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Osaka J. Math.
20 (1983), 539–552

ON $Z/2$-e-INVARIANTS

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(Received November 25, 1981)

Let G be the group $Z/2$. Denote by $\pi^S_{p,q}$ the equivariant stable homotopy group of Landweber [12]. In a similar way to the usual e-invariants we define equivariant e-invariants e_G and $e_{G,R}$ on $\pi^S_{p,2q-1}$ by using the Adams operations in the K_G- and KO_G-theories and the equivariant Chern character. And we compute these invariants, in particular $e_{G,R}$, on the image of the equivariant J-homomorphism, making use of the Adams' result for e'_R. Here we study the case when $\widetilde{KO}_G^{-1}(\Sigma^{p,2q-1})$ is torsion-free. The torsion case is discussed by Löffler [14].

1. Definitions

Let $R^{p,q}$ denote the R^{p+q} with non trivial G-action on the first p coordinates. By $B^{p,q}$ and $S^{p,q}$ we denote the unit ball and unit sphere in $R^{p,q}$ and by $\Sigma^{p,q}$ the $B^{p,q}/S^{p,q}$. If p and q are even then $R^{p,q}$ is a complex G-module. In particular, we write 1 and L for $R^{0,2}$ and $R^{2,0}$. Then $\{1, L\}$ are basis of the complex representation ring $R(G)$ of G.

For the Thom class of $R^{2p,2q}$ as a complex G-vector bundle over a point we write $\lambda_{2p,2q}$, so that $\tilde{K}_G(\Sigma^{2p,2q})=R(G)\cdot\lambda_{2p,2q}$ [16]. Here let $A\cdot x$ denote the module generated by x over a ring A. Then we have the formula

$$\psi^t(\lambda_{2p,2q}) = \rho^t(2p, 2q)\lambda_{2p,2q}, \quad \rho^t(2p, 2q)\in R(G)$$

for the t-th Adams operation ψ^t, and $\rho^t(2p, 2q)$ is computed briefly, using the result for ψ^t in $\tilde{K}(S^{2n})$, as follows.

Lemma 1.1. *$\rho^t(0, 2q)=t^q$, and if $p>0$ then*

$$\rho^t(2p, 2q) = \begin{cases} \dfrac{1}{2}\, t^{p+q}(L+1) & (t \text{ even}) \\[2ex] t^{p+q}+\dfrac{1}{2}\, t^q(t^p-1)\,(L-1) & (t \text{ odd}) . \end{cases}$$

As is easily seen, $\tilde{K}_G(\Sigma^{1,0})$ is isomorphic to the augmentation-ideal of $R(G)$. Identifying $\tilde{K}_G(\Sigma^{1,0})$ with $Z\cdot(1-L)$ it is clear that $\tilde{K}_G(\Sigma^{2p+1,2q})=Z\cdot$

$(1-L)\lambda_{2p,2q}$. Hence we have the following

Corollary 1.2. *ψ^t operates on $\tilde{K}_G(\Sigma^{2p+1,2q})$ as multiplication by 0 if t is even and by t^q if t is odd.*

For p, $q-1\geqq 0$ suppose given a base point preserving G-map $f: \Sigma^{p+2k,2q-1+2l}\to\Sigma^{2k,2l}$ for k, l large, which is fixed in this section. f yields a cofiber sequence

$$\Sigma^{p+2k,2q-1+2l}\xrightarrow{f}\Sigma^{2k,2l}\xrightarrow{i}C_f\xrightarrow{j}\Sigma^{p+2k,2q+2l}\xrightarrow{-\Sigma^{0,1}f}\Sigma^{2k,2l+1}$$

where i, j are the inclusion and projection maps and C_f is the mapping cone of f. Applying $\tilde{K}_G$ we obtain the following exact sequence.

$$\begin{array}{ccccccc} 0\leftarrow & \tilde{K}_G(\Sigma^{2k,2l}) & \overset{i^*}{\leftarrow} & \tilde{K}_G(C_f) & \overset{j^*}{\leftarrow} & \tilde{K}_G(\Sigma^{p+2k,2q+2l}) & \leftarrow 0 \\ & \approx R(G) & & & & \approx\begin{cases}R(G) & (p\ \text{even})\\ Z & (p\ \text{odd})\end{cases} & \end{array}$$

Choose generators ξ, η of $\tilde{K}_G(C_f)$ so that

$$i^*(\xi)=\lambda_{2k,2l}\quad\text{and}\quad\eta=\begin{cases}j^*(\lambda_{p+2k,2q+2l}) & (p\ \text{even})\\ j^*((1-L)\lambda_{p-1+2k,2q+2l}) & (p\ \text{odd})\,.\end{cases}$$

For any odd integer $t(\neq\pm 1)$, $\psi^t(\xi)$ must be given by the formula

$$\psi^t(\xi)=\rho^t(2k,2l)\xi+\begin{cases}(c(t)+d(t)\,(L-1))\eta & (p\ \text{even})\\ c(t)\eta & (p\ \text{odd})\,,\end{cases}$$

$c(t)$, $d(t)\in Z$. So we set

$$\lambda(f)=\frac{c(t)}{t^{p/2+k+q+l}-t^{k+l}}\qquad (p\ \text{even})$$

$$\mu(f)=\begin{cases}\dfrac{1}{2}\left(\dfrac{c(t)}{t^{p/2+k+q+l}-t^{k+l}}+\dfrac{2d(t)-c(t)}{t^{q+l}-t^l}\right) & (p\ \text{even})\\[2ex] \dfrac{c(t)}{t^{q+l}-t^l} & (p\ \text{odd})\,.\end{cases}$$

Using Lemma 1.1, Corollary 1.2 and the relation $\psi^s\psi^t=\psi^{st}$ we can check that the values $\{\lambda(f)\}$, $\{\mu(f)\}$ do not depend on the choice of an integer t where $\{\ \}$ denotes the coset in Q/Z. As in [1, IV], §7 we see that the assignment

$$f\mapsto\begin{cases}(\{\lambda(f)\},\ \{\mu(f)\}) & (p\ \text{even})\\ \{\mu(f)\} & (p\ \text{odd})\end{cases}$$

induces a group homomorphism

$$e_G\colon \pi^S_{p,2p-1} \to \begin{cases} Q/Z \oplus Q/Z & (p \text{ even}) \\ Q/Z & (p \text{ odd}) \end{cases} \quad \text{for } p,\ q-1 \geqq 0\,.$$

Regard e_G as taking values in $\tilde{K}_G(\Sigma^{p+2k,2q+2l}) \otimes Q/Z$, namely let $e_G[f]$ be $(\{\lambda(f)\} + \{\mu(f)\}(L-1))\lambda_{p+2k,2q+2l}$ or $\{\mu(f)\}(1-L)\lambda_{p-1+2k,2l}$ according as p is even or odd where $[f]$ is the stable homotopy class of f. Then we have easily the following

Proposition 1.3. *e_G is natural for stable maps from $\Sigma^{p,2q-1}$ to $\Sigma^{r,2q-1}$.*

To evaluate $\psi^t(\xi)$ we shall next describe e_G in terms of the equivariant Chern character. Let ch_G be as in [18] and ch^n_G denote the $2n$-dimensional component of ch_G which is a homomorphism of K_G to $H^{2n}_G(\ \ , R_G)$ in the notation of [18]. By the definition of equivariant Bredon cohomology [7] we have the following canonical isomorphisms

$$\begin{aligned} H^{p+2k+2q+2l}_G(C_f, R_G) &\approx H^{p+2k+2q+2l}(C_{\psi f}, Q) \\ &\approx H^{p+2k+2q+2l}(S^{p+2k+2q+2l}, Q)\,, \\ H^{2q+2l}_G(C_f, R_G) &\approx H^{2q+2l}(C_{\phi f}, Q)\cdot(1-L) \\ &\approx H^{2q+2l}(S^{2q+2l}, Q)\cdot(1-L)\,. \end{aligned}$$

Here ψ and ϕ are the forgetful and fixed point functors [3]. Under the identification of the above isomorphisms we may set

$$ch^{p/2+k+q+l}_G(\xi) = a(f)h^{p+2k+2q+2l}$$

and

$$ch^{q+l}_G(\xi) = b(f)h^{2q+2l}(1-L)\,,$$

$a(f), b(f) \in Q$ (p even) where $h^{2i} \in H^{2i}(S^{2i}, Z)$ is a canonical generator such that $ch^i(\psi\lambda_{0,2i}) = h^{2i}$. Then we obtain

Proposition 1.4. *If p even then*

$$\lambda(f) = a(f),\ \mu(f) = \frac{1}{2}\left(a(f) - \frac{b(f)}{2^{p/2+k-1}}\right)$$

and if p is odd then

$$\mu(f) = \frac{b(f)}{2^{(p-1)/2+k}}\,.$$

Proof. Consider the following commutative diagram with the exact sequence which ϕf yields as f does.

$$(*)\qquad \begin{array}{ccccccccc} 0 \leftarrow & \tilde{K}_G(\Sigma^{2k,2l}) & \overset{i^*}{\leftarrow} & \tilde{K}_G(C_f) & \overset{j^*}{\leftarrow} & \tilde{K}_G(\Sigma^{p+2k,2q+2l}) & \leftarrow 0 \\ & \downarrow h^* & & \downarrow h^* & & \downarrow h^* & \\ 0 \leftarrow & \tilde{K}_G(\Sigma^{0,2l}) & \overset{i_1^*}{\leftarrow} & \tilde{K}_G(C_{\phi f}) & \overset{j_1^*}{\leftarrow} & \tilde{K}_G(\Sigma^{0,2q+2l}) & \leftarrow 0 \end{array}$$

(Here h's are the inclusions.) Choose $\xi_1 \in \tilde{K}_G(C_{\phi f})$ so that $i_1^*(\xi_1)=\lambda_{0,2l}$ and put $\eta_1=j_1^*(\lambda_{0,2q+2l})$. Then we may write

$$h^*(\xi) = 2^{k-1}(1-L)\xi_1+x(1-L)\eta_1, \quad x\in Z$$

for a cohomological reason and the fact that $h^*(\lambda_{2k,2l})=2^{k-1}(1-L)\lambda_{0,2l}$. Applying ψ^t we have

$$\psi^t(h^*\xi) = 2^{k-1}(1-L)\psi^t(\xi_1)+xt^{q+l}(1-L)\eta_1\,. \tag{1}$$

On the other hand, apply h^* to the defining formula of $c(t)$, $d(t)$ we have

$$\begin{aligned}\psi^t(h^*\xi) &= 2^{k-1}t^l(1-L)\xi_1+xt^l(1-L)\eta_1 \\ &+\begin{cases}2^{p/2+k-1}(c(t)-2d(t))\,(1-L)\eta_1 & (p\ \text{even})\\ 2^{(p-1)/2+k}c(t)\,(1-L)\eta_1 & (p\ \text{odd})\,.\end{cases}\end{aligned} \tag{2}$$

Combining (1) and (2) shows

$$\psi^t(\xi_1) = t^l\xi_1+\frac{x(t^l-t^{q+l})}{2^{k-1}}\eta_1+\begin{cases}2^{p/2}(c(t)-2d(t))\eta_1 & (p\ \text{even})\\ 2^{(p+1)/2}c(t)\eta_1 & (p\ \text{odd})\,.\end{cases}$$

Case p even. From the definition of ch_G it follows easily that

$$ch_G^{p/2+k+q+l}(\xi) = ch^{p/2+k+q+l}(\psi\xi)$$

and

$$ch_G^{q+l}(\xi) = 2^{k-1}ch^{q+l}(\psi\xi_1)\,(1-L)+xh^{2q+2l}(1-L)\,.$$

Hence we get

$$ch^{p/2+k+q+l}(\psi\xi) = a(f)h^{p+2k+2q+2l} \quad\text{and}\quad ch^{q+l}(\psi\xi_1) = \frac{b(f)-x}{2^{k-1}}h^{2q+2l}\,.$$

Therefore [1, IV], Proposition 7.5 for ψf and ϕf leads to the equialities

$$a(f) = \frac{c(f)}{t^{p/2+k+q+l}-t^{k+l}} \quad\text{and}\quad \frac{b(f)}{2^{p/2+k-1}} = \frac{c(t)-2d(t)}{t^{q+l}-t^l}\,.$$

Case p odd. Similar to the proof of the above case. q.e.d.

2. (0, $2q-1$)-stem

Let π: $\Sigma^{2k,2q-1+2l}\to\Sigma^{2k,2q-1+2l}/\Sigma^{0,2q-1+2l}$ be the canonical projection map for k, l large. Let $\lambda^S_{p,q}$ denote the equivariant stable homotopy group introduced in [12]. Then we have by [12] a split short exact sequence

$$0\to\lambda^S_{0,2q-1}\xrightarrow{\pi^*}\pi^S_{0,2q-1}\underset{\theta}{\overset{\phi}{\rightleftarrows}}\pi^S_{2q-1}\to 0$$

where π^* is the homomorphism induced by π and θ denotes a left inverse of ϕ as in [4], §5.

By the definition we can easily describe the values of e_G on Im θ in terms of the complex e-invariant e_C in [1, IV]. So we consider e_G on Im π^* in this section.

Suppose given a base point preserving G-map $\tilde{f}\colon \Sigma^{2k,2q-1+2l}/\Sigma^{0,2q-1+2l}\to\Sigma^{2k,2l}$, so that $\tilde{f}$ and $\tilde{f}\pi$ define elements $[\tilde{f}]$ and $[\tilde{f}\pi]$ of $\lambda^S_{0,2q-1}$ and $\pi^S_{0,2q-1}$ respectively. We consider $\tilde{f}\pi$ as f in §1.

Since $\Sigma^{i,j}/\Sigma^{0,j}$ is equivariantly homeomorphic to $\Sigma^{0,j+1}S^{i,0}_+$ ([12], Lemma 4.1), we have $\tilde{K}_G(\Sigma^{i,j}/\Sigma^{0,j})\approx K^{-j-1}(RP^{i-1})$ [16] where RP^n is the real n-dimensional projective space. Let η_n be the complexification of a canonical real line bundle over RP^n and put $\tilde{\eta}_n=1-\eta_n$. We now recall [6] that

$$\tilde{K}^0(RP^{2n}) = Z/2^n\cdot\tilde{\eta}_{2n},\ K^1(RP^{2n}) = 0$$
$$\tilde{K}^0(RP^{2n+1}) = Z/2^n\cdot\tilde{\eta}_{2n+1},\ K^1(RP^{2n+1})\approx Z\,.$$

Then we can identify

$$\tilde{K}^0_G(\Sigma^{2k,2q-1+2l}/\Sigma^{0,2q-1+2l}) = Z\oplus Z/2^{k-1}\cdot(\psi\lambda_{0,2q+2l})\tilde{\eta}_{2k-1}\,.$$

Consider $\tilde{f}^*\colon \tilde{K}_G(\Sigma^{2k,2l})\to\tilde{K}_G(\Sigma^{2k,2q-1+2l}/\Sigma^{0,2q-1+2l})$. Because $[\tilde{f}]\in\lambda^S_{0,2q-1}$ for $q\geqq 1$ is of finite order ([12], Theorem 2.4 and Corollary 6.3) we may put

$$\tilde{f}^*(\lambda_{2k,2l}) = [\tilde{b}(\tilde{f})]\,(\psi\lambda_{0,2q+2l})\tilde{\eta}_{2k-1},\ \tilde{b}(\tilde{f})\in Z$$

where [ ] denotes the coset in $Z/2^{k-1}$.

Lemma 2.1. $\tilde{b}(\tilde{f})=-b(\tilde{f}\pi)$ mod 2^{k-1}
where $b(\tilde{f}\pi)$ *is as in* §1.

Proof. Observe the following commutative diagram involving (*) in §1.

$$\begin{array}{ccccccccc}
\tilde{K}_G(\Sigma^{2k,2q-1+2l}/\Sigma^{0,2q-1+2l}) & \xleftarrow{\tilde{f}^*} & \tilde{K}_G(\Sigma^{2k,2l}) & \leftarrow & \tilde{K}_G(C_{\tilde{f}}) & \leftarrow & \tilde{K}_G(\Sigma^{2k,2q+2l}/\Sigma^{0,2q+2l}) & & \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \leftarrow & \tilde{K}_G(\Sigma^{2k,2l}) & \xleftarrow{i^*} & \tilde{K}_G(C_{\tilde{f}\pi}) & \xleftarrow{j^*} & \tilde{K}_G(\Sigma^{2k,2q+2l}) & \leftarrow & 0 \\
 & & \downarrow h^* & & \downarrow h^* & & \downarrow h^* & & \\
0 & \leftarrow & \tilde{K}_G(\Sigma^{0,2l}) & \xleftarrow{i_1^*} & \tilde{K}_G(C_{\phi(\tilde{f}\pi)}) & \xleftarrow{j_1^*} & \tilde{K}_G(\Sigma^{0,2q+2l}) & \leftarrow & 0 \\
 & & & & & & \downarrow \delta & & \\
 & & & & & & K^1_G(\Sigma^{2k,2q+2l}/\Sigma^{0,2q+2l}) & &
\end{array}$$

where the right-hand sequence is the exact sequence for a pair $(\Sigma^{2k,2q+2l},\ \Sigma^{0,2q+2l})$. Clearly $C_{\phi(\tilde{f}\pi)}\approx\Sigma^{0,2q+2l}\vee\Sigma^{0,2l}$, hence we can verify that $\tilde{f}^*(\lambda_{2k,2l})=-\delta j_1^{*-1}h^*(\xi)$ where ξ is as in §1. Hence the canonical identification such that $\tilde{K}_G(\Sigma^{0,2q+2l})$ $=\tilde{K}(S^{2q+2l})\otimes R(G)=H^{2q+2l}(S^{2q+2l},\ Z)\otimes R(G)$ leads to the desired assertion. q.e.d.

Let BG denote the real infinite dimensional projective space. There is an integer $c(n)$ such that $c(n)\eta_{2n-1}$ becomes trivial (see, e.g. [9], p. 219). So we have an equivariant homeomorphism $\Sigma^{c(n),0}S_+^{n,0}\approx\Sigma^{0,c(n)}S_+^{n,0}$. This homeomorphism, the equivariant suspension theorem and the Spanier-Whitehead duality theorem yield an isomorphism

$$\lambda^S_{0,n} \underset{\approx}{\longrightarrow} \pi_n^S(BG_+),$$

denoted by I, as follows. Let τ be the tangent bundle of RP^{2k-1} and ν be a normal bundle of RP^{2k-1} for an embedding of RP^{2k-1} in R^{2m-1} for m suitably large. Note that the Thom complex $T(\nu)$ of ν is a $(2m-1)$-dual of RP_+^{2k-1} [5], and $\tau\oplus 1\approx 2k\eta'_{2k-1}$ so that $S^{2m}T((sc-k)\eta_{2k-1})\approx S^{2sc}T(\nu)$ for $sc>k$ where η'_{2k-1} denotes the underlying real vector bundle of η_{2k-1} and $c=c(k)$ is as above. Then we have the following isomorphisms.

$$\begin{aligned}
\lambda^S_{0,n} &= \varinjlim_{k,l}\,[\Sigma^{2k,n+2l}/\Sigma^{0,n+2l},\ \Sigma^{2k,2l}]^G && \text{by definition [12]}\\
&\approx \varinjlim_{k,l}\,[\Sigma^{0,n+2l+1}S_+^{2k,0},\ \Sigma^{2k,2l}]^G\\
&\approx \varinjlim_{k,l}\,[\Sigma^{2sc,n+2l-2sc+1}S_+^{2k,0},\ \Sigma^{2k,2l}]^G && \text{for some } c\\
&\approx \varinjlim_{k,l}\,[\Sigma^{2sc-2k,n+2l-2sc+1}S_+^{2k,0},\ \Sigma^{0,2l}]^G && \text{by [3], Theo. 11.9}\\
&\approx \varinjlim_{k,l}\,[S^{n+2l-2sc+1}T((sc-k)\eta_{2k-1}),\ S^{2l}]\\
&\approx \varinjlim_{k,l}\,[S^{n+2l-2m+1}T(\nu),\ S^{2l}]\\
&\approx \varinjlim_{k}\,\{S^n,\ RP_+^{2k-1}\} && \text{by [19], Cor. (7.10)}\\
&= \pi_n^S(BG_+)
\end{aligned}$$

On the other hand, the geometrical interpretation of I by Landweber [12] shows that the composite $\psi\pi^*I^{-1}$: $\pi_n^S(BG_+)\rightarrow\pi_n^S$ agrees with the $Z/2$-transfer. So we write $t=\psi\pi^*I^{-1}$ as usual.

Following the homotopical construction of I we see that $I[\tilde f]$ is represented by a stable map g: $S^{2q-1}\rightarrow RP_+^{2k-1}$. Let $\tilde g$: $S^{2q-1}\rightarrow RP^{2k-1}$ be the composite g and the canonical projection from RP_+^{2k-1} to RP^{2k-1} and let

$$\alpha_1\in\pi^S_{2q-1}(BG)$$

denote the stable homotopy class induced by $\tilde g$. Then we have

Proposition 2.2. $\left\{\dfrac{\tilde b(\tilde f)}{2^{k-1}}\right\}=e_C t(\alpha_1)$

where e_C is as in [1, IV].

We prepare a lemma for a proof of Proposition 2.2. We recall the following universal coefficient sequence for a finite CW-complex X [2]

$$0 \to \operatorname{Ext}(\tilde{K}^0(X), Z) \to K_1(X) \xrightarrow{k} \operatorname{Hom}(K^1(X), Z) \to 0$$

where k is a map induced by the Kronecker product. Here we denote by ι the injection map. Furthermore we have a natural homomorphism

$$\operatorname{Hom}(\tilde{K}^0(X), Q/Z) \to \operatorname{Ext}(\tilde{K}^0(X), Z),$$

which we denote by Δ. In particular, for $X=RP^{2k}$, ι and Δ are isomorphisms.

Denote by p the collapsing map $RP^{2k-1}\to RP^{2k-1}/RP^{2k-2}$ and identify RP^{2k-1}/RP^{2k-2} with S^{2k-1}. Then, clearly $p^*\colon \tilde{K}^0(S^{2k})=K^1(S^{2k-1}) \to K^1(RP^{2k-1})$ is an isomorphism and hence by using the universal coefficient sequence we see that $p_*\colon K_1(RP^{2k-1}) \to K_1(S^{2k-1})=\tilde{K}_0(S^{2k})$ is an epimorphism. Therefore, if we put $z'=p^*(\psi\lambda_{0,2k})\in K^1(RP^{2k-1})$ then we have an element $z\in K_1(RP^{2k-1})$ such that p_*z is a dual element of $\psi\lambda_{0,2k}$, i.e. $\langle z', z\rangle=1$, which is a fundamental class of RP^{2k-1} ([19], p. 217). By [19], Corollary (7.8) we have an isomorphism

$$P = z\cap : \tilde{K}^0(RP^{2k-1}) \to K_1(RP^{2k-1}).$$

Consider the composite

$$\tilde{K}^0(RP^{2k-1}) \xrightarrow{P} K_1(RP^{2k-1}) \xrightarrow{i'_*} K_1(RP^{2k}) \xrightarrow{(\iota\Delta)^{-1}} \operatorname{Hom}(\tilde{K}^0(RP^{2k}), Q/Z)$$

where $i'\colon RP^{2k-1}\subset RP^{2k}$ is the inclusion map. Then

Lemma 2.3. $((\iota\Delta)^{-1}i'_*P\tilde{\eta}_{2k-1})\tilde{\eta}_{2k} = -\left\{\frac{1}{2^{k-1}}\right\}$.

Proof. Let γ^* be the co-Hopf bundle on the complex $(k-1)$-dimensional projective CP^{k-1} and γ be its dual. By D and S we denote the total spaces of the unit disk and unit sphere bundles of $\gamma^*\otimes\gamma^*$ with respect to some metric. Then $D\simeq CP^{k-1}$ clearly and $S\approx RP^{2k-1}$ (see [10], IV.1.14. Example). We identify S with RP^{2k-1}. Because, if we put $\tilde{\gamma}=1-\gamma$ then $K^*(D)\approx Z[\tilde{\gamma}]/(\tilde{\gamma}^k)$ and $i^*\tilde{\gamma}=\tilde{\eta}_{2k-1}$, we have a short exact sequence

$$0 \to K^1(S) \xrightarrow{\delta} K^0(D, S) \xrightarrow{j^*} K^0(D) \xrightarrow{i^*} K^0(S) \to 0$$

where δ is a coboundary homomorphism and i, j are the inclusion maps. As is well known, $j^*\lambda=-\tilde{\gamma}^{*2}+2\tilde{\gamma}^*$ where $\tilde{\gamma}^*=1-\gamma^*$ and λ is the Thom class of $\gamma^*\otimes\gamma^*$. Hence $K^*(D, S)\approx\bigoplus_{i=0}^{k-1} Z\cdot\lambda\tilde{\gamma}^i$. Moreover, by an observation for $\tilde{\gamma}^{k-1}$ in [6], p. 100 we have

$$\delta^{-1}\lambda\tilde{\gamma}^{k-1} = z' .$$

Put $z_1' = \delta z'$ and denote by z_1 a dual element of z_1' so that we may suppose that $\partial z_1 = z$ where ∂ is the boundary homomorphism. Similarly $P_1 = z_1 \cap : K^0(D) \to K_0(D, S)$ is then an isomorphism and the diagram

$$\begin{array}{ccc} K^0(D) & \xrightarrow{i^*} & K^0(S) \\ P_1 \downarrow & & P \downarrow \\ K_0(D, S) & \xrightarrow{\partial} & K_1(S) \end{array}$$

commutes.

A routine computation shows that $\lambda\tilde{\gamma}^{k-2} \in K^0(D, S)$ is a dual element of $P_1\tilde{\gamma}$, i.e.,

$$\langle \lambda\tilde{\gamma}^{k-2}, P_1\tilde{\gamma} \rangle = 1 .$$

Let put $M = D \times S^{2k-1}$ and $i_1 \colon S \subset M$ be an embedding given by $i_1(x) = (i(x), p(x))\ x \in S$. Then we get a short exact sequence

$$0 \to K^*(M, S) \xrightarrow{j_1^*} \tilde{K}^*(M) \xrightarrow{i_1^*} \tilde{K}^*(S) \to 0 ,$$

which is a free resolution of $\tilde{K}^*(S)$, where j_1 is the inclusion map. Hence we see that

$$\tilde{K}^0(M) = \bigoplus_{i=1}^{k-1} Z \cdot q^*\tilde{\gamma}^i \quad \text{and} \quad K^0(M, S) = \bigoplus_{i=0}^{k-2} Z \cdot q^*\lambda\tilde{\gamma}^i ,$$

where q is the projection map of M to D.

Here we adopt the above resolution as a free resolution in the proof of [2], Theorem 3.1 for $K_1(S)$. Define $f \in \mathrm{Hom}(K^0(M, S), Z)$ by

$$f(q^*\lambda\tilde{\gamma}^i) = \begin{cases} 1 & \text{if } i = k-2 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathrm{Hom}(q^*, 1)f = \langle \ \ , P_1\tilde{\gamma} \rangle .$$

This implies that because Coker $\mathrm{Hom}(j_1^*, 1) = \mathrm{Ext}(\tilde{K}^0(S), Z)$,

$$\iota[f] = P\tilde{\eta}_{2k-1} ,$$

where $[f]$ denotes the equivalence class of f in Coker $\mathrm{Hom}(j_1^*, 1)$.

By the definition of Δ it is verified that

$$(\Delta^{-1}[f])\tilde{\eta}_{2k-1} = -\left\{\frac{1}{2^{k-1}}\right\} .$$

Hence,

$$(\iota\Delta)^{-1}(P\tilde{\eta}_{2k-1})\tilde{\eta}_{2k-1} = -\left\{\frac{1}{2^{k-1}}\right\}.$$

This proves the lemma because $i'_*\iota\Delta = \iota\Delta\mathrm{Hom}(i'^*, 1)$.

Proof of Proposition 2.2. We may suppose that ν is a complex vector bundle, since the stable tangent bundle of RP^{2k-1} has a complex structure.

Observing the construction of I we have the following commutative diagram.

$$\begin{array}{ccccc}
 & \tilde{K}_G^0(\Sigma^{0,2q+2l}S_+^{2k,0}) & \xleftarrow{\tilde{f}^*} & \tilde{K}_G^0(\Sigma^{2k,2l}) & \\
 & {\scriptstyle I_0}\Big\downarrow & & \Big\uparrow{\scriptstyle I_1} & \\
 & & & \tilde{K}_G^0(\Sigma^{0,2l}) = \tilde{K}^0(S^{2l})\otimes R(G) & \\
 & & & \Big\uparrow & \\
 & \tilde{K}^0(S^{2l+2q-2m}T(\nu)) & & \tilde{K}^0(S^{2l}) & \\
 & {\scriptstyle D_2}\Big\downarrow & & \Big\downarrow{\scriptstyle D_3} & \\
K_1(RP^{2k}) \xleftarrow{i'_*} & K_1(RP^{2k-1}) & \xleftarrow{\tilde{g}_*} & K_1(S^{2q-1}) &
\end{array}$$

Here D_2, D_3 are the duality isomorphisms as in [19], Corollary (7.10), and I_0, I_1 are isomorphisms given by $I_0((\psi^r\lambda_{0,2q+2l})\tilde{\eta}_{2k-1}) = (\psi^r\lambda_{0,2l+2q-2m})\lambda_\nu\tilde{\eta}_{2k-1}$, $I_1(\lambda_{0,2l}) = \lambda_{2k,2l}$ where λ_ν denotes the Thom class of ν.

By [19], Corollaries (7.8) and (7.10) we have

$$D_2I_0((\psi^r\lambda_{0,2q+2l})\tilde{\eta}_{2k-1}) = P\tilde{\eta}_{2k-1},$$

which is pointed out by Dyer in [8]. By Lemma 2.3 we therefore have

$$((\iota\Delta)^{-1}(i'\tilde{g})_*\beta)\tilde{\eta}_{2k} = -\left\{\frac{\tilde{b}(\tilde{f})}{2^{k-1}}\right\}$$

where $\beta = D_3(\psi^r\lambda_{0,2l})$.

Identifying $K_1(RP^{2k})$ with $\mathrm{Hom}(\tilde{K}^0(RP^{2k}), Q/Z)$ through the isomorphism $\iota\Delta$, we may write

$$(h\alpha_1)\tilde{\eta}_{2k} = -\left\{\frac{\tilde{b}(\tilde{f})}{2^{k-1}}\right\}$$

in terms of the Hurewicz homomorphism $h\colon \pi_{2q-1}^S(BG) \to K_1(BG)$. Hence by [11], Theorem 2.1 we obtain

$$(CH^q(\alpha_1))\tilde{\eta}_{2k} = -\left\{\frac{\tilde{b}(\tilde{f})}{2^{k-1}}\right\}$$

where CH^q is the functional Chern character. By the naturality of CH^q we get

$$e_C t(\alpha_1) = -(CH^q(\alpha_1))\tilde{\eta}_{2k} .$$

(For the sign, see Remark 4 of [11], p. 128.)
Therefore

$$\left\{\frac{\tilde{b}(\tilde{f})}{2^{k-1}}\right\} = e_C t(\alpha_1) .$$

q.e.d.

Consequently we get the following

Theorem 2.4. *For* $\alpha \in \pi^S_{0,2q-1}$ $(q \geqq 1)$,

$$e_G(\alpha) = \begin{cases} (e_C(\psi\alpha),\ 0) & \text{for } \alpha \in \operatorname{Im} \theta \\ (e_C(\psi\alpha),\ \frac{1}{2}(e_C(\psi\alpha)+e_C t(\alpha_1)+\varepsilon) & \text{for } \alpha \in \operatorname{Im} \pi^* \end{cases}$$

$(\varepsilon=0,\ 1)$ *where* α_1 *denotes the first factor of* $I\pi^{*-1}(\alpha)$ *under the identification* $\pi^S_{2q-1}(BG_+)=\pi^S_{2q-1}(BG)\oplus\pi^S_{2q-1}$.

Proof. As to the first factors this is clear from the definitions of e_G and e_C. As to the second this follows in addition from Proposition 1.4, Lemma 2.1 and Proposition 2.2. q.e.d.

3. Images of the S^1-transfer

Let $\tilde{t}$: $\pi^S_n(BS^1_+) \to \pi^S_{n+1}(BG_+)$ denote the S^1-transfer, where BS^1 is the complex infinite dimensional projective space.

Proposition 3.1. *Let* $\alpha \in \operatorname{Im}\ \{\pi^*: \lambda^S_{0,4q-1} \to \pi^S_{0,4q-1}\}$ $(q \geqq 1)$ *and* $I\pi^{*-1}(\alpha) \in \operatorname{Im} \tilde{t}$. *Then*

$$e_C t(\alpha_1) = (1-2^{2q})e_C(\psi\alpha)$$

where α_1 *is as in Theorem* 2.4.

Proof. Consider the isomorphisms

$$\lambda_{0,4q-1} \underset{\approx}{\overset{I}{\to}} \pi^S_{4q-1}(BG_+) = \pi^S_{4q-1}(BG)\oplus\pi^S_{4q-1} .$$

We may write $I\pi^{*-1}(\alpha)=(\alpha_1,\ \alpha_2)$. Applying t we have

$$\psi\alpha = t\alpha_1+2\alpha_2 ,$$

Since $t=\psi\pi^* I^{-1}$ and t operates on π^S_{4q-1} as multiplication by 2. From [13], Theorem 3.4 it follows that

$$e_C(\alpha_2) = 2^{2q-1}e_C(\psi\alpha) .$$

Therefore we get the proposition.

The following theorem follows immediately from Theorem 2.4 and Proposition 3.1.

Theorem 3.2. *For* $\alpha\in\pi^S_{0,4q-1}$ *as in Proposition* 3.1 *we have*

$$e_G(\alpha)=(e_C(\psi\alpha),\ (1-2^{2q-1})e_C(\psi\alpha)+\frac{\varepsilon}{2}),\quad (\varepsilon=0,\ 1)\,.$$

Let J_G: $\widetilde{KO}_G^{-1}(\Sigma^{0,4q-1})\to\pi^S_{0,4q-1}$ $(q\geqq 1)$ be the equivariant J-homomorphism [14, 17]. Set $\alpha=J_G(H\nu)\in\pi^S_{0,4q-1}$ where ν is a canonical generator of $\widetilde{KO}^{-1}(S^{4q-1})$ and $H=R^{1,0}$. Then $\alpha\in\mathrm{Im}\ \pi^*$ because $\phi(\alpha)=0$.

Lemma 3.3. *Let* α *be as above. Then* $I\pi^{*-1}(\alpha)$ *or* $2I\pi^{*-1}(\alpha)\in\mathrm{Im}\ \tilde{t}$ *according as q is odd or even.*

Proof. We consider the S^1-homotopy theory. Replace $R^{1,0}$ by the standard complex 1-dimensional non trivial representation V of S^1 in the $Z/2$-homotopy theory. Then by the same argument as in [12] we have the S^1-homotopy groups $\pi^{V,S}_n$, $\lambda^{V,S}_n$ and an exact sequence $\lambda^{V,S}_n\xrightarrow{\pi^*}\pi^{V,S}_n\xrightarrow{\phi}\pi^S_n$. Moreover, we have an isomorphism $\lambda^{V,S}_n\approx\pi^S_{n-1}(BS^1_+)$. Clearly the diagram

$$\begin{array}{ccccccc} & & \lambda^{V,S}_n & \xrightarrow{\pi^*} & \pi^{V,S}_n & \xrightarrow{\phi} & \pi^S_n \\ & & \downarrow r & & \downarrow r & & \| \\ 0 & \to & \lambda^S_{0,n} & \xrightarrow{\pi^*} & \pi^S_{0,n} & \xrightarrow{\phi} & \pi^S_n \end{array}$$

commutes where r denotes the restriction of S^1-actions. Identifying the left-hand groups with the cobordism groups canonically, r agrees with the S^1-transfer $\tilde{t}$.

Analogously for S^1-actions we can define the equivariant J-map J_V as follows. Denote by $U(kV+l)$ the unitary group of $kV\oplus C^l$ with the induced action and by U_V the infinite unitary group obtained by taking a limit with respect to canonical inclusions of $U(kV+l)$'s. Then we have a map J_V from the equivariant homotopy group $[S^n,\ U_V]^{S^1}$ to $\pi^{V,S}_n$ as usual.

Now a generator μ of $\tilde{K}^{-1}(S^{4q-1})$, viewed as a map from S^{4q-1} to an unitary group, comes from $[S^{4q-1},\ U_V]^{S^1}$ and so $V\mu$ does. Generally an equivariant map from S^{4q-1} to U_V defines an element of $\tilde{K}^{-1}_{S^1}(S^{4q-1})$. So we have a map $[S^{4q-1},\ U_V]^{S^1}\to\tilde{K}^{-1}_{S^1}(S^{4q-1})$.

Because $J_V(V\mu)=0$, using the same notation for $V\mu$ in $[S^{4q-1},\ U_V]^{S^1}$, there exists $x\in\lambda^{V,S}_{4q-1}$ such that $\pi^*x=J_V(V\mu)$. From the above discussion it follows that $r(J_V(V\mu))=\alpha$ or 2α, so that $r(x)=\pi^{*-1}(\alpha)$ or $2\pi^{*-1}(\alpha)$, according as q is odd or even. q.e.d.

Let J_0 be the real J-homomorphism. By [1, IV], Theorem 7.16 we may write

$$e'_R J_o(\nu) = \frac{a_q}{m(2q)} \in Q/Z, \quad (a_q, m(2q)) = 1$$

where $m(2q)$, e'_R are as in [1, II]. Then we have

Theorem 3.4. *For* $\alpha = J_G(H\nu) \in \pi^S_{0,4q-1}$ $(q \geqq 1)$,

$$e_G(\alpha) = \begin{cases} \left(\dfrac{2a_q}{m(2q)}, 2(1-2^{2q-1})\dfrac{a_q}{m(2q)} + \dfrac{\varepsilon}{2}\right) & (q \text{ odd}) \\ \left(\dfrac{a_q}{m(2q)}, \ (1-2^{2q-1})\dfrac{a_q}{m(2q)} + \dfrac{\varepsilon}{4} + \dfrac{\varepsilon'}{2}\right) & (q \text{ even}) \end{cases}$$

$(\varepsilon, \varepsilon' = 0, 1)$ *as rational numbers* mod 1 *and the order of each factor of* $e_G(\alpha)$ *is* $\dfrac{m(2q)}{2}$ *or* $m(2q)$ *according as* q *is odd or even.*

Proof. The first claim follows from Theorem 3.2, Lemma 3.3 and [1, IV], Proposition 7.14. The second follows from [1, II], Lemma (2.12) and the equality $\nu_2(m(2q)) = 3 + \nu_2(q)$ ([1, II], p. 139) immediately. e.d.q.

4. Real $Z/2$-e-invariants

We take a base point preserving G-map $f: \Sigma^{p+8k, 2q-1+8l} \to \Sigma^{8k, 8l}$ as a representative of elements of $\pi^S_{p,2q-1}$ for p, $q-1 \geqq 0$. Then the parallel argument to e_G, using the Adams operation in the KO_G-theory [12] and Table of [14], yields the following equivariant e-invariants.

$$(1) \qquad e_{G,R}: \pi^S_{8p+4\zeta+i, 8q+4\delta-1} \to \begin{cases} (Q/Z)^2 & (i = 0) \\ Q/Z & (i = 1, 2, 3) \end{cases}$$

$$(2) \qquad e_{G,R}: \pi^S_{8p+4\zeta+2, 8q+4\delta+1} \to \quad Q/Z$$

for ζ, $\delta = 0, 1$.

Theorem 4.1. *For* $\bar{\alpha} = J_G(\nu)$, $\alpha = J_G(H\nu) \in \pi^S_{0,4q-1}$ $(q \geqq 1)$,

$$e_{G,R}(\bar{\alpha}) = \left(\frac{a_q}{m(2q)}, \ 0\right),$$

$$e_{G,R}(\alpha) = \left(\frac{a_q}{m(2q)}, \ (1-2^{2q-1})\frac{a_q}{m(2q)} + \frac{\varepsilon}{4} + \frac{\varepsilon'}{2}\right)$$

$(\varepsilon, \varepsilon' = 0, 1)$ *as rational numbers* mod 1 *and the order of the second factor of* $e_{G,R}(\alpha)$ *is* $m(2q)$.

Proof. As to the first factors of the equialties this follows immediately from the definitions of $e_{G,R}$ and e'_R. As to the second this follows in addition from Theorem 3.4 and the fact that $e_G = e_{G,R}$ or $2e_{G,R}$ according as q is even or odd. The proof of the last claim is similar to that of Theorem 3.4. q.e.d.

Finally we shall consider $e_{G,R}$ on $\operatorname{Im} J_G$ for $\pi^S_{p,4q-1}$ $(p\geqq 1)$. Let χ, ρ be as in [3] and $\hat{\eta}$ be the homomorphism induced by the element of [4], (8.1). Observe χ, ρ and $\hat{\eta}$ on the groups $\widetilde{KO}_G^{-1}(\Sigma^{p,2q-1})$ (see [15], §2), then since $e_{G,R}J_G$ commutes with χ, ρ and $\hat{\eta}$ (by an analogue of Proposition 1.3), we can compute $e_{G,R}$ of (1) on $\operatorname{Im} J_G$ inductively by using Theorem 4.1. For $e_{G,R}$ of (2), considering $\psi e_{G,R}$ we get readily $e_{G,R}$ on $\operatorname{Im} J_G$. Specifically we have

Theorem 4.2. *Let* $\nu_1\in\widetilde{KO}_G^{-1}(\Sigma^{8p+4\zeta,8q+4\delta-1})$ $(8p+4>0)$, $\nu_2\in\widetilde{KO}_G^{-1}(\Sigma^{8p+4\zeta+i,8q+4\delta-1})$ $(1\leqq i\leqq 3)$ *and* $\nu_3\in\widetilde{KO}_G^{-1}(\Sigma^{8p+4\zeta+2,8q+4\delta+1})$ *be generators as modules over the real representation ring of G respectively and set* $\alpha_k=J_G(\nu_k)$ $(1\leqq k\leqq 3)$. *Then as rational numbers* mod 1

$$e_{G,R}(\alpha_1)=\Big(\frac{a_{2p+2q+\zeta+\delta}}{m(4p+4q+2\zeta+2\delta)},\ \frac{1}{2}\Big\{\frac{a_{2p+2q+\zeta+\delta}}{m(4p+4q+2\zeta+2\delta)}-(1-2^{4q+2\delta-1})\frac{a_{2q+\delta}}{m(4q+2\delta)}-\frac{\varepsilon}{4}-\frac{\varepsilon'}{2}+\varepsilon''\Big\}\Big),$$

$$e_{G,R}(\alpha_2)=(1-2^{4q+2\delta-1})\frac{a_{2q+\delta}}{m(4q+2\delta)}+\frac{\varepsilon}{4}+\frac{\varepsilon'}{2},$$

$$e_{G,R}(\alpha_3)=\frac{a_{2p+2q+\zeta+\delta+1}}{m(4p+4q+2\zeta+2\delta+2)}+\frac{\varepsilon}{2}$$

$(\varepsilon,\ \varepsilon',\ \varepsilon''=0,\ 1)$ *up to sign and*

$$\text{order } e_{G,R}(\alpha_1)=\frac{m(4p+4q+2\zeta+2\delta)m(4q+2\delta)}{2^\kappa d},$$

$$\text{order } e_{G,R}(\alpha_1)=m(4q+2\delta),$$

$$\text{order } e_{G,R}(\alpha_3)=m(4p+4q+2\zeta+2\delta+2)$$

where

$$d=\Big(\frac{m(4p+4q+2\zeta+2\delta)}{2^{\nu_2(2p+2q+\zeta+\delta)+3}},\ \frac{m(4q+2\delta)}{2^{\nu_2(2q+\delta)+3}}\Big)$$

and κ *is the following integer*:

$$\begin{array}{ll} \nu_2(2q+\zeta)+2 & \text{if } \zeta=\delta \text{ and } \nu_2(2q+\zeta)\leqq\nu_2(p+q+\zeta),\\ \nu_2(2q+\zeta)+3 & \text{if } \zeta=\delta \text{ and } \nu_2(2q+\zeta)=\nu_2(p+q+\zeta)+1,\\ \nu_2(p+q+\zeta)+3 & \text{if } \zeta=\delta \text{ and } \nu_2(2q+\zeta)\geqq\nu_2(p+q+\zeta)+2,\\ 3 & \text{if } \zeta=0 \text{ and } \delta=1,\\ 2 & \text{if } \zeta=1 \text{ and } \delta=0. \end{array}$$

Here let $\nu_2(s)$ *denote the exponent to which* 2 *occurs in s.*

By Theorems 4.1, 4.2 and the results of [15] we have

Corollary 4.3. *For* $\pi^S_{p,q}$ *in* [15], *Theorems* 3.1, 3.2 *and* 3.3,

$$\mathrm{Im}\, J_G \overset{i}{\hookrightarrow} \pi^S_{p,q} \xrightarrow{e_{G,R}} \mathrm{Im}\, e_{G,R}$$

provides a direct sum splitting.

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