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ON NON-RATIONAL NUMERICAL DEL PEZZO SURFACES

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Introduciton

In this paper we call a normal compact complex surface X a numerical Del Pezzo surface if X is a Moishezon surface, the intersection number $(-K_X)\cdot C$ is positive for every curve C on X, and the self-intersection number $(-K_X)^2$ is positive (see Definition 1.1). If X is nonsingular, such a surface is called a Del Pezzo surface and its properties are fairly well-known. Several results on such surfaces are obtained by F. Hidaka and K.-i.Watanabe [9] when X is Gorenstein, by F. Sakai [13] when X is rational $\boldsymbol{Q}$-Gorenstein, and by L. Badescu [3] when X is non-rational $\boldsymbol{Q}$-Gorenstein.

The purpose of this paper is to study the structure of non-rational numerical Del Pezzo surfaces. In the present paper the canonical divisor K_X is not necessarily $\boldsymbol{Q}$-Cariter. In section 1 we define the notion of a numerical Del Pezzo surface and study its basic properties. In section 2 we describe the structure of a non-rational numerical Del Pezzo surface (Theorem 2.1). Our results are similar to those in L. Badescu [3], where the surface is assumed to be $\boldsymbol{Q}$-Gorenstein. In section 3 we define the notion of a minimal contraction of a ruled surface, and the notion of a DP1-ruled surface, which is a ruled surface whose singular fibers are of special type (see Definition 3.2 and 3.7). Then we obtain a criterion for the minimal contraction of a DP1-ruled surface to be a non-rational numerical Del Pezzo surface (Theorem 3.11). This is one of the main results of this paper. In section 4 we define the notion of indices of non-rational numerical Del Pezzo surfaces, and show that they are the minimal contractions of DP1-ruled surfaces if the Picard numbers are equal to 1 and their indices are prime numbers (Theorem 4.9). In appendix we prepare several results on weighted graphs. As for terminologies on weighted graphs, the reader may consult T. Fujita [7] and P. Orlik-P. Wagreich [12].

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Notations

A normal surface means an irreducible reduced normal compact complex space of dimension 2.

By a ruled surface we mean a projective surface birational to a product of a complete nonsingular curve and the projective line $\boldsymbol{P}^1$. A geometrically ruled surface means a $\boldsymbol{P}^1$-bundle over a complete nonsingular curve. The minimal section of a geometrically ruled surface measns the section of the surface whose self-intersection number is minimal among all sections.

The intersection number of two $\boldsymbol{Q}$-Weil divisors on a normal surface is defined as in F. Sakai [14].

For a $\boldsymbol{Q}$-Weil divisor $D=\sum a_V V (a_V \in \boldsymbol{Q})$ where V's are prime divisors, we define $Supp D$, the support of D, by $Supp D = \bigcup_{a_V \neq 0} V$. By [D] we denote the integral part of a $\boldsymbol{Q}$-divisor D, that is, $[D]=\sum [a_V] V$ where $[\cdot]$ means the Gauss symbol. We define $\ulcorner D \urcorner = -[-D]$ and call it a round up of D.

The set of the positive integers is denoted by $\boldsymbol{Z}_{>0}$. We use $\boldsymbol{Z}_{\geq 0}$, $\boldsymbol{Q}_{>0}$ etc. in the similar meanings.

In this paper we use the following notations:

$p_a(Z) = \frac{1}{2} Z \cdot (Z+K)+1$

: virtual genus of an effective integral divisor Z on a nonsingular surface.

(K is a canonical divisor of the surface.)

$p_g(x, X) = \dim_C R^1 f_* \mathcal{O}_{\tilde{X}}$ for a resolution $f: \tilde{X} \longrightarrow X$

: geometric genus of a singular point x on X.

$\equiv$: numerically equivalence of divisor.

$f_* D$: push-forward of a $\boldsymbol{Q}$-Weil divisor D by a morphism f.

1. Basic properties of numerical Del Pezzo surfaces

In this section we define the notion of a numerical Del Pezzo surface and study its basic properties. Our resuults are similar to L. Badescu's (compare [3] Theorem 2 and Corollary 8), where the surface is assumed to be $\boldsymbol{Q}$-Gorenstein.

DEFINITION 1.1. *A normal surface X is called a numerical Del Pezzo surface if X is a Moishezon surface and its anti-canonical divisor $-K_X$ satisfies the following two conditions:*

(1.1.1) $(-K_X) \cdot C > 0$ for every irreducible curve C on X.

(1.1.2) $(-K_X)^2 > 0$.

Let X be a normal surface and $f: \tilde{X} \longrightarrow X$ be the minimal resolution of X.

We denote by $\{E_i\}_{i\in I}$ the exceptional curves of f and put $E=\sum_{i\in I}E_i$. We define a $\boldsymbol{Q}$-divisor $\Delta=\sum_{i\in I}a_iE_i(a_i\in\boldsymbol{Q})$ by the following equalities (F. Sakai [14],(4.1)):

(1.2) $$(K_{\tilde{X}}+\Delta)\cdot E_i=0 \text{ for every } i\in I.$$

It is well known that Δ is well-defined and that $\Delta\geq 0$. According to F. Sakai [14], we define a $\boldsymbol{Q}$-divisor f^*K_X as

(1.3) $$f^*K_X=K_{\tilde{X}}+\Delta.$$

Proposition 1.4. *Let X be a numerical Del Pezzo surface and $f: \tilde{X}\longrightarrow X$ be the minimal resolution of X. Then we have the following equalities:*

(1.4.1) $$H^i(X, \mathcal{O}_X)=0 \text{ for } i>0$$
(1.4.2) $$H^i(X, \mathcal{O}_X(mK_X))=0 \text{ for } i<2 \text{ and } m\in\boldsymbol{Z}_{>0}$$
(1.4.3) $$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}}))=0 \text{ for } m\in\boldsymbol{Z}_{>0}$$

Proof. The equalities (1.4.1) and (1.4.2) are the direct consequences of the vanishing theorem due to F. Fakai (F. Sakaki [14], Theorem (5.1)). The last equality follows from (1.4.2) using the projection formula (F. Sakai [14], Theorem (2.1)) and by the fact that Δ is efficitve. Q.E.D.

Corollary 1.5. *The minimal resolution of a numerical Del Pezzo surface is a ruled surface.*

Theorem 1.6. *If a normal surface X is a numerical Del Pezzo surface, then X is a projective surface.*

Proof. Because X is a Moishezon surface with $H^2(X, \mathcal{O}_X)=0$ by (1.4.1), we obtain the conclusion from Brenton's results (L. Brenton [4], 7. Proposition). Q.E.D.

Lemma 1.7. *We have the following isomorphisms:*

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})\cong H^0(X, R^1f_*\mathcal{O}_{\tilde{X}})\cong H^1(E, \mathcal{O}_E)$$

where $E=\sum_{i\in I}E_i$ is the exceptional divisor of the minimal resolution f: $\tilde{X}\longrightarrow X$.

Proof. Using Leray's spectral sequence we conclude the first isomorphism immediately by (1.4.1). We obtain the second isomorphism from Brenton's result (L. Brenton [5] 10. Thorem), because $H^2(X,\mathcal{O}_X)=0$ by (1.4.1). Q.E.D.

REMARK 1.8. *A normal surface singularity is called a Du Bois singularity, if the second isomorphism holds (cf. P. Du Bois [6], S. Ishii [10] and J.H.M. Steenbrink [15]).*

Corollary 1.9.

$$\sum_{x\in Sing X} p_g(x,\ X)=h^1(\tilde{X},\ \mathcal{O}_{\tilde{X}})$$

Especially, X is a rational surface if and only if all singular points of X are rational singularities.

Lemma 1.10.

$$H^i(\tilde{X},\ \mathcal{O}_{\tilde{X}}(-[\Delta]))=0 \textit{ for } i>0$$

where [ ] *means the integral part of a* **Q**-*divisor.*

Proof. By Sakai's vanishing theorem we obtain the result. Q.E.D.

Theorem 1.11. *Let X be a numerical Del Pezzo surface then singular points of X which are not quotient singularities are at most one point.*

Proof. We can write $[\Delta]=\sum_{j=1}^{l} D_j$, where D_j's are non zero effective divisors whose supports, $Supp D_j$'s, are contained in inverse images of mutually distinct singular points. To show the theorem, it is sufficient to prove that $l\leq 1$ (Ki. Watanabe [16], Proposition 3.5 and Theorem 3.9). So it is trivial when $[\Delta]=0$. We may assume $[\Delta]>0$. Then we have $H(\tilde{X},\mathcal{O}_{\tilde{X}}(-[\Delta]))=0$. By the Riemann-Roch theorem, Proposition 1.4 and Lemma 1.9

$$p_a([\Delta])=h^1(\tilde{X},\ \mathcal{O}_{\tilde{X}})$$

where $p_a(\cdot)$ is the virtual genus of an effective divisor. Because D_i and D_j do not intersect for distinct i, j,

$$p_a([\Delta])=\sum_{j=1}^{l} p_a(D_j)-l+1.$$

It is well known that $\sum_{j=1}^{l} p_a(D_j)\leq\sum_{x\in Sing X} p_g(x,\ X)$. Considering Corollary 1.8 we have

$$h^1(\tilde{X},\mathcal{O}_{\tilde{X}})=p_a([\Delta])\leq\sum_{x\in Sing X} p_a(x,\ X)-l+1=h^1(\tilde{X},\mathcal{O}_X)-l+1.$$

Then we conclude $l\leq 1$. Q.E.D.

2. Structure of non-rational numerical Del Pezzo surfaces

Let X be a non-rational numerical Del Pezzo surface and $f:\ \tilde{X}\longrightarrow X$ be the minimal resolution of X. By Corollary 1.5 $\tilde{X}$ is a ruled surface. We choose an arbitrary relatively minimal model $\sigma:\ \tilde{X}\longrightarrow Y$ of $\tilde{X}$, then Y is a geometrically ruled surface. Let C be the base curve of Y, g be the genus of C, and $\psi:\ Y\longrightarrow C$ be the projection from Y to C. We put $\varphi=\psi\circ\sigma:\ \tilde{X}\longrightarrow C$. Since X is not a rational surface, $\widetilde{X}$ and Y are not rational surfaces and then $g=h^1(Y, \mathcal{O}_Y)=h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})\geq 1$.

For the sake of convenience we write $\overline{D}$ instead of $\sigma_* D$ for a $\boldsymbol{Q}$-divisor D on $\widetilde{X}$, where σ_* means push-forward of a $\boldsymbol{Q}$-Weil divisor.

The following theorem has been proved by L. Badescu under the assumption that X is $\boldsymbol{Q}$-Gorenstein (L. Badescu [3], Theorem 2). In the remainder of this section we prove the same result without the $\boldsymbol{Q}$-Gorenstein assumption.

Theorem 2.1. *Under the above notations*

(2.1.1) *Among the exceptional curves $\{E_i\}_{i\in I}$ there is a curve which is a section of $\varphi: \widetilde{X}\longrightarrow C$. The others are irreducible components of singular fibers of φ.*
We denote the curve which is a section of φ by E_0 and the otheres by $E_1, \ldots, E_n$.

(2.1.2) *The geometrically ruled surface Y is decomposable and $\sigma(E_0)$ coincides with the minimal section C_0 of Y. Moreover we have an inequality $-C_0^2>2g-2$.*

(2.1.3) *Let a_0 be the coefficient of E_0 in Δ, then $1\leq a_0<2$.*

(2.1.4) $\{E_i\}_{i=1}^n=\{E$ *; non-singular rational curve on X such that $E^2\leq -2\}$*

Proof. If we assume that all E_i's are irreducible components of singular fibers of φ, then $E=\sum_{i\in I}E_i$ are trees of nonsingular rational curves, and we have $H^1(E, \mathcal{O}_E)=0$. This contradicts Lemma 1.7 and the fact that $h^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})\geq 1$. Therefore among $\{E_i\}_{i\in I}$ there is a curve E_i whose image of $\varphi: \widetilde{X}\longrightarrow C$ is C. We denote it by E_0, the otheres by $E_1, \ldots, E_n$.

Let $\pi: \widetilde{E}_0\longrightarrow E_0$ be the normalization of the curve E_0. Then $g(\widetilde{E}_0)\leq h^1(E_0, \mathcal{O}_{E_0})$, and the equality holds if and only if E_0 is nonsingular. Because C is the image of E_0 by φ, $\varphi\circ\pi: \widetilde{E}_0\longrightarrow C$ is surjective. Therefore $g=g(C)\leq g(\widetilde{E}_0)$ by Hurwitz' formula.

On the other hand, we have a short exat sequence

$$0\longrightarrow \mathcal{O}_E\longrightarrow \oplus_{i=0}^n \mathcal{O}_{E_i}\longrightarrow \mathcal{F}\longrightarrow 0.$$

Because $Supp\mathcal{F}$ are finite points we have a surjection

$$H^1(E, \mathcal{O}_E)\longrightarrow \oplus_{i=0}^n H^1(E_i, \mathcal{O}_{E_i})$$

from the above short exact sequence. Therefore we have following inequalities:

$$g\leq g(\widetilde{E}_0)\leq h^1(E_0, \mathcal{O}_{E_0})\leq \sum_{i=0}^n h^1(E_i, \mathcal{O}_{E_i})\leq h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})=g.$$

So we have

$$g=g(\widetilde{E}_0)=h^1(E_0,\ \mathcal{O}_{E_0})=\sum_{i=0}^{n} h^1(E_i,\ \mathcal{O}_{E_i})$$

and then $h^1(E_i, \mathcal{O}_{E_i})=0$ for $i=1, \ldots, n$. Therefore E_0 is a nonsingular curve of genus g and E_i is a nonsingular rational curve which is an irreducible component of a singular fiber for every $i=1, \ldots, n$.

By the definition of $\Delta=\sum_{i=0}^{n}\alpha_i E_i$, we have

$$\begin{aligned}0&=f^*K_X\cdot E_0\\&=(K_{\tilde{X}}+\Delta)\cdot E\\&=2g-2+(\alpha_0-1)E_0^2+\sum_{i=1}^{n}\alpha_i E_i\cdot E_0\\&\geq 2g-2+(\alpha_0-1)E_0^2.\end{aligned}$$

Then we can easily see that $\alpha_0\geq 1$, using the fact that $g\geq 1$ and $E_0^2<0$.

Let C_0 be the minimal section of a geometrically ruled surface Y. We put $e=-C_0^2$.

We will show that $\sigma(E_0)$ coincides with C_0.

It is well known that we can write

$$\overline{E}_0\equiv aC_0+bF \qquad a,\ b\in \boldsymbol{Z}$$

on a ruled surface Y where F is the fiber of Y, and $\equiv$ means numerically equivalent.

If we assume $\overline{E}_0\neq C_0$, then

$$0\leq \overline{E}_0\cdot C_0=-ae+b. \tag{2.1.5}$$

On the other hand we can write

$$\overline{\Delta}\equiv\alpha_0\overline{E}_0+\beta F$$

for some $\beta\in \boldsymbol{Q}_{\geq 0}$, and

$$K_Y\equiv -2C_0+(2g-2-e)F.$$

Then

$$K_X+\overline{\Delta}\equiv(\alpha_0 a-2)C_0+(2g-2-e+\alpha_0 a+\beta)F.$$

By the assumption $\overline{E}_0\neq C_0$, σ^*C_0 contains an irreducible component which is not contracted by $f: \widetilde{X}\longrightarrow X$. Then we have

$$\begin{aligned}0&>(K_{\tilde{X}}+\Delta)\cdot\sigma^*C_0\\&=\sigma^*(K_Y+\overline{\Delta})\cdot\sigma^*C_0\\&=(K_Y+\overline{\Delta})\cdot C_0\\&=\alpha_0(b-ae)+2g-2+e+\beta.\end{aligned} \tag{2.1.6}$$

From $\alpha_0\geq 1$, $\beta\geq 0$ and (2.1.5) we conclude

(2.1.7) $$2g-2+e<0.$$

On the other hand, the inequality $e \geq -g$ is well-known (M. Nagata [11], Theorem 1). Therefore $g=1$ by (2.1.7). Once again by (2.1.7) and $e \geq -g$ we conclude that $e=-1$. Then $b+a \geq 0$ by (2.1.5). Combining the above fact, $\alpha_0 \geq 1$ and (2.1.6) we obtain that $b+a=0$. Then

$$\overline{E}_0 \equiv aC_0 - aF \text{ for some } a \in \boldsymbol{Z}.$$

This contradicts that $\overline{E}_0$ is an irreducible curve (e.g. R. Hartshorne [8], Proposition V.2.21). As a consequence we obtain that $\sigma(E_0)=\overline{E}_0=C_0$. So E_0 turns out to be a section of φ: $\widetilde{X} \longrightarrow C$ and (2.1.1) has been shown.

Furthermore the inequality

$$0>(K_{\widetilde{X}}+\Delta)\cdot F=-2+\alpha_0$$

holds for the general fiber of φ. Therefore we have $\alpha_0<2$. Thus (2.1.3) has been shown.

Next, we will show that $e>2g-2$.

From the fact that $\overline{E}_0=C_0$

$$\overline{\Delta} \equiv \alpha_0 C_0 + \beta F$$

for some $\beta \in \boldsymbol{Q}_{\geq 0}$, and

$$K_Y+\overline{\Delta} \equiv (\alpha_0-2)C_0+(2g-2-e+\beta)F.$$

If we assume that $e=-C_0^2 \leq 0$, then $Supp(\sigma^* C_0)$ contains an exceptional curve of the first kind. So we have

$$\begin{aligned} 0 &> (K_{\widetilde{X}}+\Delta)\cdot \sigma^* C_0 \\ &= (K_Y+\overline{\Delta})\cdot C_0 \\ &= (1-\alpha_0)e+2g-2+\beta \end{aligned}$$

and this contradicts to the fact that $\alpha_0 \geq 1$, $e \leq 0$, $g \geq 1$ and $\beta \geq 0$. Then we have $e>0$. Then using the following inequalities:

$$\begin{aligned} 0 &\geq (K_{\widetilde{X}}+\Delta)\cdot \sigma^* C_0 \\ &= (K_Y+\overline{\Delta})\cdot C_0 \\ &= (1-\alpha_0)e+2g-2+\beta \end{aligned}$$

we obtain $e>2g-2$, and then Y turns out decomposable (e.g. R. Hartshorne [8]). Thus we have proved (2.1.2).

Lastly we will show (2.1.4).

Let E be a nonsingular rational curve which is different from $E_1, \dots, E_n$. Then

$$0<(K_{\widetilde{X}}+\Delta)\cdot E=-2-E^2+\Delta\cdot E \geq -2-E^2.$$

So we have shown (2.1.4). Q.E.D.

3. DP1-ruled surfaces and their minimal contractions

In the remainder of this paper we use terminologies concerning normal crossing divisors on nonsingular projective surfaces and weighted graphs interchangeably. As for terminologies on weighted graphs, see T. Fujita [7], P. Orlik-P. Wagreich [12] and Appendix.

NOTATION 3.1. *Let $\tilde{X}$ be a non-rational ruled surface, and Y be a relatively minimal model of X.*

The curve on $\tilde{X}$ whose image in Y coincides with the minimal section of Y is called the minimal section of $\tilde{X}$ (with respect to Y) and denoted by C_0.

The (total) singular fibers of $\tilde{X}$, considered as Weil divisors, are denoted by $\tilde{F}_1, \dots, \tilde{F}_r$, and we set $F_j=(\tilde{F}_j)_{red}$.

By $\mathcal{E}_j$ we denote the set of curves which are irreducible components of a singular fiber F_j and whose self-intersection numbers are not equal to -1. And by $\mathcal{E}'_j$ we denote the set of curves which are irreducible components of a singular fiber F_j with the self-intersection number -1. Furthermore we set $\mathcal{E}=\bigcup_{j=1}^{r}\mathcal{E}_j$ and $\mathcal{E}'=\bigcup_{j=1}^{r}\mathcal{E}'_j$.

We use the above terminologies freely in the remainder of this paper.

DEFINITION 3.2. *Under the above situation, the pair of a normal surface X and a morphism $f: \tilde{X} \longrightarrow X$ is said to be a minimal contaction of $\tilde{X}$ (with respect to Y) if the following conditions are satisfied:*

(3.2.1) f *is the minimal resolution of* X.

(3.2.2) $$f^{-1}(Sing X)_{red}=C_0+\sum_{E\in\mathcal{E}} E$$

We sometimes say X is a minimal contraction of $\tilde{X}$ (with respect to Y) for short.

By definition if there exists a minimal contraction of X, it is unique up to isomorophism.

REMARK 3.3. *If X is a non-rational numerical Del Pezzo surface and f: $\tilde{X} \longrightarrow X$ is the minimal resolution of $\tilde{X}$ (with respect to an arbitrary minimal model of $\tilde{X}$) because of* Theorem 2.1.

REMARK 3.4. If the minimal section of $\tilde{X}$ is independent of the choice of relatively minimal models, we will not specify the minimal models.

Now we will define a certain kind of non-rational ruled surfaces. At first we prepare several notations. We will use terminologies on weighted graphs in T.

Fujita [7] and Appendix.

DEFINITION 3.5. *Let $A=[a_1, \ldots, a_r]$ be an admissible twig (see Definition A. 3), and $A^*=[b_1, \ldots, b_s]$ the adjoint of A (see Definition A.6). Let n be a non negative integer. We call a weighted graph Γ the simple tree of type (A, n), if Γ is as follows:*

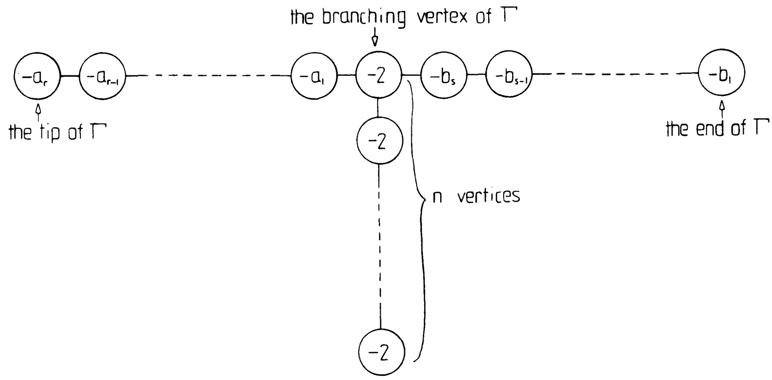

figure (3.5.1)

If $n=0$, then Γ is regarded as follows:

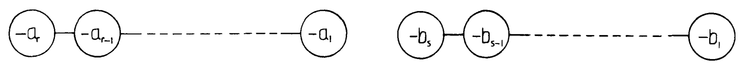

figure (3.5.2)

where the numbers in the circles are the weights of the vertices.

For a simple tree Γ of type (A, n) the vertex correspoding to a_r is called tip of Γ, and the vertex corresponding to b_1 is called the end of Γ. When a simple tree Γ is as in the figure (3.5.1), the vertex which is joined to A and A^* is called the branching vertex of Γ. Sometimes we say that a simple tree Γ has no branching vertex if $n=0$, that is, Γ is a simple tree of type $(A, 0)$ (figure (3.5. 2)).

DEFINITION 3.6. *Let Γ be a simple tree and Γ' an arbitrary weighted graph. By $\Gamma+v+\Gamma'$ we denote the following weighted graph:*

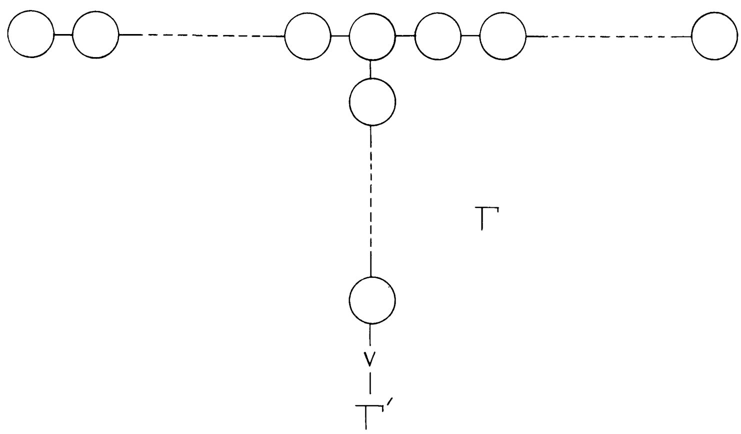

figure (3.6.1)

When Γ is a simple tree without branching vertex, $\Gamma+v+\Gamma'$ denotes the following :

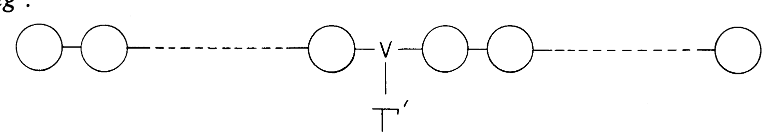

figure (3.6.2)

Especially, by $\Gamma+(-1)$ we denote the following weighted graph :

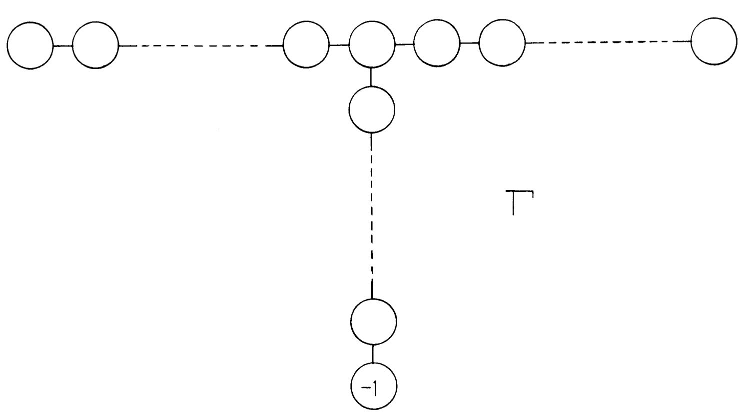

figure (3.6.3)

If Γ has no branching vertex $\Gamma+(-1)$ denotes the following:

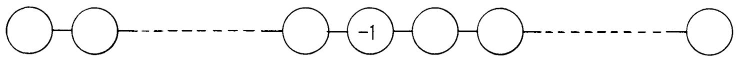

figure (3.6.4)

DEFINITION 3.7. *Let $(p_j, q_j, n_j)(j=1, \dots, r)$ be sets of integers such that p_j, q_j are positive and coprime to each other and n_j non-negative for every j. Furthermore, let e, g be positive integers.*

Let A_j be the admissible twig whose inductance (*T. Fujita* [7], (3.5)) *is equal to q_j/p_j, and Γ_j the simple tree of type (A_j, n_j) for every j.*

A nonsingular projective surface $\widetilde{X}$ is said to be a DP1-ruled surface of type $\{e, g, (p_j, q_j, n_j)_{j=1}^{r}\}$ if the surface satisfies the following conditons:

(3.7.0) *$\widetilde{X}$ is a ruled surface.*

(3.7.1) *The genus of the base curve is equal to g (hence the base curve is not a rational curve).*

(3.7.2) *$\widetilde{X}$ has the singular fibers $F_1, \dots, F_r$ and every F_j is of the form Γ_j $+(-1)$.*

(3.7.3) *The minimal section C_0 of $\widetilde{X}$ is joined to F_j at the tip of F_j for every j.*

(3.7.4) $C_0^2=-e-r$.

(3.7.5) $e>2g-2$.

We call $\widetilde{X}$ a DP1-ruled surface if we need not mention the type.

REMARK 3.8. *By the conditions* (3.7.4) *and* (3.7.5) *the minimal section of $\widetilde{X}$ is independent of the choice of a minimal model of $\widetilde{X}$.*

REMARK 3.9. *We can construct a DP1-ruled surface of type $\{e, g, (p_j, q_j, n_j)_{j=1}^{r}\}$ form a geometrically ruled surface over a complete nonsingular curve of genus g such that the self-intersection number of its minimal section is equal to $-e$, and that the inequality $e>2g-2$ holds* (*T. Fujita* [7], (4.7) *Proposition*).

REMARK 3.10. *If $\widetilde{X}$ is a DP1-ruled surface of type $\{e, g, (p_j, q_j, n_j)_{j=1}^{r}\}$ then $C_0+\sum_{E\in\mathcal{E}}E$ is a DP1-graph of type $\{e, g, (p_j, q_j, n_j)_{j=1}^{r}\}$ (see Appendix, Definition* A.13).

Theorem 3.11. *Let $\widetilde{X}$ be a DP1-ruled surface, then the following two conditions are equivalent*:

(3.11.1) *There exists a minimal contraction X of $\widetilde{X}$ such that X is a non-rational numerical Del Pezzo surface.*

(3.11.2) *There exists an effective* $\boldsymbol{Q}$*-divisor* $\Delta = aC_0 + \sum_{E\in\mathcal{E}} a_E E$ *such that*

(3.11.2.1) $(K_{\tilde{X}} + \Delta)\cdot C_0 = 0$

(3.11.2.2) $(K_{\tilde{X}} + \Delta)\cdot E = 0$ *for every element* E *of* $\mathcal{E}$

(3.11.2.3) $0 < a < 2$.

Proof. It was shown in Theorem 2.1 that the condtion (3.11.2) follows from (3.11.1). Then we assume (3.11.2).

By the conditions (3.11.2.1) and (3.11.2.2), it can be easily seen that $\Delta\cdot C_0 < 0$, $\Delta\cdot E \leq 0$ for every element E of $\mathcal{E}$ and $\Delta^2 < 0$. Then the intersection matrix of the divisor $C_0 + \sum_{E\in\mathcal{E}} E$ is negative definite (M. Artin [2], Proposition 2). Therefore there exists the minimal contraction $f: \tilde{X} \longrightarrow X$ of $\tilde{X}$, and $f^* K_X = K_{\tilde{X}} + \Delta$ by the condition (3.11.2).

To show that the surface X is a non-rational numerical Del Pezzo surface, it is seffucient to prove the following two inequalities:

(3.11.3) $(K_{\tilde{X}} + \Delta)\cdot C < 0$ for every irreducible curve on $\tilde{X}$ which is not contracted to a point by f.

(3.11.4) $(K_{\tilde{X}} + \Delta)^2 > 0$.

By the assumption (3.11.2.3) we can easily check that (3.11.3) is the case for the general fiber F of $\tilde{X}$

Take an irreducible curve C and we can write $C = aC_0 + \sum_{E\in\mathcal{E}} a_E E + \varphi^* D$ as an element of $Pic\tilde{X} \otimes \boldsymbol{Q}$ where a and a_E are rational numbers, D is a divisor on the base curve and φ is the projection of $\tilde{X}$ to the base curve. Then $a \geq 0$ because $a = C\cdot F$.

If we assume that $degD$, the degree of D, is not positive, then we can see that $degD = C\cdot C_0 = 0$, $a = C\cdot F = 0$, and $C\cdot E = 0$ for every element E of $\mathcal{E}$ which is contained in the connected component of $C_0 + \sum_{E\in\mathcal{E}} E$ containing C_0 by the similar argument as in Ki. Watanabe [16] (Lemma 3.1 and 3.2). But it is impossible.

Then $degD$ is positive, and it turns out that $(K_{\tilde{X}} + \Delta)\cdot C < 0$. Therefore (3.11.3) has been proved.

Next we will show the inequality (3.11.4). We write $K_{\tilde{X}} + \Delta = \Delta_1 + \varphi^* D'$ as an element of $Pic\tilde{X} \otimes \boldsymbol{Q}$, where Δ_1 is a $\boldsymbol{Q}$-weil divisor whose support is contained in $C_0 \cup (\bigcup_{E\in\mathcal{E}} E)$ and D' is a divisor on the base curve. Then we can easily see that $\Delta_1 \neq 0$ and that $(K_{\tilde{X}} + \Delta)^2 = -\Delta_1^2$ by the conditions (3.11.2.1) and (3.11.2.2). Because the intersection matrix of the divisor $C_0 + \sum_{E\in\mathcal{E}} E$ is negative definite we obtain the inequality $-\Delta_1^2 > 0$, and then (3.11.4) has been shown. Q.E.D.

The conditon (3.11.2) in the above theorem essentially depends only on the DP1-graph Γ of type $\{e, g, (p_j, q_j, n_j)_{j=1}^r\}$ and $\boldsymbol{Q}(\Gamma)$. Therefore we can use results on weighted graphs in Appendix (Proposition A.14) and obtain the following theorem.

Theorem 3.12. *Let $\tilde{X}$ be a DP1-ruled surface of type $\{e, g, (p_j, q_j, n_j)_{j=1}^r\}$. Then the following two conditions are equivalent.*

(3.12.1) *There exists the minimal contraction X of $\tilde{X}$ and X is a non-rational numerical Del Pezzo surface.*

(3.12.2) *The following inequality holds* :

$$e-(2g-2)>\sum_{j=1}^{r}\frac{q_j+n_j-1}{p_j}.$$

4. A certain kind of non-rational numerical Del Pezzo surfaces

In this section we will characterize a certain kind of non-rational numerical Del Pezzo surfaces.

Lemma 4.1. *Let $\tilde{X}$ be a non-rational ruled surface. We assume that there exists the minimal contraction $f: \tilde{X}\longrightarrow X$ of $\tilde{X}$ (with respect to some minimal model of $\tilde{X}$), such that all singular points of X are Du Bois singularities. Then a Weil divisor D on X is a Cartier divisor if and only if the following two conditions are satisfied* :

(4.1.1) $\quad f^*D$ *is integral.*

(4.1.2) $\quad \mathcal{O}(f^*D)\otimes\mathcal{O}_{C_0}\cong\mathcal{O}_{C_0}.$

Proof. It is clear that the conditions (4.1.1) and (4.1.2) are necessary. Conversely we assume the conditions (4.1.1) and (4.1.2).

Because $F_1, \ldots, F_r$ are trees of rational curved and all singular points of X are Du Bois, X has only one non-rational singular point, and we denote it by x_0.

From the result of M. Artin (M. Artin [1], Corollary (2.6)) D is a Cartier divisor around all rational singular points.

Let V be a sufficiently small Stein neighborhood of x_0. We set $M=f^{-1}(V)$ and $N=f^{-1}(x_0)_{red}$. Then $PicM\cong PicN$ (S. Ishii [10], the proof of Proposition 4. 2). So it is sufficient to prove that $\mathcal{O}(f^*D)\otimes\mathcal{O}_N\cong\mathcal{O}_N$. We put $N_1=N-C_0$. Then N_1 can be contracted rational singularities, because N_1 can be contractied to Du Bois sungularities (S. Ishii [10], Theorem 2.2) and N_1 is trees of rational curves. Again from Artin's result $\mathcal{O}(f^*D)\otimes\mathcal{O}_{N_1}\cong\mathcal{O}_{N_1}$. The conclusion follows from this isomorphism and the conditon (4.1.2) because N is a normal crossing divisor and every connected component of N_1 intersects C_0 at most one point. Q.E.D.

Corollary 4.2. *Under the situation as in Lemma 4.1, an invertible sheaf $\mathcal{L}$ on $\tilde{X}$ is contained in $f^*(PicX)$ if and only if $\mathcal{L}$ satisfies the following two conditions* :

(4.2.1) $\quad \mathcal{L}\cdot E=0$ *for every element E of $\mathcal{E}$*

(4.2.2) $\quad \mathcal{L}\otimes\mathcal{O}_{C_0}\cong\mathcal{O}_{C_0}.$

Proof. It is obvious that the conditions (4.2.1) and (4.2.2) are necessary. We will prove the converse. We can take a Weil divisor D on $\tilde{X}$ such that $\mathcal{O}(D)$ is isomorphic to $\mathcal{L}$. By the assumptions (4.2.1) and (4.2.2) $SuppD$ is not contained in $C_0 \cup (\bigcup_{E \in \mathcal{E}} E)$. Then $D' = f_* D$ is a non-zero Weil divisor on X.

Because the support of a $\boldsymbol{Q}$-Weil divisor $D - f^* D'$ contained in $C_0 \cup (\bigcup_{E \in \mathcal{E}} E)$ and $C_0 + \sum_{E \in \mathcal{E}} E$ is contractible, it turns out that $D = f^* D'$ from the equalities (4.2.1) and (4.2.2).

Therefore D' satisfies the conditions (4.1.1) and (4.1.2), and then D' is a Cartier divisor on X. Thus we obtain the result. Q.E.D.

REMARK 4.3. *Because X is a normal surface, the homomorohism f^*: $PicX \longrightarrow Pic\tilde{X}$ is injective by the projection formula.*

Proposition 4.4. *Let $\tilde{X}$ be a non-rational ruled surface which has r singular fibers. We assume that there exists the minimal contraciton $f: \tilde{X} \longrightarrow X$ (with respect to some minimal model of $\tilde{X}$). Then $PicX$ is torsion free. Furthermore if all singular points of X are Du Bois singularities, then*

$$\rho(X) = \#\mathcal{E}' + 1 - r.$$

In particular, $\rho(X) = 1$ if and only if every singular fiber has only one exceptional curve of the first kind.

Proof. Let C be the base curve of the reled surface $\tilde{X}$, and φ the projection of $\tilde{X}$ to C. We can identify C and C_0 because C_0 is a section. By Remark 4.3 $PicX$ can be considered as subgroup of $Pic\tilde{X}$ Because $Pic\tilde{X}/\varphi^*(PicC)$ is a free $\boldsymbol{Z}$-module, torsion elements of $PicX$ are contained in $\varphi^*(PicC)$. Then $PicX$ is torsion free by the condition (4.2.2).

Next we assume that all singular points of X are Du Bois singularities.

By Corollary 4.2, we have known that $f^*(PicX \otimes \boldsymbol{Q})/\varphi^*(PicC \otimes \boldsymbol{Q})$ is a linear subspace of $(Pic\tilde{X} \otimes \boldsymbol{Q})/\varphi^*(PicC \otimes \boldsymbol{Q})$ defined by the equalities (4.2.1). We can easily check that the dimension of $f^*(PicX \otimes \boldsymbol{Q})/\varphi^*(PicC \otimes \boldsymbol{Q})$ is equal to $\#\mathcal{E}' + 1 - r$ because the intesection matrix of the divisor $C_0 + \sum_{E \in \mathcal{E}} E$ is negqative definite. The condition (4.2.2) determines the part contained in $\varphi^*(PicC \otimes \boldsymbol{Q})$. Therefore we have the conclusion. Q.E.D.

Corollary 4.5. *Let X be a non-rational numerical Del Pezzo surface with $\rho(X) = 1$, $f: \tilde{X} \longrightarrow X$ be the minimal resolution of X, and F_j ($j = 1, \dots, r$) be the singular fibers of the ruled surface $\tilde{X}$. Then F_j is of the form $\Gamma_j + E_j + \Gamma_j'$ where Γ_j is some simple tree without branching vertex, E_j is an irreducible curve and Γ_j' is some weighted graph for every j.*

Proof. It is obvious because every singular fiber has only one exceptional curve of the first kind by Proposition 4.4. Q.E.D.

DEFINETION 4.6. Let X be a non-rational numerical Del Pezzo surface with $\rho(X)=1$, $f: \tilde{X} \longrightarrow X$ the minimal resolution of X, and F the general fiber of X. Let H be an ample Cartier divisor which is a generator of $PicX$. We call the intersection number $H \cdot f_* F$ the index of X.

Lemma 4.7. *Under the assumption in Corollary 4.5 Let X, $\tilde{X}$ and so on be as in Corollary 4.5. By $\tilde{F}_j$ $(j=1, \ldots, r)$ we denote the (total) singular fibers of $\tilde{X}$ such that $(\tilde{F}_j)_{red}=F_j$ for every j. Then the coefficient of E_j in $\tilde{F}_j$ divides the index of X for every j.*

Proof. Let Γ_j be as in Corollary 4.5, and its type $(A_j, 0)$. Then the coefficients of E_j in $\tilde{F}_j$ coincide with $d(A_j)$(T. Fujita[7], (4.8) Proposition).

The set $C_0 \cup (\cup_{j=1}^{r} \mathcal{E}_j'')$ is a basis of the vector space $(Pic\tilde{X} \otimes \boldsymbol{Q})/\varphi^*(PicC \otimes \boldsymbol{Q})$ where $\mathcal{E}_j''$ are the sets which consist of the elements of $\mathcal{E}_j \cup \mathcal{E}_j'$ different from the curves corresponding to the ends of Γ_j. We regard f^*H as an element of $(Pic\tilde{X} \otimes \boldsymbol{Q})$ and write it in a lenear combination of $C_0 \cup (\cup_{j=1}^{r} \mathcal{E}_j'')$ with rational coefficients modulo $\varphi^*(PicC \otimes \boldsymbol{Q})$. Let α be the coefficient of C_0 in the linear combination. Then $H \cdot f_* F = f^*H \cdot F = \alpha$. Because f^*H satisfies the following conditions:

(4.7.1) $\quad f^*H \cdot E = 0$ for every element E of $\mathcal{E}$

(4.7.2) $\quad f^*H \cdot C_0 = 0$

we can describe the coefficients by Lemma A.7 and obtain the conclusion. Q.E.D.

Lemma 4.8. *Let X be a non-rational numerical Del Pezzo surface, and x_0 its non-rational singular point. We assume that $\rho(X)=1$. Then there exits an effective Cartier divisor H on X such that its support does not contain x_0 and $PicX = \boldsymbol{Z}H$.*

Proof. By Proposition 4.4 $PicX$ is torsion free. Then we can take an ample Cartier divisor H such that $PicX = \boldsymbol{Z}H$.

By the definition of numerical Del Pezzo surface and by the fact that H is ample, it can be easily seen that $H - K_X$ is nef and big. Therefore we have $H^0(X, \mathcal{O}(H)) \neq 0$ by the Sakai's vanishing theorem and the Riemann-Roch theorem. Thus we may assume that H is effective.

Next we will show that $x_0 \notin B_s|H|$.

Let $f: \tilde{X} \longrightarrow X$ be the minimal resolution of X. A $\boldsymbol{Q}$-divisor Δ is defined as in (1.2). A divisor $D = [\Delta] - C_0$, where $[\Delta]$ means a integral part of Δ, is an effective integral divisor on $\tilde{X}$ by Theorem 2.1. We denote the ideal sheaf of $\{x_0\}$

by $\mathcal{J}$. Then $\mathcal{J}$ and $f_*\mathcal{O}_{\tilde{X}}(-C_0)$ are ideal sheaves of $\mathcal{O}_X$ and $\mathcal{J}$ is contained in $f_*\mathcal{O}_{\tilde{X}}(-C_0)$. Because $\mathcal{J}_{x_0}$ is the maximal ideal of $\mathcal{O}_{X,x_0}$, $\mathcal{J}$ coincides with $f_*\mathcal{O}_{\tilde{X}}(-C_0)$.

On the other hand we have a short exact sequence

$$0\longrightarrow\mathcal{O}_{\tilde{X}}(-[\Delta])\longrightarrow\mathcal{O}_{\tilde{X}}(-C_0)\longrightarrow\mathcal{O}_D(-C_0)\longrightarrow 0 \tag{4.8.1}$$

and from it we have

$$0\longrightarrow f_*\mathcal{O}_{\tilde{X}}(-[\Delta])\longrightarrow\mathcal{J}\longrightarrow f_*\mathcal{O}_D(-C_0). \tag{4.8.2}$$

By Lemma 1.9 we have $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-[\Delta]))=0$. Then we have $H^0(D, \mathcal{O}_D(-C_0))=0$ by the long exact sequence obtained from (4.8.1). Because $Supp f_*\mathcal{O}_D(-C_0)=x_0$ and $(f_*\mathcal{O}_D(-C_0))_{x_0}=H^0(D, \mathcal{O}_D(-C_0))=0$ we have $\mathcal{J}\cong f_*\mathcal{O}_{\tilde{X}}(-[\Delta])$, and then $f_*\mathcal{O}_{\tilde{X}}(f^*H-[\Delta])\cong\mathcal{J}\cdot\mathcal{O}_X(H)$ by the projection formula. Then we have an injective homomorphism

$$H^1(X, \mathcal{J}\cdot\mathcal{O}_X(H))\longrightarrow H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(f^*H-[\Delta]))$$

by Laray's spectral sequence for $\mathcal{O}_{\tilde{X}}(f^*H-[\Delta])$. It can be easily seen that $f^*H-K_{\tilde{X}}-\Delta$ is nef and big. Then we obtain $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(f^*H-[\Delta]))=0$ by Sakai's vanishing theorem and then $H^1(X, \mathcal{J}\cdot\mathcal{O}(H))=0$ by the above injection. So we obtain the following exact sequence:

$$0\longrightarrow H^0(X, \mathcal{J}\cdot\mathcal{O}(H)\longrightarrow H^0(X, \mathcal{J}\cdot\mathcal{O}_X(H))\longrightarrow \boldsymbol{C}\longrightarrow 0$$

from the short exact sequence

$$0\longrightarrow\mathcal{J}\cdot\mathcal{O}_X(H)\longrightarrow\mathcal{O}_X(H)\longrightarrow\boldsymbol{C}\longrightarrow 0.$$

Therefore we have

$$H^0(X, \mathcal{J}\cdot\mathcal{O}_X(H))\neq H^0(X, \mathcal{O}_X(H))$$

and we obtain the conclusion $x_0\notin Bs|H|$. Q.E.D.

The following theorem is the main result of this paper.

Theorem 4.9. *If a normal surface X is a non-rational numerical Del Pezzo surface whose Picard number is equal to 1 and whose index is a prime number p, then the minimal resolution $\tilde{X}$ of X is a DP1-ruled surface of type $\{e, g, (p_j, q_j, n_j)_{j=1}^r\}$ satisfying the following conditions:*

$$p_j=p \text{ for } j=1, \dots, r \tag{4.9.1}$$

$$e-(2g-2)>\sum_{j=1}^r\frac{q_j+n_j-1}{p_j} \tag{4.9.2}$$

Furthermore X is the minimal contraciton of $\tilde{X}$.

Conversely if $\tilde{X}$ is a DP1-ruled surface of type $\{e, g, (p_j, q_j, n_j)_{j=1}^r\}$

satisfying the conditions (4.9.1) *and* (4.9.2), *then there exists the minimal contraction X of $\tilde{X}$ and X is a non-rational numerical Del Pezzo surface whose Picard number is equal to 1 and whose index is equal to p.*

Proof. Let X be a non-rational numerical Del Pezzo surface whose Picard number is equal to 1 and whose index is a prime number p. Let $f: \tilde{X} \longrightarrow X$ be the minimal resolution of X. Then $\tilde{X}$ is a non-rational ruled surface and $f: \tilde{X} \longrightarrow X$ is the minimal contraction of $\tilde{X}$ by Theorem 2.1. We denote the projection from $\tilde{X}$ to the base curve C by φ. By Corollary 5.7 the singular fibers F_j of $\tilde{X}$ is of the form $\Gamma_j+E_j+\Gamma_j'$ for every j, where Γ_j is a simple tree without branching vertex, E_j is a irreducible curve and Γ_j' is some weighted graph. Then there is a birational morphism $\tau: \tilde{X} \longrightarrow Z$ from $\tilde{X}$ to a non-rational ruled surface Z, such that the singular fibers of Z are of the form Γ_j+E_j $(j=1, \ldots, r)$. On the other hand the coefficients of E_j in $\tilde{F}_j$ divide the given prime number p by Lemma 4.7, and it is not equal to 1 for every j. Then it is equal to p for every j.

Take a generator H of $PicX$ as in the conclusion of Lemma 4.8, that is, H is effective and $SuppH$ dose not contain the non-rational singular point x_0 of X. By $H'=\sum_{i=1}^{t} a_i H_i'$ we denote the proper transform of H by $f: \tilde{X} \longrightarrow X$ where a_i's are positive integers, and by D the connected components of $C_0+\sum_{E\in\mathscr{E}} E$ containing C_0. Then every H_i' does not intersect D, and every H_i' is not contained in a fiber of φ because every singular fiber has only one irreducible component with self-intersection number -1. Then $\tau_* H_i' \neq 0$ for every i. Furthermore $H'\cdot F=f^*H\cdot F =H\cdot f_*F=p$, where F is the general fiber. For $Supp(f^*H-H')$ dose not contain C_0 because H' does not intersect D, and then $(f^*H-H')\cdot F=0$.

We chose an arbitrary singular fiber, say, F_1. Then there are two cases.

If there exists some H_i such that $\tau_* H_i \cdot \tau_* E_1 \geq 1$. Then we have inequalities

$$\tau_* H_i \cdot \tau_* \tilde{F}_1 \geq p = H'\cdot F = \tau_* H' \cdot \tau_* F \geq \tau_* H_i \cdot \tau_* \tilde{F}_1$$

and then $\tau_* H_i \cdot \tau_* E_1=1$, $a_i=1$ and $a_j=0$ for every j different from i, that is, H' is irreducible and reduced and $\tau_* H' \cdot \tau_* E_1=1$. Because $\tilde{X}$ is obtained from Z by blowing-ups and H' dose not intersect D, the singular fiber F_1 turns out to be a simple tree.

If there are no H_i such that $\tau_* H_i \cdot \tau_* E_1 \neq 0$, then by the fact that H' dose not intersect D we can easily see that F_1 is a simple tree without branching vertex.

Thus F_1 is a simple tree for both two cases. Similarly the other singular fibers are simple trees. Then we have known that $\tilde{X}$ is a DP1-ruled surface.

Let the type of $\tilde{X}$ be $\{e, g, (p_j, q_j, n_j)_{j=1}^{r}\}$. As in the proof of Lemma 4.7, p_j coincides with the coefficient of E_j in the total fiver F_j. Then we have already shown in the above argument that p_j is equal to p. Furthermore the minimal contraction X of $\tilde{X}$ is a non-rational numerical Del Pezzo surface, then the inequality (4.9.2) holds by Theorem 3.12.

Conversely let $\tilde{X}$ be a DP1-ruled surface of type $\{e, g, (p_j, q_j, n_j)_{j=1}^{r}\}$ satisfying the conditions (4.9.1) and (4.9.2). By Theorem 3.10 there exists the minimal contraction X of $\tilde{X}$ and X is a non-rational numerical Del Pezzo surface such that $\rho(X)=1$. Therefore we only have to show that the index of X is equal to p.

Take an ample Cartier divisor H on X which generates $PicX$. We take a basis of $(PicX \otimes \boldsymbol{Q})/\varphi^*(PicC)$ as in the proof of Lemma 4.7. Then it can be seen by Lemma A.7 that $f^*H=a\tilde{H}$ modulo $\varphi^*(PicC)$ for some positive integer a and for some integral divisor $\tilde{H}$, and that $H \cdot f_*F = f^*H \cdot F = ap$ by the conditions (4. 7.1) and (4.7.2).

Because $a\tilde{H} \cdot E = f^*H \cdot E = 0$ for every element of $\mathcal{E}$, $D=\tilde{H}-\varphi^*(\tilde{H}|_{C_0})$ satisfies the conditions (4.2.1) and (4.2.2), where we identify the base curve C of $\tilde{X}$ and C_0. Hence D is contained in $f^*(PicX)$ by Lemma 4.2, and then $D=bf^*H$ for some integer b. Then $f^*H \cdot F = a\tilde{H} \cdot F = aD \cdot F = abf^*H \cdot F$. Therefore $a=b=1$ because a is positive. And then $H \cdot f_*F = ap = p$. Q.E.D.

Appendix. Weighted graphs

In this appendix we will prepare several facts on weighted graphs. As for the terminologies on weighted graphs, the reader may consult T. Fujita [7] and P. Orlik-P. Wagreich [12].

DEFINITION A.1. *We will call a 1-dimensional (not necessarily connected) simplicial complex with finite vertices a graph. A weighted graph means a graph to each vertex of which is assigned an integer called the weight.*

For the sake of convenience we recall several definitions in T. Fujita [7].

DEFINITION A.2. (T. FUJITA [7] (3.3)) *Let Γ be a graph. By $\boldsymbol{Q}(\Gamma)$ we denote the $\boldsymbol{Q}$-vector space of formal linear conbinations of vertices of Γ with coefficients being rational numbers. If in addition Γ is a weighted graph then we define a pairing I on $\boldsymbol{Q}(\Gamma)$ as follows. Let v and w be distinct vertices, $I(v, w)$ is equal to the number of the segments joining v and w by difinition. For a vertex v, $I(v, v)$ is equal to the weight of v by definition. We denote $I(v, w)$(resp. $I(v, v)$) by $v \cdot w$ (resp. v^2) for stort. $d(\Gamma)$ denotes the determinant of the matrix with entries $-I(v, w)$.*

DEFINITION A.3. (LOC. CIT., (3.2)) *A twig is a connected linear graph together with a total ordening $v_1 > \cdots > v_r$ among its vertices such that v_j and v_{j-1} are connected by a segment for each j. Such a twig is denoted by $[-w_1, \ldots, -w_r]$, where w_j is the weight of v_j. A twig is said to be admissible if $-w_j \geq 2$ for every j.*

DEFINITION A.4. (LOC. CIT., (3.5)) *Let A be a twig $[a_1, \ldots, a_r]$. The twig*

$[a_r, \ldots, a_1]$ *is called the trasposal of* A *and denoted by* ${}^tA$ *We define also* $\bar{A} = [a_2, \ldots, a_r]$ *and* $\underline{A} = {}^t(\overline{{}^tA}) = [a_1, \ldots, a_{r-1}]$. $e(A) = d(\bar{A})/d(A)$ *is called the inductance of* A.

Proposition A.5. (LOC. CIT., (3.8) Corollary) *e defines a one-to-one correspondence from the set of all the admissible twigs to the set of rational numbers in the interval* (0,1).

DEFINITION A.6. (LOC. CIT., (3.9)) *Let A be an admissible twig. The unique admissible twig whose inductance is equal to $1-e({}^tA)$ is called the adjoint of A and denoted by A^*. So $e({}^tA)+e(A^*)=1$.*

Lemma A.7. *Let $A=[a_1, \ldots, a_r]$ be an admissible twig, and $(x_j)_{j=0}^{r+1}$ and $(y_j)_{j=0}^{r+1}$ be two sequences of real numbers satisfying the following conditions*:

(A.7.1) $x_0=0,\ x_1=1$

(A.7.2) $x_{j+1}-a_jx_j+x_{j-1}=0$ *for* $1\leq j\leq r$.

(A.7.3) $y_{j+1}-a_jy_j+y_{j-1}=0$ *for* $1\leq j\leq r$.

Then all x_j's are integers and we have the followings:

(A.7.4) $x_j>0$ *for* $j\geq 1$

(A.7.5) $x_{j+1}>x_j$ *for* $0\leq i\leq r$

(A.7.6) $x_r=d(\underline{A}),\ x_{r+1}=d(A)$

(A.7.7) $x_{j+1}y_j-x_jy_{j+1}=y_0$ *for* $0\leq j\leq r$

(A.7.8) $d(A)y_r-d(\underline{A})y_{r+1}=y_0$.

Furthermore if $y_0=0$, then we have $y_j=x_jy_1$.

Proof. The proof is easy by induction.

Corollary A.8. *Let $A=[a_1, \ldots, a_r]$ be an admissible twig, and $(y_j)_{j=0}^{r+1}$ be a sequence of real numbers satisfying the equation* (A.7.3). *Then the following equality holds*:

$$d(A)y_1-d(\bar{A})y_0=y_{r+1}.$$

Proof. Applying (A.7.8) to ${}^tA$ and $(y_j)_{j=r+1}^{0}$, we obtain the conclusion because $d(A)=d({}^tA)$ and $d(\bar{A})=d({}^t\underline{A})$.

Lemma A.9. *Let $A=[a_1, \ldots, a_r]$ be an admissible twig, and $(x_j)_{j=0}^{r+1}$ and $(z_j)_{j=0}^{r+1}$ be two sequences of real numbers such that $(x_j)_{j=0}^{r+1}$ satifies the conditions* (A.7.1) *and* (A.7.2) *and that $(z_j)_{j=0}^{r+1}$ satisfies the following condition*:

(A.9.1) $z_{j+1}-a_jz_j+z_{j-1}=2-a_j$ *for* $1\leq j\leq r$.

Then we have

(A.9.2) $$x_{j+1}z_j - x_j z_{j+1} = z_0 - 1 + x_{j+1} - x_j$$

(A.9.3) $$d(A)z_r - d(\underline{A})z_{r+1} = z_0 - 1 + d(A) - d(\underline{A}).$$

Moreover if $z_0 \geq 0$ *and* $z_{r+1} \geq 0$ (*resp.* >0), *then we have* $z_j \geq 0$ (*resp.* >0) *for* $1 \leq j \leq r$.

Proof. Putting $y_j = z_j - 1$, $(x_j)_{j=0}^{r+1}$ and $(y_j)_{j=0}^{r+1}$ satisfy the assumption of Lemma A.7. Then we obtain (A.9.2) and (A.9.3) from (A.7.7) and (A.7.8). Now that we have (A.9.2), then

$$x_{j+1}z_j - x_j z_{j+1} \geq 0 \text{ for } 0 \leq j \leq r$$

because x_j's are positive integers and $x_{j+1} > x_j$ for $j = 1, \dots, r$. Then we can show the result inductively. Q.E.D.

Corollary A.10. *Let* $A = [a_1, \dots, a_r]$ *be an admissible twig. We assume that a sequence of real numbers* $(z_j)_{j=0}^{r+1}$ *satisfies the condition* (A.9.1). *Then*

$$d(A)z_1 - d(\bar{A})z_0 = z_{r+1} - 1 + d(A) - d(\bar{A}).$$

Proof. It is as same as Corollary A.8. Q.E.D.

We have already defined simple trees in section 3.

Lemma A.11. *Let* $A = [a_1, \dots, a_r]$ *be an admissible twig,* n *a non negative integer, and* Γ *a simple tree of type* (A, n). *Then for any rational number* α, *there is a unique element* $\Delta_{\Gamma,\alpha}$ *of* $\boldsymbol{Q}(\Gamma)$ *which satisfies the following two conditions* :

(A.11.1) $\Delta_{\Gamma,\alpha} \cdot v = 2 + v^2$ *for every vertex* v *of* Γ *different from the tip*

(A.11.2) $\Delta_{\Gamma,\alpha} \cdot v + \alpha = 2 + v^2$ *for the tip* v *of* Γ.

Moreover if $\alpha \geq 0$, *then* $\Delta_{\Gamma,\alpha}$ *is effective.*

Proof. We put $A^* = [b_1, \dots, b_s]$. Then

(A.11.3) $$d(A) = d(A^*) = d(\underline{A}) + d(\overline{A^*}) = d(\bar{A}) + d(\underline{A^*})$$

by Fujita [7], (3.9) and Corollary (3,7).

Let $u_1, \dots, u_r$ and $v_1, \dots, v_s$ be the vertices corresponding to $a_1, \dots, a_r$ and $b_1, \dots, b_s$ respectively. And let $w_1, \dots, w_n$ be the other vertices with the ordering $w_n < w_{n-1} < \cdots < w_1$ such that w_n is the branching vertex of Γ.

We define three sequences $(\alpha_j)_{j=0}^{r+1}$, $(\beta_j)_{j=0}^{r+1}$and $(\gamma_j)_{j=0}^{r+1}$ as follows :

$$\gamma_0=0, \gamma_1=1+\frac{a-2}{d(A)}$$
$$\gamma_j=j\gamma_1 \text{ for } j=2, \ldots, n$$
$$\alpha_0=\gamma_n=n\gamma_1$$
$$\alpha_1=\frac{d(\bar{A})}{d(A)}n\gamma_1+\gamma_1+\frac{1}{d(A)}-\frac{d(\bar{A})}{d(A)}$$
$$\alpha_{j+1}=a_j\alpha_j-\alpha_{j-1}+2-a_j \text{ for } j=1, \ldots, r$$
$$\beta_0=0$$
$$\beta_1=\frac{1}{d(A)}n\gamma_1-\frac{1}{d(A)}+\frac{d(A)}{d(A)}$$
$$\beta_{j+1}=b_j\beta_j-\beta_{j-1}+2-b_j \text{ for } j=1, \ldots, r.$$

Then it can be easily seen that all α_j, $\beta_j$and γ_j's are rational numbers. Moreover we have

$$\alpha_{r+1}=a$$
$$\beta_s=\left(1-\frac{d(\bar{A})}{d(A)}\right)n\gamma_1-\frac{1}{d(A)}+\frac{d(\bar{A})}{d(A)}$$
$$\beta_{s+1}=\gamma_n=n\gamma_1$$

by using Lemma A.9, Corollary A.10 and (A.11.3) to $(\alpha_j)_{j=0}^{r+1}$ and $(\beta_j)_{j=0}^{r+1}$.

Putting

$$\Delta_{\Gamma,a}=\sum_{j=1}^{r}\alpha_j u_j+\sum_{j=1}^{s}\beta_j v_j+\sum_{j=1}^{n}\gamma_j w_j$$

we can easily show the equalities (A.11.1) and (A.11.2).

If a is non negative, then γ_1 is also non negative because $d(A)\geq 2$. Therefore $\gamma_j\geq 0$ for $j=2, \ldots, n$ and $\alpha_0=\alpha_{s+}=\gamma_n\geq 0$. By Lemma A.9 we have $\alpha_j\geq 0$ for $j=1, \ldots, r$ and $\beta_j\geq 0$ for $j=1, \ldots, s$.

From the above argument there is an element $\Delta_{\Gamma,a}$ of $\boldsymbol{Q}(\Gamma)$ satisfying the conditions (A.11.1) and (A.11.2) where a is replaced by 1. We can easily check the following two inequalities:

$$\Delta_{\Gamma,1}\cdot v\leq 0 \text{ for every vertex } v \text{ of } \Gamma$$
$$(\Delta_{\Gamma,1})^2<0$$

and then Γ turns out to be contractible, that is, the bilinear form associated to Γ is negative definite (M. Artin [1]). The uniqueness of $\Delta_{\Gamma,a}$ for every rational number follows from the fact that Γ is contractible. Q.E.D.

From the above proof we have shown the following:

Corollary A.12. *Under the situation as in* Lemma A.11 *and its* proof, *we have*

$$\alpha_r=\frac{1}{d(A)}\left(d(\underline{A})+\frac{n}{d(A)}\right)\alpha+\frac{n}{d(A)}\left(1-\frac{2}{d(A)}\right)-\frac{1}{d(A)}+\left(1-\frac{d(A)}{d(A)}\right).$$

Proof. Applying Lemma A.7 to A and $(\alpha_j)_{j=0}^{r+1}$ we have the conclusion by the fact that $\alpha_{r+1}=\alpha$. Q.E.D.

DEFINITION A.13. Let $(p_j, q_j, n_j)(j=1, \dots, r)$ be sets of three integers such that p_j and q_j are positive integers coprime to each other and n_j is non negative for every j. Let $A_j=\Gamma(q_j/p_j)$ be the admissible twigs whose indectance is equal to q_j/p_j, and Γ_j simple trees of type(A_j, n_j).

A weighted graph Γ is called a DP1-graph of type $\{e, g, (p_j, q_j, n_j)_{j=1}^r\}$ when Γ consists of one distinguished vertex u, called the center of Γ, and of the Γ_j's, where u is joined only to the tip of Γ_j for every j and the weight of u is equal to $e+r$, that is,$u^2=-e-r$ (figure A.13).

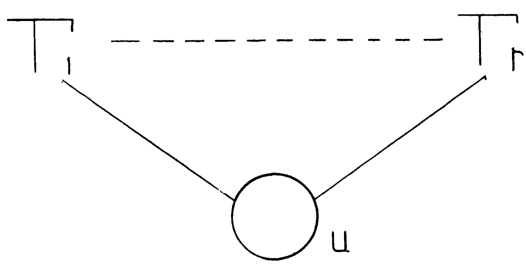

figure (A.13)

Proposition A.14. *Let Γ be a DP1-graph of type $\{e, g, (p_j, q_j, n_j)_{j=1}^r\}$, and g be a positive integer. Then the following two conditions are equivalent*:

(A.14.1) *There is an effective element Δ_Γ of $\boldsymbol{Q}(\Gamma)$ satisfying the following three conditions*:

(1) *The coefficient α of u in Δ_Γ satisfies* $0<\alpha<2$.

(2) $\Delta_\Gamma\cdot v=2+v^2$ *for every vertices of Γ different from u.*

(3) $\Delta_\Gamma\cdot u=-e-r-(2g-2)$.

(A.14.2) *The following inequality holds*:

$$e-(2g-2)>\sum_{j=1}^{r}\frac{q_j+n_j-1}{p_j}.$$

Proof. At first we will show that (A.14.2) follows from (A.14.1). By the definition of Δ_Γ it can be easily seen that $\Delta_\Gamma|_{\Gamma_j}$, the image of Δ_Γ by the projection to the direct summand $\boldsymbol{Q}(\Gamma_j)$ of $\boldsymbol{Q}(\Gamma)$, satisfies the conditions (A.11.1) and (A.11.2) for every j. Then we have

$$a^{(j)}=\frac{1}{p_j}\left(q_j+\frac{n_j}{p_j}\right)a+\frac{n_j}{p_j}\left(1-\frac{2}{p_j}\right)-\frac{1}{p_j}+\left(1-\frac{q_j}{p_j}\right) \text{ for every } j$$

where $a^{(j)}$'s are coefficients of the tips of Γ_j in Δ_Γ. By this equality and the condition (3) in (A.14.1) we have the following equality:

(A.14.3) $$\left\{e+r-\sum_{j=1}^{r}\frac{1}{p_j}\left(q_j+\frac{n_j}{p_j}\right)\right\}a=e+r+2g-2+\sum_{j=1}^{r}\left\{\frac{n_j}{p_j}\left(1-\frac{2}{p_j}\right)-\frac{1}{p_j}+1-\frac{q_j}{p_j}\right\}.$$

Because $p_j\geq 2$, $p_j>q_j$ from (A.10.4) and $n_j\geq 0$, the right hand side of the equality (A.14.3) turns out to be positive. Then we have $e+r-\sum_{j=1}^{r}\frac{1}{p_j}\left(q_j+\frac{n_j}{p_j}\right)>0$ because $a>0$. Therefore by the condition (1) in (A.14.1) we obtain the following inequality:

$$e+r+2g-2+\sum_{j=1}^{r}\left\{\frac{n_j}{p_j}\left(1-\frac{2}{p_j}\right)-\frac{1}{p_j}+1-\frac{q_j}{p_j}\right\}<2\left\{e+r-\sum_{j=1}^{r}\frac{1}{p_j}\left(q_j+\frac{n_j}{p_j}\right)\right\}.$$

And then we obtain the inequality (A.14.2) from the above inequality.

Conversely we will assume the condition (A.14.2).

By the assumption we can easily check $e+r-\sum_{j=1}^{r}\frac{1}{p_j}\left(q_j+\frac{n_j}{p_j}\right)>0$ therefore we can define a rational number a by the equality (A.14.3). Then we have $0<a<2$ by similar calculations as before.

By Lemma A.11 there is unique element $\Delta_{\Gamma_j,a}$ of $\boldsymbol{Q}(\Gamma_j)$ satisfying the conditions (A.11.1) and (A.11.2) for every j. We define Δ_Γ by $\Delta_\Gamma=au+\sum_{j=1}^{r}\Delta_{\Gamma_j,a}$. Using Corollary A.12, by similar calculations as before, we can show that Δ_Γ satisfies the conditions in (A.14.1). Q.E.D.

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