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IRREDUCIBLE REPRESENTATIONS OF THE PARTY ALGEBRA

Dedicated to Professor Noriaki Kawanaka on his sixtieth birthday

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Abstract

In this paper we construct a complete set of representatives of the irreducible representations of the party algebra. which is the centralizer of the unitary reflection group $G(r, 1, k)$ in the endomorphism ring of the tensor space $V^{\otimes n}$ under the condition that $k \geq n$ and $r > n$.

1. Introduction

Let G be a group of linear transformations on a k-dimensional vector space V. Suppose that G diagonally acts on the n-times tensor space $V^{\otimes n}$. Then the question how the tensor space $V^{\otimes n}$ decomposes into irreducible representations of G is a basic problem of the classical invariant theory. One way of studying this problem is to consider the centralizer algebra $\mathrm{End}_G\left(V^{\otimes n}\right)$. This approach was successfully done in cases $G = GL_k(\mathbb{C})$ and $O_k(\mathbb{C})$. These classical groups produced the centralizers $\mathbb{C}\mathfrak{S}_n$ and $B_n(k)$ (Brauer algebra [2, 13]) respectively, and the decompositions of the tensor representations of the original groups were obtained as well as the decompositions of their centralizers. In the 1980s, the q-deformation of these centralizers were discovered and the various connections between the centralizers and other areas (such as knot theory, conformal field theory, etc.) were clarified [10, 14].

In the early 1990s, Jones and Martin independently defined the partition algebra $P_n(Q)$ as the generalization of the Temperley-Lieb algebra and the Potts model in statistical mechanics. This algebra corresponds to the case $G = \mathfrak{S}_k$ in the classical invariant theory above; if the parameter Q of $P_n(Q)$ is specialized to a positive integer k, the partition algebra $P_n(k)$ surjectively mapped to the centralizer $\mathrm{End}_G\left(V^{\otimes n}\right)$ where $G = \mathfrak{S}_k$, and if further k is large enough, ($2k \geq n$ is sufficient), this map becomes injective [5]. In the paper [5, 9], they considered $\prod_{2n}$, the set of all the set partitions of $\{d_1, \dots, d_n, r_1, \dots, r_n\}$, as a basis of $P_n(Q)$ and defined the product among each element of $\prod_{2n}$. Further, they showed that $P_n(Q)$ is generated by the symmetric

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group $\mathfrak{S}_n$—acting on $V^{\otimes n}$ by tensor factors permutation—and two special elements A_1 and A_{12}.

Inspired by the work of Jones, Tanabe considered the case G is a unitary reflection group of type $G(r, p, k)$ where $G(r, p, k)$ is an index-p subgroup of $G(r, 1, k)$, and $G(r, 1, k)$ is a group of $k \times k$ monomial matrices whose non-zero entries are r-th roots of unity [11]. In the paper [12], Tanabe showed that $\mathrm{End}_{G(r,p,k)}(V^{\otimes n})$ is generated by the symmetric group $\mathfrak{S}_n$ together with three further special operators, E_2, F_1^r and $H_{r,p,k}$. (Note that the unitary reflection groups $\{G(r, p, k)\}$ include the symmetric group $\mathfrak{S}_k = G(1, 1, k)$. The operators E_2 and F_1^1 become A_{12} and A_1 respectively in $P_n(Q)$, and the operator $H_{r,p,k}$ is not defined in case $p = 1$.)

In this paper, we study further about the case $G = G(r, 1, k)$ ($k \geq n$, $r > n$): we construct a complete set of irreducible representations, which corresponds to "Hoefsmit-analogues" of Young's seminormal representations of the symmetric group [4].

This paper is organized as follows. First we define the *party algebra* $\mathcal{A}_n$ as an abstract algebra generated by the symmetric group $\mathfrak{S}_n$ and one of the above operators $f = E_2 = A_{12}$, which will turn out to be a subalgebra of the partition algebra $P_n(Q)$. In fact, a basis of $\mathcal{A}_n$ has one to one correspondence with a subset of $\prod_{2n}$ called the set of *seat-plan*s (see Section 1.1). We showed in the previous paper [6] that any word of $\mathcal{A}_n$ is reduced to one of the *standard* words of the generators under the defining relations and each standard word corresponds to one of the seat-plans (see Definition 1.1). Similarly to the partition algebra, there exists a surjective homomorphism from $\mathcal{A}_n$ to $\mathrm{End}_{G(r,1,k)}(V^{\otimes n})$. Moreover, if $k \geq n$ and $r > n$, this homomorphism becomes injective (Proposition 1.2 and 1.3). Next, we explicitly construct a complete set of representatives of the irreducible representations of the party algebra $\mathcal{A}_n$ drawing the Bratteli diagram of the tower $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n$ and defining the tableaux on it. Finally we check that these representations are irreducible and non-equivalent each other. Comparing the square sum of the degrees of the irreducible representations with the number of the seat plans (standard words of the generators) we find that $\mathcal{A}_n$ is semisimple.

1.1. Definition of the party algebra.

First, we define the party algebra $\mathcal{A}_n$.

DEFINITION 1.1. Let $\mathbb{Z}$ be the ring of rational integers. We put $\mathcal{A}_0 = \mathcal{A}_1 = \mathbb{Z}$. For an integer $n > 1$, the party algebra $\mathcal{A}_n$ is defined over $\mathbb{Z}$ by the following generators:

$$f, s_1, s_2, \ldots, s_{n-1}$$

and relations:

$$s_i^2 = 1 \qquad (1 \leq i \leq n-1), \qquad (P1)$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad (1 \leq i \leq n-2), \qquad (P2)$$

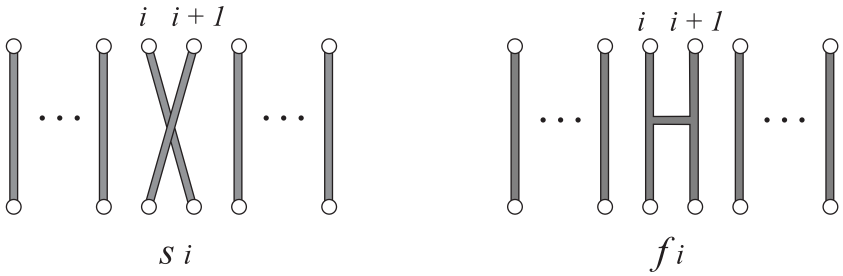

Fig. 1. Generators of $\mathcal{A}_n$

$$s_i s_j = s_j s_i \qquad (|i-j| \geq 2,\ \ 1 \leq i, j \leq n-1), \tag{P3}$$
$$f^2 = f, \tag{P4}$$
$$f s_1 = s_1 f = f, \tag{P5}$$
$$f s_i = s_i f \qquad (3 \leq i \leq n-1), \tag{P6}$$
$$f s_2 f s_2 = s_2 f s_2 f, \tag{P7}$$
$$f s_2 s_1 s_3 s_2 f s_2 s_1 s_3 s_2 = s_2 s_1 s_3 s_2 f s_2 s_1 s_3 s_2 f. \tag{P8}$$

Putting

$$\begin{aligned} f_1 &= f \\ f_i &= (s_{i-1} \cdots s_2 s_1)(s_i \cdots s_3 s_2) f_1 (s_2 s_3 \cdots s_i)(s_1 s_2 \cdots s_{i-1}) \\ &\qquad\qquad (i = 2, 3, \ldots, n-1), \end{aligned}$$

we obtain another presentation of $\mathcal{A}_n$ by f_is and s_is.

$$s_i^2 = 1 \qquad (1 \leq i \leq n-1), \tag{P1$'$}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad (1 \leq i \leq n-2), \tag{P2$'$}$$
$$s_i s_j = s_j s_i \qquad (|i-j| \geq 2,\ \ 1 \leq i, j \leq n-1), \tag{P3$'$}$$
$$f_i^2 = f_i \qquad (1 \leq i \leq n-1), \tag{P4$'$}$$
$$f_i s_i = s_i f_i = f_i \qquad (1 \leq i \leq n-1), \tag{P5$'$}$$
$$f_i f_j = f_j f_i \qquad (1 \leq i, j \leq n-1), \tag{P6$'$}$$
$$s_i f_{i+1} s_i = s_{i+1} f_i s_{i+1} \qquad (1 \leq i \leq n-2), \tag{P7$'$}$$
$$f_i s_j = s_j f_i \qquad (|i-j| \geq 2,\ \ 1 \leq i, j \leq n-1). \tag{P8$'$}$$

For the new generators $\{s_i, f_i \mid 1 \leq i \leq n-1\}$, we give the diagrams figured in Fig. 1. In the following, to each word of the generators of $\mathcal{A}_n$, we give a diagram explanation.

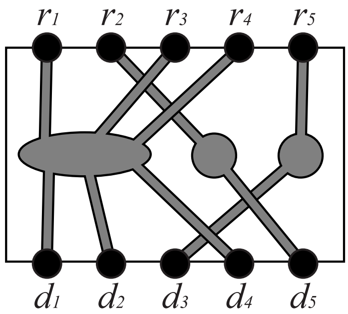

Fig. 2. A seat-plan

Let $D = \{d_1, d_2, \ldots, d_n\}$ and $R = \{r_1, r_2, \ldots, r_n\}$ be two sets, each of which consists of n distinct elements. We further assume that $D \cap R = \emptyset$. We decompose $D \sqcup R$ into subsets $B_1, B_2, \ldots, B_n$ (some of B_js might be empty) so that they satisfy

$$\bigcup_{j=1}^{n} B_j = D \sqcup R,$$
$$B_i \cap B_j = \emptyset \quad \text{if} \quad i \neq j,$$
$$|B_1| \geq |B_2| \geq \cdots \geq |B_n|,$$
$$\left|B_j \cap D\right| = \left|B_j \cap R\right| \quad \text{for} \quad j = 1, 2, \ldots, n.$$

We call such a partition into subsets a *seat-plan* of size n. Let $P(n)$ be the set of partitions of an integer n. Then there exists a partition $\lambda \in P(n)$ such that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) = (|B_1|/2, |B_2|/2, \ldots, |B_n|/2)$. The number of seat-plans is

$$\sum_{\lambda \in P(n)} \left(\frac{n!}{\lambda_1!\, \lambda_2! \cdots \lambda_n!} \right)^2 \cdot \frac{1}{\alpha_1!\, \alpha_2! \cdots \alpha_n!}, \tag{1}$$

where $\alpha_i = |\{\lambda_k\, ;\, \lambda_k = i\}|$.

A seat-plan of size n is illustrated as in Fig. 2. Consider a rectangle with n marked points on the bottom and the same n on the top. The n marked points on the bottom are labeled by $d_1, d_2, \ldots d_n$ from left to right. Similarly, the n marked points on the top is labeled by $r_1, r_2, \ldots, r_n$ from left to right. If $D \sqcup R$ is divided into non-empty m subsets, then put m shaded circles in the middle of the rectangle so that they have no intersections. Each of the circles corresponds to one of the non-empty B_js. Then we join the $2n$ marked points and the m circles with $2n$ shaded bands so that the marked points labeled by the elements of B_j are connected to the corresponding circle with $\left|B_j\right|$ bands. We associate generators $\{s_i, f_i \mid 1 \leq i \leq n-1\}$ of $\mathcal{A}_n$ to the following special seat-plans

$$\{d_1, r_1\}, \ldots, \{d_{i-1}, r_{i-1}\}, \{d_i, r_{i+1}\}, \{d_{i+1}, r_i\}, \{d_{i+2}, r_{i+2}\}, \ldots, \{d_n, r_n\}$$

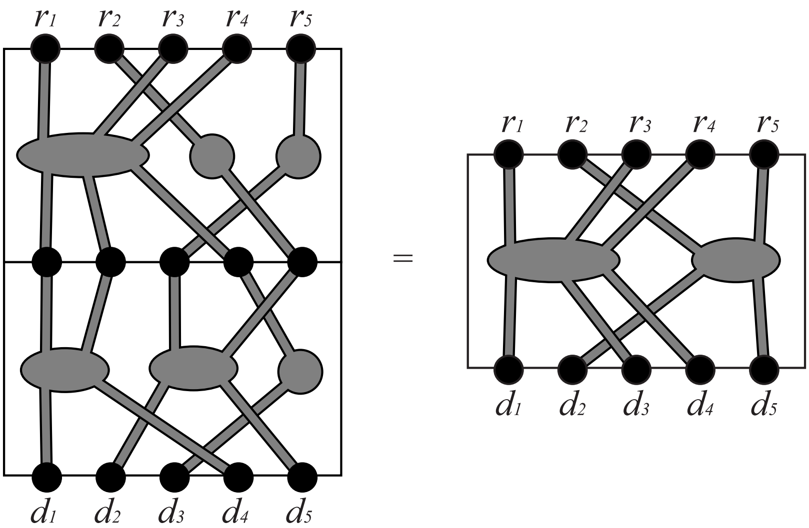

Fig. 3. The product of seat-plans

and

$$\{d_1, r_1\}, \ldots, \{d_{i-1}, r_{i-1}\}, \{d_i, d_{i+1}, r_i, r_{i+1}\}, \{d_{i+2}, r_{i+2}\}, \ldots, \{d_n, r_n\}$$

respectively, which are illustrated in Fig. 1.

Now we define the product $w_1 w_2$ between two of rectangles w_1, w_2 (each of which corresponds to a seat-plan) by placing w_1 on w_2, gluing the corresponding boundaries and shrinking half along the vertical axis as in Fig. 3. We then have a new diagram possibly containing some closed loops. The product is the resulting diagram, with the closed loops removed. It is easy to define this product in terms of seat-plans (see for example Martin's paper [9]). The set of the seat-plans satisfies the relation $(P1')$–$(P8')$. Moreover, in the paper [6], the author showed that there exist one to one correspondences between the set of seat-plans of size n and the set of standard words of $\mathcal{A}_n$ and that using only the relations $(P1')$–$(P8')$ any word of the generators becomes a standard word. This means that the linear combination of seat-plans is a surjective image of $\mathcal{A}_n$ and it makes a finite dimensional algebra whose dimension is given by the expression (1).

The following proposition given by Tanabe [12] shows the relation between the party algebras and the centralizer algebras of the unitary reflection groups.

Proposition 1.2 (Tanabe [12, Theorem 3.1]). *Let $G(r, 1, k)$ be the group of all the monomial matrices of size n whose non-zero entries are r-th roots of unity. Let V be the $\mathbb{C}$-vector space of dimension k with the basis elements $e_1, e_2, \ldots, e_k$ on which $G(r, 1, k)$ acts naturally. Let ϕ be the representation of the symmetric group $\mathfrak{S}_n$ on*

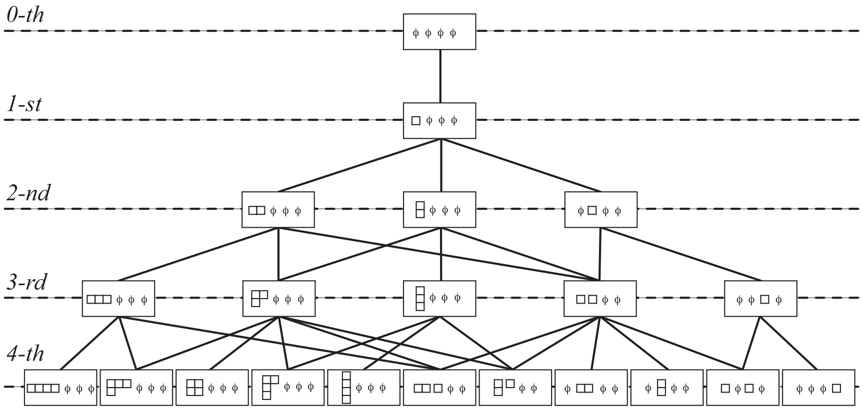

Fig. 4. Γ_4—The Bratteli diagram for the sequence $\{\mathcal{A}_i \otimes \mathbb{C}\}_{i=0}^4$

$V^{\otimes n}$ obtained by permuting the tensor product factors, i.e., for $v_1, v_2, \ldots, v_n \in V$ and for $w \in \mathfrak{S}_n$,

$$\phi(w)\big(v_1 \otimes v_2 \otimes \cdots \otimes v_n\big) := v_{w^{-1}(1)} \otimes v_{w^{-1}(2)} \otimes \cdots \otimes v_{w^{-1}(n)}.$$

Define further $\phi(f)$ as follows:

$$\phi(f)\big(e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_n}\big) := \begin{cases} e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_n} & \textit{if} \quad p_1 = p_2, \\ 0 & \textit{otherwise}. \end{cases}$$

If $r > n$, then $\mathrm{End}_{G(r,1,k)}\big(V^{\otimes n}\big)$ is generated by $\phi(\mathfrak{S}_n)$ and $\phi(f)$ and ϕ defines a homomorphism from $\mathcal{A}_n \otimes \mathbb{C}$ to $\mathrm{End}_{G(r,1,k)}\big(V^{\otimes n}\big)$.

Proposition 1.3. *Let ϕ be the map previously defined. If $k \geq n$, then ϕ is injective.*

Proof. Using Schur-Weyl reciprocity and counting the dimension, the proposition will be easily checked. □

1.2. Bratteli diagram of the party algebras. In this subsection, first we make a diagram Γ_n, which will turn out to be the Bratteli diagram of the sequence $\{\mathcal{A}_i \otimes \mathbb{C}\}_{i=0}^n$. Then we define the sets of the *tableaux* on the diagram. Fig. 4 will help the reader to understand the recipe.

Fix a positive integer n. Let

$$\boldsymbol{\alpha} = [\alpha(1), \ldots, \alpha(n)]$$

be an n-tuple of Young diagrams. The j-th coordinate of the tuple is referred to *the*

j-th board. The *height* $\|\boldsymbol{\alpha}\|$ of $\boldsymbol{\alpha}$ is defined as the weight sum of the sizes of all the $|\alpha(j)|$s. Namely, $\|\boldsymbol{\alpha}\|$ is defined by

$$\|\boldsymbol{\alpha}\| = \sum_{j=1}^{n} j|\alpha(j)|.$$

Let

$$\boldsymbol{\Lambda}_n(i) = \{\boldsymbol{\alpha} = [\alpha(1), \ldots, \alpha(n)] \mid \|\boldsymbol{\alpha}\| = i\}$$

be a set of n-tuples of height i. For $\boldsymbol{\alpha} \in \boldsymbol{\Lambda}_n(i)$, we set $\alpha(0) = n - i$ (the horizontal Young diagram of depth 1 and of width $n - i$) if necessary. Let $\boldsymbol{\alpha} \underset{1}{\prec} \tilde{\boldsymbol{\alpha}}$ or $\tilde{\boldsymbol{\alpha}} \underset{1}{\succ} \boldsymbol{\alpha}$ denote that $\tilde{\boldsymbol{\alpha}}$ is obtained from $\boldsymbol{\alpha}$ by removing one box from the Young diagram on the j-th board and adding the box to the Young diagram on the $(j+1)$-st board for some j $(0 \leq j \leq n-1)$. The diagram Γ_n is defined as the Hasse diagram Γ_n of $\bigsqcup_{i=0,\ldots,n} \boldsymbol{\Lambda}_n(i)$ with respect to the order generated by $\underset{1}{\succ}$s.

Finally we define the sets of the tableaux on Γ_n. For $\boldsymbol{\alpha} \in \boldsymbol{\Lambda}_n(n)$, The set $\mathbb{T}(\boldsymbol{\alpha})$ of *tableaux of shape* $\boldsymbol{\alpha}$ is defined by

$$\mathbb{T}(\boldsymbol{\alpha}) = \left\{ P = \left(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(n)}\right) \,\middle|\, \begin{array}{l} \boldsymbol{\alpha}^{(0)} = [\emptyset, \ldots, \emptyset], \ \ \boldsymbol{\alpha}^{(n)} = \boldsymbol{\alpha}, \\ \boldsymbol{\alpha}^{(i)} \underset{1}{\prec} \boldsymbol{\alpha}^{(i+1)} \quad \text{for} \quad 0 \leq i \leq n-1 \end{array} \right\}.$$

1.3. Construction of the irreducible representation. Now we have defined the sets of tableaux on Γ_n, we define linear transformations of the tableaux. Let $\mathbb{Q}$ be the field of rational numbers and $K_0 = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \ldots, \sqrt{n})$ its extension. In the following, the linear transformations are defined over K_0. They will turn out to be a complete set of representatives of the irreducible representations of $\mathcal{A}_n(K_0) = \mathcal{A}_n \otimes K_0$. Similar methods are used for example in the references [1, 3, 10, 13, 14].

Let $\mathbb{V}(\boldsymbol{\alpha}) = \bigoplus_{P \in \mathbb{T}(\boldsymbol{\alpha})} K_0 v_P$ be a vector space over K_0 with the standard basis $\{v_P | P \in \mathbb{T}(\boldsymbol{\alpha})\}$.

For a generator s_i of $\mathcal{A}_n(K_0)$, we define a linear map $\rho_{\boldsymbol{\alpha}}(s_i)$ on $\mathbb{V}(\boldsymbol{\alpha})$ giving the matrix M_i with respect to the basis $\{v_P \mid P \in \mathbb{T}(\boldsymbol{\alpha})\}$. Namely, for a tableaux $P = (\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(n)})$ of $\mathbb{T}(\boldsymbol{\alpha})$, define $\rho_{\boldsymbol{\alpha}}(s_i)(v_P) = \sum_{Q \in \mathbb{T}(\boldsymbol{\alpha})} (M_i)_{QP} v_Q$. Let $Q = (\boldsymbol{\alpha}'^{(0)}, \boldsymbol{\alpha}'^{(1)}, \ldots, \boldsymbol{\alpha}'^{(n)})$. If there is an $i_0 \in \{1, 2, \ldots, n-1\} \setminus \{i\}$ such that $\boldsymbol{\alpha}^{(i_0)} \neq \boldsymbol{\alpha}'^{(i_0)}$, then we put

$$(M_i)_{QP} = 0.$$

In the following, we consider the case that $\boldsymbol{\alpha}^{(i_0)} = \boldsymbol{\alpha}'^{(i_0)}$ for $i_0 \in \{1, 2, \ldots, n-1\} \setminus \{i\}$.

CASE 1. First, we assume that $\boldsymbol{\alpha}^{(i-1)}$ and $\boldsymbol{\alpha}^{(i+1)}$ of the tableau P coincide with each other except on the j-th and the $(j+1)$-st boards. In this case, $\boldsymbol{\alpha}^{(i)}$ is obtained from $\boldsymbol{\alpha}^{(i-1)}$ by moving a box in the Young diagram on the j-th board to the Young diagram on the $(j+1)$-st board and $\boldsymbol{\alpha}^{(i+1)}$ is obtained from $\boldsymbol{\alpha}^{(i)}$ by moving another box in the Young diagram on the j-th board to the Young diagram on the $(j+1)$-st board. Denote the Young diagram on the j-th board of $\boldsymbol{\alpha}^{(i-1)}$ (resp. $\boldsymbol{\alpha}^{(i)}$, $\boldsymbol{\alpha}^{(i+1)}$) by $\lambda^{(i-1)}$ (resp. $\lambda^{(i)}$, $\lambda^{(i+1)}$) and denote the Young diagram on the $(j+1)$-st board of $\boldsymbol{\alpha}^{(i-1)}$ (resp. $\boldsymbol{\alpha}^{(i)}$, $\boldsymbol{\alpha}^{(i+1)}$) by $\zeta^{(i-1)}$ (resp. $\zeta^{(i)}$, $\zeta^{(i+1)}$). Let $\mu \lhd \lambda$ or $\lambda \rhd \mu$ denote that μ is obtained from λ by removing one box. Recall that if $\nu \lhd \mu \lhd \lambda$ then we can define the *axial distance* $d = d(\nu, \mu, \lambda)$. Namely, if μ differs from ν in its r_0-th row and c_0-th column only, and if λ differs from μ in its r_1-th row and c_1-th column only, then $d = d(\nu, \mu, \lambda)$ is defined by

$$(2) \qquad d = d(\nu, \mu, \lambda) = (c_1 - r_1) - (c_0 - r_0) = \begin{cases} h_\lambda(r_1, c_0) - 1 & \text{if} \quad r_0 \geq r_1, \\ 1 - h_\lambda(r_0, c_1) & \text{if} \quad r_0 < r_1. \end{cases}$$

Here $h_\lambda(i, j)$ is the *hook-length* at (i, j) in λ and for $\lambda = (\lambda_1, \lambda_2, \dots)$ the hook-length $h_\lambda(i, j)$ is defined by

$$h_\lambda(i, j) = \lambda_i - j + |\{\lambda_k\,;\, \lambda_k \geq j\}| - i + 1.$$

Since $\lambda^{(i-1)} \rhd \lambda^{(i)} \rhd \lambda^{(i+1)}$, we can define the axial distance $d_1 = d(\lambda^{(i+1)}, \lambda^{(i)}, \lambda^{(i-1)})$. Similarly, since $\zeta^{(i-1)} \lhd \zeta^{(i)} \lhd \zeta^{(i+1)}$, we can define the axial distance $d_2 = d(\zeta^{(i-1)}, \zeta^{(i)}, \zeta^{(i+1)})$. If $|d_1| \geq 2$ (resp. $|d_2| \geq 2$), then there is a unique Young diagram $\lambda' \neq \lambda^{(i)}$ (resp. $\zeta' \neq \zeta^{(i)}$) which satisfies $\lambda^{(i-1)} \rhd \lambda' \rhd \lambda^{(i+1)}$ (resp. $\zeta^{(i-1)} \lhd \zeta' \lhd \zeta^{(i+1)}$). Let Q_1, Q_2, Q_3 be tableaux of shape $\boldsymbol{\alpha}$ which are obtained from P by replacing $(\lambda^{(i)}, \zeta^{(i)})$ on the j-th and the $(j+1)$-st board of $\boldsymbol{\alpha}^{(i)}$ with $(\lambda^{(i)}, \zeta')$, $(\lambda', \zeta^{(i)})$, (λ', ζ') respectively. For the basis elements given by the above tableaux, we define the linear map $\rho_{\boldsymbol{\alpha}}(s_i)$ by the following matrix:

$$\rho_{\boldsymbol{\alpha}}(s_i)\colon (v_P, v_{Q_1}, v_{Q_2}, v_{Q_3}) \mapsto (v_P, v_{Q_1}, v_{Q_2}, v_{Q_3})M_i,$$

where

$$M_i = \begin{pmatrix} \frac{1}{d_1 d_2} & \frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} & \sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} & \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} \\ \frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\frac{1}{d_1 d_2} & \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} \\ \sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} & \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\frac{1}{d_1 d_2} & -\frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} \\ \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} & -\frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} & \frac{1}{d_1 d_2} \end{pmatrix}.$$

If we put

$$a_d = \frac{1}{d} \quad \text{and} \quad b_d = \sqrt{1 - a_d^2}, \tag{3}$$

then M_i is written as follows:

$$M_i = \begin{pmatrix} a_{d_1} & b_{d_1} \\ b_{d_1} & -a_{d_1} \end{pmatrix} \otimes \begin{pmatrix} a_{d_2} & b_{d_2} \\ b_{d_2} & -a_{d_2} \end{pmatrix}. \tag{4}$$

Even if $|d_1| = 1$ (resp. $|d_2| = 1$), we still adopt the matrix (4) since $b_{d_1} = 0$ (resp. $b_{d_2} = 0$).

CASE 2. Next, we consider the case that $\boldsymbol{\alpha}^{(i+1)}$ is obtained from $\boldsymbol{\alpha}^{(i-1)}$ by removing one box from the Young diagram on the j-th board and adding the box to the Young diagram on the $(j+2)$-nd board. Let α, λ, β be the Young diagrams on the j-th, the $(j+1)$-st and the $(j+2)$-nd boards of $\boldsymbol{\alpha}^{(i-1)}$ respectively and $\alpha^-, \lambda, \beta^+$ the Young diagrams on the corresponding three boards of $\boldsymbol{\alpha}^{(i+1)}$ respectively. Let $\{\lambda^+_{(r)} \mid r = 1, 2, \ldots, b(\lambda)\}$ (resp. $\{\lambda^-_{(r')} \mid r' = 1, 2, \ldots, b'(\lambda)\}$) be the set of all the Young diagrams which satisfy $\lambda^+_{(r)} \rhd \lambda$ (resp. $\lambda^-_{(r')} \lhd \lambda$) and let $P_1, P_2, \ldots, P_{b(\lambda)}$ (resp. $Q_1, Q_2, \ldots, Q_{b'(\lambda)}$) be all the tableaux which are obtained from P by replacing the Young diagrams on the j-th, the $(j+1)$-st and the $(j+2)$-nd board of $\boldsymbol{\alpha}^{(i)}$ with $\alpha^-, \lambda^+_{(r)}, \beta$ (resp. $\alpha, \lambda^-_{(r')}, \beta^+$). For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$\left.\begin{aligned} (M_i)_{P_{r'}, P_r} &= \sqrt{\frac{h(\lambda)^2}{h\left(\lambda^+_{(r')}\right) h\left(\lambda^+_{(r)}\right)}}, \\ (M_i)_{P_r, Q_{r'}} = (M_i)_{Q_{r'}, P_r} &= \sqrt{\frac{h(\lambda)^2}{h\left(\lambda^-_{(r')}\right) h\left(\lambda^+_{(r)}\right)} \frac{1}{d\left(\lambda^-_{(r')}, \lambda, \lambda^+_{(r)}\right)}}, \\ (M_i)_{Q_r, Q_{r'}} &= 0. \end{aligned}\right\} \tag{5}$$

Here $h(\lambda)$ is the product of all the hook-lengths in λ:

$$h(\lambda) = \prod_{(i,j)\in\lambda} h_\lambda(i, j).$$

Putting

$$H\left(\frac{\kappa\lambda}{\mu\nu}\right) = \sqrt{\frac{h(\kappa)h(\lambda)}{h(\mu)h(\nu)}}, \tag{6}$$

and combining the expression (2) and (3), we can write M_i as follows:

$$(5') \qquad \left. \begin{aligned} (M_i)_{P_{r'},P_r} &= H\left(\frac{\lambda^2}{\lambda^+_{(r')}\lambda^+_{(r)}}\right), \\ (M_i)_{Q_{r'},P_r} = (M_i)_{P_r,Q_{r'}} &= H\left(\frac{\lambda^2}{\lambda^-_{(r')}\lambda^+_{(r)}}\right) a_{d\left(\lambda^-_{(r')},\lambda,\lambda^+_{(r)}\right)}, \\ (M_i)_{Q_r,Q_{r'}} &= 0. \end{aligned} \right\}$$

CASE 3. Finally, we consider the remaining cases. In these cases, if $\boldsymbol{\alpha}^{(i)}$ is obtained from $\boldsymbol{\alpha}^{(i-1)}$ by moving one box to the next board and if $\boldsymbol{\alpha}^{(i+1)}$ is obtained from $\boldsymbol{\alpha}^{(i)}$ by moving another box to the next board in a tableau P, then exchanging the i-th step and the $(i+1)$-st step, we have another tableau Q. For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$(7) \qquad (v_P, v_Q) \mapsto (v_P, v_Q)M_i = (v_P, v_Q)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

1.4. Main theorem. Now we have completed the preparation, we state the following main result.

Theorem 1.4. *Let* $\boldsymbol{\alpha} = [\alpha(1), \ldots, \alpha(n)]$ *be an* n*-tuple of Young diagrams in* $\boldsymbol{\Lambda}_n(n)$. *Let* $\mathbb{Q}$ *be the field of rational numbers and* $K_0 = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \ldots, \sqrt{n})$ *its extension*.

1. *Define* $\rho_{\boldsymbol{\alpha}}$ *as follows*:

$$\rho_{\boldsymbol{\alpha}}(s_i)v_P = \sum_{Q\in\mathbb{T}(\boldsymbol{\alpha})} (M_i)_{QP}v_Q,$$

$$\rho_{\boldsymbol{\alpha}}(f)v_P = \begin{cases} v_P & \textit{if} \quad \boldsymbol{\alpha}^{(2)} = [\emptyset, \square, \emptyset, \ldots, \emptyset] \\ 0 & \textit{otherwise}. \end{cases}$$

Then $(\rho_{\boldsymbol{\alpha}}, \mathbb{V}(\boldsymbol{\alpha}))$ *defines an absolutely irreducible representation of* $\mathcal{A}_n(K_0)$.

2. *For* $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \boldsymbol{\Lambda}_n(n)$, *the irreducible representations* $\rho_{\boldsymbol{\alpha}}$ *and* $\rho_{\boldsymbol{\alpha}'}$ *of* $\mathcal{A}_n(K_0)$ *are equivalent if and only if* $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$.

3. *Conversely, for any irreducible representation* ρ *of* $\mathcal{A}_n(K_0)$, *there exists an* $\boldsymbol{\alpha} \in \boldsymbol{\Lambda}_n(n)$ *such that* ρ *and* $\rho_{\boldsymbol{\alpha}}$ *are equivalent*.

In other words, $\{\rho_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \boldsymbol{\Lambda}_n(n)\}$ *make a complete set of the representatives of the irreducible representations of* $\mathcal{A}_n(K_0)$.

Corollary 1.5. *The party algebras* $\{\mathcal{A}_n(K_0)\}$ *are absolutely semisimple, and the Bratteli diagram of the sequence* $\{\mathcal{A}_i(K_0)\}_{i=0,1,\ldots,n}$ *is given by the graph* Γ_n.

Let $\boldsymbol{\alpha}'$ be an r-tuple of Young diagrams such that $\boldsymbol{\alpha}' \underset{1}{\prec} \boldsymbol{\alpha}$ and

$$\mathbb{T}(\boldsymbol{\alpha}';\boldsymbol{\alpha}) = \left\{ \left(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\alpha}^{(n-1)}\boldsymbol{\alpha}^{(n)}\right) \in \mathbb{T}(\boldsymbol{\alpha}) \;\middle|\; \boldsymbol{\alpha}^{(n-1)} = \boldsymbol{\alpha}' \right\}.$$

Let $\mathbb{V}(\boldsymbol{\alpha}';\boldsymbol{\alpha})$ be a subspace of $\mathbb{V}(\boldsymbol{\alpha})$ spanned by $\{v_Q \mid Q \in \mathbb{T}(\boldsymbol{\alpha}';\boldsymbol{\alpha})\}$. Since $\mathcal{A}_{n-1}(K_0)$ is isomorphic to the subalgebra $\mathcal{A}' = \langle f, s_1, \dots, s_{n-2}\rangle$ of $\mathcal{A}_n(K_0)$, considering the definition of $\rho_{\boldsymbol{\alpha}}$ we find that the subspace $\mathbb{V}(\boldsymbol{\alpha}';\boldsymbol{\alpha})$ is stable under the action of $\mathcal{A}'$. Further, applying the theorem above replacing n with $n-1$, we find that $\mathbb{V}(\boldsymbol{\alpha}';\boldsymbol{\alpha})$ affords an irreducible representation of $\mathcal{A}_{n-1}(K_0)$. In the proof of the theorem above, we will further obtain the following restriction rule.

Corollary 1.6. *For $\boldsymbol{\alpha} \in \boldsymbol{\Lambda}_n(n)$, the branching rule of the restriction of irreducible representation of $\mathcal{A}_n(K_0)$ to the subalgebra $\mathcal{A}' = \mathcal{A}_n(K_0)$ is given as follows*:

$$\mathbb{V}(\boldsymbol{\alpha}) = \bigoplus_{\boldsymbol{\alpha}'_p \underset{1}{\prec} \boldsymbol{\alpha}} \mathbb{V}(\boldsymbol{\alpha}'_p;\boldsymbol{\alpha}) \quad \textit{as} \quad \mathcal{A}_{n-1}(K_0)\textit{-modules}.$$

2. Preliminary results for the axial distances and the hook-lengths

To prove the main theorem, our main task is to show the well-definedness of the representations $\{\rho_{\boldsymbol{\alpha}}\}$. Since $\rho_{\boldsymbol{\alpha}}(s_i)$ is defined by the matrix M_i in the theorem and the entries of M_i are written in terms of a_d, b_d and $H(\kappa\lambda/(\mu\nu))$, the task will be done by showing the various relations among them. In this section, we show miscellaneous relations among $a_d, b_d, H(\kappa\lambda/(\mu\nu))$ defined by the expressions (2), (3) and (6).

First we note that by the definition of a_d and b_d we immediately have $a_d^2 + b_d^2 = 1$. Using this we obtain the following relations among $\{a_d\}$ and $\{b_d\}$ by direct calculation:

Lemma 2.1. *Let d_0, d_1, d_2 be non-zero integers such that $d_0 = d_1 + d_2$. Then we have the following.*

1. $-a_{d_1}a_{d_2} + a_{d_0}a_{d_1} + a_{d_2}a_{d_0} = 0,$
2. $a_{d_1}a_{d_2}^2 + a_{d_0}b_{d_2}^2 = a_{d_2}a_{d_1}^2 + a_{d_0}b_{d_1}^2.$

Let $\{\alpha^+_{(r)}\}_{r=1,\dots,b(\alpha)}$ (resp. $\{\alpha^-_{(r')}\}_{r'=1,\dots,b'(\alpha)}$) be the set of all the Young diagrams which satisfy $\alpha^+_{(r)} \rhd \alpha$ (resp. $\alpha^-_{(r')} \lhd \alpha$). If λ and μ are a pair of Young diagrams such that $\lambda \rhd \mu$, then we have $\{\mu^+_{(r)}\} \ni \lambda$ and $\{\lambda^-_{(r')}\} \ni \mu$.

In the following, we assume that $\mu^+_{(1)} = \lambda$ and $\lambda^-_{(1)} = \mu$.

For $s = 2, \dots, b(\mu)$, we put $\kappa_{(s)} = \mu^+_{(1)} \cup \mu^+_{(s)} = \lambda \cup \mu^+_{(s)}$. Further, let $\lambda_{(r)} = \lambda^+_{(r)} \backslash (\lambda \backslash \mu)$. If $\lambda_{(r)}$ is a Young diagram, then there exists an index s such that $\lambda_{(r)} = \mu^+_{(s)}$. More precisely, we have the following:

Lemma 2.2. *Let* $\{e_r\}$ *be axial distances defined by* $e_r = d\big(\mu, \lambda, \lambda^+_{(r)}\big)$. *Then we have*

$$\big\{\big(\lambda^+_{(r)}, \lambda_{(r)}\big) \mid b_{e_r} \neq 0,\ \ r = 1, \ldots, b(\lambda)\big\} = \big\{\big(\kappa_{(s)}, \mu^+_{(s)}\big) \mid s = 2, \ldots, b(\mu)\big\}.$$

In other words, there exists a bijection τ *from the set* $\{s \mid s = 2, \ldots, b(\mu)\}$ *to the set* $\big\{r \mid r = 1, \ldots, b(\lambda),\ \ b_{e_r} \neq 0\big\}$ *such that* $\big(\lambda^+_{(\tau(s))}, \lambda_{(\tau(s))}\big) = \big(\kappa_{(s)}, \mu^+_{(s)}\big)$.

Similarly, for $r' = 2, \ldots, b'(\lambda)$, we put $\nu_{(r')} = \lambda^-_{(1)} \cap \lambda^-_{(r')} = \mu \cap \lambda^-_{(r')}$ and $\mu_{(s')} = \mu^-_{(s')} \cup (\lambda \setminus \mu)$. Then we have the following:

Lemma 2.3. *Let* $\{d_{s'}\}$ *be axial distances defined by* $d_{s'} = d\big(\mu^-_{(s')}, \mu, \lambda\big)$. *Then we have*

$$\Big\{\Big(\mu^-_{(s')}, \mu_{(s')}\Big) \ \Big|\ b_{d_{s'}} \neq 0,\ \ s' = 1, \ldots, b'(\mu)\Big\} = \Big\{\Big(\nu_{(r')}, \lambda^-_{(r')}\Big) \ \Big|\ r' = 2, \ldots, b'(\lambda)\Big\}.$$

In other words, there exists a bijection σ *from the set* $\{r' \mid r' = 2, \ldots, b'(\lambda)\}$ *to the set* $\{s' \mid s' = 1, \ldots, b'(\mu),\ \ b_{d_{s'}} \neq 0\}$ *such that* $\big(\mu^-_{(\sigma(r'))}, \mu_{(\sigma(r'))}\big) = \big(\nu_{(r')}, \lambda^-_{(r')}\big)$.

We have also the following relations among $\{b_d\}$ and $\{H\}$:

Lemma 2.4. *Let* ν, μ, λ *be Young diagrams such that* $\nu \lhd \mu \lhd \lambda$ *and* $d = d(\nu, \mu, \lambda)$ *their axial distance. If* $d \neq \pm 1$, *then there exists a Young diagram* μ' *such that* $\nu \lhd \mu' \lhd \lambda$ *which differs from* μ. *Further in this case, we have the following*:

$$b_d = H\left(\frac{\nu\lambda}{\mu\mu'}\right).$$

Let Λ be the set of all the Young diagrams of any size. Consider the vector space $K_0\Lambda$ whose natural basis is indexed by the set $\{[\lambda] \mid \lambda \in \Lambda\}$. Combining the result of the previous three lemmas, we have the following:

$$\begin{aligned}&\left\{b_{e_r}\left[\lambda^+_{(r)}\right] = H\left(\frac{\mu\lambda^+_{(r)}}{\lambda\lambda_{(r)}}\right)\left[\lambda^+_{(r)}\right] \ \middle|\ b_{e_r} \neq 0,\ \ r = 1, \ldots, b(\lambda)\right\}\\ &= \left\{b_{e_{\tau(s)}}\left[\kappa_{(s)}\right] = H\left(\frac{\mu\kappa_{(s)}}{\lambda\mu^+_{(s)}}\right)\left[\kappa_{(s)}\right] \ \middle|\ s = 2, \ldots, b(\mu)\right\},\end{aligned} \tag{8}$$

$$\begin{aligned}&\left\{b_{d_{s'}}\left[\mu^-_{(s')}\right] = H\left(\frac{\mu^-_{(s')}\lambda}{\mu\mu_{(s')}}\right)\left[\mu^-_{(s')}\right] \ \middle|\ b_{d_{s'}} \neq 0,\ \ s' = 1, \ldots, b'(\mu)\right\}\\ &= \left\{b_{d_{\sigma(r')}}\left[\nu_{(r')}\right] = H\left(\frac{\nu_{(r')}\lambda}{\mu\lambda^-_{(r')}}\right)\left[\nu_{(r')}\right] \ \middle|\ r' = 2, \ldots, b'(\lambda)\right\}.\end{aligned} \tag{9}$$

Similarly, let $K_0(\Lambda \times \Lambda)$ be the vector space whose natural basis is indexed by the set

$$\left\{ \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = (\lambda, \mu) \,\middle|\, \lambda, \mu \in \Lambda \right\}.$$

Then we have the following:

$$\begin{aligned} &\left\{ b_{e_r} \begin{bmatrix} \lambda^+_{(r)} \\ \lambda_{(r)} \end{bmatrix} = H\left(\frac{\mu\lambda^+_{(r)}}{\lambda\lambda_{(r)}}\right) \begin{bmatrix} \lambda^+_{(r)} \\ \lambda_{(r)} \end{bmatrix} \,\middle|\, b_{e_r} \neq 0,\ \ r = 1, \dots, b(\lambda) \right\} \\ &= \left\{ b_{e_{\tau(s)}} \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix} = H\left(\frac{\mu\kappa_{(s)}}{\lambda\mu^+_{(s)}}\right) \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix} \,\middle|\, s = 2, \dots, b(\mu) \right\}, \end{aligned} \tag{10}$$

$$\begin{aligned} &\left\{ b_{d_{s'}} \begin{bmatrix} \mu_{(s')} \\ \mu^-_{(s')} \end{bmatrix} = H\left(\frac{\mu^-_{(s')}\lambda}{\mu\mu_{(s')}}\right) \begin{bmatrix} \mu_{(s')} \\ \mu^-_{(s')} \end{bmatrix} \,\middle|\, b_{d_{s'}} \neq 0,\ \ s' = 1, \dots, b'(\mu) \right\} \\ &= \left\{ b_{d_{\sigma(r')}} \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix} = H\left(\frac{\nu_{(r')}\lambda}{\mu\lambda^-_{(r')}}\right) \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix} \,\middle|\, r' = 2, \dots, b'(\lambda) \right\}. \end{aligned} \tag{11}$$

Under the notation in Lemma 2.2, 2.3, in case $\lambda_{(r)}$ is a Young diagram, we have

$$e_r = d\left(\mu, \lambda, \lambda^+_{(r)}\right) = -d\left(\mu, \lambda_{(r)}, \lambda^+_{(r)}\right).$$

By Lemma 2.2, if $e_r \neq \pm 1$, this is also equal to the following:

$$e_r = e_{\tau(s)} = d\left(\mu, \lambda, \kappa_{(s)}\right) = -d\left(\mu, \mu^+_{(s)}, \kappa_{(s)}\right) \quad (s \geq 2). \tag{12}$$

Similarly, in case $\mu_{(s')}$ is a Young diagram, we have

$$d_{s'} = d\left(\mu^-_{(s')}, \mu, \lambda\right) = -d\left(\mu^-_{(s')}, \mu_{(s')}, \lambda\right).$$

By Lemma 2.3, if $d_{s'} \neq \pm 1$, this is also equal to the following:

$$d_{s'} = d_{\sigma(r')} = d\left(\nu_{(r')}, \mu, \lambda\right) = -d\left(\nu_{(r')}, \lambda^-_{(r')}, \lambda\right) \quad (r' \geq 2). \tag{13}$$

On the other hand, in case $\mu_{(1)}$ is a Young diagram, put

$$\mu_{(1)} = \mu' \quad \text{and} \quad e'_r = d\left(\mu', \lambda, \lambda^+_{(r)}\right). \tag{14}$$

If $\lambda_{(r)}$ is a Young diagram, then we have

$$e'_r = d\left(\mu', \lambda, \lambda^+_{(r)}\right) = d\left(\mu^-_{(1)}, \mu, \lambda_{(r)}\right).$$

Further if $e'_r \neq \pm 1$, then we have

$$e'_r = e'_{\tau(s)} = d\left(\mu', \lambda, \kappa_{(s)}\right) = d\left(\mu^-_{(1)}, \mu, \mu^+_{(s)}\right) \quad (s \geq 2). \tag{15}$$

In case $s = 1$, we have

$$e'_{\tau(1)} = d_1 = d\left(\mu^-_{(1)}, \mu, \lambda\right). \tag{16}$$

Similarly, in case $\lambda_{(1)}$ is a Young diagram, put

$$\lambda_{(1)} = \lambda' \quad \text{and} \quad d'_{s'} = d\left(\mu^-_{(s')}, \mu, \lambda'\right). \tag{17}$$

If $\mu_{(s')}$ is a Young diagram, then we have

$$d'_{s'} = d\left(\mu^-_{(s')}, \mu, \lambda'\right) = d\left(\mu_{(s')}, \lambda, \lambda^+_{(1)}\right)$$

and

$$d'_{s'} = d'_{\sigma(r')} = d\left(\nu_{(r')}, \mu, \lambda'\right) = d\left(\lambda^-_{(r')}, \lambda, \lambda^+_{(1)}\right) \quad (r' \geq 2), \tag{18}$$

$$d'_{\sigma(1)} = e_1 = d\left(\mu, \lambda, \lambda^+_{(1)}\right). \tag{19}$$

Finally, we put

$$d_{s',s} = d\left(\mu^-_{(s')}, \mu, \mu^+_{(s)}\right), \tag{20}$$

$$e_{r',r} = d\left(\lambda^-_{(r')}, \lambda, \lambda^+_{(r)}\right). \tag{21}$$

Using the notation above we finally obtain the following relations among $\{a_d\}$ and $\{h(\lambda)\}$:

Lemma 2.5. 1. $\sum_{r=1}^{b(\lambda)} h(\lambda) \big/ h\left(\lambda^+_{(r)}\right) = 1$,

2. $\sum_{r=1}^{b(\lambda)} \left(h(\lambda) \big/ h\left(\lambda^+_{(r)}\right)\right) a_{e_r} = 0$,

3. $\sum_{r=1}^{b(\lambda)} \left(h(\lambda) \big/ h\left(\lambda^+_{(r)}\right)\right) a_{e_r} a_{e_{r',r}} = \begin{cases} h(\mu)/h(\lambda) & (r' = 1), \\ 0 & (r' \neq 1), \end{cases}$

4. $\sum_{s'=1}^{b(\mu)} \left(h(\mu) \big/ h\left(\mu^-_{(s')}\right)\right) a_{d_{s'}} a_{d_{s',s}} = \begin{cases} h(\lambda)/h(\mu) - 1 & (s' = 1), \\ -1 & (s' \neq 1), \end{cases}$

5. $h(\mu)^3 \sum_{s'=1}^{b'(\mu)} a^3_{d_{s'}} \big/ h\left(\mu^-_{(s')}\right) = h(\lambda)^3 \sum_{r=1}^{b(\lambda)} a^3_{e_r} \big/ h\left(\lambda^+_{(r)}\right)$.

Proof. The above relations are proved by specializing the parameter q to 1 in the equations of Theorem 0.1–0.2 in the paper [7]. □

3. Well-definedness of the representations

In this section, we show that $\{\rho_{\boldsymbol{\alpha}}\}$ in the main theorem preserve the defining relations of the party algebra. First in Section 3.1 we check that $\{\rho_{\boldsymbol{\alpha}}\}$ preserve the braid relation $(P2)$ in Definition 1.1. Actually, this is the main part of the paper. Next in Section 3.2, we check that they also preserve the other relations.

3.1. Preservation of the braid relation. Let $P = (\boldsymbol{\alpha}^{(0)}, \ldots, \boldsymbol{\alpha}^{(n)})$ be a tableau of shape $\boldsymbol{\alpha}$. As we defined in Section 1.3 and Section 1.4, the linear map $\rho_{\boldsymbol{\alpha}}(s_i)$ is defined by the matrix M_i and it is defined by the way how $\boldsymbol{\alpha}^{(i+1)}$ is obtained from $\boldsymbol{\alpha}^{(i-1)}$ (see the expressions (4), (5) and (7)). Suppose that $\boldsymbol{\alpha}^{(i)}$ (resp. $\boldsymbol{\alpha}^{(i+1)}$, $\boldsymbol{\alpha}^{(i+2)}$) is obtained from $\boldsymbol{\alpha}^{(i-1)}$ (resp. $\boldsymbol{\alpha}^{(i)}$, $\boldsymbol{\alpha}^{(i+1)}$) by moving a box on the j_0-th (resp. j_1-th, j_2-th) board of $\boldsymbol{\alpha}^{(i-1)}$ (resp. $\boldsymbol{\alpha}^{(i)}$, $\boldsymbol{\alpha}^{(i+1)}$) to the (j_0+1)-st (resp. (j_1+1)-st, (j_2+1)-st) board. Then to know the actions of s_i and s_{i+1}, it will be sufficient to know that how they alter the following matrix whose entries are Young diagrams:

$$\left(\alpha^{(i')}(j)\right)_{i-1\le i'\le i+2,\ \ j=j_0,j_0+1,j_1,j_1+1,j_2,j_2+1}. \tag{22}$$

Since column indices of the matrix above do not necessarily distinct depending on the differences among j_0, j_1 and j_2, we have to consider Table 1.

In Table 1, if we replace $(j_0, j_0+1, j_1, j_1+1, j_2, j_2+1)$ with the reverse $(j_2+1, j_2, j_1+1, j_1, j_0+1, j_0)$, then we find Case 3.1 (resp. Case 3.2) turns Case 3.3 (resp. Case 3.4). This means $s_i s_{i+1} s_i$-action on v_P in Case 3.3 (resp. Case 3.4)

Table 1. Classification by differences among j_0, j_1 and j_2

Case 1	$\lvert j_0-j_1\rvert \ge 2,\quad \lvert j_1-j_2\rvert \ge 2,\quad \lvert j_2-j_0\rvert \ge 2$
Case 2	$j_0=j_1=j_2$
Case 3.1 (Case 3.3)	$\lvert j_1-j_2\rvert \ge 2,\quad \lvert j_2-j_0\rvert \ge 2,\quad j_1=j_0+1$
Case 3.2 (Case 3.4)	$\lvert j_1-j_2\rvert \ge 2,\quad \lvert j_2-j_0\rvert \ge 2,\quad j_0=j_1+1$
(Case 3.3)	$\lvert j_2-j_0\rvert \ge 2,\quad \lvert j_0-j_1\rvert \ge 2,\quad j_2=j_1+1$
(Case 3.4)	$\lvert j_2-j_0\rvert \ge 2,\quad \lvert j_0-j_1\rvert \ge 2,\quad j_1=j_2+1$
Case 3.5	$\lvert j_0-j_1\rvert \ge 2,\quad \lvert j_1-j_2\rvert \ge 2,\quad j_0=j_2+1$
Case 3.6	$\lvert j_0-j_1\rvert \ge 2,\quad \lvert j_1-j_2\rvert \ge 2,\quad j_2=j_0+1$
Case 4.1 (Case 4.2)	$j_0=j_1,\quad \lvert j_2-j_0\rvert = \lvert j_1-j_2\rvert \ge 2$
(Case 4.2)	$j_1=j_2,\quad \lvert j_0-j_1\rvert = \lvert j_2-j_0\rvert \ge 2$
Case 4.3	$j_2=j_0,\quad \lvert j_1-j_2\rvert = \lvert j_0-j_1\rvert \ge 2$
Case 5.1	$j_1=j_0+1,\quad j_2=j_1+1$
Case 5.2 (Case 5.3)	$j_2=j_0+1,\quad j_1=j_2+1$
(Case 5.3)	$j_0=j_1+1,\quad j_2=j_0+1$
Case 5.4 (Case 5.5)	$j_2=j_1+1,\quad j_0=j_2+1$
(Case 5.5)	$j_0=j_2+1,\quad j_1=j_0+1$
Case 5.6	$j_1=j_2+1,\quad j_0=j_1+1$
Case 6.1 (Case 6.4)	$j_2=j_1=j_0+1$
Case 6.2 (Case 6.5)	$j_1=j_0=j_2+1$
Case 6.3 (Case 6.6)	$j_0=j_2=j_1+1$
(Case 6.4)	$j_0=j_1,\quad j_2=j_0+1=j_1+1$
(Case 6.5)	$j_1=j_2,\quad j_0=j_1+1=j_2+1$
(Case 6.6)	$j_2=j_0,\quad j_1=j_2+1=j_0+1$

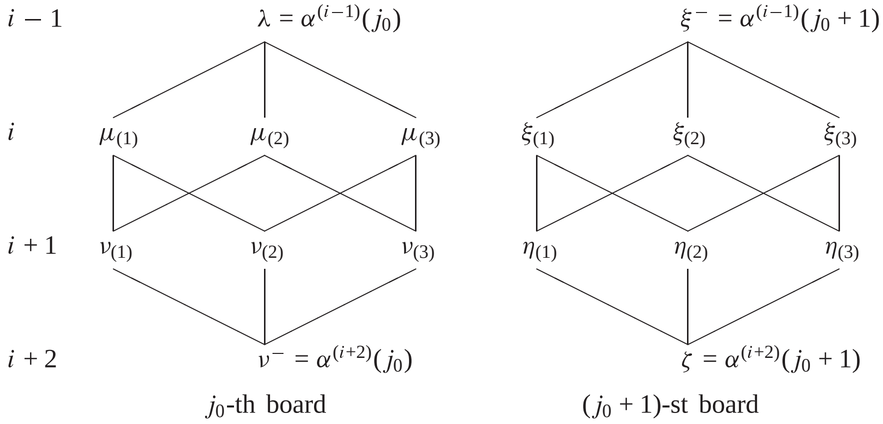

Fig. 5. Case 2 Hasse diagrams of Young diagrams on j_0-th and (j_0+1)-st board

is obtained from $s_{i+1}s_is_{i+1}$-action on v_P in Case 3.1 (resp. Case 3.2) by changing indices of tableaux and vice versa. Similarly, the conditions Case 4.2, 5.3, 5.5, 6.4, 6.5, and Case 6.6 are obtained from the conditions Case 4.1, 5.2, 5.4, 6.1, 6.2 and Case 6.3, respectively. Hence, we have only to consider Case 1, 2, 3.1, 3.2, 3.5, 3.6, 4.1, 4.3, 5.1, 5.2, 5.4, 5.6 and Case 6.1–6.3.

In the following, to simplify the notation, we write merely $s_i v_P$ instead of $\rho_{\boldsymbol{\alpha}}(s_i)(v_P)$. Further, for a Young diagram λ, we use the notation λ^- (resp. λ^+) to denote a Young diagram which is obtained from λ by removing (resp. adding) a box from (resp. to) λ.

CASE 1. First we consider Case 1, the most general case. In this case, column indices j_0, j_0+1, j_1, j_1+1, j_2 and j_2+1 of the matrix (22) are all distinct each other. Since in this case, the actions of s_i and s_{i+1} are both presented by the matrix (7), they merely exchange the entries of the matrix (22). By direct calculation, we can check that $s_is_{i+1}s_iv_P = s_{i+1}s_is_{i+1}v_P$ in this case.

CASE 2. Next we consider Case 2. The assumption $j_0 = j_1 = j_2$ means that on the way from $\boldsymbol{\alpha}^{(i-1)}$ to $\boldsymbol{\alpha}^{(i+2)}$ of P, three boxes of $\lambda = \alpha^{(i-1)}(j_0)$ are removed and they are attached to $\xi^- = \alpha^{(i-1)}(j_0+1)$ one by one. Let Fig. 5 be Hasse diagrams which respectively describe how Young diagrams on the j_0-th and the (j_0+1)-st boards of P would transform themselves (Some of the Young diagrams may be virtual ones. However, we do not have to care about that, since the associate coefficients would be zero). Let

$$\begin{bmatrix} \mu_{(r)} \\ \nu_{(q)} \end{bmatrix} \otimes \begin{bmatrix} \xi_{(s)} \\ \eta_{(t)} \end{bmatrix}$$

be the vector which corresponds to a tableau obtained from P by replacing Young diagrams on the j_0-th and the (j_0+1)-st boards of $\boldsymbol{\alpha}^{(i)}$ (resp. $\boldsymbol{\alpha}^{(i+1)}$) with $\mu_{(r)}$ and $\xi_{(s)}$ (resp. $\nu_{(q)}$ and $\eta_{(t)}$). For this vector, s_i gives the following matrix;

$$\begin{pmatrix} a_{e(q,r)} & b_{e(q,r)} \\ b_{e(q,r)} & a_{-e(q,r)} \end{pmatrix} \otimes \begin{pmatrix} a_{d(s,t)} & b_{d(s,t)} \\ b_{d(s,t)} & a_{-d(s,t)} \end{pmatrix},$$

and s_{i+1} gives the following matrix;

$$\begin{pmatrix} a_{e'(q,r)} & b_{e'(q,r)} \\ b_{e'(q,r)} & a_{-e'(q,r)} \end{pmatrix} \otimes \begin{pmatrix} a_{d'(s,t)} & b_{d'(s,t)} \\ b_{d'(s,t)} & a_{-d'(s,t)} \end{pmatrix}.$$

Here $e(q,r) = d(\nu_{(q)}, \mu_{(r)}, \lambda)$, $d(s,t) = d(\xi^-, \xi_{(s)}, \eta_{(t)})$, $e'(q,r) = d(\nu^-, \nu_{(q)}, \mu_{(r)})$ and $d'(s,t) = d(\xi_{(s)}, \eta_{(t)}, \zeta)$. Hence, using a result of the paper [14], we can easily check that in this case $s_i s_{i+1} s_i v_P = s_{i+1} s_i s_{i+1} v_P$ holds.

CASE 3. We consider mainly Case 3.1 (Case 3.3). For the explanation below, see Table 2.

Put $(\alpha^{(i-1)}(j_0), \alpha^{(i-1)}(j_1), \alpha^{(i-1)}(j_1+1)) = (\alpha, \lambda, \beta^-)$. If $\alpha^{(i-1)}(j_1) \neq \alpha^{(i+1)}(j_1)$ in the tableau P, then s_i and s_{i+1} both give linear transformations defined by the matrix of type (7) and we can apply Case 1. Hence we have only to consider the case $\alpha^{(i-1)}(j_1) = \alpha^{(i+1)}(j_1)$, namely, we assume the following:

1) One box in α on the j_0-th board is removed and it is attached to λ on the j_1-th board.

2) Then the attached box on the j_1-th board is again removed and it is attached to β^- on the (j_1+1)-st board.

We keep our eyes on the i-th and the $(i+1)$-st coordinates of Table 2. Let

$$\left[\begin{array}{c|c|ccc} \alpha(j_0) & \alpha(j_1) & \alpha(j_1+1) & \alpha(j_2) & \alpha(j_2+1) \\ \beta(j_0) & \beta(j_1) & \beta(j_1+1) & \beta(j_2) & \beta(j_2+1) \end{array}\right] \tag{23}$$

be a vector which corresponds to a tableau obtained from P by replacing Young diagrams on the j_0-th, j_1-th, (j_1+1)-st, j_2-th and (j_2+1)-st boards of the i-th (resp. $(i+1)$-st) coordinate with $\alpha(j_0), \alpha(j_1), \alpha(j_1+1), \alpha(j_2), \alpha(j_2+1)$ (resp. $\beta(j_0), \beta(j_1), \beta(j_1+1), \beta(j_2), \beta(j_2+1)$). By the definition of tableaux, we find that all the entries of the

Table 2. Case 3.1 (Case 3.3) $|j_1 - j_2| \geq 2$, $|j_2 - j_0| \geq 2$, $j_1 = j_0 + 1$

coordinate	board				
	j_0	j_1	j_1+1	j_2	j_2+1
$i-1$	α	λ	β^-	$\gamma_{(1)}$	$\gamma_{(2)}^-$
i	α^-	$\lambda_{(1)}^+$	β^-	$\gamma_{(1)}$	$\gamma_{(2)}^-$
$i+1$	α^-	λ	β	$\gamma_{(1)}$	$\gamma_{(2)}^-$
$i+2$	α^-	λ	β	$\gamma_{(1)}^-$	$\gamma_{(2)}$

matrix (23) will be recovered from the entries of the second column. Hence instead of using the matrix (23) we merely write

$$\begin{bmatrix} \alpha(j_1) \\ \beta(j_1) \end{bmatrix}.$$

Let $\left\{\lambda^+_{(r)} \;\middle|\; r = 1, \ldots, b(\lambda)\right\}$ be the set of Young diagrams such that $\lambda^+_{(r)} \rhd \lambda$ and $\left\{\lambda^-_{(r')} \;\middle|\; r' = 1, \ldots, b'(\lambda)\right\}$ the set of Young diagrams such that $\lambda^-_{(r')} \lhd \lambda$. Let

$$v_P = \begin{bmatrix} \lambda^+_{(1)} \\ \lambda \end{bmatrix}$$

be the vector indexed by the tableau P. Then by definition (5′) and (7) and using notation (21), we can check that both $s_i s_{i+1} s_i v_P$ and $s_{i+1} s_i s_{i+1} v_P$ are equal to the following:

$$\sum_{r=1}^{b(\lambda)} H\left(\frac{\lambda^2}{\lambda^+_{(r)}\lambda^+_{(1)}}\right) \begin{bmatrix} \lambda \\ \lambda^+_{(r)} \end{bmatrix} + \sum_{r'=1}^{b'(\lambda)} H\left(\frac{\lambda^2}{\lambda^-_{(r')}\lambda^+_{(1)}}\right) a_{e_{r',1}} \begin{bmatrix} \lambda \\ \lambda^-_{(r')} \end{bmatrix}.$$

As for Case 3.2 (3.4), 3.5, and Case 3.6, using Table 3, 4 and Table 5 respectively we can more easily check that $s_i s_{i+1} s_i v_P = s_{i+1} s_i s_{i+1} v_P$.

Table 3. Case 3.2 (Case 3.4) $|j_1 - j_2| \geq 2$, $|j_2 - j_0| \geq 2$, $j_0 = j_1 + 1$

coordinate	board				
	j_1	j_0	j_0+1	j_2	j_2+1
$i-1$	α	λ	β^-	$\gamma_{(1)}$	$\gamma^-_{(2)}$
i	α	$\boxed{\lambda^-_{(1)}}$	β	$\gamma_{(1)}$	$\gamma^-_{(2)}$
$i+1$	α^-	$\boxed{\lambda}$	β	$\gamma^-_{(1)}$	$\gamma_{(2)}$
$i+2$	α^-	λ	β	$\gamma^-_{(1)}$	$\gamma_{(2)}$

Table 4. Case 3.5 $|j_0 - j_1| \geq 2$, $|j_1 - j_2| \geq 2$, $j_0 = j_1 + 1$

coordinate	board				
	j_2	j_0	j_0+1	j_1	j_1+1
$i-1$	α	λ	β^-	$\gamma_{(1)}$	$\gamma^-_{(2)}$
i	α	$\boxed{\lambda^-_{(1)}}$	β	$\gamma_{(1)}$	$\gamma^-_{(2)}$
$i+1$	α	$\boxed{\lambda^-_{(1)}}$	β	$\gamma^-_{(1)}$	$\gamma_{(2)}$
$i+2$	α^-	λ	β	$\gamma^-_{(1)}$	$\gamma_{(2)}$

Table 5. Case 3.6 $|j_0 - j_1| \geq 2$, $|j_1 - j_2| \geq 2$, $j_2 = j_0 + 1$

coordinate	board				
	j_0	j_2	j_2+1	j_1	j_1+1
$i-1$	α	λ	β^-	$\gamma_{(1)}$	$\gamma^-_{(2)}$
i	α^-	$\boxed{\lambda^+_{(1)}}$	β^-	$\gamma_{(1)}$	$\gamma^-_{(2)}$
$i+1$	α^-	$\boxed{\lambda^+_{(1)}}$	β^-	$\gamma^-_{(1)}$	$\gamma_{(2)}$
$i+2$	α^-	λ	β	$\gamma^-_{(1)}$	$\gamma_{(2)}$

CASE 4. We consider mainly Case 4.1 (Case 4.2). For the explanation below, see Table 6.

Put $(\alpha^{(i-1)}(j_0), \alpha^{(i-1)}(j_0+1)) = (\lambda, \xi)$. The assumption $j_0 = j_1$ means that on the way from $\boldsymbol{\alpha}^{(i-1)}$ of P to $\boldsymbol{\alpha}^{(i+1)}$ of P, two boxes of $\lambda = \alpha^{(i-1)}(j_0)$ are removed and they are attached to $\xi = \alpha^{(i-1)}(j_0+1)$ one by one. Hence we can put $(\alpha^{(i)}(j_0), \alpha^{(i)}(j_0+1)) = (\mu, \eta)$ and $(\alpha^{(i+1)}(j_0), \alpha^{(i+1)}(j_0+1)) = (\nu, \zeta)$ using the Young diagrams such that $\lambda \rhd \mu \rhd \nu$ and $\xi \lhd \eta \lhd \zeta$. Further, put $(\alpha^{(i-1)}(j_2), \alpha^{(i-1)}(j_2+1)) = (\beta, \gamma^-)$. The assumption $|j_2 - j_0| = |j_1 - j_2| \geq 2$ means that we can put $(\alpha^{(i)}(j_2), \alpha^{(i)}(j_2+1)) = (\alpha^{(i+1)}(j_2), \alpha^{(i+1)}(j_2+1)) = (\beta, \gamma^-)$ and $(\alpha^{(i+2)}(j_2), \alpha^{(i+2)}(j_2+1)) = (\beta^-, \gamma)$ using the Young diagrams such that $\beta \rhd \beta^-$ and $\gamma^- \lhd \gamma$. Let

$$\left[\begin{array}{cc|cc} \alpha(j_0) & \alpha(j_0+1) & \alpha(j_2) & \alpha(j_2+1) \\ \beta(j_0) & \beta(j_0+1) & \beta(j_2) & \beta(j_2+1) \end{array}\right] \tag{24}$$

be a vector which corresponds to a tableau obtained from P by replacing Young diagrams on the j_0-th, (j_0+1)-st, j_2-th and (j_2+1)-st boards of the i-th (resp. $(i+1)$-st) coordinate of $\boldsymbol{\alpha}$ with $\alpha(j_0), \alpha(j_0+1), \alpha(j_2), \alpha(j_2+1)$ (resp. $\beta(j_0), \beta(j_0+1), \beta(j_2), \beta(j_2+1)$). By the definition of tableaux, we find that all the entries of the matrix (24) will be recovered from the first two columns. Hence instead of using the matrix (24) we merely write

$$\begin{bmatrix} \alpha(j_0) & \alpha(j_0+1) \\ \beta(j_0) & \beta(j_0+1) \end{bmatrix}.$$

Let

$$v_P = \begin{bmatrix} \mu & \eta \\ \nu & \zeta \end{bmatrix}$$

be the vector indexed by the tableau P. Put $d = d(\nu, \mu, \lambda)$ and $f = d(\xi, \eta, \zeta)$. Let μ' (resp. η') be a (possibly virtual) Young diagram which satisfies $\lambda \rhd \mu' \rhd \nu$ (resp. $\zeta \rhd \eta' \rhd \xi$) and $\mu' \neq \mu$ (resp. $\eta' \neq \eta$). Then by definition (4) and (7), we can check that both $s_i s_{i+1} s_i v_P$ and $s_{i+1} s_i s_{i+1} v_P$ are equal to the following:

$$a_d a_f \begin{bmatrix} \lambda & \xi \\ \mu & \eta \end{bmatrix} + a_d b_f \begin{bmatrix} \lambda & \xi \\ \mu & \eta' \end{bmatrix} + b_d a_f \begin{bmatrix} \lambda & \xi \\ \mu' & \eta \end{bmatrix} + b_d b_f \begin{bmatrix} \lambda & \xi \\ \mu' & \eta' \end{bmatrix}.$$

Table 6. Case 4.1 (Case 4.2) $j_0 = j_1$, $|j_2 - j_0| = |j_1 - j_2| \geq 2$

coordinate	board			
	j_0	j_0+1	j_2	j_2+1
$i-1$	λ	ξ	β	γ^-
i	μ	η	β	γ^-
$i+1$	ν	ζ	β	γ^-
$i+2$	ν	ζ	β^-	γ

As for Case 4.3, using Table 7 we can check that $s_i s_{i+1} s_i v_P = s_{i+1} s_i s_{i+1} v_P$.

CASE 5. Let $\{\lambda^+_{(r)}\}$ and $\{\lambda^-_{(r')}\}$ be the sets of Young diagrams defined in Case 3.1. We also use the notation $e_{r',r}$ as in (21). Similarly, let $\{\zeta^+_{(s)}\}$ and $\{\zeta^-_{(s')}\}$ be the sets of Young diagrams such that $\zeta^+_{(s)} \rhd \zeta$ and $\zeta^-_{(s')} \lhd \zeta$ respectively. Further, let $\{f_{s',1}\}$ be the set of axial distances defined by $f_{s',1} = d\big(\zeta^-_{(s')}, \zeta, \zeta^+_{(1)}\big)$.

First we consider Case 5.1. For the explanation below, see Table 8.

Put

$$\big(\alpha^{(i-1)}(j_0), \alpha^{(i-1)}(j_1), \alpha^{(i-1)}(j_2), \alpha^{(i-1)}(j_2)\big) = (\alpha, \lambda, \zeta, \beta^-).$$

As we saw in Case 3.1, under the assumption $j_1 = j_0 + 1$, if $\alpha^{(i-1)}(j_1) \neq \alpha^{(i+1)}(j_1)$, then we can attribute this case to one of the previous ones. Hence we may assume $\alpha^{(i-1)}(j_1) = \alpha^{(i+1)}(j_1)$. The same things also hold for $\alpha^{(i)}(j_2)$ and $\alpha^{(i+2)}(j_2)$. Hence we may further assume that $\alpha^{(i)}(j_2) = \alpha^{(i+2)}(j_2)$. In other words, we have only to consider a tableau P of the form presented by the data in Table 8. As we saw in the previous cases, we keep our eyes on the i-th and the $(i+1)$-st coordinates of Table 8. Let

$$\left[\begin{array}{c|cc|c} \alpha(j_0) & \alpha(j_1) & \alpha(j_2) & \alpha(j_2+1) \\ \beta(j_0) & \beta(j_1) & \beta(j_2) & \beta(j_2+1) \end{array}\right] \tag{25}$$

be a vector which corresponds to a tableau obtained from P by replacing Young diagrams on the j_0-th, j_1-th, j_2-th and (j_2+1)-st board of the i-th (resp. $(i+1)$-st) coordinate with $\alpha(j_0), \alpha(j_1), \alpha(j_2), \alpha(j_2+1)$ (resp. $\beta(j_0), \beta(j_1), \beta(j_2), \beta(j_2+1)$). By the definition of tableaux, we find that all the entries of the matrix (25) will be recovered from the entries of the second and the third columns. Hence instead of using

Table 7. Case 4.3 $j_2 = j_0,\ |j_1 - j_2| = |j_0 - j_1| \geq 2$

coordinate	board			
	j_0	j_0+1	j_1	j_1+1
$i-1$	λ	ξ	β	γ^-
i	μ	η	β	γ^-
$i+1$	μ	η	β^-	γ
$i+2$	ν	ζ	β^-	γ

Table 8. Case 5.1 $j_1 = j_0 + 1,\ j_2 = j_1 + 1$

coordinate	board			
	j_0	j_1	j_2	j_2+1
$i-1$	α	λ	ζ	β^-
i	α^-	$\lambda^+_{(1)}$	ζ	β^-
$i+1$	α^-	λ	$\zeta^+_{(1)}$	β^-
$i+2$	α^-	λ	ζ	β

the matrix (25) we merely write

$$\begin{bmatrix} \alpha(j_1) & \alpha(j_2) \\ \beta(j_1) & \beta(j_2) \end{bmatrix}.$$

Using these notation, we write

$$v_P = \begin{bmatrix} \lambda^+_{(1)} & \zeta \\ \lambda & \zeta^+_{(1)} \end{bmatrix}.$$

Then by definition (5′) and (7) we find that $s_i s_{i+1} s_i v_P$ is equal to the following:

$$\begin{aligned}
&\sum_{s=1}^{b(\zeta)} \sum_{q=1}^{b(\lambda)} H\left(\frac{\lambda^2}{\lambda^+_{(q)}\lambda^+_{(1)}}\right) H\left(\frac{\zeta^2}{\zeta^+_{(s)}\zeta^+_{(1)}}\right) \begin{bmatrix} \lambda^+_{(q)} & \zeta \\ \lambda & \zeta^+_{(s)} \end{bmatrix} \\
&+ \sum_{r=1}^{b(\lambda)} \sum_{s'=1}^{b'(\zeta)} H\left(\frac{\lambda^2}{\lambda^+_{(r)}\lambda^+_{(1)}}\right) H\left(\frac{\zeta^2}{\zeta^-_{(s')}\zeta^+_{(1)}}\right) a_{f_{s',1}} \begin{bmatrix} \lambda & \zeta^-_{(s')} \\ \lambda^+_{(r)} & \zeta^-_{(s')} \end{bmatrix} \\
&+ \sum_{r'=1}^{b'(\lambda)} \sum_{s=1}^{b(\zeta)} H\left(\frac{\lambda^2}{\lambda^-_{(r')}\lambda^+_{(1)}}\right) H\left(\frac{\zeta^2}{\zeta^+_{(1)}\zeta^+_{(s)}}\right) a_{e_{r',1}} \begin{bmatrix} \lambda^-_{(r')} & \zeta^+_{(s)} \\ \lambda^-_{(r')} & \zeta \end{bmatrix} \\
&+ \sum_{r'=1}^{b'(\lambda)} \sum_{s'=1}^{b'(\zeta)} H\left(\frac{\lambda^2}{\lambda^-_{(r')}\lambda^+_{(1)}}\right) H\left(\frac{\zeta^2}{\zeta^-_{(s')}\zeta^+_{(1)}}\right) a_{e_{r',1}} a_{f_{s',1}} \begin{bmatrix} \lambda & \zeta^-_{(s')} \\ \lambda^-_{(r')} & \zeta \end{bmatrix}.
\end{aligned}$$

Here we used Lemma 2.5 1, 2 to obtain the first line of the equation above. In this equation, if we exchange λ and ζ, exchange the first and the second columns of matrices, and exchange the first and the second rows of the matrices, then we have the same equation. This means $s_{i+1}s_i s_{i+1}$-action and $s_i s_{i+1} s_i$-action on v_P coincide.

Next, we consider Case 5.2 (Case 5.3). According to Table 9, we put

$$v_P = \begin{bmatrix} \lambda^+_{(1)} & \zeta \\ \lambda^+_{(1)} & \zeta^-_{(1)} \end{bmatrix}.$$

Table 9. Case 5.2 (Case 5.3) $j_2 = j_0 + 1,\ j_1 = j_2 + 1$

coordinate	board			
	j_0	j_2	j_1	$j_1 + 1$
$i - 1$	α	λ	ζ	β^-
i	α^-	$\lambda^+_{(1)}$	ζ	β^-
$i + 1$	α^-	$\lambda^+_{(1)}$	$\zeta^-_{(1)}$	β
$i + 2$	α^-	λ	ζ	β

Then by definition (5′) and (7) we find that $s_{i+1}s_is_{i+1}v_P$ is equal to the following:

$$\sum_{r=1}^{b(\lambda)} H\left(\frac{\lambda^2}{\lambda^+_{(r)}\lambda^+_{(1)}}\right)\begin{bmatrix} \lambda^+_{(r)} & \zeta \\ \lambda^+_{(r)} & \zeta^-_{(1)} \end{bmatrix}$$
$$+\sum_{s=1}^{b(\zeta)}\sum_{r'=1}^{b'(\lambda)} H\left(\frac{\zeta^2}{\zeta^-_{(1)}\zeta^+_{(s)}}\right) a_{f_{1,s}} H\left(\frac{\lambda^2}{\lambda^-_{(r')}\lambda^+_{(1)}}\right) a_{e_{r',1}}\begin{bmatrix} \lambda^-_{(r')} & \zeta^+_{(s)} \\ \lambda^-_{(r')} & \zeta \end{bmatrix}.$$

Here we applied Lemma 2.5 2, 3 to obtain the first line of the equation above. By direct calculation, we find that $s_is_{i+1}s_{i+1}v_P$ coincides with this equation.

As for Case 5.4 (5.5) and Case 5.6, using Table 10 and Table 11 respectively, we can more easily check that $s_is_{i+1}s_iv_P = s_{i+1}s_is_{i+1}v_P$.

CASE 6. Throughout Case 6 we adopt the notation introduced in Section 2. Further let f be an axial distance defined by $f = d(\xi, \eta, \zeta)$.

CASE 6.1. (Case 6.4.) $j_2 = j_1 = j_0 + 1$.

For the explanation below, see Table 12 and Table 13. Put

$$\left(\alpha^{(i-1)}(j_0), \alpha^{(i-1)}(j_1), \alpha^{(i-1)}(j_1+1)\right) = (\alpha^+, \lambda, \xi).$$

The assumption $j_2 = j_1 = j_0 + 1$ means the following:

1) On the way from $\boldsymbol{\alpha}^{(i-1)}$ to $\boldsymbol{\alpha}^{(i)}$, one box of $\alpha^+ = \alpha^{(i-1)}(j_0)$ on the j_0-th board is removed and it is attached to $\lambda = \alpha^{(i-1)}(j_1)$ on the j_1-th board. We put

$$\left(\alpha^{(i)}(j_0), \alpha^{(i)}(j_1), \alpha^{(i)}(j_1+1)\right) = (\alpha, \lambda^+_{(1)}, \xi).$$

2) Then on the way from $\boldsymbol{\alpha}^{(i)}$ to $\boldsymbol{\alpha}^{(i+2)}$, two boxes of $\lambda^+_{(1)} = \alpha^{(i)}(j_1)$ on the j_1-th board are moved to the next board. If the two boxes are both distinct from the box which is attached at the former step, then we can attribute this case to one of the previous cases. So we may assume that $\alpha^{(i+1)}(j_1) = \lambda$ (Case 6.1.1) or $\alpha^{(i+1)}(j_1) = \lambda'$ (Case 6.1.2). Here λ' is a Young diagram which satisfies $\lambda^+_{(1)} \rhd \lambda' \rhd \mu$ and $\lambda \neq \lambda'$. Hence we

Table 10. Case 5.4 (Case 5.5) $j_2 = j_1 + 1$, $j_0 = j_2 + 1$

coordinate	board			
	j_1	j_2	j_0	j_0+1
$i-1$	α	λ	ζ	β^-
i	α	λ	$\zeta^-_{(1)}$	β
$i+1$	α^-	$\lambda^+_{(1)}$	$\zeta^-_{(1)}$	β
$i+2$	α^-	λ	ζ	β

Table 11. Case 5.6 $j_1 = j_2 + 1$, $j_0 = j_1 + 1$

coordinate	board			
	j_2	j_1	j_0	j_0+1
$i-1$	α	λ	ζ	β^-
i	α	λ	$\zeta^-_{(1)}$	β
$i+1$	α	$\lambda^-_{(1)}$	ζ	β
$i+2$	α^-	λ	ζ	β

may put

$$\big(\alpha^{(i+1)}(j_0), \alpha^{(i+1)}(j_1), \alpha^{(i+1)}(j_1+1)\big) = \begin{cases} (\alpha, \lambda, \eta) & \text{(Case 6.1.1)}, \\ (\alpha, \lambda', \eta) & \text{(Case 6.1.2)}, \end{cases}$$
$$\big(\alpha^{(i+2)}(j_0), \alpha^{(i+2)}(j_1), \alpha^{(i+2)}(j_1+1)\big) = (\alpha, \mu, \zeta).$$

Note that $\xi \lhd \eta \lhd \zeta$ and there may exist a Young diagram η' such that $\xi \lhd \eta' \lhd \zeta$ and $\eta' \neq \eta$.

CASE 6.1.1. $j_2 = j_1 = j_0 + 1$, $\alpha^{(i-1)} = \alpha^{(i+1)}$.

For the explanation below, see Table 12. Let

$$\left[\begin{array}{c|c|c} \alpha(j_0) & \alpha(j_1) & \alpha(j_1+1) \\ \beta(j_0) & \beta(j_1) & \beta(j_1+1) \end{array}\right] \tag{26}$$

be a vector which corresponds to a tableau obtained from P by replacing Young diagrams on the j_0-th, j_1-th and (j_1+1)-st boards of the i-th (resp. $(i+1)$-st) coordinate with $\alpha(j_0), \alpha(j_1), \alpha(j_1+1)$ (resp. $\beta(j_0), \beta(j_1), \beta(j_1+1)$). As we show below, in the process of $s_i s_{i+1} s_i$-action or $s_{i+1} s_i s_{i+1}$-action to v_P, a set of entries $\{\alpha(j_1+1), \beta(j_1+1)\}$ of the matrix (26) contains η or η'. Hence under the assumption that the matrix (26) denotes a tableau which appears in the process of $s_i s_{i+1} s_i$-action or $s_{i+1} s_i s_{i+1}$-action on v_P, we merely write

(27)
$$\begin{bmatrix} \alpha(j_1) \\ \beta(j_1) \end{bmatrix} \quad \text{if} \quad \{\alpha(j_1+1), \beta(j_1+1)\} \ni \xi, \qquad \begin{bmatrix} \alpha(j_1) \\ \beta(j_1) \end{bmatrix}' \quad \text{if} \quad \{\alpha(j_1+1), \beta(j_1+1)\} \ni \xi'.$$

For example,

$$\begin{bmatrix} \lambda^+_{(1)} \\ \lambda \end{bmatrix} = \left[\begin{array}{c|c|c} \alpha & \lambda^+_{(1)} & \xi \\ \alpha & \lambda & \eta \end{array}\right] \quad \text{and} \quad \begin{bmatrix} \lambda^+_{(1)} \\ \lambda \end{bmatrix}' = \left[\begin{array}{c|c|c} \alpha & \lambda^+_{(1)} & \xi \\ \alpha & \lambda & \eta' \end{array}\right].$$

We note that all the tableaux which appear in the following calculation are distinguished by the notation (27). Let

$$v_P = \begin{bmatrix} \lambda^+_{(1)} \\ \lambda \end{bmatrix}.$$

Table 12. Case 6.1.1 $j_2 = j_1 = j_0 + 1$, $\alpha^{(i-1)} = \alpha^{(i+1)}$

coordinate	board		
	j_0	$j_1 = j_2$	$j_1 + 1$
$i-1$	α^+	λ	ξ
i	α	$\lambda^+_{(1)}$	ξ
$i+1$	α	λ	η
$i+2$	α	μ	ζ

be the vector indexed by a tableau P. Then we have the following:

$$\begin{aligned}
s_i s_{i+1} s_i v_P = {} & H\left(\frac{\mu}{\lambda^+_{(1)}}\right)\left(a_f \begin{bmatrix} \mu \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \lambda \end{bmatrix}'\right) \\
& + \sum_{r=1}^{b(\lambda)} H\left(\frac{\lambda^2}{\lambda^+_{(r)}\lambda^+_{(1)}}\right) b_{e_r} \left(a_f \begin{bmatrix} \mu \\ \lambda_{(r)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \lambda_{(r)} \end{bmatrix}'\right) \\
& + a_{e_1} \sum_{r=1}^{b(\lambda)} H\left(\frac{\lambda^2}{\lambda^+_{(1)}\lambda^+_{(r)}}\right) a_{e_r} \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + a_{e_1} \sum_{s=2}^{b(\mu)} H\left(\frac{\lambda\mu}{\lambda^+_{(1)}\mu^+_{(s)}}\right) \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix} \\
& + \sum_{s'=1}^{b'(\mu)} a_{e_1} H\left(\frac{\lambda\mu}{\lambda^+_{(1)}\mu^-_{(s')}}\right) a^2_{d_{s'}} \left(a_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix}'\right) \\
& + a_{e_1} \sum_{s'=1}^{b'(\mu)} H\left(\frac{\lambda\mu}{\lambda^+_{(1)}\mu^-_{(s')}}\right) a_{d_{s'}} b_{d_{s'}} \left(a_f \begin{bmatrix} \mu_{(s')} \\ \mu^-_{(s')} \end{bmatrix} + b_f \begin{bmatrix} \mu_{(s')} \\ \mu^-_{(s')} \end{bmatrix}'\right) \\
& + \sum_{r'=2}^{b'(\lambda)} H\left(\frac{\lambda^2}{\lambda^-_{(r')}\lambda^+_{(1)}}\right) a_{d'_{\sigma(r')}} \left(-a_{d_{\sigma(r')}}\right) \left(a_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix} + b_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix}'\right) \\
& + \sum_{r'=2}^{b'(\lambda)} H\left(\frac{\lambda^2}{\lambda^-_{(r')}\lambda^+_{(1)}}\right) a_{d'_{\sigma(r')}} b_{d_{\sigma(r')}} \left(a_f \begin{bmatrix} \mu \\ \nu_{(r')} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \nu_{(r')} \end{bmatrix}'\right).
\end{aligned}$$

Here we applied Lemma 2.5 2, 3 to obtain the first line of the equation above. Applying Lemma 2.2 and 2.4 to the second line, the equation (13) and Lemma 2.3 to the third line from the bottom, and the equation (9) and Lemma 2.3 to the last line, we find that this equation is equal to the following:

$$\begin{aligned}
& \sum_{s=1}^{b(\mu)} H\left(\frac{\lambda\mu}{\lambda^+_{(1)}\mu^+_{(s)}}\right)\left(a_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix}'\right) \\
& + a_{e_1} \sum_{r=1}^{b(\lambda)} H\left(\frac{\lambda^2}{\lambda^+_{(r)}\lambda^+_{(1)}}\right) a_{e_r} \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + a_{e_1} \sum_{s=2}^{b(\mu)} H\left(\frac{\lambda\mu}{\lambda^+_{(1)}\mu^+_{(s)}}\right) \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix} \\
& + \sum_{s'=1}^{b'(\mu)} H\left(\frac{\lambda\mu}{\lambda^+_{(1)}\mu^-_{(s')}}\right) \left(a_{e_1} a^2_{d_{s'}} + a_{d'_{s'}} b^2_{d_{s'}}\right) \left(a_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix}'\right) \\
& + \sum_{r'=2}^{b'(\lambda)} H\left(\frac{\lambda^2}{\lambda^-_{(r')}\lambda^+_{(1)}}\right) \left(a_{d_{\sigma(r')}} \left(a_{e_1} - a_{d'_{\sigma(r')}}\right)\right) \left(a_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix} + b_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix}'\right).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
s_{i+1} s_i s_{i+1} v_P = {} & a_{e_1} \sum_{r=1}^{b(\lambda)} H\left(\frac{\lambda^2}{\lambda^+_{(r)}\lambda^+_{(1)}}\right)\left(a_{e_r} \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_{e_r} \begin{bmatrix} \lambda^+_{(r)} \\ \lambda_{(r)} \end{bmatrix}\right) \\
& + \sum_{s=1}^{b(\mu)} H\left(\frac{\lambda\mu}{\lambda^+_{(1)}\mu^+_{(s)}}\right) a^2_{e_1} \left(a_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix}'\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{s'=1}^{b'(\mu)} H\left(\frac{\lambda\mu}{\lambda^+_{(1)}\mu^-_{(s')}}\right) a^2_{e_1} a_{d_{s'}} \left(a_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix}'\right) \\
&+ a_{e_1} \sum_{r'=2}^{b'(\lambda)} H\left(\frac{\lambda^2}{\lambda^-_{(r')}\lambda^+_{(1)}}\right) a_{d'_{\sigma(r')}} \left(a_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix} + b_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix}'\right) \\
&+ b_{e_1} \sum_{s=1}^{b(\mu)} H\left(\frac{\mu^2}{\mu^+_{(s)}\lambda'}\right) \left(a_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix}'\right) \\
&+ b_{e_1} \sum_{s'=1}^{b'(\mu)} H\left(\frac{\mu^2}{\mu^-_{(s')}\lambda'}\right) a_{d'_{s'}} \left(a_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix}'\right).
\end{aligned}
$$

Now we apply the equation (10) and Lemma 2.2 to the first line, and multiply the last two lines of the equation above by

$$
b_{e_1} H\left(\frac{\lambda\lambda'}{\mu\lambda^+_{(1)}}\right) = \begin{cases} 1 & \text{if} \quad b_{e_1} \neq 0 \\ 0 & \text{if} \quad b_{e_1} = 0 \end{cases}.
$$

Then this equation will be equal to the following:

$$
\begin{aligned}
&a_{e_1} \sum_{r=1}^{b(\lambda)} H\left(\frac{\lambda^2}{\lambda^+_{(r)}\lambda^+_{(1)}}\right) a_{e_r} \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + a_{e_1} \sum_{s=2}^{b(\mu)} H\left(\frac{\lambda\mu}{\lambda^+_{(1)}\mu^+_{(s)}}\right) \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix} \\
&+ \sum_{s=1}^{b(\mu)} H\left(\frac{\lambda\mu}{\lambda^+_{(1)}\mu^+_{(s)}}\right) \left(a_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix}'\right) \\
&+ \sum_{s'=1}^{b'(\mu)} H\left(\frac{\lambda\mu}{\lambda^+_{(1)}\mu^-_{(s')}}\right) \left(a^2_{e_1} a_{d_{s'}} + b^2_{e_1} a_{d'_{s'}}\right) \left(a_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix}'\right) \\
&+ \sum_{r'=2}^{b'(\lambda)} H\left(\frac{\lambda^2}{\lambda^-_{(r')}\lambda^+_{(1)}}\right) a_{e_1} a_{d'_{\sigma(r')}} \left(a_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix} + b_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix}'\right).
\end{aligned}
$$

Applying Lemma 2.1 2, 1 to the last two lines of the equation above, we obtain the same equation as $s_i s_{i+1} s_i v_P$.

CASE 6.1.2. $j_2 = j_1 = j_0 + 1,\ \alpha^{(i-1)} \neq \alpha^{(i+1)}$.

Table 13. Case 6.1.2 $j_2 = j_1 = j_0 + 1,\ \alpha^{(i-1)} \neq \alpha^{(i+1)}$

coordinate	board		
	j_0	$j_1 = j_2$	$j_1 + 1$
$i-1$	α^+	λ	ξ
i	α	$\lambda^+_{(1)}$	ξ
$i+1$	α	λ'	η
$i+2$	α	μ	ζ

In this case we assumed $\lambda = \alpha^{(i-1)}(j_1+1)$ and $\lambda' = \alpha^{(i+1)}(j_1+1)$ are distinct. So we find $e_1 \neq \pm 1$ or $b_{e_1} \neq 0$. By the same method shown in Case 6.1.1, using Table 13 we can put

$$v_P = \begin{bmatrix} \lambda^+_{(1)} \\ \lambda' \end{bmatrix}. \tag{28}$$

Then we have

$$s_i v_P = \begin{bmatrix} \mu \\ \lambda' \end{bmatrix}. \tag{29}$$

Using the notation (17), we have

$$s_{i+1}s_i v_P = \sum_{s=1}^{b(\mu)} H\left(\frac{\mu^2}{\mu^+_{(s)}\lambda'}\right)\begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix} + \sum_{s'=1}^{b'(\mu)} H\left(\frac{\mu^2}{\mu^-_{(s')}\lambda'}\right) a_{d'_{s'}} \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix}. \tag{30}$$

Hence using the notation in Lemma 2.2 and 2.3, we have

$$\begin{aligned} s_i s_{i+1} s_i v_P = &\sum_{r=1}^{b(\lambda)} H\left(\frac{\mu\lambda}{\lambda'\lambda^+_{(r)}}\right)\left(a_{e_r}\begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_{e_r}\begin{bmatrix} \lambda^+_{(r)} \\ \lambda_{(r)} \end{bmatrix}\right) \\ &+ \sum_{s'=1}^{b'(\mu)} H\left(\frac{\mu^2}{\mu^-_{(s')}\lambda'}\right) a_{d'_{s'}} a_{d_{s'}} \left(a_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix}'\right) \\ &+ \sum_{r'=2}^{b'(\lambda)} H\left(\frac{\mu\lambda}{\lambda^-_{(r')}\lambda'}\right) a_{d'_{\sigma(r')}} \left(a_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix} + b_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix}'\right). \end{aligned} \tag{31}$$

Here we used the equation (10) and Lemma 2.2 (resp. the equations (11), (18) and Lemma 2.3) to obtain the second (resp. bottom) line of the equation above.

On the other hand, we have

$$s_{i+1}v_P = -a_{e_1}\left(a_f\begin{bmatrix} \lambda^+_{(1)} \\ \lambda' \end{bmatrix} + b_f \begin{bmatrix} \lambda^+_{(1)} \\ \lambda' \end{bmatrix}\right) + b_{e_1}\left(a_f\begin{bmatrix} \lambda^+_{(1)} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^+_{(1)} \\ \lambda \end{bmatrix}\right). \tag{32}$$

Since $b_{e_1} \neq 0$ in this case, using (18), (19) and $b_{e_1} = H\left(\mu\lambda^+_{(1)} \big/ (\lambda\lambda')\right)$ we have

$$\begin{aligned} s_i s_{i+1} v_P = &-a_{e_1}\left(a_f\begin{bmatrix} \mu \\ \lambda' \end{bmatrix} + b_f\begin{bmatrix} \mu \\ \lambda' \end{bmatrix}'\right) \\ &+ \sum_{r=1}^{b(\lambda)} H\left(\frac{\mu\lambda}{\lambda'\lambda^+_{(r)}}\right)\left(a_f\begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_f\begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix}'\right) \\ &+ \sum_{r'=1}^{b'(\lambda)} H\left(\frac{\mu\lambda}{\lambda'\lambda^-_{(r')}}\right) a_{d'_{\sigma(r')}}\left(a_f\begin{bmatrix} \lambda^-_{(r')} \\ \lambda \end{bmatrix} + b_f\begin{bmatrix} \lambda^-_{(r')} \\ \lambda \end{bmatrix}'\right). \end{aligned} \tag{33}$$

Hence we have

$$\begin{aligned} s_{i+1}s_is_{i+1}v_P = &\sum_{s'=1}^{b'(\mu)} H\left(\frac{\mu^2}{\mu^-_{(s')}\lambda'}\right)\left(-a_{e_1}a_{d'_{s'}} + a_{e_1}a_{d_{s'}}\right)\left(a_f\begin{bmatrix}\mu\\ \mu^-_{(s')}\end{bmatrix} + b_f\begin{bmatrix}\mu\\ \mu^-_{(s')}\end{bmatrix}'\right)\\ &+\sum_{r=1}^{b(\lambda)} H\left(\frac{\mu\lambda}{\lambda'\lambda^+_{(r)}}\right)\left(a_{e_r}\begin{bmatrix}\lambda^+_{(r)}\\ \lambda\end{bmatrix} + b_{e_r}\begin{bmatrix}\lambda^+_{(r)}\\ \lambda_{(r)}\end{bmatrix}\right)\\ &+\sum_{r'=2}^{b'(\lambda)} H\left(\frac{\mu\lambda}{\lambda^-_{(r')}\lambda'}\right)a_{d'_{\sigma(r')}}\left(a_f\begin{bmatrix}\lambda^-_{(r')}\\ \nu_{(r')}\end{bmatrix} + b_f\begin{bmatrix}\lambda^-_{(r')}\\ \nu_{(r')}\end{bmatrix}'\right). \end{aligned}$$

Since $e_1 + d_{s'} = d'_{s'}$, using Lemma 2.1 we have $-a_{e_1}a_{d'_{s'}} + a_{e_1}a_{d_{s'}} = a_{d'_{s'}}a_{d_{s'}}$. Comparing the equation (31), in this case we find $s_is_{i+1}s_iv_P = s_{i+1}s_is_{i+1}v_P$.

CASE 6.2. (Case 6.5.) $j_1 = j_0 = j_2 + 1$.

For the explanation below, see Table 14 and Table 15. Put

$$\left(\alpha^{(i-1)}(j_2), \alpha^{(i-1)}(j_0), \alpha^{(i-1)}(j_0+1)\right) = (\alpha, \lambda, \xi).$$

The assumption $j_1 = j_0 = j_2 + 1$ means the following.

1) On the way from $\boldsymbol{\alpha}^{(i-1)}$ to $\boldsymbol{\alpha}^{(i+1)}$, two boxes of $\lambda = \alpha^{(i-1)}(j_0)$ on the j_0-th board are moved to the (j_0+1)-st board. We put

$$\left(\alpha^{(i+1)}(j_2), \alpha^{(i+1)}(j_0), \alpha^{(i+1)}(j_0+1)\right) = \left(\alpha, \mu^-_{(1)}, \zeta\right).$$

2) Then on the way from $\boldsymbol{\alpha}^{(i+1)}$ to $\boldsymbol{\alpha}^{(i+2)}$, one box of $\alpha = \alpha^{(i+1)}(j_2)$ on the j_2-th board is attached to $\alpha^{(i+1)}(j_0) = \mu^-_{(1)}$ on the j_0-th board. We put

$$\left(\alpha^{(i+2)}(j_2), \alpha^{(i+2)}(j_0), \alpha^{(i+2)}(j_0+1)\right) = (\alpha^-, \mu, \zeta).$$

Similarly as in Case 6.1, we have only to consider the case $\lambda \rhd \alpha^{(i)}(j_0) \rhd \mu^-_{(1)}$. Namely, if we put $\mu' = \mu^-_{(1)} \cup (\lambda \setminus \mu)$, the following cases should be considered:

$$\left(\alpha^{(i)}(j_2), \alpha^{(i)}(j_0), \alpha^{(i)}(j_0+1)\right) = \begin{cases}(\alpha, \mu', \eta) & \text{Case 6.2.1},\\ (\alpha, \mu, \eta) & \text{Case 6.2.2}.\end{cases}$$

CASE 6.2.1. $j_1 = j_0 = j_2 + 1$, $\alpha^{(i)}(j_0) \neq \alpha^{(i+2)}(j_0)$.

Table 14. Case 6.2.1 $j_1 = j_0 = j_2 + 1$, $\alpha^{(i)}(j_0) \neq \alpha^{(i+2)}(j_0)$

coordinate	board		
	j_2	$j_0 = j_1$	$j_0 + 1$
$i-1$	α	λ	ξ
i	α	μ'	η
$i+1$	α	$\mu^-_{(1)}$	ζ
$i+2$	α^-	μ	ζ

Since $\mu \neq \mu'$ we have $d_1 \neq \pm 1$ and $b_{d_1} \neq 0$. By the same method shown in Case 6.1.1, using Table 14 we can put

$$v_P = \begin{bmatrix} \mu' \\ \mu^-_{(1)} \end{bmatrix}. \tag{34}$$

Then we have

$$s_i v_P = -a_{d_1}\left(a_f \begin{bmatrix} \mu' \\ \mu^-_{(1)} \end{bmatrix} + b_f \begin{bmatrix} \mu' \\ \mu^-_{(1)} \end{bmatrix}'\right) + b_{d_1}\left(a_f \begin{bmatrix} \mu \\ \mu^-_{(1)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^-_{(1)} \end{bmatrix}'\right). \tag{35}$$

Since $b_{d_1} \neq 0$, using the notation (15), (16) and $b_{d_1} = H\left(\mu^-_{(1)}\lambda \,\big/\, (\mu\mu')\right)$ we have

$$\begin{aligned} s_{i+1}s_i v_P = &-a_{d_1}\left(a_f \begin{bmatrix} \mu' \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \mu' \\ \lambda \end{bmatrix}'\right) \\ &+ \sum_{s=1}^{b(\mu)} H\left(\frac{\mu\lambda}{\mu'\mu^+_{(s)}}\right) a_{e'_{\tau(s)}} \left(a_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix}'\right). \end{aligned} \tag{36}$$

Hence using the notation (14), we obtain

$$\begin{aligned} s_i s_{i+1} s_i v_P = &\sum_{r=1}^{b(\lambda)} H\left(\frac{\lambda^2}{\lambda^+_{(r)}\mu'}\right)\left(-a_{d_1}a_{e'_r} + a_{d_1}a_{e_r}\right)\left(a_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix}'\right) \\ &+ \sum_{s=2}^{b(\mu)} H\left(\frac{\mu\lambda}{\mu'\mu^+_{(s)}}\right) a_{e'_{\tau(s)}} \left(a_f \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix}'\right). \end{aligned}$$

On the other hand we have

$$s_{i+1} v_P = \begin{bmatrix} \mu' \\ \lambda \end{bmatrix}. \tag{37}$$

Hence we have

$$\begin{aligned} s_{i+1}s_i s_{i+1} v_P = &\sum_{r=1}^{b(\lambda)} H\left(\frac{\lambda^2}{\mu'\lambda^+_{(r)}}\right) a_{e'_r}a_{e_r}\left(a_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix}'\right) \\ &+ \sum_{s=2}^{b(\mu)} H\left(\frac{\mu\lambda}{\mu'\mu^+_{(s)}}\right) a_{e'_{\tau(s)}} \left(a_f \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix}'\right). \end{aligned} \tag{38}$$

Here we used the equations (10), (15) and Lemma 2.2 to obtain the second line of the equation above. Since $d_1 + e_r = e'_r$, we have $-a_{d_1}a_{e'_r} + a_{d_1}a_{e_r} = a_{e'_r}a_{e_r}$ by Lemma 2.1. Hence in this case we obtain $s_i s_{i+1} s_i v_P = s_{i+1} s_i s_{i+1} v_P$.

CASE 6.2.2. $j_1 = j_0 = j_2 + 1,\ \alpha^{(i)}(j_0) = \alpha^{(i+2)}(j_0)$.

By the same method shown in Case 6.1.1, using Table 15 we can put

$$v_P = \begin{bmatrix} \mu \\ \mu^-_{(1)} \end{bmatrix}.$$

Then using the notation (15) and (16), we have

$$\begin{aligned} s_{i+1}s_i v_P &= a_{d_1} \sum_{s=1}^{b(\mu)} H\left(\frac{\mu^2}{\mu^-_{(1)}\mu^+_{(s)}}\right) a_{e'_{\tau(s)}} \left(a_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix}'\right) \\ &\quad + b_{d_1}\left(a_f \begin{bmatrix} \mu' \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \mu' \\ \lambda \end{bmatrix}'\right). \end{aligned}$$

By Lemma 2.4 we have $b^2_{d_1} = b_{d_1} H\big(\mu\mu' \,/\, (\mu^-_{(1)}\lambda)\big)$. Using this, we find that

$$\begin{aligned} s_i s_{i+1} s_i v_P &= \sum_{r=1}^{b(\lambda)} H\left(\frac{\mu\lambda}{\mu^-_{(1)}\lambda^+_{(r)}}\right) (a_{e_r} a^2_{d_1} + a_{e'_r} b^2_{d_1}) \left(a_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix}'\right) \\ &\quad + \sum_{s=2}^{b(\mu)} H\left(\frac{\mu^2}{\mu^-_{(1)}\mu^+_{(s)}}\right) a_{d_1} a_{e'_{\tau(s)}} \left(a_f \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix}'\right). \end{aligned}$$

On the other hand we have

$$\begin{aligned} s_{i+1} s_i s_{i+1} v_P &= \sum_{r=1}^{b(\lambda)} H\left(\frac{\mu\lambda}{\mu^-_{(1)}\lambda^+_{(r)}}\right) a_{d_1} a^2_{e_r} \left(a_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix}'\right) \\ &\quad + \sum_{r=1}^{b(\lambda)} H\left(\frac{\mu\lambda}{\mu^-_{(1)}\lambda^+_{(r)}}\right) a_{d_1} a_{e_r} b_{e_r} \left(a_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda_{(r)} \end{bmatrix} + b_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda_{(r)} \end{bmatrix}'\right) \\ &\quad + \sum_{s=2}^{b(\mu)} H\left(\frac{\mu^2}{\mu^-_{(1)}\mu^+_{(s)}}\right) a_{e'_{\tau(s)}} (-a_{e_{\tau(s)}}) \left(a_f \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \kappa_{(s)} \\ \mu^+_{(s)} \end{bmatrix}'\right) \\ &\quad + \sum_{s=2}^{b(\mu)} H\left(\frac{\mu^2}{\mu^-_{(1)}\mu^+_{(s)}}\right) a_{e'_{\tau(s)}} b_{e_{\tau(s)}} \left(a_f \begin{bmatrix} \kappa_{(s)} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \kappa_{(s)} \\ \lambda \end{bmatrix}'\right). \end{aligned}$$

Table 15. Case 6.2.2 $j_1 = j_0 = j_2 + 1,\ \alpha^{(i)}(j_0) = \alpha^{(i+2)}(j_0)$

coordinate	board		
	j_2	$j_0 = j_1$	$j_0 + 1$
$i-1$	α	λ	ξ
i	α	μ	η
$i+1$	α	$\mu^-_{(1)}$	ζ
$i+2$	α^-	μ	ζ

Applying the equations (10), (12) (resp. the equations (15), (8) and Lemma 2.2) to the second (resp. bottom) line of the equation above, we obtain

$$\begin{aligned} s_{i+1}s_is_{i+1}v_P &= \sum_{r=1}^{b(\lambda)} H\left(\frac{\mu\lambda}{\mu^-_{(1)}\lambda^+_{(r)}}\right)\left(a_{d_1}a^2_{e_r} + a_{e'_r}b^2_{e_r}\right)\left(a_f\begin{bmatrix}\lambda^+_{(r)}\\ \lambda\end{bmatrix} + b_f\begin{bmatrix}\lambda^+_{(r)}\\ \lambda\end{bmatrix}'\right) \\ &\quad + \sum_{s=2}^{b(\mu)} H\left(\frac{\mu^2}{\mu^-_{(1)}\mu^+_{(s)}}\right)\left(a_{d_1}a_{e_{\tau(s)}} - a_{e'_{\tau(s)}}a_{e_{\tau(s)}}\right)\left(a_f\begin{bmatrix}\kappa_{(s)}\\ \mu^+_{(s)}\end{bmatrix} + b_f\begin{bmatrix}\kappa_{(s)}\\ \mu^+_{(s)}\end{bmatrix}'\right). \end{aligned}$$

Since $d_1 + e_r = e'_r$ and $d_1 + e_{\tau(s)} = e'_{\tau(s)}$, using Lemma 2.1 2, 1 respectively we find $s_is_{i+1}s_iv_P = s_{i+1}s_is_{i+1}v_P$ in this case.

CASE 6.3. (Case 6.6.) $j_0 = j_2 = j_1 + 1$.

For the explanation below, see Table 16, 17 and 18. Put

$$\left(\alpha^{(i-1)}(j_1), \alpha^{(i-1)}(j_0), \alpha^{(i-1)}(j_0+1)\right) = (\alpha, \lambda, \xi).$$

The assumption $j_0 = j_2 = j_1 + 1$ means the following:

1) On the way from $\boldsymbol{\alpha}^{(i-1)}$ to $\boldsymbol{\alpha}^{(i)}$, one box of $\lambda = \alpha^{(i-1)}(j_0)$ on the j_0-th board is removed and it is attached to $\xi = \alpha^{(i-1)}(j_1)$ on the j_1-th board. We put $\alpha^{(i)}(j_0+1) = \eta$.
2) Then on the way from $\boldsymbol{\alpha}^{(i)}$ to $\boldsymbol{\alpha}^{(i+1)}$, one box of $\alpha = \alpha^{(i)}(j_1)$ on the j_1-th board is removed and it is attached to $\alpha^{(i)}(j_0)$ on the j_0-th board. We put $\alpha^{(i+1)}(j_1) = \alpha^-$.
3) Then on the way from $\boldsymbol{\alpha}^{(i+1)}$ to $\boldsymbol{\alpha}^{(i+2)}$, one box of $\alpha^{(i+1)}(j_0)$ on the j_0-th board are moved to the next board. We put

$$\left(\alpha^{(i+2)}(j_1), \alpha^{(i+2)}(j_0), \alpha^{(i+2)}(j_0+1)\right) = (\alpha^-, \mu, \zeta).$$

As well as the Case 6.1, we find that the only following cases should be considered:

$$\left(\alpha^{(i)}(j_0), \alpha^{(i+1)}(j_0)\right) = \begin{cases} (\mu', \lambda) & \text{(Case 6.3.1)}, \\ (\mu, \lambda') & \text{(Case 6.3.2)}, \\ (\mu, \lambda) & \text{(Case 6.3.3)}. \end{cases}$$

Here μ' and λ' are Young diagrams such that $\lambda \rhd \mu'$, $\mu \lhd \lambda'$, $\lambda' \neq \lambda$ and $\mu' \neq \mu$. Note also that in these cases $\lambda \rhd \mu$.

CASE 6.3.1. $j_0 = j_2 = j_1 + 1$, $\alpha^{(i+1)}(j_0) = \alpha^{i-1}(j_0)$, $\alpha^{(i+2)}(j_0) \neq \alpha^{(i)}(j_0)$.

By the same method shown in Case 6.1.1, using Table 16 we can put

$$v_P = \begin{bmatrix}\mu'\\ \lambda\end{bmatrix}. \tag{39}$$

Using the calculation (37), and (38) in Case 6.2.1, in this case we have

(40)
$$\begin{aligned} s_{i+1}s_i v_P &= \sum_{r=1}^{b(\lambda)} H\left(\frac{\lambda^2}{\mu'\lambda^+_{(r)}}\right) a_{e_r}a_{e'_r}\left(a_f\begin{bmatrix}\lambda^+_{(r)}\\ \lambda\end{bmatrix} + b_f\begin{bmatrix}\lambda^+_{(r)}\\ \lambda\end{bmatrix}'\right)\\ &+\sum_{s=2}^{b(\mu)} H\left(\frac{\mu\lambda}{\mu'\mu^+_{(s)}}\right) a_{e'_{\tau(s)}}\left(a_f\begin{bmatrix}\kappa_{(s)}\\ \mu^+_{(s)}\end{bmatrix} + b_f\begin{bmatrix}\kappa_{(s)}\\ \mu^+_{(s)}\end{bmatrix}'\right). \end{aligned}$$

Hence we have

(41)
$$\begin{aligned} s_is_{i+1}s_i v_P &= \sum_{q=1}^{b(\lambda)} H\left(\frac{\lambda^2}{\mu'\lambda^+_{(q)}}\right)\left(\sum_{r=1}^{b(\lambda)}\frac{h(\lambda)}{h(\lambda^+_{(r)})}a_{e_r}a_{e'_r}\right)\left(a_f\begin{bmatrix}\lambda^+_{(q)}\\ \lambda\end{bmatrix} + b_f\begin{bmatrix}\lambda^+_{(q)}\\ \lambda\end{bmatrix}'\right)\\ &+\sum_{q'=1}^{b'(\lambda)} H\left(\frac{\lambda^2}{\mu'\lambda^-_{(q')}}\right)\left(\sum_{r=1}^{b(\lambda)}\frac{h(\lambda)}{h(\lambda^+_{(r)})}a_{e_r}a_{e'_r}a_{e_{q',r}}\right)\left(a_f\begin{bmatrix}\lambda^-_{(q')}\\ \lambda\end{bmatrix} + b_f\begin{bmatrix}\lambda^-_{(q')}\\ \lambda\end{bmatrix}'\right)\\ &+\sum_{s=2}^{b(\mu)} H\left(\frac{\mu\lambda}{\mu'\mu^+_{(s)}}\right) a_{e'_{\tau(s)}}\left(a_f\begin{bmatrix}\mu\\ \mu^+_{(s)}\end{bmatrix} + b_f\begin{bmatrix}\mu\\ \mu^+_{(s)}\end{bmatrix}'\right) \end{aligned}$$

The first line of the equation (41) should be 0 by Lemma 2.5 3. As for the second line, since $d_1 + e_r = e'_r$, we have $a_{e_r}a_{e'_r} = a_{d_1}(a_{e_r} - a_{e'_r})$ by Lemma 2.1. Hence as well as the first line, using Lemma 2.5 3 we have

$$\begin{aligned} \sum_{r=1}^{b(\lambda)}\frac{h(\lambda)}{h(\lambda^+_{(r)})}a_{e_r}a_{e'_r}a_{e_{q',r}} &= a_{d_1}\sum_{r=1}^{b(\lambda)}\frac{h(\lambda)}{h(\lambda^+_{(r)})}(a_{e_r}a_{e_{q',r}} - a_{e'_r}a_{e_{q',r}})\\ &= \begin{cases} a_{d_1}\dfrac{h(\mu)}{h(\lambda)} & \text{if } \ q' = 1 \text{ so } \lambda^-_{(q')} = \mu,\\ -a_{d_1}\dfrac{h(\mu')}{h(\lambda)} & \text{if } \ \lambda^-_{(q')} = \mu',\\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

Table 16. Case 6.3.1 $j_0 = j_2 = j_1 + 1$, $\alpha^{(i+1)}(j_0) = \alpha^{(i-1)}(j_0)$, $\alpha^{(i+2)}(j_0) \neq \alpha^{(i)}(j_0)$.

coordinate	board		
	j_1	$j_0 = j_2$	$j_0 + 1$
$i-1$	α	λ	ξ
i	α	μ'	η
$i+1$	α^-	λ	η
$i+2$	α^-	μ	ζ

Hence we obtain

$$
\begin{aligned}
s_i s_{i+1} s_i v_P = H\left(\frac{\mu}{\mu'}\right) a_{d_1} \left(a_f \begin{bmatrix} \mu \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \lambda \end{bmatrix}'\right) - a_{d_1} \left(a_f \begin{bmatrix} \mu' \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \mu' \\ \lambda \end{bmatrix}'\right) \\
+ \sum_{s=2}^{b(\mu)} H\left(\frac{\mu\lambda}{\mu' \mu^+_{(s)}}\right) a_{e'_{\tau(s)}} \left(a_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix}'\right).
\end{aligned}
$$

On the other hand, since we have

$$
s_{i+1} v_P = \begin{bmatrix} \mu' \\ \mu^-_{(1)} \end{bmatrix},
$$

using the calculation (34), (35) and (36) in Case 6.2.1 we have

$$
\begin{aligned}
s_{i+1} s_i s_{i+1} v_P = -a_{d_1} \left(a_f \begin{bmatrix} \mu' \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \mu' \\ \lambda \end{bmatrix}'\right) \\
+ \sum_{s=1}^{b(\mu)} H\left(\frac{\mu\lambda}{\mu' \mu^+_{(s)}}\right) a_{e'_{\tau(s)}} \left(a_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix}'\right).
\end{aligned}
$$

Since $e'_{\tau(1)} = d_1$ by (16), we find that in this case $s_i s_{i+1} s_i v_P = s_{i+1} s_i s_{i+1} v_P$.

CASE 6.3.2. $j_0 = j_2 = j_1 + 1$, $\alpha^{(i+1)}(j_0) \neq \alpha^{(i-1)}(j_0)$, $\alpha^{(i+2)}(j_0) = \alpha^{(i)}(j_0)$.

By the same method shown in Case 6.1.1, using Table 17 we can put

$$
v_P = \begin{bmatrix} \mu \\ \lambda' \end{bmatrix}.
$$

Since

$$
s_i v_P = \begin{bmatrix} \lambda^+_{(1)} \\ \lambda' \end{bmatrix},
$$

Table 17. Case 6.3.2 $j_0 = j_2 = j_1 + 1$, $\alpha^{(i+1)}(j_0) \neq \alpha^{(i-1)}(j_0)$, $\alpha^{(i+2)}(j_0) = \alpha^{(i)}(j_0)$.

coordinate	board		
	j_1	$j_0 = j_2$	$j_0 + 1$
$i-1$	α	λ	ξ
i	α	μ	η
$i+1$	α^-	λ'	η
$i+2$	α^-	μ	ζ

using the calculation (28), (32) and (33) in Case 6.1.2 we have

$$
\begin{aligned}
s_i s_{i+1} s_i v_P = & -a_{e_1}\left(a_f \begin{bmatrix} \mu \\ \lambda' \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \lambda' \end{bmatrix}'\right) \\
& + \sum_{r=1}^{b(\lambda)} H\left(\frac{\mu\lambda}{\lambda'\lambda^+_{(r)}}\right)\left(a_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix}'\right) \\
& + \sum_{r'=1}^{b'(\lambda)} H\left(\frac{\mu\lambda}{\lambda^-_{(r')}\lambda'}\right) a_{d'_{\sigma(r')}}\left(a_f \begin{bmatrix} \lambda^-_{(r')} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^-_{(r')} \\ \lambda \end{bmatrix}'\right).
\end{aligned}
$$

On the other hand, using the calculation (29), (30) and (31) in Case 6.1.2, in this case we have

$$
\begin{aligned}
s_i s_{i+1} v_P = & \sum_{r=1}^{b(\lambda)} H\left(\frac{\mu\lambda}{\lambda'\lambda^+_{(r)}}\right)\left(a_{e_r} \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_{e_r} \begin{bmatrix} \lambda^+_{(r)} \\ \lambda_{(r)} \end{bmatrix}\right) \\
& + \sum_{s'=1}^{b'(\mu)} H\left(\frac{\mu^2}{\mu^-_{(s')}\lambda'}\right) a_{d'_{s'}} a_{d_{s'}}\left(a_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix}\right) \\
& + \sum_{r'=2}^{b'(\lambda)} H\left(\frac{\mu\lambda}{\lambda^-_{(r')}\lambda'}\right) a_{d'_{\sigma(r')}}\left(a_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix} + b_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix}'\right).
\end{aligned}
$$

Hence using the notation (20), we have

$$
\begin{aligned}
s_{i+1} s_i s_{i+1} v_P = & \sum_{r=1}^{b(\lambda)} H\left(\frac{\mu\lambda}{\lambda'\lambda^+_{(r)}}\right)\left(a_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix}'\right) \\
& + \sum_{s=1}^{b(\mu)} H\left(\frac{\mu^2}{\mu^+_{(s)}\lambda'}\right) \sum_{s'=1}^{b'(\mu)} \frac{h(\mu)}{h(\mu^-_{(s')})} a_{d'_{s'}} a_{d_{s'}} a_{d_{s',s}}\left(a_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix}'\right) \\
& + \sum_{r'=2}^{b'(\lambda)} H\left(\frac{\mu\lambda}{\lambda^-_{(r')}\lambda'}\right) a_{d'_{\sigma(r')}}\left(a_f \begin{bmatrix} \lambda^-_{(r')} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^-_{(r')} \\ \lambda \end{bmatrix}'\right).
\end{aligned}
$$

As for the second line of the equation above, since $e_1 + d_{s'} = d'_{s'}$, by Lemma 2.1 1 we have the following:

$$
\left(a_{d'_{s'}} a_{d_{s'}}\right) a_{d_{s',s}} = \left(a_{e_1} a_{d_{s'}} - a_{e_1} a_{d'_{s'}}\right) a_{d_{s',s}} = a_{e_1}\left(a_{d_{s'}} a_{d_{s',s}} - a_{d_{s'}} a_{d_{s',s}}\right).
$$

Hence using Lemma 2.5 4, we have

$$\sum_{s'=1}^{b'(\mu)} \frac{h(\mu)}{h\left(\mu^-_{(s')}\right)} a_{d'_{s'}} a_{d_{s'}} a_{d_{s',s}} = a_{e_1} \sum_{s'=1}^{b'(\mu)} \frac{h(\mu)}{h\left(\mu^-_{(s')}\right)} \left(a_{d_{s'}} a_{d_{s',s}} - a_{d'_{s'}} a_{d_{s',s}} \right)$$
$$= \begin{cases} a_{e_1} \dfrac{h(\lambda)}{h(\mu)} & \text{if} \quad s = 1 \quad \text{so} \quad \mu^+_{(s)} = \lambda, \\ -a_{e_1} \dfrac{h(\lambda')}{h(\mu)} & \text{if} \quad \mu^+_{(s)} = \lambda', \\ 0 & \text{otherwise.} \end{cases}$$

Thus we obtain

$$\begin{aligned} s_{i+1}s_i s_{i+1} v_P = & \sum_{r=1}^{b(\lambda)} H\left(\frac{\mu\lambda}{\mu'\lambda^+_{(r)}}\right) \left(a_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix}' \right) \\ & + H\left(\frac{\lambda}{\lambda'}\right) a_{e_1} \left(a_f \begin{bmatrix} \mu \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \lambda \end{bmatrix}' \right) \\ & - a_{e_1} \left(a_f \begin{bmatrix} \mu \\ \lambda' \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \lambda' \end{bmatrix}' \right) \\ & + \sum_{r'=2}^{b'(\lambda)} H\left(\frac{\mu\lambda}{\lambda^-_{(r')}\lambda'}\right) a_{d'_{\sigma(r')}} \left(a_f \begin{bmatrix} \lambda^-_{(r')} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^-_{(r')} \\ \lambda \end{bmatrix}' \right). \end{aligned}$$

Since $d'_{\sigma(1)} = e_1$, in this case we obtain $s_i s_{i+1} s_i v_P = s_{i+1} s_i s_{i+1} v_P$.

CASE 6.3.3. $j_0 = j_2 = j_1 + 1$, $\alpha^{(i+1)}(j_0) = \alpha^{(i-1)}(j_0)$, $\alpha^{(i+2)}(j_0) = \alpha^{(i)}(j_0)$.

By the same method shown in Case 6.1.1, using Table 18 we can put

$$v_P = \begin{bmatrix} \mu \\ \lambda \end{bmatrix}.$$

Table 18. Case 6.3.3 $j_0 = j_2 = j_1 + 1$, $\alpha^{(i+1)}(j_0) = \alpha^{(i-1)}(j_0)$, $\alpha^{(i+2)}(j_0) = \alpha^{(i)}(j_0)$.

coordinate	board		
	j_1	$j_0 = j_2$	$j_0 + 1$
$i - 1$	α	λ	ξ
i	α	μ	η
$i + 1$	α^-	λ	η
$i + 2$	α^-	μ	ζ

Replacing μ' with μ in the calculation (39), (40) and (41) in Case 6.3.1, we have

(42)
$$\begin{aligned} s_i s_{i+1} s_i v_P &= \sum_{q=1}^{b(\lambda)} H\left(\frac{\mu}{\lambda^+_{(q)}}\right)\left(a_f \begin{bmatrix} \lambda^+_{(q)} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^+_{(q)} \\ \lambda \end{bmatrix}'\right) \\ &+ \frac{h(\lambda)^2}{h(\mu)} \sum_{r=1}^{b(\lambda)} \frac{a_{e_r}^3}{h\left(\lambda^+_{(r)}\right)} \left(a_f \begin{bmatrix} \mu \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \lambda \end{bmatrix}'\right) \\ &+ \sum_{q'=2}^{b'(\lambda)} H\left(\frac{\lambda^2}{\mu\lambda^-_{(q')}}\right) \sum_{r=1}^{b(\lambda)} \frac{h(\lambda)}{h\left(\lambda^+_{(r)}\right)} a_{e_r}^2 a_{e_{q',r}} \left(a_f \begin{bmatrix} \lambda^-_{(q')} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^-_{(q')} \\ \lambda \end{bmatrix}'\right) \\ &+ \sum_{r=1}^{b(\lambda)} a_{e_{\tau(s)}} H\left(\frac{\lambda}{\mu^+_{(s)}}\right)\left(a_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix}'\right). \end{aligned}$$

Here we used Lemma 2.5 3 to obtain the first line of the equation above.

On the other hand, replacing λ' with λ in the calculation (29), (30) and (31) in Case 6.1.2, we have

$$\begin{aligned} s_i s_{i+1} v_P &= \sum_{r=1}^{b(\lambda)} H\left(\frac{\mu}{\lambda^+_{(r)}}\right)\left(a_{e_r} \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_{e_r} \begin{bmatrix} \lambda^+_{(r)} \\ \lambda_{(r)} \end{bmatrix}\right) \\ &+ \sum_{s'=1}^{b'(\mu)} H\left(\frac{\mu^2}{\mu^-_{(s')}\lambda}\right) a_{d_{s'}}^2 \left(a_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^-_{(s')} \end{bmatrix}'\right) \\ &+ \sum_{s'=1}^{b'(\mu)} H\left(\frac{\mu}{\lambda^-_{(r')}}\right) a_{d_{\sigma(r')}} \left(a_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix} + b_f \begin{bmatrix} \lambda^-_{(r')} \\ \nu_{(r')} \end{bmatrix}'\right). \end{aligned}$$

Hence we have

(43)
$$\begin{aligned} s_{i+1} s_i s_{i+1} v_P &= \sum_{r=1}^{b(\lambda)} H\left(\frac{\mu}{\lambda^+_{(r)}}\right)\left(a_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^+_{(r)} \\ \lambda \end{bmatrix}'\right) \\ &+ \frac{h(\mu)^2}{h(\lambda)} \sum_{s=1}^{b(\mu)} \frac{a_{d_{s'}}^3}{h\left(\mu^-_{(s')}\right)} \left(a_f \begin{bmatrix} \mu \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \lambda \end{bmatrix}'\right) \\ &+ \sum_{s=2}^{b(\mu)} H\left(\frac{\mu^2}{\mu^+_{(s)}\lambda}\right) \sum_{s'=1}^{b'(\mu)} \frac{h(\mu)}{h\left(\mu^-_{(s')}\right)} a_{d_{s'}}^2 a_{d_{s',s}} \left(a_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix} + b_f \begin{bmatrix} \mu \\ \mu^+_{(s)} \end{bmatrix}'\right) \\ &+ \sum_{r'=2}^{b'(\lambda)} H\left(\frac{\mu}{\lambda^-_{(r')}}\right) a_{d_{\sigma(r')}} \left(a_f \begin{bmatrix} \lambda^-_{(r')} \\ \lambda \end{bmatrix} + b_f \begin{bmatrix} \lambda^-_{(r')} \\ \lambda \end{bmatrix}'\right). \end{aligned}$$

Now we see that the expressions (42) and (43) coincide. The first lines of them are obvious. As for the second lines of them, by Lemma 2.5 5 they coincide. We show that the third line of the expression (42) and the last line of the expression (43) are equivalent. To prove this, we have only to show the following:

$$\frac{h(\lambda)}{h(\mu)} \sum_{r=1}^{b(\mu)} \frac{h(\lambda)}{h\left(\lambda^+_{(r)}\right)} a^2_{e_r} a_{e_{r',r}} = a_{d_{\sigma(r')}}. \tag{44}$$

Since $e_r + d_{\sigma(r')} = e_{r',r}$, by Lemma 2.1 1 we have

$$a^2_{e_r} a_{e_{r',r}} = a_{e_r} \left(a_{e_r} a_{d_{\sigma(r')}} - a_{e_{r',r}} a_{d_{\sigma(r')}}\right) = \left(a^2_{e_r} - a_{e_r} a_{e_{r',r}}\right) a_{d_{\sigma(r')}}.$$

Hence if we use Lemma 2.5 3, we can show that the equation (44) holds. Similarly, we can show that the last line of the expression (42) and the third line of the expression (43) are equal. Hence in this case, we obtain $s_i s_{i+1} s_i v_p = s_{i+1} s_i s_{i+1} v_p$.

3.2. Preservation of the other relations. In the following, we check that $\{\rho_{\boldsymbol{\alpha}}\}$ preserve other relations $(P1)$, $(P3)$–$(P8)$. By the definition of $\{\rho_{\boldsymbol{\alpha}}\}$, the relations $(P3)$, $(P4)$, $(P5)$, $(P6)$ are immediately checked. We focus on the relations $(P1)$, $(P7)$ and $(P8)$.

Proposition 3.1. *$\rho_{\boldsymbol{\alpha}}$ preserves the relation* $(P1)$.

Proof. Let M_i be the matrix defined in Section 1. We have only to show that $M_i^2 = I$ (the identity matrix). In case that M_i is given by the matrix (4) or (7) this is easily checked.

Consider the case that M_i is given by the matrix (5). Since the matrix is symmetric, in order to show that $M_i^2 = I$, we find that the following equations must be checked:

$$\sum_{r=1}^{b(\lambda)} \frac{h(\lambda)}{h\left(\lambda^+_{(r)}\right)} + \sum_{r'=1}^{b'(\lambda)} \frac{1}{d\left(\lambda^-_{(r')}, \lambda, \lambda^+_{(r_0)}\right)^2} \cdot \frac{h(\lambda)}{h\left(\lambda^-_{(r')}\right)} = \frac{h\left(\lambda^+_{(r_0)}\right)}{h(\lambda)},$$

$$\sum_{r=1}^{b(\lambda)} \frac{1}{d\left(\lambda^-_{(r'_0)}, \lambda, \lambda^+_{(r)}\right)^2} \cdot \frac{h(\lambda)}{h\left(\lambda^+_{(r)}\right)} = \frac{h\left(\lambda^-_{(r'_0)}\right)}{h(\lambda)},$$

$$\sum_{r=1}^{b(\lambda)} \frac{h(\lambda)}{h\left(\lambda^+_{(r)}\right)} + \sum_{r'=1}^{b'(\lambda)} \frac{1}{d\left(\lambda^-_{(r')}, \lambda, \lambda^+_{(r_0)}\right) d\left(\lambda^-_{(t')}, \lambda, \lambda^+_{(r_1)}\right)} \cdot \frac{h(\lambda)}{h\left(\lambda^-_{(r')}\right)} = 0,$$

$$\sum_{r=1}^{b(\lambda)} \frac{1}{d\left(\lambda^-_{(r'_0)}, \lambda, \lambda^+_{(r)}\right) d\left(\lambda^-_{(r'_1)}, \lambda, \lambda^+_{(r)}\right)} \cdot \frac{h(\lambda)}{h\left(\lambda^+_{(r)}\right)} = 0,$$

$$\sum_{r=1}^{b(\lambda)} \frac{1}{d\left(\lambda^-_{(r'_0)}, \lambda, \lambda^+_{(r)}\right)} \cdot \frac{h(\lambda)}{h\left(\lambda^+_{(r)}\right)} = 0.$$

Here $1 \leq r_0, r_1 \leq b(\lambda)$, $1 \leq r'_0, r'_1 \leq b'(\lambda)$ and $r_0 \neq r_1$, $r'_0 \neq r'_1$. Applying Lemma 2.5, we immediately obtain the equations above. $\square$

Proposition 3.2. *$\rho_{\boldsymbol{\alpha}}$ preserves the relation $(P7)$.*

Proof. Consider the subalgebra $\langle f, s_1, s_2 \rangle$ of $\mathcal{A}_n(K_0)$. This algebra is isomorphic to the algebra $\mathcal{A}_3(K_0)$. Hence, it is sufficient to prove that the proposition holds for $\mathcal{A}_3(K_0)$.

Put $\boldsymbol{\alpha} = [\alpha(1), \alpha(2), \alpha(3)]$. Fig. 4 will help the reader to understand the following argument.

If $|\alpha(1)| = 3$, then none of the tableau of shape $\boldsymbol{\alpha}$ has $[\emptyset, \square, \emptyset]$ at its second coordinate Hence in this case we have $\rho_{\boldsymbol{\alpha}}(f) = 0$ and obviously the proposition holds.

Next, consider the case $\boldsymbol{\alpha} = [\square, \square, \emptyset]$ and let Q_1, Q_2, Q_3 be all the tableaux of shape $\boldsymbol{\alpha}$ defined by

$$\begin{aligned} Q_1 &= \left(\boldsymbol{\alpha}^{(0)}, [\square, \emptyset, \emptyset], [\square\square, \emptyset, \emptyset], \boldsymbol{\alpha}\right), \\ Q_2 &= \left(\boldsymbol{\alpha}^{(0)}, [\square, \emptyset, \emptyset], \left[\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \emptyset, \emptyset\right], \boldsymbol{\alpha}\right), \\ Q_3 &= \left(\boldsymbol{\alpha}^{(0)}, [\square, \emptyset, \emptyset], [\emptyset, \square, \emptyset], \boldsymbol{\alpha}\right). \end{aligned}$$

Then the representation matrices of f and s_2 with respect to this basis become as follows:

$$\rho_{\boldsymbol{\alpha}}(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_{\boldsymbol{\alpha}}(s_2) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}.$$

Hence in this case, we can check that the proposition holds by direct calculation.

Finally, consider the case $\boldsymbol{\alpha} = [\emptyset, \emptyset, \square]$. In this case, there exists only one tableau of shape $\boldsymbol{\alpha}$. The generators s_2 and f identically act on this tableau. Hence in this case the proposition holds. $\square$

Proposition 3.3. *$\rho_{\boldsymbol{\alpha}}$ preserves the relation $(P8)$.*

Proof. As we saw in the proof of the previous proposition, it is sufficient to prove the proposition holds for $\mathcal{A}_4(K_0)$.

Put $\boldsymbol{\alpha} = [\alpha(1), \alpha(2), \alpha(3), \alpha(4)]$. Again Fig. 4 will help the reader to understand the following argument. Similarly as in the proof of the previous proposition, we may assume that $|\alpha(1)| < 4$.

Consider the case $\boldsymbol{\alpha} = [\yng(2), \yng(1), \emptyset, \emptyset]$. Let $P_1, \ldots, P_6$ be all the tableaux of shape $\boldsymbol{\alpha}$ defined by

$$\begin{aligned}
P_1 &= \left(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, [\yng(2), \emptyset, \emptyset, \emptyset], [\yng(3), \emptyset, \emptyset, \emptyset], \boldsymbol{\alpha}\right), \\
P_2 &= \left(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, [\yng(2), \emptyset, \emptyset, \emptyset], \left[\yng(2,1), \emptyset, \emptyset, \emptyset\right], \boldsymbol{\alpha}\right), \\
P_3 &= \left(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, \left[\yng(1,1), \emptyset, \emptyset, \emptyset\right], \left[\yng(2,1), \emptyset, \emptyset, \emptyset\right], \boldsymbol{\alpha}\right), \\
P_4 &= \left(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, [\yng(2), \emptyset, \emptyset, \emptyset], [\yng(1), \yng(1), \emptyset, \emptyset], \boldsymbol{\alpha}\right), \\
P_5 &= \left(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, \left[\yng(1,1), \emptyset, \emptyset, \emptyset\right], [\yng(1), \yng(1), \emptyset, \emptyset], \boldsymbol{\alpha}\right), \\
P_6 &= (\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, [\emptyset, \yng(1), \emptyset, \emptyset], [\yng(1), \yng(1), \emptyset, \emptyset], \boldsymbol{\alpha}).
\end{aligned}$$

Then the representation matrices of f, s_1, s_2 and s_3 with respect to this basis become as follows:

$$\begin{aligned}
\rho_{\boldsymbol{\alpha}}(f) &= \mathrm{diag}(0, 0, 0, 0, 0, 1), \\
\rho_{\boldsymbol{\alpha}}(s_1) &= \mathrm{diag}(1, 1, -1, 1, -1, 1), \\
\rho_{\boldsymbol{\alpha}}(s_2) &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\
0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-\sqrt{2}}{2} \\
0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0
\end{pmatrix}, \\
\rho_{\boldsymbol{\alpha}}(s_3) &= \begin{pmatrix}
\frac{1}{3} & \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{6}}{3} & 0 & 0 \\
\frac{\sqrt{2}}{3} & \frac{2}{3} & 0 & \frac{-\sqrt{3}}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\frac{\sqrt{6}}{3} & \frac{-\sqrt{3}}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\end{aligned}$$

Using these matrices we can check that the relation $(P8)$ holds.

Similarly, for the cases $\boldsymbol{\alpha} = \left[\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square, \emptyset, \emptyset\right]$, $[\emptyset, \square\square, \emptyset, \emptyset]$, $\left[\emptyset, \begin{smallmatrix}\square\\\square\end{smallmatrix}, \emptyset, \emptyset\right]$, $[\square, \emptyset, \square, \emptyset]$, $[\emptyset, \emptyset, \emptyset, \square]$, we can concretely obtain the representation matrices and the relation $(P8)$ will be checked by the direct calculation. $\square$

4. Proof of the main theorem

This section is devoted to prove the main theorem.

In order to know whether two representations of $\mathcal{A}_n(K_0)$ are equivalent or not, it is useful to check that how they split into irreducible ones as $\mathcal{A}_{n-1}(K_0)$-modules. So we consider the following set. Let $\boldsymbol{\gamma}$ be an n-tuple of Young diagrams of height n. Then the restriction $\boldsymbol{\gamma}|_{n-1}$ of $\boldsymbol{\gamma}$ to $\mathcal{A}_{n-1}(K_0)$ is defined by

$$\boldsymbol{\gamma}|_{n-1} = \left\{ \boldsymbol{\gamma}'_p \;\middle|\; \boldsymbol{\gamma}'_p \underset{1}{\prec} \boldsymbol{\gamma} \right\}.$$

Lemma 4.1. *Let* $\boldsymbol{\alpha} = [\alpha(1), \dots, \alpha(n)]$ *and* $\boldsymbol{\beta} = [\beta(1), \dots, \beta(n)]$ *be two n-tuples of Young diagrams whose heights are both n. For these* $\boldsymbol{\alpha}, \boldsymbol{\beta}$, *define two integers k_0 and k_1 by*

$$k_0 = \max\{j \mid \alpha(j) \neq \emptyset\}, \quad k_1 = \max\{j \mid \beta(j) \neq \emptyset\}.$$

Assume that $n \geq 3$. If $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, *then the following statement holds*:

1) $\boldsymbol{\alpha}|_{n-1} \neq \boldsymbol{\beta}|_{n-1}$, *or else*
2) $k_0 = k_1$, $\alpha(j) = \beta(j) = \emptyset$ *for* $1 \leq j < k_0$ *and* $\{\alpha(k_0), \beta(k_0)\} = \left\{\square\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$.

Proof. Assuming that $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ and $\boldsymbol{\alpha}|_{n-1} = \boldsymbol{\beta}|_{n-1}$, we show that the second statement holds. Without loss of generality, we may further assume that $k_1 < k_0$, or else $k_1 = k_0$ and $|\beta(k_0)| \leq |\alpha(k_0)|$. Since if $k_1 < k_0$ then $\beta(k_0) = \emptyset$ and $|\beta(k_0)| < |\alpha(k_0)|$, we may always assume that $k_1 \leq k_0$ and $|\beta(k_0)| \leq |\alpha(k_0)|$.

First we show that $|\beta(k_0)| = |\alpha(k_0)|$ (accordingly we have $k_1 = k_0$). Assume that $|\beta(k_0)| < |\alpha(k_0)|$. If there exists a j $(0 < j < k_0)$ such that $\alpha(j) \neq \emptyset$, then there exists an n-tuple of Young diagrams $\boldsymbol{\alpha}' \in \boldsymbol{\alpha}|_{n-1}$ such that

$$\boldsymbol{\alpha}' = \left[\alpha(1), \dots, \alpha(j')^+, \alpha(j)^-, \dots, \alpha(k_0), \emptyset, \dots, \emptyset\right],$$

where $j' = j - 1$, $\alpha(j')^+ \rhd \alpha(j')$ and $\alpha(j)^- \lhd \alpha(j)$. The assumption $|\beta(k_0)| < |\alpha(k_0)|$ makes us unable to obtain $\boldsymbol{\beta}$ from $\boldsymbol{\alpha}'$. This contradicts the assumption $\boldsymbol{\alpha}|_{n-1} = \boldsymbol{\beta}|_{n-1}$. Hence if $|\beta(k_0)| < |\alpha(k_0)|$, then $\boldsymbol{\alpha}$ must be of the form

$$\boldsymbol{\alpha} = [\emptyset, \cdots, \emptyset, \alpha(k_0), \emptyset, \dots, \emptyset].$$

However, this requires that

$$\boldsymbol{\alpha}|_{n-1} = \left\{ \left[\emptyset, \emptyset, \dots, \emptyset, \square, \alpha(k_0)^-_{(p)}, \emptyset, \dots, \emptyset\right] \;\middle|\; \alpha(k_0)^-_{(p)} \lhd \alpha(k_0) \right\}. \tag{45}$$

Since $k_1 \leq k_0$, $\boldsymbol{\beta}$ must be written as one of the following forms:

1) $k_1 = k_0$ and $\boldsymbol{\beta} = [\emptyset, \emptyset \ldots, \emptyset, \emptyset, \beta(k_0), \emptyset, \ldots, \emptyset]$,

2) $3 \leq k_0$ and $\boldsymbol{\beta} = \left[\square, \emptyset, \ldots, \emptyset, \square, \alpha(k_0)^-_{(p)}, \emptyset, \ldots, \emptyset\right]$,

3) $k_0 = 2$ and $\boldsymbol{\beta} = \left[\square\square, \alpha(k_0)^-_{(p)}, \emptyset \ldots, \emptyset\right]$,

4) $k_0 = 2$ and $\boldsymbol{\beta} = \left[\begin{smallmatrix}\square\\\square\end{smallmatrix}, \alpha(k_0)^-_{(p)}, \emptyset \ldots, \emptyset\right]$.

The first one contradicts the assumption $|\beta(k_0)| < |\alpha(k_0)|$. In the second case, there exists an n-tuple of Young diagrams $\boldsymbol{\beta}' \underset{1}{\prec} \boldsymbol{\beta}$ which is not contained in the set (45). In the remaining cases, in order that $\boldsymbol{\beta}|_{n-1} = \boldsymbol{\alpha}|_{n-1}$ holds, $\alpha(k_0)^-_{(p)}$ must be the empty partition and this contradicts the assumption $n \geq 3$. Hence we obtain $|\alpha(k_0)| = |\beta(k_0)|$. Accordingly, $k_1 = k_0$ follows.

Next we show that $\{\alpha(k_0), \beta(k_0)\} = \left\{\square\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$. Since $\boldsymbol{\alpha}|_{n-1} = \boldsymbol{\beta}|_{n-1}$ requires $\left\{\alpha(k_0)^-_{(p)} \;\middle|\; \alpha(k_0)^-_{(p)} \lhd \alpha(k_0)\right\} = \left\{\beta(k_0)^-_{(q)} \;\middle|\; \beta(k_0)^-_{(q)} \lhd \beta(k_0)\right\}$, we find that $\alpha(k_0) = \beta(k_0)$ or $|\alpha(k_0)| = |\beta(k_0)| \leq 2$. If $\alpha(k_0) = \beta(k_0)$, then $\boldsymbol{\alpha}|_{n-1} = \boldsymbol{\beta}|_{n-1}$ requires $\alpha(k_0 - 1) = \beta(k_0 - 1)$ and inductively we obtain $\alpha(j) = \beta(j)$ for $1 \leq j < k_0$. This contradicts the assumption $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$. Hence we obtain $\{\alpha(k_0), \beta(k_0)\} = \left\{\square\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$.

Finally, we show that $\alpha(j) = \beta(j) = \emptyset$ for $1 \leq j < k_0$. Note that since $n \geq 3$, we can assume that $2 \leq k_0$. If there exists an $\alpha(j) \neq \emptyset$ for $1 \leq j < k_0$, then there exists an n-tuple of Young diagrams $\boldsymbol{\alpha}' \in \boldsymbol{\alpha}|_{n-1}$ such that

$$\boldsymbol{\alpha}' = [\alpha(1), \ldots, \alpha(j')^+, \alpha(j)^-, \ldots, \alpha(k_0), \emptyset, \ldots, \emptyset].$$

However since $|\alpha(k_0)| = |\beta(k_0)|$ and $\alpha(k_0) \neq \beta(k_0)$, we can not obtain $\boldsymbol{\beta}$ from $\boldsymbol{\alpha}'$. This contradicts the assumption $\boldsymbol{\alpha}|_{n-1} = \boldsymbol{\beta}|_{n-1}$. The same argument also holds for $\boldsymbol{\beta}$. Hence we have $\alpha(j) = \beta(j) = \emptyset$ for $1 \leq j < k_0$. Thus we have proved the lemma. □

Lemma 4.2. *Let* $\boldsymbol{\alpha} = [\alpha(1), \ldots, \alpha(n)]$ *be an* n*-tuple of Young diagrams of height* n. *For an arbitrary distinct pair* $\{\boldsymbol{\alpha}'_0, \boldsymbol{\alpha}'_1\} \subset \boldsymbol{\alpha}|_{n-1}$, *there exists an* n*-tuple of Young diagrams* $\boldsymbol{\gamma}''$ *of height* $n-2$ *such that* $\boldsymbol{\gamma}'' \underset{1}{\prec} \boldsymbol{\alpha}'_0$ *and* $\boldsymbol{\gamma}'' \underset{1}{\prec} \boldsymbol{\alpha}'_1$.

Proof. Assume that $\boldsymbol{\alpha}'_0$ (resp. $\boldsymbol{\alpha}'_1$) is obtained from $\boldsymbol{\alpha}$ by moving a box on the j_0-th (resp. j_1-th) board. Namely, $\boldsymbol{\alpha}'_0$ and $\boldsymbol{\alpha}'_1$ are written as follows:

$$\boldsymbol{\alpha}'_0 = \left[\alpha(1), \ldots, \alpha(j_0 - 2), \alpha(j'_0)^+_{(p)}, \alpha(j_0)^-_{(p')}, \alpha(j_0 + 1), \ldots, \alpha(n)\right],$$
$$\boldsymbol{\alpha}'_1 = \left[\alpha(1), \ldots, \alpha(j_1 - 2), \alpha(j'_1)^+_{(q)}, \alpha(j_1)^-_{(q')}, \alpha(j_1 + 1), \ldots, \alpha(n)\right].$$

Here $j'_0 = j_0 - 1$ and $j'_1 = j_1 - 1$. Since without loss of generality we may assume that $j_1 \leq j_0$, the following cases should be considered.

CASE 1. $j_1 \leq j_0 - 2$.

In this case,

$$\boldsymbol{\gamma}'' = \left[\alpha(1), \ldots, \alpha(j'_1)^+_{(q)}, \alpha(j_1)^-_{(q')}, \ldots, \alpha(j'_0)^+_{(p)}, \alpha(j_0)^-_{(p')}, \ldots, \alpha(n)\right]$$

satisfies the required condition.

CASE 2. $j_1 = j_0 - 1$.

In this case,

$$\boldsymbol{\gamma}'' = \left[\alpha(1), \ldots, \alpha(j_1')^{+}_{(q)}, \alpha(j_1), \alpha(j_0)^{-}_{(p')}, \ldots, \alpha(n)\right]$$

satisfies the required condition.

CASE 3. $j_1 = j_0$.

In this case,

$$\boldsymbol{\gamma}'' = \left[\alpha(1), \ldots, \alpha(j_0')^{++}, \alpha(j_0)^{--}, \ldots, \alpha(n)\right].$$

satisfies the required condition. Here

$$\alpha(j_0')^{++} = \begin{cases} \alpha(j_0')^{+}_{(p)} \cup \alpha(j_0')^{+}_{(q)} & \text{if} \quad \alpha(j_0')^{+}_{(p)} \neq \alpha(j_0')^{+}_{(q)}, \\ \text{one of the Young diagrams} & \\ \text{such that} \quad \alpha(j_0')^{++} \rhd \alpha^{+}_{(p)}(j_0') & \text{if} \quad \alpha(j_0')^{+}_{(p)} = \alpha(j_0')^{+}_{(q)}, \end{cases}$$

$$\alpha(j_0)^{--} = \begin{cases} \alpha(j_0)^{-}_{(p')} \cap \alpha(j_0)^{-}_{(q')} & \text{if} \quad \alpha(j_0)^{-}_{(p')} \neq \alpha(j_0)^{-}_{(q')}, \\ \text{one of the Young diagrams} & \\ \text{such that} \quad \alpha(j_0)^{--} \lhd \alpha(j_0')^{-}_{(p')} & \text{if} \quad \alpha(j_0)^{-}_{(p')} = \alpha(j_0)^{-}_{(q')}. \end{cases}$$

□

Proof of Theorem 1.4. Since we have shown that the representations $\{\rho_{\boldsymbol{\alpha}}\}$ are well-defined, we have only to show that they are also absolutely irreducible and mutually non-equivalent. We do this by induction on n.

If $n = 0$, then the result is obvious. So is the case $n = 1$. If $n = 2$, we can easily check that the (three) representations are mutually non-isomorphic. Since they are all one-dimensional and $\dim \mathcal{A}_{n-1}(K_0) = 3$, we find that they make a complete set of absolutely irreducible representations. Assume that $n \geq 3$ and the theorem holds for $n - 1$. Let $\mathcal{A}' = \langle f, s_1, \ldots, s_{n-2} \rangle$ be the subalgebra of $\mathcal{A}_n(K_0)$. This algebra is isomorphic to the algebra $\mathcal{A}_{n-1}(K_0)$. Consider the restriction of the representation $\rho_{\boldsymbol{\alpha}}$ of $\mathcal{A}_n(K_0)$ to the subalgebra $\mathcal{A}'$. Suppose that $\boldsymbol{\alpha}|_{n-1} = \{\boldsymbol{\alpha}_1', \ldots, \boldsymbol{\alpha}_k'\}$. We divide the set $\mathbb{T}(\boldsymbol{\alpha})$ of the standard tableaux of shape $\boldsymbol{\alpha}$ into subsets $\mathbb{T}\left(\boldsymbol{\alpha}_1'\right), \ldots, \mathbb{T}\left(\boldsymbol{\alpha}_k'\right)$. Here $\mathbb{T}\left(\boldsymbol{\alpha}_p'\right)$ is the subset of $\mathbb{T}(\boldsymbol{\alpha})$ whose $(n-1)$-st coordinate is $\boldsymbol{\alpha}_p'$. We define subspaces $\mathbb{V}\left(\boldsymbol{\alpha}_p'\right)$ of $\mathbb{V}(\boldsymbol{\alpha})$ corresponding to these subsets $\mathbb{T}\left(\boldsymbol{\alpha}_p'\right)$. Namely,

$$\mathbb{V}\left(\boldsymbol{\alpha}_p'\right) = \sum_{P \in \mathbb{T}\left(\boldsymbol{\alpha}_p'\right)} K_0 v_P.$$

Then the definition of the action of $s_1, \ldots, s_{n-2}$ and f implies that $\mathbb{V}\left(\boldsymbol{\alpha}_p'\right)$ is stable under the action of $\mathcal{A}'$ and induction hypothesis shows that $\mathbb{V}\left(\boldsymbol{\alpha}_p'\right)$ gives an absolutely irreducible representation of $\mathcal{A}_{n-1}(K_0)$ (hence $\mathcal{A}'$) and that if $p \neq q$ then $\mathbb{V}\left(\boldsymbol{\alpha}_p'\right)$ is not isomorphic to $\mathbb{V}\left(\boldsymbol{\alpha}_q'\right)$ as $\mathcal{A}_{n-1}(K_0)$-modules.

Let W be a non-zero subspace of $\mathbb{V}(\boldsymbol{\alpha}) \otimes \overline{K_0}$ as $\mathcal{A}_n\left(\overline{K_0}\right)$-modules, where $\overline{K_0}$ denotes the algebraic closure of the field K_0 and $\mathcal{A}_n\left(\overline{K_0}\right) = \mathcal{A}_n \otimes \overline{K_0}$. If we consider W as an $\mathcal{A}' \otimes \overline{K_0}$-module, then it contains some irreducible component $\mathbb{V}\left(\boldsymbol{\alpha}'_p\right)$ of $\mathcal{A}' \otimes \overline{K_0}$. Let $\boldsymbol{\alpha}'_q$ ($q \neq p$) be another n-tuple of Young diagrams contained in $\boldsymbol{\alpha}|_{n-1}$. Then by Lemma 4.2, there exists an n-tuple of Young diagrams $\boldsymbol{\gamma}''$ contained in both $\boldsymbol{\alpha}'_p|_{n-2}$ and $\boldsymbol{\alpha}'_q|_{n-2}$. Let P be a tableau of shape $\boldsymbol{\alpha}$ whose $(n-2)$-nd and $(n-1)$-st coordinates are $\boldsymbol{\gamma}''$ and $\boldsymbol{\alpha}'_p$ respectively. We can obtain another tableau Q of shape $\boldsymbol{\alpha}$ from P by replacing the $(n-1)$-st coordinate of P with $\boldsymbol{\alpha}'_q$.

Now we claim that there exists a projection e_{PQ} of $\mathcal{A}_n(K_0)$ such that $e_{PQ}v_P = v_Q$. By induction assumption, $\mathcal{A}'$ is an absolutely semisimple algebra with the minimal central idempotents $\{z_{\boldsymbol{\alpha}'}\}$, labeled by n-tuples of Young diagrams of height $(n-1)$. According to the classification in the proof of Lemma 4.2, consider the action of $s_{n-1} \in \mathcal{A}_n(K_0)$ one by one.

CASE 1. $j_1 \leq j_0 - 2$.

In this case, we have $s_{n-1}v_P = v_Q$ and the claim is proved.

CASE 2. $j_1 = j_0 - 1$.

In this case we have

$$z_{\boldsymbol{\alpha}'_p} s_{n-1} v_P = \frac{1}{d\left(\alpha(j_1)^-_{(q')}, \alpha(j_1), \alpha(j_1)^+_{(p)}\right)} \sqrt{\frac{h(\alpha(j_1))^2}{h\left(\alpha(j_1)^-_{(q')}\right) h\left(\alpha(j_1)^+_{(p)}\right)}}\, v_Q.$$

Since the coefficient of v_Q is not equal to zero, the claim is proved.

CASE 3. $j_1 = j_0$.

In this case, the following four cases are considered. In each case, $z_{\boldsymbol{\alpha}'_p} s_{n-1}$ sends v_P non-zero scalar multiple of v_Q, so the claim is proved. In the following, we put $d = d\left(\alpha(j'_0), \alpha(j'_0)^+_{(p)}, \alpha(j'_0)^{++}\right)$ and $e = d\left(\alpha(j_0)^{--}, \alpha(j_0)^-_{(p')}, \alpha(j_0)\right)$.

CASE 3.1. $\alpha(j'_0)^+_{(p)} \neq \alpha(j'_0)^+_{(q)}$ *and* $\alpha(j_0)^-_{(p')} \neq \alpha(j_0)^-_{(q')}$.

In this case we have

$$z_{\boldsymbol{\alpha}'_p} s_{n-1} v_P = \sqrt{\frac{d^2-1}{d^2}} \sqrt{\frac{e^2-1}{e^2}}\, v_Q.$$

CASE 3.2. $\alpha(j'_0)^+_{(p)} \neq \alpha(j'_0)^+_{(q)}$ *and* $\alpha(j_0)^-_{(p')} = \alpha(j_0)^-_{(q')}$.

In this case we have

$$z_{\boldsymbol{\alpha}'_p} s_{n-1} v_P = \sqrt{\frac{d^2-1}{d^2}} \frac{1}{e}\, v_Q.$$

CASE 3.3. $\alpha(j_0')^+_{(p)} = \alpha(j_0')^+_{(q)}$ *and* $\alpha(j_0)^-_{(p')} \neq \alpha(j_0)^-_{(q')}$.
In this case we have

$$z_{\boldsymbol{\alpha}'_p} s_{n-1} v_P = \frac{1}{d}\sqrt{\frac{e^2-1}{e^2}}\, v_Q.$$

CASE 3.4. $\alpha(j_0')^+_{(p)} = \alpha(j_0')^+_{(q)}$ *and* $\alpha(j_0)^-_{(p')} = \alpha(j_0)^-_{(q')}$.
In this case we have

$$z_{\boldsymbol{\alpha}'_p} s_{n-1} v_P = \frac{1}{d}\frac{1}{e} v_Q.$$

The claim implies that the irreducible $\mathcal{A}'$-module W also contains $\mathbb{V}\big(\boldsymbol{\alpha}'_q\big)$ as well as $\mathbb{V}\big(\boldsymbol{\alpha}'_p\big)$. Since the choice of $\boldsymbol{\alpha}'_p$ was arbitrary, we obtain

$$W \supset \bigoplus_{\boldsymbol{\alpha}'_p \underset{1}{\prec} \boldsymbol{\alpha}} \mathbb{V}(\boldsymbol{\alpha}'_p) \otimes \overline{K_0} = \mathbb{V}(\boldsymbol{\alpha}) \otimes \overline{K_0} \quad (\text{as } \mathcal{A}' \otimes \overline{K_0}\text{-modules}).$$

In case $n \geq 3$, Lemma 4.1 asserts that if $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ either $\mathbb{V}(\boldsymbol{\alpha})$ and $\mathbb{V}(\boldsymbol{\beta})$ are non-isomorphic already as $\mathcal{A}'$-modules, or else

$$\{\boldsymbol{\alpha}, \boldsymbol{\beta}\} = \left\{[\emptyset, \ldots, \emptyset, \emptyset, \square\square], \left[\emptyset, \ldots, \emptyset, \emptyset, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right]\right\}.$$

We show that even in the latter case, $\rho_{\boldsymbol{\alpha}}$ and $\rho_{\boldsymbol{\beta}}$ are mutually non-isomorphic. In the latter case, we have

$$\boldsymbol{\alpha}|_{n-1} = \boldsymbol{\beta}|_{n-1} = \{\boldsymbol{\gamma}' = [\emptyset, \ldots, \emptyset, \square, \square]\}.$$

Hence we can take $\boldsymbol{\gamma}'' \in \boldsymbol{\gamma}'|_{n-2}$ so that

$$\boldsymbol{\gamma}'' = [\emptyset, \ldots, \emptyset, \square\square, \emptyset].$$

If we choose a tableau P of shape $\boldsymbol{\alpha}$ so that its $(n-2)$-nd and $(n-1)$-st coordinates coincide with $\boldsymbol{\gamma}''$ and $\boldsymbol{\gamma}'$ respectively, then the tableau Q obtained from P by replacing the n-th coordinate $\boldsymbol{\alpha}$ with $\boldsymbol{\beta}$ is a tableau of shape $\boldsymbol{\beta}$. The generator s_{n-1} of $\mathcal{A}_n(K_0)$ acts differently on v_P and v_Q. Hence $\mathbb{V}(\boldsymbol{\alpha})$ and $\mathbb{V}(\boldsymbol{\beta})$ are non-isomorphic as $\mathcal{A}_n(K_0)$-modules.

Since

$$\dim\left(\bigoplus_{\boldsymbol{\alpha}\in\Lambda_n} \mathbb{V}(\boldsymbol{\alpha})\right)^2 = \sum_{\lambda\in P(n)} \left(\frac{n!}{\lambda_1!\,\lambda_2!\cdots\lambda_n!}\right)^2 \cdot \frac{1}{\alpha_1!\,\alpha_2!\cdots\alpha_n!} = \dim \mathcal{A}_n(K_0),$$

$\{\rho_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \Lambda_n(n)\}$ define a complete set of the representative of the irreducible representations of $\mathcal{A}_n(K_0)$. In particular, the party algebra $\mathcal{A}_n(K_0)$ is absolutely semisimple and the Bratteli diagram of the sequence $\{\mathcal{A}_i(K_0)\}_{i=0,1,\ldots,n}$ is given by the graph Γ_n. $\square$

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