(\kappa)$ means $\underbrace{\pi(\pi(\pi(\dots \pi(\kappa))))\dots)}_{i\text{-times}}$. If we take such an r that $\pi^r(\kappa) = \kappa$, then the above inequalities lead that $[Ae_\kappa]_l = [Ae_{\pi^r(\kappa)}]_l \geqq \dots \geqq [Ae_\kappa]_l$, therefore we have that all $[Ae_{\pi^i(\lambda_j)}]_l$'s are the same, and moreover $Ae_{\pi^i(\lambda_j)}$ has only $\bar{A}\bar{e}_{\pi^h(\lambda_k)}$'s as the composition factor groups of it. This shows that we can classify $(1, \dots, n)$ into classes C_σ as follows;

(α) If $\lambda \in C_\sigma$, then any composition factor group of Ae_λ is isomorphic to $\bar{A}\bar{e}_\mu$ with some $\mu \in C_\sigma$.

(β) If λ, $\mu \in C_\sigma$, then $[Ae_\lambda]_l = [Ae_\mu]_l$.

(α) shows that we have a direct decomposition of A into two-sided ideals $A_\sigma = E_\sigma A E_\sigma$, where we denote $\sum_{\mu \in C_\sigma; i=1,\dots,f(\mu)} e_{\mu,i}$ by E_σ.[11] Since A satisfies (II), every A_σ satisfies (II) also. From (β), if we denote the radical of A_σ by N_σ, then $A_\sigma e_\kappa (= Ae_\kappa) \supset N_\sigma e_\kappa \dots \supset N_\sigma^{\rho(\sigma)-1} e_\kappa \supset 0$ is the unique composition series of Ae_κ, and since $r(N_\sigma)\, e_\kappa$ is the largest completely reducible subideal of Ae_κ, we have $r(N_\sigma)\, e_\kappa = N_\sigma^{\rho(\sigma)-1}\, e_\kappa$, and since $r(N_\sigma^2)\, e_\kappa/r(N_\sigma)\, e_\kappa$ is the largest completely reducible left submodule of $Ae_\kappa/N_\sigma^{\rho(\sigma)-1}e_\kappa$, we have $r(N_\sigma^2)\, e_\kappa = N_\sigma^{\rho(\sigma)-2}\, e_\kappa$, finally we have $r(N_\sigma^t)\, e_\kappa = N_\sigma^{\rho(\sigma)-t}\, e_\kappa$ and $r(N_\sigma^t) = \sum_{\kappa \in C_\sigma; i=1,\dots,f(\kappa)} r(N_\sigma^t)\, e_{\kappa,i} = \sum_{\kappa \in C_\sigma; i=1,\dots,f(\kappa)} N_\sigma^{\rho(\sigma)-t}\, e_{\kappa,i} = N_\sigma^{\rho(\sigma)-t}$ for $t = 1, 2, \dots, \rho(\sigma)$. Moreover, as was shown in the proof of Theorem 2, $r(N_\sigma) = l(N_\sigma)$ and consequently $r(N_\sigma^t) = l(N_\sigma^t) = N_\sigma^{\rho(\sigma)-t}$ for $t = 1, 2, \dots, \rho(\sigma)$.

Theorem 5. *Let A be a generalized left uni-serial ring, then A is a D_l-ring if and only if A satisfies (II) and (*).*

Proof. "if" part follow from Theorem 1 and Theorem 4 directly. Let $\mathfrak{l}$ be a left ideal of a ring A which satisfies (II) and (*). We may assume that A is a *generalized left uni-serial ring* in which $r(N^t) = l(N^t) = N^{\rho-t}$, where $N^\rho = 0$ and $N^{\rho-1} \neq 0$. Let $\mathfrak{l} = \bigcup_{i=1}^{n} Ab_i$ and $A = \sum_{j=1}^{m} Ae_j$ be the direct decomposition of A into indecomposable left ideals, then

11) This decomposition is so called "Block decomposition".

$Ab_i = \bigcup_{j=1}^{m} Ae_jb_i$ and consequently $\mathfrak{l} = \bigcup_{i,j} Ae_jb_i$.

Since every Ae_jb_i is homomorphic to Ae_j and consequently it has a unique simple left subideal, Ae_jb_i is contained in an indecomposable left ideal $Ae^{(j,i)}$ which is generated by a suitable primitive idempotent $e^{(j,i)}$, by (∗). Therefore $Ae_jb_i = N^{r(j,i)}e^{(j,i)}$. Let $r(1,1)$ be the minimum of $r(j,i)$. Then $\mathfrak{l}e^{(1,1)} = \bigcup_{j,i} N^{r(j,i)}e^{(i,j)}e^{(1,1)} = N^{r(1,1)}e^{(1,1)} \subseteqq \mathfrak{l}$. Therefore we have the direct decomposition of $\mathfrak{l}$ into $\mathfrak{l}e^{(1,1)}$ and $\mathfrak{l}(1-e^{(1,1)})$; $\mathfrak{l} = \mathfrak{l}e^{(1,1)} + \mathfrak{l}(1-e^{(1,1)})$.

Let $\mathfrak{l}_0$ be a simple left ideal then $\mathfrak{l}_0$ is contained in an indecomposable left ideal Ae_0 and consequently $\mathfrak{l}_0 = N^{\rho-1}e_0$ by (∗). Then it can easily be seen that $l(r(\mathfrak{l}_0)) = \mathfrak{l}_0$.

Now we shall apply induction. Assume that the duality relation holds for left ideals which have shorter composition lengths than that of $\mathfrak{l}$.

If $\mathfrak{l} = \mathfrak{l}e^{(1,1)}$, then $\mathfrak{l} = N^{r(1,1)}e^{(1,1)}$, hence $r(\mathfrak{l}) = (1-e^{(1,1)})A + e^{(1,1)}N^{\rho-r(1,1)}$ and $l(r(\mathfrak{l})) = N^{r(1,1)}e^{(1,1)} = \mathfrak{l}$. Thus in this case the duality relation holds for $\mathfrak{l}$. If $\mathfrak{l} = \mathfrak{l}(1-e^{(1,1)})$ then $\mathfrak{l}e^{(1,1)} = 0$, but this is a contradiction.

Therefore we can assume that $\mathfrak{l}e^{(1,1)}$ and $\mathfrak{l}(1-e^{(1,1)})$ have shorter composition lengths than that of $\mathfrak{l}$. Then, by the induction assumption, the duality relation holds for $\mathfrak{l}e^{(1,1)}$ and $\mathfrak{l}(1-e^{(1,1)})$.

We see readily $r(\mathfrak{l}e^{(1,1)}) = (1-e^{(1,1)})A + (e^{(1,1)}A \frown r(\mathfrak{l}))$ and $r(\mathfrak{l}(1-e^{(1,1)})) = e^{(1,1)}A + ((1-e^{(1,1)})A \frown r(\mathfrak{l}))$. We shall denote $l(r(\mathfrak{l}))$ by $\bar{\mathfrak{l}}$. Of course $r(\mathfrak{l}) = r(\bar{\mathfrak{l}})$.

On the other hand we have $r(\bar{\mathfrak{l}}e^{(1,1)}) = (1-e^{(1,1)})A + (e^{(1,1)}A \frown r(\bar{\mathfrak{l}})) = r(\mathfrak{l}e^{(1,1)})$, hence $l(r(\bar{\mathfrak{l}}e^{(1,1)})) = l(r(\mathfrak{l}e^{(1,1)})) = \mathfrak{l}e^{(1,1)}$, but since $\bar{\mathfrak{l}}e^{(1,1)} \supseteqq \mathfrak{l}e^{(1,1)}$, we have $\bar{\mathfrak{l}}e^{(1,1)} = \mathfrak{l}e^{(1,1)}$. Similarly we have $\bar{\mathfrak{l}}(1-e^{(1,1)}) = \mathfrak{l}(1-e^{(1,1)})$. Since $\bar{\mathfrak{l}} \subseteqq \bar{\mathfrak{l}}e^{(1,1)} + \bar{\mathfrak{l}}(1-e^{(1,1)})$, we have $\bar{\mathfrak{l}} \subseteqq \mathfrak{l}e^{(1,1)} + \mathfrak{l}(1-e^{(1,1)}) = \mathfrak{l}$. This shows $\bar{\mathfrak{l}} = \mathfrak{l}$.

For a *generalized left uni-serial ring* A, we can take the following condition (∗′), in place of (∗);

(∗′): *Every left ideal which is homorphic to an indecomposable left ideal Ae which is generated by a primitive idempotent e, is contained in an indecomposable left ideal Ae′ for a suitable primitive idempotent e′.*

By Theorem 5 and Corollary 2 of Theorem 1 we have

Theorem 6. *Let A be a left uni-serial ring then the following four conditions are equivalent.*

(1) *A is an S. D_l-ring.*

(2) *A is a D_l-ring.*

(3) *A satisfies* (∗).

(4) *A is a direct sum of a uni-serial ring and completely primary left uni-serial rings.*

Proof. (1)→(4) can easily be seen by Corollary 1 of Theorem 1 and by the fact that a left uni-serial ring is a quasi-Frobenius ring if it is a uni-serial ring.

(4)→(2) from the fact that a completely primary left uni-serial ring is a D_l-*ring*.

(2)→(3) is trivial.

(3)→(1) it is sufficient to show that (II) is satisfied.
But this can easily be seen from the fact that $r(N) = l(N) = N^{\rho-1}$ is the unique minimal two-sided ideal of a primary left uni serial ring A, where $N^{\rho} = 0$ and $N^{\rho-1} \neq 0$.

3. *Rings whose residue class rings are all S. D_l-rings.*

It is well known that A is a uni-serial ring if and only if every residue class ring of A is a Frobenius ring.[12]

We shall generalize this theorem as follows.

Theorem 7. *If every residue class ring of a ring A is an S. D_l-ring, then A is a left uni-serial S. D_l-ring, that is, it is the direct sum of a uni-serial ring and completely primary left uni-serial rings. The converse is also true.*

Proof. Let A be a ring such that every residue class ring of it is an *S. D_l-ring*. Now we shall decompose A into a bound ring A_1 and a semi-simple ring A_2.[13] Then, of course, A_2 is a uni-serial ring. Therefore it is sufficient to prove that A_1 is a left uni-serial *S. D_l-ring*. Since A_1 is an *S. D_l-ring*, it has an identity element, hence satisfies the maximum condition (not only the minimum condition). Therefore A_1 has a composition series. Now let us assume that our assertion is true for rings with smaller composition lengths. Since A_1 is a bound ring, we see readily that $\tilde{A}_1\tilde{e}_\lambda$ is an indecomposable left ideal of $\tilde{A}_1 = A_1/E_{\pi(\kappa)}ME_\kappa$ for each λ ($\sim$denotes residue classes mod. $E_{\pi(\kappa)}ME_\kappa$). By our assumption, A_1 is a left uni-serial ring. Hence $\tilde{A}_1\tilde{e}_\lambda = A_1e_\lambda \cup E_{\pi(\kappa)}ME_\kappa/E_{\pi(\kappa)}ME_\kappa$ has a unique composition series and the composition factor groups of it are isomorphic to $\bar{A}_1\bar{e}_\lambda$. Since $A_1e_\lambda \cup E_{\pi(\kappa)}ME_\kappa/E_{\pi(\kappa)}ME_\kappa \cong A_1e_\lambda$ for $\lambda \neq \kappa$, we have that A_1e_λ has a unique composition series composition factor groups are isomorphic to $\bar{A}_1\bar{e}_\lambda$. Since κ is arbitrary, it follows that A_1e_κ has a unique composition series and the composition factor

12) T. Nakayama: Note on uni-serial rings and generalized uni.serial rings, Proc. Imp. Acad. 16 (1940).

13) A ring A is called as a bound ring, if $r(N) \cap l(N) \subseteq N$. M. Hall: The position of the radical in an algebra, Trans. Amer. Math. Soc. 48.

groups of it are isomorphic to $\bar{A}\bar{e}_\kappa$ for all κ. This shows that A_1 is a left uni-serial *S. D_l-ring*. Therefore, by Theorem 6, we have our Theorem. The converse is trivial.

As a corollary of this theorem, we have

Corollary. *If every residue class ring of a ring A is quasi-Frobenius ring, then A is a uni-serial ring, and conversely.*

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