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ON THE BIALGEBRAS OF GROUP SCHEMES

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Let G be an algebraic group scheme over an algebraically closed field k. We shall first show that the set $\mathfrak{H}(G)$ of left invariant high order derivations on G will have a natural structure of bialgebra over k with only one grouplike element. If α is a surjective homomorphism of a group variety G onto a group variety G', the kernel H of α in the category of algebraic k-group schemes is well defined. Moreover we have a bialgebra homomorphism $d\alpha$ of $\mathfrak{H}(G)$ into $\mathfrak{H}(G')$. H. Yanagihara showed surjectivity of $d\alpha$ and investigated k-vector space structure of the kernel of $d\alpha$ in the category of bialgebras using the semi-derivations in [13]. In this paper it will be proved that the kernel of $d\alpha$ in the category of bialgebras coincides with the bialgebra of H and we have an exact sequence

$$0 \longrightarrow \mathfrak{H}(H) \longrightarrow \mathfrak{H}(G) \longrightarrow \mathfrak{H}(G') \longrightarrow 0$$

in the category of bialgebras, while the bialgebra of H is not defined in general using the semi-derivations. Thus the bialgebra $\mathfrak{H}(G)$ may be a good substitute of Lie algebras in the case of positive characteristic. The next problem which we are interested is the characterization of sub-bialgebra of $\mathfrak{H}(G)$ which arises from a closed subgroup scheme. Unfortunately we have no general solution, but a solution will be given when G is a commutative group variety over k. Our results have close connection with the work of H. Yanagihara and our bialgebra $\mathfrak{H}(G)$ coincides with the bialgebra used by H. Yanagihara in [12] when G is a group variety.

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1. Local high order derivations of a local ring

Let O be a noetherian local ring containing a field k such that $O/\mathfrak{m}$ is canonically isomorphic to k, where $\mathfrak{m}$ is the unique maximal ideal of O. We denote by $x(o)$ the element of k representing the class of x in O modulo $\mathfrak{m}$. A k-linear homomorphism D of O into k is called a local n-th order derivation of O if we have

$$D(x_0x_1\cdots x_n)=\sum_{s=1}^{n}(-1)^{s-1}\sum_{i_1<\cdots<i_s}x_{i_1}(o)\cdots x_{i_s}(o)D(x_0\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_n)$$

for any sequence $x_0, x_1,\cdots, x_n$ of (n+1)-elements in O. We denote by $\mathfrak{D}_n(O)$ the set of local n-th order derivations of O and set $\mathfrak{D}(O)=k\oplus\bigcup_{n=1}^{\infty}\mathfrak{D}_n(O)$, where $a(x)$ is defined by $ax(\mathrm{o})$ for $a\in k$ and $x\in O$. Then it is easily seen that $\mathfrak{D}(O)$ is a subspace of $\mathrm{Hom}_k(O,k)$.

Proposition 1. *Let the situation be as above. Then we have*
(1) $\mathfrak{D}_n(O)$ *is canonically isomorphic to* $Hom_k(\mathfrak{m}/\mathfrak{m}^{n+1}, k)$ *as a k-vector space.*
(2) $\bigcup_{n=1}^{\infty}\mathfrak{D}_n(O)$ *is the set of k-linear homomorphisms of O into k vanishing on some power of* $\mathfrak{m}$.
(3) $\mathfrak{D}(O)$ *has a cocommutative coalgebra structure over k.*

Proof. (1) The mapping Φ of $\mathfrak{D}_n(O)$ into $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^{n+1}, k)$ is defined as follows. If $D\in\mathfrak{D}_n(O)$, we set $\Phi(D)(\bar{x})=D(x)$ for $x\in\mathfrak{m}$, where $\bar{x}$ is the class of x in $\mathfrak{m}$ modulo $\mathfrak{m}^{n+1}$. Since D vanishes on $\mathfrak{m}^{n+1}$, $\Phi(D)$ is well defined. Clearly Φ is k-linear and injective. We shall prove that Φ is surjective. Let $f\in\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^{n+1}, k)$. We put $D(x)=f(\overline{x\text{-}x(\mathrm{o})})$ for x in O. It will suffice to show $D\in\mathfrak{D}_n(O)$. Then D is k-linear and $[D, a+x]=[D, x]$ for a in k and x in $\mathfrak{m}$. (For the definition of $[D, x]$, see [8].) Hence we have $[\cdots[[D, a_1+x_1], a_2+x_2],\cdots, a_n+x_r]=[\cdots[[D, x_1], x_2],\cdots, x_n]$ for any $a_i\in k$ and any $x_i\in\mathfrak{m}$. Now $[\cdots[[D, x_1], x_2],\cdots, x_n](a+x)=0$ for any $a\in k$ and any $x, x_i\in\mathfrak{m}$ since D is k-linear and vanishes on $\mathfrak{m}^{n+1}$. Hence D is in $\mathfrak{D}_n(O)$.
(2) Obvious from (1).
(3) Let $\mu: O\otimes_k O\to O$ be the homomorphism induced by the multiplication of O. Then we have the dual mapping $\mu^*: \mathrm{Hom}_k(O, k)\to\mathrm{Hom}(O\otimes_k O, k)$. We shall prove $\mu^*(\mathfrak{D}(O))\subset\mathfrak{D}(O)\otimes_k\mathfrak{D}(O)$ ($\subset\mathrm{Hom}_k(O\otimes_k O,k)$). To this purpose, we have only to show $\mu^*(\mathfrak{D}_n(O))\subset\mathfrak{D}(O)\otimes_k\mathfrak{D}(O)$. Since $O/\mathfrak{m}\cong k$, $O/\mathfrak{m}^{n+1}$ is a finite dimensional k-vector space. We assume that the classes of $u_0=1, u_1,\cdots, u_m$ modulo $\mathfrak{m}^{n+1}$ form a k-basis of $O/\mathfrak{m}^{n+1}$. We denote by $\bar{u}_i$ the class of u_i in $O/\mathfrak{m}^{n+1}$ and $\bar{u}_0^*, \bar{u}_1^*,\cdots, \bar{u}_m^*$ its dual basis. Then $\bar{u}_1^*\circ\omega,\cdots, \bar{u}_m^*\circ\omega$ form a k-basis of $\mathfrak{D}_n(O)$, where ω is the canonocal homomorphism of O onto $O/\mathfrak{m}^{n+1}$. If $D\in\mathfrak{D}_n(O)$, an easy computation shows $\mu^*(D)=\sum_{i,j=1}^{m}D(u_iu_j)(\bar{u}_i^*\circ\omega\otimes\bar{u}_j^*\circ\omega)+\sum_{i=1}^{m}D(u_i)$ $\bar{u}_i^*\circ\omega\otimes\bar{u}_0^*\circ\omega+\bar{u}_0^*\circ\omega\otimes\bar{u}_i^*\circ\omega)+\bar{u}_0^*\circ\omega\otimes\bar{u}_0^*\circ\omega$. Thus $\mu^*(\mathfrak{D}_n(O))\subset\mathfrak{D}(O)\otimes_k\mathfrak{D}(O)$. We set $\Delta=\mu^*|\mathfrak{D}(O)$, the restriction of μ^* on $\mathfrak{D}(O)$. Since O is commutative, Δ is cocommutative. Augmentation $\varepsilon: \mathfrak{D}(O)\to k$ is defined by $\varepsilon(D)=D(1)$ for D in $\mathfrak{D}(O)$. Then it is easily seen that $(\mathfrak{D}(O), \Delta,\varepsilon)$ is a coalgebra over k.

2. The bialgebras of group schemes

Let S be a prescheme and X be an S-prescheme. We denote by f the structure morphism: $X \to S$. An n-th order derivation D of X/S is, by definition, an endomorphism of $f^{-1}(O_S)$-Module O_X satisfying the following identity:

$$D(\varphi_0 \varphi_1 \cdots \varphi_n) = \sum_{s=1}^{n}(-1)^{s-1} \sum_{i_1<\cdots<i_s} \varphi_{i_1} \cdots \varphi_{i_s} D(\varphi_0 \cdots \hat{\varphi}_{i_1} \cdots \hat{\varphi}_{i_s} \cdots \varphi_n)$$

for every open set U of X and every sequence $\varphi_0, \varphi_1 \cdots, \varphi_n$ of $\Gamma(U, O_X)$. $\mathfrak{D}_0^{(n)}(X/S)$ denotes the set of n-th order derivations of X/S. We set $\mathfrak{D}_0(X/S) = \bigcup_{n=1}^{\infty} \mathfrak{D}_n(X/S)$ and $\mathfrak{D}(X/S) = \Gamma(X, O_X) \oplus \mathfrak{D}_0(X/S)$. We see easily that $DE \in \mathfrak{D}_0(X/S)$ and $[D,\varphi] = D\varphi - \varphi D - D(\varphi)$ is an (m–1)-th order derivation for $D \in \mathfrak{D}_0^{(m)}(X/S)$, $E \in \mathfrak{D}_0^{(n)}(X/S)$ and $\varphi \in \Gamma(X, O_X)$ (cf. [8]). From these we can see that $\mathfrak{D}(X/S)$ is a $\Gamma(X, O_X)$-algebra. If u is a morphism of preschemes : $X \to Y$, we denote by $\tilde{u}$ the homomorphism of O_Y into $u_*(O_X)$.

Let G be an S-group scheme and let g: $S \to G$ be a section. The morphism g_G: $G \simeq S \times_S G \xrightarrow{g \times_S 1_G} G \times_S G \xrightarrow{m} G$ is the left translation by g of G, where 1_G (resp. m) is the identity morphism of G (resp. the multiplication of G). If D is a high order derivation of G/S, then we set $D^g = \tilde{g}_G^{-1}(g_G)_*(D)\tilde{g}_G$. D^g is also a high order derivation of G/S. A high order derivation D of G/S is called left invariant if we have $(D_T)^g = D_T$ for any base change $t: T \to S$ and any section $g : T \to T \times_S G$, where D_T is the high order derivation of $T \times_S G/T$ induced by D. Let k be a field and G be an algebraic k-group scheme. From now on we shall mean by a k-group scheme an algebraic k-group scheme. In this case we say a high order derivation of G/Spec(k) simply a high order derivation of G/k. We shall denote by $\mathfrak{G}(G)$ the set of left invariant high order derivations of G/k and set $\mathfrak{H}(G) = k \oplus \mathfrak{G}(G)$. It is clear that $\mathfrak{H}(G)$ is a k-algebra. Then $\mathfrak{H}(G)$ coincides with the algebra of left invariant differential operators on G defined in 2B of [3].

Hereafter we assume that k is an algebraically closed field of positive characteristic p.

Proposition 2. *Let G be a k-group scheme. Then $\mathfrak{D}(O_{G,e})$ is a bialgebra over k, where e is the origin of G.*

Proof. We set $O = O_{G,e}$ and denote by $\mathfrak{m}$ the maximal ideal of O. If we put $\mathfrak{n} = O \otimes_k \mathfrak{m} + \mathfrak{m} \otimes_k O (\subset O \otimes_k O)$, then we have the canonical isomorphism φ : $O_{G\times G,\, e\times e} \simeq (O \otimes_k O)_{\mathfrak{n}}$. Let $D \in \mathfrak{D}_m(O)$ and $E \in \mathfrak{D}_n(O)$, then $D \otimes E : O \otimes_k O \to k$ is an $(m+n)$-th order derivation. $D \otimes E$ is uniquely extended to an element of $\mathfrak{D}_{m+n}((O \otimes_k O)_{\mathfrak{n}})$ ([8] Theorem 15). We denote it $D \otimes E$ again. The product of D and E is given by ;

$$(D * E)(x)=(D\otimes E)(\varphi m^*(x))$$

for x in O, where m^* is the homomorphism of $O=O_{G,e}$ into $O_{G\times G,e\times e}$ associated with the multiplication m of G. Clearly we have $D*E\in\mathfrak{D}_{m+n}(O)$. We define $\alpha*D=D*\alpha=\alpha D$ and $\alpha*\beta=\beta*\alpha=\alpha\beta$ for α, β in k and D in $\bigcup_{n=1}^{\infty}\mathfrak{D}_n(O)$. Then $\mathfrak{D}(O)$ is a k-algebra with respect to this multiplication $*$ and ordinary addition. Let $(\mathfrak{D}(O), \Delta, \mathcal{E})$ be the coalgebra defined in Proposition 1. Obviously $\mathcal{E}$ is an algebra homomorphism. To complete our proof, it suffices to show that Δ is an algebra homomorphism, i.e. to see the following diagram is commutative

$$\begin{array}{ccc}
\mathfrak{D}(O)\otimes\mathfrak{D}(O) \xrightarrow{\ \nu\ } \mathfrak{D}(O) \xrightarrow{\ \Delta\ } & & \mathfrak{D}(O)\otimes\mathfrak{D}(O) \\
\Big\downarrow \Delta\otimes\Delta & & \Big\uparrow \nu\otimes\nu \\
\mathfrak{D}(O)\otimes\mathfrak{D}(O)\otimes\mathfrak{D}(O)\otimes\mathfrak{D}(O) & \xrightarrow{\ 1\otimes T\otimes 1\ } & \mathfrak{D}(O)\otimes\mathfrak{D}(O)\otimes\mathfrak{D}(O)\otimes\mathfrak{D}(O)
\end{array}$$

where ν is the mapping induced by the multiplication $*$ and T is a twisting homomorphism: $D\otimes E\to E\otimes D$. Let $\Delta(D)=\sum_i D_i\otimes D'_i$ and $\Delta(E)=\sum_j E_j\otimes E'_j$. Then we have $\Delta(D*E)(x\otimes y)=(D\otimes E)(\varphi m^*(xy))$. On the other hand we see $(\nu\otimes\nu)(1\otimes T\otimes 1)(\Delta\otimes\Delta)(D\otimes E)(x\otimes y)=\sum_{i,j}(D_i\otimes E_j)(\varphi m^*(x))(D'_i\otimes E'_j)(\varphi m^*(y))$. Since $\varphi m^*(xy)=\varphi m^*(x)\varphi m^*(y)$ and a high order derivation is uniquely extended to a quotient ring, we have only to show the following identity:

$(D\otimes E)(xu\otimes yv)=\sum_{i,j}(D_i\otimes E_j)(x\otimes y)(D'_i\otimes E'_j)(u\otimes v)$ for $x\otimes y$, $u\otimes v\in O\otimes_k O$. Being $\Delta(D)=\sum_i D_i\otimes D'_i$ and $\Delta(E)=\sum_j E_j\otimes E'_j$, we get $D(xu)=\sum_i D_i(x)D'_i(u)$ and $E(yv)=\sum_j E_j(y)E'_j(v)$. This proves our assertion.

REMARK 1. It is easily seen that $\mathfrak{D}(O_{G,e})$ is a Hopf algebra, i.e. $\mathfrak{D}(O_{G,e})$ has an antipode.

Proposition 3. *Let the situation be the same as in Proposition 2. Then $\mathfrak{D}(O_{G,e})$ is canonically isomorphic to $\mathfrak{H}(G)$ as a k-algebra.*

Proof. We set $O=O_{G,e}$. If D is in $\mathfrak{G}(G)$, D induces a high order derivation of O into itself. We shall denote it D again. Then we define $\Phi(D)=\pi\circ D$, where π is the canonical homomorphism of O onto k, and $\Phi(a)=a$ for $a\in k$. Thus we have defined a mapping $\Phi:\mathfrak{H}(G)\to\mathfrak{D}(O)$. Φ is k-linear. To show Φ is an algebra homomorphism, we must prove $\Phi(DE)=\Phi(D)*\Phi(E)$ for D, E in $\mathfrak{G}(G)$. Since D is left invariant, the diagram:

$$\begin{array}{ccc}
O_{G,e} & \xrightarrow{\ m^*\ } & O_{G\times G,e\times e} \\
\Big\downarrow D & & \Big\downarrow D_G \\
O_{G,e} & \xrightarrow{\ m^*\ } & O_{G\times G,e\times e}
\end{array}$$

is commutative, where m^* is the homomorphism associated with the multiplication m of G. (cf. [3] 2B, A) Lemma). Hence we have $(1\otimes\pi)\, D_G m^* = (1\otimes\pi)\, m^* D = D$, i.e. $(1\otimes\Phi(D))m^* = D$ where 1 denotes the identity mapping of O, and $1\otimes\pi$ and $1\otimes\Phi(D)$ are given as follows. Let $\mathfrak{m}$ be the maximal ideal of O and put $\mathfrak{n} = O\otimes_k\mathfrak{m} + \mathfrak{m}\otimes_k O(\subset O\otimes_k O)$. Then we see easily that the mapping: $O\otimes_k O \ni f\otimes g \to f\pi(g) \in O$ (resp. $O\otimes_k O \ni f\otimes g \to f\,\Phi(D)(g) \in O$) can be extended to the mapping: $(O\otimes_k O)_{\mathfrak{n}} \to O$ uniquely. We also denote by $1\otimes\pi$ and $1\otimes\Phi(D)$ these mappings composed with the canonical isomorphism: $O_{G\times G, e\times e} \cong (O\otimes_k O)_{\mathfrak{n}}$ respectively. We have $(1\otimes\Phi(D))m^*\,(1\otimes\Phi(E))m^* = DE$. On the other hand $\pi(1\otimes\Phi(D))m^* = \Phi(D)$. Thus we get $\Phi(DE) = \Phi(D) * \Phi(E)$. To prove Φ is an isomorphism, we exhibit the inverse mapping Ψ. Let $D_0 \in \mathfrak{D}_n(O)$ and let ε be the unit section: $\mathrm{Spec}(k) \to G$. Then D_0 induces a high order derivation of O_G into $\varepsilon_*(k)$ by adjointness with respect to ε. We denote it D_0 again. We set $h = 1_G \times \varepsilon : G\times k \to G\times G$ and define $\Psi(D_0)$ to be $O_G \xrightarrow{\tilde{m}} m_*(O_{G\times G}) \xrightarrow{m_*(D_{0G})} m_* h_*(O_{G\times k}) \cong O_G$. It is easily seen that Φ and Ψ are inverse to each other.

REMARK 2. This proof is a version of that of 2.4 of [3] 2B, A).
(*) A high order derivation: $O_G \to \varepsilon_*(k)$ is a k-linear homomorphism satisfying the similar identity as a high order derivation of G/k.

We transform the bialgabra structure of $\mathfrak{D}(O_{G,e})$ into $\mathfrak{H}(G)$ by the isomorphism defined in Proposition 3. Thus $\mathfrak{H}(G)$ is a bialgebra over k.

Theorem 1. *If G is a k-group scheme, then $\mathfrak{H}(G)$ is a bialgebra with only one grouplike element $1\in k$.*

Proof. We shall show the assertion for $\mathfrak{D}(O)$, where $O = O_{G,e}$. Assume that $a+D(a\in k,\ D\in \bigcup_{n=1}^{\infty} \mathfrak{D}_n(O))$ is grouplike. Since $\Delta(a+D) = (a+D)\otimes(a+D)$, we have $(a+D)(xy) = (a+D)(x)\,(a+D)(y)$ for x, y in O. Hence $D(xy) = D(x)\,D(y)$ for x, y in $\mathfrak{m}$ because $a(x)=0$ by the definition of operation of elements in k on O. Let $\mathfrak{m}^i$ be the least power of $\mathfrak{m}$ on which D vanishes. We assume $i>1$. Since $D \neq 0$ there is an element x in $\mathfrak{m}$ satisfying $D(x) \neq 0$. For $x_1,\cdots, x_{i-1}\in\mathfrak{m}$ we have $D(xx_1\cdots x_{i-1}) = D(x)D(x_1\cdots x_{i-1}) = 0$ and so $D(x_1\cdots x_{i-1}) = 0$. Now D vanishes on $\mathfrak{m}^{i-1}$ contrary to the assumption on i and hence $D=0$. We obtain $a=1$ immediately.

Proposition 4.(1) *We assume that G and G' are group varieties defined over k, and α is a surjective k-homomorphism of G onto G'. We set $O=O_{G,e}$ and $O'=O_{G',e'}$, where e(resp.e') is the neutral element of G(resp. G'). Then there exists a regular system of parameters $\{t_1,\cdots, t_n\}$ for O such that $\{t_1^{p^{e_1}}, \cdots, t_m^{p^{e_m}}\}$ is a regular system of parameters for O', where we identity the rational function field of G' with a subfield of the rational function field of G by the cohomomorphism α^*.*

(1) The author knew that H. Yanagihara obtained this result in [13].

Proof. We decompose α: $G \rightarrow G'$ as follows:

$$G \xrightarrow{\beta} G/\mathrm{Ker}(\alpha)_{\mathrm{red}} \xrightarrow{\gamma} G',$$

where β is the canonical epimorphism and γ is the homomorphism induced by α. Since β is separable and γ is a purely inseparable isogeny, we get the assertion using Theorem in [6].

Let H, K be bialgebras over k and let π: $H \rightarrow K$ be a homomorphism of bialgebras. Then we define HKer $(\pi) = \{x \in H \mid 1 \otimes x = (\pi \otimes 1)\, \Delta_H(x)$ in $K \otimes_k H\}$. If H is cocommutative we see that HKer (π) is a sub-bialgebra of H ([11] Lemma 16. 1. 1.).

We let $\alpha : G \rightarrow G'$ denote a homomorphism of k-group schemes. Since the induced homomorphism $\alpha^* : O_{G',e'} \rightarrow O_{G,e}$ is local, it gives a homomorphism of k-vector spaces $d\alpha$: $\mathfrak{D}(O_{G,e}) \rightarrow \mathfrak{D}(O_{G',e'})$, where e(resp. e') is the origin of G(resp. G'). Then we have

Proposition 5. *$d\alpha$ is a homomorphism of bialgebras.*

Proof. We shall first show that $d\alpha$ is an algebra homomorphism. To this purpose, we have only to prove $d\alpha(D * E) = d\alpha(D) * d\alpha(E)$ for D, E in $\bigcup_{n=1}^{\infty} \mathfrak{D}_n(O_{G,e})$. Let $x \in O_{G',e'}$. Then we have $d\alpha(D * E)\,(x) = (D \otimes E)\,(\varphi m^* \alpha^*(x))$, where φ is the canonical isomorphism: $O_{G \times G, e \times e} \xrightarrow{\sim} (O \otimes_k O)_{\mathfrak{n}}$ used in the proof of Proposition 2, and m^* is the homomorphism: $O_{G,e} \rightarrow O_{G \times G, e \times e}$ associated with the multiplication m of G. On the other hand we have $(d\alpha(D) * d\,\alpha(E))\,(x) = (D \otimes E)\,(\alpha_1^*\, \varphi' m^*\,(x))$, where φ': $O_{G' \times G', e' \times e'} \xrightarrow{\sim} (O' \otimes_k O')_{\mathfrak{n}'}$ and m'^*: $O_{G',e'} \rightarrow O_{G' \times G', e' \times e'}$ are defined similarly for G' and α_1^* is the homomorphism : $(O' \otimes_k O')_{\mathfrak{n}'} \rightarrow (O \otimes_k O)_{\mathfrak{n}}$ induced by α^*: $O' \rightarrow O$. We obtain $\varphi m^* \alpha^* = \alpha_1^*\, \varphi' m'^*$, since α is a homomorphism of G into G'. Thence $d\alpha$ is an algebra homomorphism. Next we shall prove that $d\alpha$ is a coalgebra homomorphism. Let $\Delta(D) = \sum_i D_i \otimes D_i'$. Then we get $(d\alpha \otimes d\alpha)\,(\Delta(D))\,(x \otimes y) = \sum_i D_i(\alpha^*(x))\, D_i'(\alpha^*(y))$ for $x, y \in O_{G',e'}$. On the other hand $\Delta(d\alpha(D))\,(x \otimes y) = D(\alpha^*(x)\alpha^*(y))$. Since $\Delta(D) = \sum_i D_i \otimes D_i$, we see $D(\alpha^*(x)\, \alpha^*(y)) = \sum_i D_i(\alpha^*(x))\, D_i'(\alpha^*(y))$. This completes our proof.

Thus $d\alpha$ induces a homomorphism of bialgebfas: $\mathfrak{H}(G) \rightarrow \mathfrak{H}(G')$. We also denote it $d\alpha$.

We assume that G is a group variety defined over k and $\{t_1, \cdots, t_n\}$ is a regular system of parameters for $O_{G,e}$. Let $f \in O_{G,e}$ and we express $f \equiv \sum a_{i_1 \cdots i_n} t_1^{i_1} \cdots t_n^{i_n}$ mod. $\mathfrak{m}_{G,e}^N$ with $a_{i_1 \cdots i_n} \in k$ for sufficiently large N, where $\mathfrak{m}_{G,e}$ is the maximal ideal of $O_{G,e}$. Then the elements $a_{i_1 \cdots i_n}$ are uniquely determined by f and a regular system of pareamters $\{t_1, \cdots t_n\}$. We set $I_{i_1 \cdots i_n, e}(f) = a_{i_1 \cdots i_n}$. If $\sum_{j=1}^{n} i_j > 0$, $I_{i_1 \cdots i_n, e}$ vanishes on 1 and on $\mathfrak{m}_{G,e}^{\sum i_j + 1}$. Thence we see $I_{i_1 \cdots i_n, e} \in \mathfrak{D}_m(O_{G,e})$ for some m by Proposition 1, (2). Since $\mathfrak{D}(O_{G,e})$ is canonically isomorphic to $\mathfrak{H}(G)$ by Proposition 3, $I_{i_1\cdots i_n,e}$ corresponds to the unique left invariant high order derivation $I_{i_1\cdots i_n}$ of G. We say that the $I_{i_1\cdots i_n}$ are the *canonical* left invariant high order derivations with respect to a regular system of parameters $\{t_1,\cdots,t_n\}$ for $O_{G,e}$(2).

Proposition 6. *In the above situation the $I_{i_1\cdots i_n}$ form a basis of the $k(G)$-vector space of all high order derivations of $k(G)/k$, where $k(G)$ is the rational function field of G over k.*

Proof. Following [8] we denote by $\mathfrak{D}_0^{(q)}(k(G)/k)$ the set of all q-th order derivations of $k(G)/k$. We have only to show that the $I_{i_1\cdots i_n}$ $(0<\sum_{j=1}^{n} i_j \leq q)$ form a $k(G)$-basis of $\mathfrak{D}_0^{(q)}(k(G)/k)$. From the proof of Proposition 18 in [9] we know the dimension of $\mathfrak{D}_0^{(q)}(k(G)/k)$ over $k(G)$. Thus it is sufficient to see that the $I_{i_1\cdots i_n}$ are independent over $k(G)$. Let $\sum a_{i_1\cdots i_n} I_{i_1\cdots i_n}=0$ with $a_{i_1\cdots i_n}\in k(G)$. There is a closed point g in G such that non-zero $a_{i_1\cdots i_n}$ are unit in $O_{G,g}$. We have $\sum a_{i_1\cdots i_n} I_{i_1\cdots i_n}(L^*_{g^{-1}}(t_1^{j_1}\cdots t_n^{j_n}))=\sum a_{i_1\cdots i_n} L^*_{g^{-1}} I_{i_1\cdots i_n}(t_1^{j_1}\cdots t_n^{j_n})=0$ where $L^*_{g^{-1}}$ is the automorphism of $k(G)$ associated with the left translation by g^{-1} of G. By the definition of $I_{i_1\cdots i_n}$ we see that $L^*_{g^{-1}} I_{i_1\cdots i_n}(t_1^{j_1}\cdots t_n^{j_n})$ is unit in $O_{G,g}$ for $i_1=j_1,\cdots, i_n=j_n$ and is non-unit in $O_{G,g}$ otherwise. If $a_{j_1\cdots j_n}\neq 0$, we have $a_{j_1\cdots j_n} L^*_{g^{-1}} I_{j_1\cdots j_n}(t_1^{j_1}\cdots t_n^{j_n})=-\sum_{\substack{(i_1,\cdots,i_n)\\ \neq(j_1,\cdots,j_n)}} a_{i_1\cdots i_n} L^*_{g^{-1}} I_{i_1\cdots i_n}(t_1^{j_1}\cdots t_n^{j_n})$. In this equality the left hand side is unit in $O_{G,g}$ while the right hand side is non-unit in $O_{G,g}$. This is contradiction.

Let $\alpha\colon G\to G'$ be surjective homomorphism of group varieties defined over k. By Proposition 4 we can choose a regular system of parameters $\{t_1,\cdots,t_n\}$ for $O_{G,e}$ such that $\{t_1^{p^{e_1}},\cdots, t_m^{p^{e_m}}\}$ is a regular system of parameters for $O_{G',e'}$. We let $\{I_{j_1\cdots j_n}\}$ denote the *canonical* left invariant high order derivations of G with respect to $\{t_1,\cdots, t_n\}$ and $\{I'_{l_1\cdots l_m}\}$ be the *canonical* left invariant high order derivations of G' with respect to $\{t_1^{p^{e_1}},\cdots, t_m^{p^{e_m}}\}$. Then we have

Theorem 2.(3) (1) *$d\alpha\colon \mathfrak{H}(G)\to\mathfrak{H}(G')$ is surjective.*
(2) *$\mathfrak{H}(Ker(\alpha))=HKer\,(d\alpha)$ and moreover as a k-vector space $\mathfrak{H}(Ker\,(\alpha))$ has a k-basis $\{I_{j_1\cdots j_n}\}_{j_l<p^{e_l}}\,(1\leq l\leq m)$.*
(3) *$Ker\,(d\alpha)$ is a k-vector space with a basis $\{I_{j_1\cdots j_m 0\cdots 0}\}_{\exists_i(1\leq i\leq m)\,p^{e_i}\nmid j_i}\cup\{I_{j_1\cdots j_n}\}$ at least one of $j_{m+1},\cdots,j_n>0$ and in fact $Ker\,(d\alpha)$ is a left ideal of $\mathfrak{H}(G)$ generated by $\mathfrak{H}(Ker(\alpha))^+=\{D\in\mathfrak{H}(Ker\,(\alpha))\mid \varepsilon(D)=0\}$, where ε is the augmentation of bialgebra $\mathfrak{H}(Ker\,(\alpha))$.*

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(2) These are the same as the canonical left invariant semiderivations of G with respect to $\{t_1,\cdots, t_n\}$ defined in [11].

(3) The author knew that H. Yanagihara obtained (1) and the latter part of (2) in [13].

Proof. (1) We see that $\{I'_{l_1\cdots l_m}\}$ is a k-basis of $\mathfrak{H}(G')$, since the $I'_{l_1\cdots l_m,e'}$ form a k-basis of $\mathfrak{D}(O_{G',e'})$. An easy calculation shows $d\alpha(I_{l_1p^{e_1}\cdots l_mp^{e_m}0\cdots 0}) = I'_{l_1\cdots l_m}$ and so $d\alpha$ is surjective.

(2) Since Ker (α) is a closed subgroup scheme of G, it is clear that $\mathfrak{H}$ (Ker (α)) is a sub-bialgebra of $\mathfrak{H}(G)$. We see Ker $(\alpha) = G \underset{G'}{\times} \mathrm{Spec}(k)$. Hence if $\mathfrak{m}'$ is the maximal ideal of $O_{G',e'}$ we have $O_{\mathrm{Ker}(\alpha),e} = O_{G,e}/\alpha^*(\mathfrak{m}')O_{G,e}$ where α^* is the homomorphism: $O_{G',e'} \to O_{G,e}$ induced by α. Now it is immediate to see that $\mathfrak{H}$ (Ker (α)) coincides with HKer $(d\alpha)$ as sub-bialgebras of $\mathfrak{H}(G)$. Next we prove the second part. If $I_{j_1\cdots j_n} \in$ HKer $(d\alpha)$, we have $I_{j_1\cdots j_n,e}(\alpha^*(x')y) = \alpha^*(x')\,(o)\, I_{j_1\cdots j_n,e}(y)$ for any $x' \in O_{G',e'}$ and any $y \in O_{G,e}$ and coversely. We see easily $I_{j_1\cdots j_n,e}(\alpha^*(x')y) = \sum_{l_i+l'_i=j_i} I_{l_1\cdots l_n,e}(\alpha^*(x'))\, I_{l_1'\cdots l_n',e}(y)$. Hence we obtain $I_{j_1\cdots j_n} \in$ HKer $(d\alpha)$ if and only if $\sum_{l_i+l_i'=j_i\,\sum_i j_i>0} I_{l_1\cdots l_n,e}(\alpha^*(x'))\, I_{l_1'\cdots l_n',e}(y) = 0$ for any $x' \in O_{G',e'}$ and any $y \in O_{G,e}$. Since $I_{l_1\cdots l_n,e}(t_1^{l_1'}\cdots t_n^{l_n'}) = 1$ for $l_i = l_i'(1 \leq i \leq n)$ and 0 otherwise, we see $I_{l_1\cdots l_n,e}(\alpha^*(x')) = 0$ for any $x' \in O_{G',e'}$ and any integers $l_1,\ldots,l_n$ satisfying $0 \leq l_i \leq j_i (1 \leq i \leq n)$ and $\sum_i l_i > 0$. Thence we must have $j_l < p^{e_l}$ for $1 \leq l \leq m$. Since the $I_{j_1\cdots j_n}$ form a k-basis of $\mathfrak{H}(G)$, our assertion is now immediate.

(3) we have $d\alpha(I_{l_1p^{e_1}\cdots l_mp^{e_m}0\cdots 0}) = I'_{l_1\cdots l_m}$ and $d\alpha(I_{j_1\cdots j_n}) = 0$ if $(j_1,\cdots,j_n)$ is not of the form $(l_1p^{e_1},\cdots, l_mp^{e_m}, 0,\cdots,0)$. Now the first assertion is obvious. We have $\varphi m^*(t_i) \equiv t_i \otimes 1 + 1 \otimes t_i$ mod. $\mathfrak{m}^2$(cf. chap. IX in [7]), where m^* is the homomorphism : $O_{G,e} \to O_{G\times G,e\times e}$ associated with the multiplication m of G and φ is the canonical isomorphism : $O_{G\times G,e\times e} \xrightarrow{\sim} (O_{G,e}\otimes_k O_{G,e})_{\mathfrak{n}}$ and $\mathfrak{m}$ denotes the maximal ideal of $(O_{G,e}\otimes_k O_{G,e})_{\mathfrak{n}}$. Then an easy computation shows $I_{i_1\cdots i_n,e} * I_{j_1\cdots j_n,e} \equiv \binom{i_1+j_1}{i_1}\cdots\binom{i_n+j_n}{i_n} I_{i_1+j_1\cdots i_n+j_n,e}$ mod. $\mathfrak{D}^{(\sum_l (i_l+j_l)-1)}(O_{G,e})$. Hence we get $I_{i_1\cdots i_n} I_{j_1\cdots j_n} \equiv \binom{i_1+j_1}{i_1}\cdots\binom{i_n+j_n}{i_n} I_{i_1+j_1\cdots i_n+j_n}$ mod. $\mathfrak{H}(G)\cap\mathfrak{D}_0^{(\sum_l (i_l+j_l)-1)}(G/k)$. If we express $i_j = a_jp^{e_j}+b_j$ with $0 \leq b_j < p^{e_j}$ for j$=1,\cdots,$ m, we hvae $I_{i_1\cdots i_m0\cdots 0} \equiv I_{a_1p^{e_1}\cdots a_mp^{e_m}0\cdots 0}\, I_{b_1\cdots b_m0\cdots 0}$ mod. $\mathfrak{H}(G)\cap\mathfrak{D}_0^{(\sum_j i_j-1)}(G/k)$, since $\binom{a_ip^{e_i}+b_i}{a_ip^{e_i}} \equiv 1$ mod. p. We see $I_{b_1\cdots b_m0\cdots 0} \in \mathfrak{H}(\mathrm{Ker}(\alpha))^+$ by (2) if some of b_j is positive. Moreover we have $I_{j_1\cdots j_n} \equiv I_{j_1\cdots j_m0\cdots 0}\, I_{0\cdots 0j_{m+1}\cdots j_n}$ mod. $\mathfrak{H}(G)\cap\mathfrak{D}_0^{(\sum_l j_l-1)}(G/k)$. If at least one of $j_{m+1},\ldots,j_n$ is positive, $I_{0\cdots 0j_{m+1}\cdots j_n} \in \mathfrak{H}(\mathrm{Ker}(\alpha))^+$ by (2). Now the induction on the order of high order derivations completes our proof.

If G is a k-group scheme and G' is a closed subgroup scheme of G, it is immediate that $\mathfrak{H}(G')$ is a sub-bialgebra of $\mathfrak{H}(G)$. We consider which sub-bialgebras of $\mathfrak{H}(G)$ arise from closed subgroup schemes of G. We obtain a characterization in the case G is a commutative group variety.

Let G be a group variety defined over k and let $\mathfrak{H}$ be a sub-bialgebra of $\mathfrak{H}(G)$. Then we define $k(G)^{\mathfrak{H}}$ to be the set of elements x in $k(G)$ such that $D(x)=$

0 for every D in $\mathfrak{H}$ satisfying $\varepsilon(D)=0$ where $k(G)$ denotes the field of rational functions on G over k. We see that $k(G)^{\mathfrak{H}}$ is a subfield of $k(G)$.

Proposition 7. *We asume that G and G' are group varieties defined over k and α is a surjective homomorphism of G onto G' defined over k. Then we have $k(G)^{\mathrm{HKer}(d\alpha)}=k(G')_s$, where we identify $\alpha^*(k(G'))$ with $k(G')$ and $k(G')_s$ denotes the separably algebraic closure of $k(G')$ in $k(G)$.*

Proof. We shall first show that $k(G')$ is contained in $k(G)^{\mathrm{HKer}(d\alpha)}$. Let $D \in \mathrm{HKer}\,(d\alpha)$. Then D vanishes on $k(G')$. Since an high order derivation can be uniquely extended to an high order derivation of separably algebraic extension field ([9] Theorem 17), D vanishes on $k(G')_s$. Hence we have $k(G')_s \subset k(G)^{\mathrm{HKer}(d\alpha)}$. We assume $k(G')_s \subsetneqq k(G)^{\mathrm{HKer}(d\alpha)}$. Then there exists an element x in $k(G)^{\mathrm{HKer}(d\alpha)}$ satisfying $x \notin k(G')_s$. We shall show that this will lead to contradiction. Since $x \notin k(G')_s$, x is either transcendental over $k(G')_s$ or purely inseparable over $k(G')_s$. In any case there exists an ordinary derivation D of $k(G')_s(x)$ such that D vanishes on $k(G')_s$ and $D(x)=1$. Then D can be extended to a high order derivation $\tilde{D}$ of $k(G)$ ([9] Proposition 13, Theorem 17). Let $\{t_1,\cdots,t_n\}$ be a regular system of parameters for $O_{G,e}$ as in Proposition 4. We assume that the $I_{j_1\cdots j_n}$ are the *canonical* left invariant high order derivations of G with respect to $\{t_1,\cdots,t_n\}$. The $I_{j_1\cdots j_n}$ form a basis of the $k(G)$-vector space of all high order derivations of $k(G)/k$ by Proposition 6. Thence we have $\tilde{D}=\sum a_{j_1\cdots j_n} I_{j_1\cdots j_n}$ with $a_{j_1\cdots j_n}$ in $k(G)$. We shall show $a_{l_1p^{e_1}\cdots l_mp^{e_m}0\cdots 0}=0$. To the contrary we assume $a_{l_1p^{e_1}\cdots l_mp^{e_m}0\cdots 0}\neq 0$. There exists a closed point g in G such that every non zero $a_{j_1\cdots j_n}$ is a unit in $O_{G,g}$. We have $\tilde{D}(L^*_{g^{-1}}(t_1^{l_1p^{e_1}}\cdots t_m^{l_mp^{e_m}}))=\sum a_{j_1\cdots j_n} L^*_{g^{-1}}(I_{j_1\cdots j_n}(t_1^{l_1p^{e_1}}\cdots t_m^{l_mp^{e_m}}))$, where $L^*_{g^{-1}}$ is the automorphism of $k(G)$ associated with the left translation by g^{-1}. $\tilde{D}$ vanishes on $k(G')$ by our construction and $\sum a_{j_1\cdots j_n} L^*_{g^{-1}}(I_{j_1\cdots j_n}(t_1^{l_1p^{e_1}}\cdots t_m^{l_mp^{e_m}}))$ is a unit in $O_{G,g}$ because $I_{j_1\cdots j_n}(t_1^{l_1p^{e_1}}\cdots t_m^{l_mp^{e_m}})$ is a unit for $j_i=l_ip^{e_i}(1\leq i\leq m)$, $j_{m+1}=\cdots=j_n=0$ and a non unit otherwise. This is contradiction. Hence we have $a_{l_1p^{e_1}\cdots l_mp^{e_m}0\cdots 0}=0$. Since $D(x)=1$, there is a set of integers $\{j_1,\cdots,j_n\}$ satisfying $I_{j_1\cdots j_n}(x)\neq 0$. The above argument means that either some j_i of $j_1,\cdots,j_m$ is not divisible by p^{e_i} or at least one of $j_{m+1},\cdots,j_n$ is positive. Consequently we have $I_{j_1\cdots j_n}\in \mathrm{Ker}\,(d\alpha)$ by Theorem 2, (3) and so there exists D' in $\mathrm{HKer}\,(d\alpha)^+$ such that $D'(x)\neq 0$, because $\mathrm{Ker}\,(d\alpha)$ is a left ideal generated by $\mathrm{HKer}\,(d\alpha)^+$ (Theorem 2, (3)). This contradicts to $x\in k(G)^{\mathrm{HKer}(d\alpha)}$.

Lemma 1 ([14] Lemma 2). *Let K be a field of positive characteristic and $\{D_0=1, D_1, D_2,\cdots\}$ be a higher derivation of K in the sense of* [4]. *If we set $K_\infty=\{x\in K \mid D_i(x)=0$ for any $i\geq 1\}$, then K is a separable extension of K_∞.*

For the results of bialgebras with one grouplike element we refer to [10]. Let H be a cocommutative bialgebra over a perfect field k of positive characteristic p. We assume that H has only one grouplike element and set $H' = \mathrm{Hom}_k (H,k)$. Then H' is a commutative algebra with respect to convolution (Cf. [11]). We define $F(a') = a'^p$ for $a' \in H'$. The transposed mapping $F' : H'' \to H''$ is given by $\langle a', F'(b'')\rangle = \langle F(a'), b''\rangle^{1/p}$ for $a' \in H'$ and $b'' \in H''$. Identifying H with subspace of H'' we have $F'(H) \subset H$. Let V denote the restriction of F' on H and let V^n be $V \cdots V$ (n times). We put $V^\infty(H) = \bigcap_{n=1}^{\infty} V^n(H)$. It is shown that $V^\infty(H)$ is a sub-bialgebra of H. We denote by $L(H)$ the set of primitive elements in H, i. e. $x \in H$ satisfying $\Delta(x) = x \otimes 1 + 1 \otimes x$, where Δ is the comultiplication of H. Moreover we set $L_i(H) = L(H) \cap V^i(H)$ for i $= 0, 1, \cdots, \infty$.

REMARK 3. If G is a k-group scheme, then we have $V^\infty(\mathfrak{H}(G)) = \mathfrak{H}(G_{\mathrm{red}})$, and G is reduced if and only if $\mathfrak{H}(G) = V^\infty(\mathfrak{H}(G))$. This follows immediately from 6.4 of [2] III §3.

Lemma 2. *Let G be a group variety defined over k of dimension n. Then we see that $L(\mathfrak{H}(G)) = L_\infty(\mathfrak{H}(G))$ and this is n-dimensional as a k-vector space.*

Proof. We note that $L(\mathfrak{H}(G))$ is the set of left invariant (ordinary) derivations of G and is of dimension n over k as a k-vector space. Thus we have only to prove $L(\mathfrak{H}(G)) \subset L_\infty(\mathfrak{H}(G))$. Let $\{I_{j_1 \cdots j_n}\}$ be the *canonical* left invariant high order derivations of G with respect to a regular system of parameters for $O_{G,e}$. Then it is easily seen that $\{1, I_{0\cdots 0\overset{i}{1}0\cdots 0}, I_{0\cdots 0\overset{i}{2}0\cdots 0}, \cdots, I_{0\cdots 0\overset{i}{r}0\cdots 0}, \cdots\}$ is an infinite higher derivation in the sense of [4]. Thence we have $I_{0\cdots 0\overset{i}{1}0\cdots 0} \in L_\infty(\mathfrak{H}(G))$ by Theorem 2 of [10]. On the other hand the $I_{0\cdots 0\overset{i}{1}0\cdots 0}$ form a k-basis of $L(\mathfrak{H}(G))$ and so our proof is complete.

Theorem 3. *Let G be a commutative group variety defined over an algebraically closed field k of positive characteristic and $\mathfrak{H}$ be a sub-bialgebra of $\mathfrak{H}(G)$. Then $\mathfrak{H}$ is the bialgebra of a closed subgroup scheme of G if and only if we have tr. $\deg_k k(G)^{\mathfrak{H}} = \dim G - \dim_k L_\infty(\mathfrak{H})$, where tr. $\deg_k k(G)^{\mathfrak{H}}$ denotes the transcendence degree of $k(G)^{\mathfrak{H}}$ over k.*

Proof. We assume $\mathfrak{H} = \mathfrak{H}(G')$ for some closed subgroup scheme G' of G. We consider the canonical epimorphism $\alpha: G \to G/G'$ of group varieties. Then we have $\mathrm{HKer}(d\alpha) = \mathfrak{H}(G')$ by Theorem 2, (2). Hence $k(G)^{\mathfrak{H}} = k(G/G')_s$ by Proposition 7 and so $\mathrm{tr.deg}_k k(G)^{\mathfrak{H}} = \dim G - \dim G'$. On the other hand $L_\infty(\mathfrak{D}(O_{G',e'})) = L_\infty(\mathfrak{D}(O_{G'_{\mathrm{red}},e'}))$ by Theorem 2 of [10], since $O_{G',e'} = O_{G'_{\mathrm{red}},e'} \otimes_k H$ for some finite bialgebra H over k ([2]III 3, 6.4) and so $\mathfrak{D}(O_{G',e'}) \cong \mathfrak{D}(O_{G'_{\mathrm{red}},e'}) \otimes_k \mathrm{Hom}_k(H, k)$. Being G'_{red} smooth over k, we have $\dim_k L_\infty(\mathfrak{D}(O_{G'_{\mathrm{red}},e'})) = \dim_k L_\infty(\mathfrak{H}(G'_{\mathrm{red}})) = \dim G'_{\mathrm{red}} = \dim G'$ by Lemma 2. Hence we have $\mathrm{tr.deg}_k k(G)^{\mathfrak{H}} = \dim G - \dim_k L_\infty(\mathfrak{H})$. Conversely we assume $\mathrm{tr.deg}_k k(G)^{\mathfrak{H}} = \dim G - \dim_k L_\infty(\mathfrak{H})$. Since $\mathfrak{H}(G)$ has only one grouplike element 1, $\mathfrak{H}$ is so. Thus we can apply Theorem 3 of [10] to see the coalgebra structure of $\mathfrak{H}$. Since G is commutative, $\mathfrak{H}(G)$ is commutative. An element of $\mathfrak{H}$ therefore induces a high order derivation of $k(G)^{V^\infty(\mathfrak{H})}$ into itself. We assert that $k(G)^{V^\infty(\mathfrak{H})}$ is a finite modular purely inseparable extension of $k(G)^{\mathfrak{H}}$, for the latter is the constant field of higher derivations of finite rank in the sense of [4] by the coalgebra structure of $\mathfrak{H}$ ([10] Theorem 3). We see that $k(G)^{V^\infty(\mathfrak{H})}$ (resp. $k(G)^{\mathfrak{H}}$) is the function field of some group variety G_0(resp. G_1) defined over k by Proposition 8 of [1], because $\mathfrak{H}\subset\mathfrak{H}(G)$ and G is commutative. We also have epimorphisms $\beta: G\to G_0$ and $\gamma: G_0\to G_1$. Clearly γ is purely inseparable isogeny. Since $V^\infty(\mathfrak{H})$ is commutative and is generated by the components of infinite higher derivations by Theorem 3 in [10], β is separable by Lemma 1. We set $\alpha=\gamma\circ\beta$. We shall prove $\mathfrak{H}=\mathrm{HKer}(d\alpha)$. To this purpose it suffices to show $L_i(\mathfrak{H})=L_i(\mathrm{HKer}(d\alpha))$ ($i=0, 1, 2,\cdots, \infty$) by Theorem 3 of [10]. By our assumption $\dim_k L_\infty(\mathfrak{H})=\dim G-\mathrm{tr.deg}_k k(G)^{\mathfrak{H}}=\dim G-\dim G_1$. Since β is separable and γ is purely inseparable, there exists a regular system of parameters $\{t_1,\cdots, t_n\}$ for $O_{G,e}$ such that $\{t_1,\cdots, t_m\}$ (resp. $\{t_1^{p^{e_1}},\cdots, t_m^{p^{e_m}}\}$) is a regular system of parameters for the local ring of G_0 at the origin (resp. the local ring of G_1 at the origin). Then $\dim G-\dim G_1=n-m$ and on the other hand $\dim_k L_\infty(\mathrm{HKer}(d\alpha))=n-m$ by Theorem 2,(2). Being $\mathfrak{H}\subset\mathrm{HKer}(d\alpha)$ we get $L_\infty(\mathfrak{H})=L_\infty(\mathrm{HKer}(d\alpha))$. We see $\dim_k L_1(\mathrm{HKer}(d\alpha))=(n-m)+$ (the number of l satisfying $i+1\leq e_l (1\leq l\leq m)$) from Theorem 2 in [10] and Theorem 2,(2). Thus we have $\dim_k L_i(\mathrm{HKer}(d\gamma))=\dim_k L_i(\mathrm{HKer}(d\alpha))-\dim_k L_\infty(\mathrm{HKer}(d\alpha))$ for $i=0, 1, 2,\cdots$. We also see that $\mathrm{HKer}(d\gamma)=\{D|_{K(G)^{V^\infty(\mathfrak{H})}}$ for some D in $\mathfrak{H}\}$ by Jacobson-Bourbaki Theorem (cf. [5]), where $D|_{K(G)^{V^\infty(\mathfrak{H})}}$ denotes the restriction of D on $k(G)^{V^\infty(\mathfrak{H})}$. Since $L_\infty(\mathfrak{H})=L_\infty(\mathrm{HKer}(d\alpha))$ we have $\dim_k L_i(\mathfrak{H})-\dim_k L_\infty(\mathfrak{H})\leq\dim_k L_i(\mathrm{HKer}(d\alpha))-\dim_k L_\infty(\mathrm{HKer}(d\alpha))=\dim_k L_i(\mathrm{HKer}(d\gamma))$. We set $H=\{D|_{k(G)^{V^\infty(\mathfrak{H})}}$ for some D in $\mathfrak{H}\}$. By Theorem 3 of [10] we see $\dim_k \mathrm{HKer}(d\gamma)=p^{\sum_i \dim_k L_i(\mathrm{HKer}(d\gamma))}$ and $\dim_k H\leq p^{\sum_i(\dim_k L_i(\mathfrak{H})-\dim_k L_\infty(\mathfrak{H}))}$. Since $\mathrm{HKer}(d\gamma)=H$ we get $\dim_k L_i(\mathfrak{H})-\dim_k L_\infty(\mathfrak{H})=\dim_h L_i(\mathrm{HKer}(d\gamma))$ for $i=0, 1, 2,\cdots$. Hence we have $\dim_k L_i(\mathfrak{H})=\dim_k L_i(\mathrm{HKer}(d\alpha))$. Since $\mathfrak{H}\subset\mathrm{HKer}(d\alpha)$ we obtain $L_i(\mathfrak{H})=L_i(\mathrm{HKer}(d\alpha))$ for $i=0, 1, 2,\cdots$. Thus we have $\mathfrak{H}=\mathrm{HKer}(d\alpha)$, i. e. $\mathfrak{H}=\mathfrak{H}(\mathrm{Ker}(\alpha))$ and we are done.

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