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K-GROUPS OF SYMMETRIC SPACES II

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1. Introduction

Let $M=G/K$ be a symmetric homogeneous space such that G is a simply connected compact Lie group. In [I] the author showed that the unitary K -group of M is isomorphic to the tensor product of $R(K) \otimes_{R(G)} Z$ and an exterior algebra E over Z , where $R(G)$ and $R(K)$ are the complex representation rings of G and K respectively, and in particular described the generators of E as an exterior algebra explicitly.

The purpose of this paper is to present a structure of $R(K) \otimes_{R(G)} Z$ as a group in the following nine cases:

*Type of $M = AIII, BDI(a)(Spin(2p+2q+2)/Spin(2p+1) \cdot Spin(2q+1)),$
 $BDII(b)(Spin(2n+1)/Spin(2n)), DIII, CII, EI, FI, FII$ or G .*

Now let us denote by $n(L)$ the order of the Weyl group of a compact connected Lie group L . We know that if U is a closed connected subgroup of G of maximal rank then $R(U) \otimes_{R(G)} Z$ is a free module of rank $n(G)/n(U)$ and is isomorphic to $K^*(G/U)$ [12]. Throughout this paper we shall identify $R(U) \otimes_{R(G)} Z$ with the K -group of G/U in the above situation and denote by the same letter ρ the element of $K^*(G/U)$ defined by an element ρ of $R(U)$ in the natural way. Furthermore we shall denote by $Z(g)$ the free module generated by an element g .

2. Representation rings

In this section we recall the structure of the complex representation rings of classical groups.

Write ρ_n for the canonical representations $SU(n) \rightarrow U(n)$, $U(n) \rightarrow U(n)$, $Sp(n) \rightarrow U(2n)$ and $Spin(n) \rightarrow SO(n) \rightarrow U(n)$ for each n , and write $\lambda^i \rho_n$ for the i -th exterior product of ρ_n . According to [10] we have

$$\begin{aligned}
 R(SU(n)) &= Z[\lambda^1 \rho_n, \dots, \lambda^{n-1} \rho_n], \\
 R(U(n)) &= Z[\lambda^1 \rho_n, \dots, \lambda^n \rho_n, (\lambda^n \rho_n)^{-1}], \\
 (2.1) \quad R(Sp(n)) &= Z[\lambda^1 \rho_n, \dots, \lambda^n \rho_n] = Z[\sigma_1, \dots, \sigma_n], \\
 R(Spin(2n+1)) &= Z[\lambda^1 \rho_{2n+1}, \dots, \lambda^{n-1} \rho_{2n+1}, \Delta_{2n+1}^+], \\
 R(Spin(2n)) &= Z[\lambda^1 \rho_{2n}, \dots, \lambda^{n-2} \rho_{2n}, \Delta_{2n}^+, \Delta_{2n}^-].
 \end{aligned}$$

Here we denote by $\sigma_1, \dots, \sigma_n$ the elementary symmetric functions in the n variables $t_1+t_1^{-1}, \dots, t_n+t_n^{-1}$ when we set $R(T)=Z[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ for a maximal torus T of $Sp(n)$, and denote by $\Delta_{2n}^+, \Delta_{2n}^-$ and Δ_{2n+1} the half-spin representations of $Spin(2n)$ and the spin representation of $Spin(2n+1)$, respectively.

Proposition 2.1 (See [17], p. 120). *If G is a compact Lie group, N is a finite normal subgroup of G and $\pi: G \rightarrow G/N$ is the canonical map, then there is a homomorphism of $R(G/N)$ -modules $\pi_*: R(G) \rightarrow R(G/N)$ such that $\pi_*(1)=1$.*

Proof. It is easy to see that the correspondence $V \rightarrow V^N$, where V is a G -module and V^N the N -invariant submodule of V , defines the homomorphism π_* , as desired. q.e.d.

Using Proposition 2.1 we can calculate the representation rings of some quotient groups. For example,

$$(2.2) \quad R(PSp(4)) = Z[\lambda^2 \rho_4, \lambda^4 \rho_4, (\rho_4)^2, (\lambda^3 \rho_4)^2, \rho_4 \lambda^3 \rho_4]$$

as a subring of $R(Sp(4))$ and

$$R(Sp(3) \times_{Z_2} SU(2)) = Z[\lambda^2 \rho_3, (\rho_3)^2, (\lambda^3 \rho_3)^2, \rho_3 \lambda^3 \rho_3, (\rho_2)^2, \rho_2 \rho_3, \rho_2 \lambda^3 \rho_3]$$

as a subring of $R(Sp(3) \times SU(2))$ where Z_2 is the intersection of the centers of $Sp(3)$ and $SU(2)$.

Using the relations of [10], §13, Theorem 10.3 we get

$$\begin{aligned}
 (2.3) \quad &R(Spin(2m+1) \times_{Z_2} Spin(2n+1)) \\
 &= Z[\lambda^1 \rho_{2m+1}, \dots, \lambda^m \rho_{2m+1}, \lambda^1 \rho_{2n+1}, \dots, \lambda^n \rho_{2n+1}, \Delta_{2m+1} \Delta_{2n+1}] / I
 \end{aligned}$$

as a subalgebra of $R(Spin(2m+1) \times Spin(2n+1))$, where Z_2 is the intersection of the centers of $Spin(2m+1)$ and $Spin(2n+1)$, and I is the ideal generated by the element

$$(\Delta_{2m+1} \Delta_{2n+1})^2 - (\lambda^m \rho_{2m+1} + \dots + \lambda^1 \rho_{2m+1} + 1)(\lambda^n \rho_{2n+1} + \dots + \lambda^1 \rho_{2n+1} + 1).$$

3. AIII, BDI(a), BDII(b) and CII

Type AIII $(U(m+n)/U(m) \times U(n))$. Let $T=S_1^1 \times \dots \times S_n^1$ be the canonical

maximal torus of $U(n)$ where $S_i^1, 1 \leq i \leq n$, are the circle groups, and set $R(S_i^1) = Z[t_i, t_i^{-1}]$ for each i where t_i is a standard 1-dimensional non-trivial representation of S_i^1 . Moreover let us define F_k to be the free module generated by $1, t_k, \dots, t_k^{m+k-1}$ for $k=1, \dots, n$.

Lemma 3.1. $R(U(m) \times T)$ is a free $R(U(m+n))$ -module (by restriction) generated by $t_1^{a_1} \dots t_n^{a_n}$ ($0 \leq a_k \leq m+k-1$). Namely,

$$R(U(m) \times T) \cong R(U(m+n)) \otimes F_1 \otimes \dots \otimes F_n$$

with the above notation.

Proof. $R(U(m) \times U(1))$ is freely generated as an $R(U(m+1))$ -module by $1, t, \dots, t^m$, when we put $R(U(1)) = Z[t, t^{-1}]$ ([9], Lemma 7.3). Let

$$U_k = U(m+k) \times S_{k+1}^1 \times \dots \times S_n^1 \quad \text{for } k = 0, \dots, n-1$$

and $U_n = U(m+n)$.

Then we have

$$R(U_k) \cong R(U_{k+1}) \otimes F_{k+1}$$

for $k=0, \dots, n-1$ and this implies Lemma 3.1.

Theorem 3.2.

$$K^*(U(m+n)/U(m) \times U(n)) \cong \bigoplus_{\substack{p_1 \geq 0, \dots, p_n \geq 0 \\ p_1 + \dots + p_n \leq m}} Z((\lambda^1 \rho_n)^{p_1} \dots (\lambda^n \rho_n)^{p_n})$$

for $m, n \geq 1$.

Proof. Put

$$G_k = U(m+n)/U(m) \times U(k) \times S_{k+1}^1 \times \dots \times S_n^1 \quad \text{for } k = 1, \dots, n-1,$$

$$G_n = U(m+n)/U(m) \times U(n),$$

and

$$W_k = \bigoplus_{\substack{p_1 \geq 0, \dots, p_k \geq 0 \\ p_1 + \dots + p_k \leq m}} Z((\lambda^1 \rho_k)^{p_1} \dots (\lambda^k \rho_k)^{p_k})$$

for $k=1, \dots, n$.

$K^*(G_k)$ is a free module of rank $(m+n)!/m!k!$ and identified with $R(U(m) \times U(k) \times S_{k+1}^1 \times \dots \times S_n^1) \otimes_{R(U(m+n))} Z$ for each k . In particular, from Lemma 3.1 we have

$$K^*(G_1) = F_1 \otimes \dots \otimes F_n.$$

Therefore we see that $K^*(G_k)$ contains $W_k \otimes F_{k+1} \otimes \dots \otimes F_n$ as a free subgroup by considering the injective homomorphism $K^*(G_k) \rightarrow K^*(G_1)$ for each k ([I], Proposition 7.1).

We have

$$\begin{aligned} \text{rank } W_k \otimes F_{k+1} \otimes \cdots \otimes F_n &= (\sum_{s=0}^m H_s)(m+k+1) \cdots (m+n) \\ &= \binom{m+k}{k} (m+k+1) \cdots (m+n) \\ &= (m+n)! / m! k! \end{aligned}$$

where ${}_k H_s = \binom{k+s-1}{k-1}$ is the number of the repeated combination. This proves

(a) $K^*(G_k) \otimes Q = W_k \otimes F_{k+1} \otimes \cdots \otimes F_n \otimes Q$ for $k = 1, \dots, n-1$,
 $K^*(G_n) \otimes Q = W_n \otimes Q$.

Next we shall prove by induction on k

(b) $K^*(G_k) = W_k \otimes F_{k+1} \otimes \cdots \otimes F_n$ for $k = 1, \dots, n-1$,
 $K^*(G_n) = W_n$.

Since $W_1 = F_1$, (b) in case of $k=1$ follows by Lemma 3.1. Suppose that (b) is true when $k=l$. For any element $x \in K^*(G_{l+1})$ there is an integer $N > 0$ such that

$$Nx = \sum_{\substack{p_1 \geq 0, \dots, p_{l+1} \geq 0 \\ p_1 + \dots + p_{l+1} \leq m}} a_{p_1 \dots p_{l+1}} (\lambda^1 \rho_{l+1})^{p_1} \cdots (\lambda^{l+1} \rho_{l+1})^{p_{l+1}}$$

where $a_{p_1 \dots p_{l+1}} \in F_{l+2} \otimes \cdots \otimes F_n$ by (a). Let

$$i^*: K^*(G_{l+1}) \rightarrow K^*(G_l)$$

be the natural injective homomorphism. Since

$$i^*(\lambda^i \rho_{l+1}) = \lambda^i \rho_l + (\lambda^{i-1} \rho_l) t_{l+1} \quad \text{for } i = 1, \dots, l$$

and $i^*(\lambda^{l+1} \rho_{l+1}) = (\lambda^l \rho_l) t_{l+1}$,

we have

$$\begin{aligned} &i^*((\lambda^1 \rho_{l+1})^{p_1} \cdots (\lambda^{l+1} \rho_{l+1})^{p_{l+1}}) \\ &= (\lambda^1 \rho_l)^{p_2} \cdots (\lambda^l \rho_l)^{p_{l+1}} t_{l+1}^{p_1 + \dots + p_{l+1}} + \text{lower monomials} \end{aligned}$$

where the lower monomial implies a monomial whose degree with respect to the variable t_{l+1} is lower than $p_1 + \dots + p_{l+1}$. Observe the image of Nx by i^* then we see by the inductive hypothesis that $a_{p_1 \dots p_{l+1}}$ is divisible by N . Thus we have $x \in W_{l+1} \otimes F_{l+2} \otimes \cdots \otimes F_n$. This completes the induction. q.e.d.

Type CII ($Sp(m+n)/Sp(m) \times Sp(n)$). Let $Sp_i(1) \times \cdots \times Sp_n(n)$, where $Sp_i(1) = Sp(1)$ ($1 \leq i \leq n$), be the subgroup of $Sp(n)$ embedded diagonally, and put $R(Sp_i(1)) = Z[\theta_i]$ for each i where $\theta_i = t_i + t_i^{-1}$ and t_i is the standard 1-dimensional non-trivial representation of a maximal torus of $Sp_i(1)$.

By replacing S_k^1 and t_k in case of Type AIII by $Sp_k(1)$ and θ_k for

$k=1, \dots, n$ respectively, we obtain analogously the following results.

Lemma 3.3. *Let E_k be the free module generated by $1, \theta_k, \dots, \theta_k^{m+k-1}$ for $k=1, \dots, n$. Then we have an isomorphism*

$$R(Sp(m) \times Sp_1(1) \times \dots \times Sp_n(1)) \cong R(Sp(m+n)) \otimes E_1 \otimes \dots \otimes E_n$$

with the above notation.

Theorem 3.4.

$$\begin{aligned} K^*(Sp(m+n)/Sp(m) \times Sp(n)) &\cong \bigoplus_{\substack{p_1 \geq 0, \dots, p_n \geq 0 \\ p_1 + \dots + p_n \leq m}} Z(\sigma_1^{p_1} \dots \sigma_n^{p_n}) \\ &= \bigoplus_{\substack{p_1 \geq 0, \dots, p_n \geq 0 \\ p_1 + \dots + p_n \leq m}} Z((\lambda^1 \rho_n)^{p_1} \dots (\lambda^n \rho_n)^{p_n}) \end{aligned}$$

for $m, n \geq 1$.

The equality in Theorem 3.4 is obtained immediately by the formula

$$\lambda^k \rho_n = \sigma_k + \sum_{i < k} a_i \sigma_i$$

for $a_i \in Z$ and $k=1, \dots, n$ ([10], 13, Proposition 5.4).

Type $BDI(a)(Spin(2m+2n+2)/Spin(2m+1) \cdot Spin(2n+1))$. From the relations of [10], §13, Theorem 10.3 and (2.3) we see that

$$\begin{aligned} R(Spin(2m+1) \cdot Spin(2n+1)) &\otimes_{R(Spin(2m+2n+2))} Z \\ &= Z[\lambda^1 \rho_{2m+1}, \dots, \lambda^m \rho_{2m+1}, \lambda^1 \rho_{2n+1}, \dots, \lambda^n \rho_{2n+1}]/I \end{aligned}$$

where I is the ideal generated by the elements

$$\sum_{i+j=l} (\lambda^i \rho_{2m+1}) (\lambda^j \rho_{2n+1}) - \binom{2m+2n+2}{i}$$

for all l .

On the other hand, when we put $\lambda_i' = \lambda^i \rho_m + \lambda^{i-1} \rho_m$ ($1 \leq i \leq m$) and $\lambda_j = \lambda^j \rho_n + \lambda^{j-1} \rho_n$ ($1 \leq j \leq n$)

$$R(Sp(m) \times Sp(n)) \otimes_{R(Sp(m+n))} Z = Z[\lambda_1', \dots, \lambda_m', \lambda_1, \dots, \lambda_n]/J$$

where J is the ideal generated by the elements

$$\sum_{i+j=l} \lambda_i' \lambda_j - \binom{2m+2n+2}{i}$$

for all l .

Hence we see that the correspondences $\lambda_i' \rightarrow \lambda^i \rho_{2m+1}$ and $\lambda_j \rightarrow \lambda^j \rho_{2n+1}$ ($1 \leq i \leq m, 1 \leq j \leq n$) induce an isomorphism of algebras $R(Sp(m) \times Sp(n)) \otimes_{R(Sp(m+n))} Z$ and $R(Spin(2m+1) \cdot Spin(2n+1)) \otimes_{R(Spin(2m+2n+2))} Z$. Thus we have by Theorem 3.4

$$\begin{aligned}
 & R(\text{Spin}(2m+1) \cdot \text{Spin}(2n+1)) \otimes_{R(\text{Spin}(2m+2n+2))} \mathbb{Z} \\
 & \cong \bigoplus_{\substack{p_1 \geq 0, \dots, p_n \geq 0 \\ p_1 + \dots + p_n \leq m}} Z(\lambda_1^{p_1} \cdots \lambda_n^{p_n}) \\
 & \cong \bigoplus_{\substack{p_1 \geq 0, \dots, p_n \geq 0 \\ p_1 + \dots + p_n \leq m}} Z((\lambda^1 \rho_{2n+1})^{p_1} \cdots (\lambda^n \rho_{2n+1})^{p_n}).
 \end{aligned}$$

This and [I], Proposition 7.1 prove the following

Theorem 3.5.

$$\begin{aligned}
 & K^*(\text{Spin}(2m+2n+2)/\text{Spin}(2m+1) \times \text{Spin}(2n+1)) \\
 & = \left\{ \bigoplus_{\substack{p_1 \geq 0, \dots, p_n \geq 0 \\ p_1 + \dots + p_n \leq m}} Z((\lambda^1 \rho_{2n+1})^{p_1} \cdots (\lambda^n \rho_{2n+1})^{p_n}) \right\} \otimes \wedge (\beta(\Delta_{2m+2n+2}^+ - \Delta_{2m+2n+2}^-)).
 \end{aligned}$$

for $m, n \geq 0$.

Type *BDII*(b) ($\text{Spin}(2n+1)/\text{Spin}(2n)$). The following is an immediate result of [10], §13, Theorem 10.3.

Theorem 3.6. $K^*(\text{Spin}(2n+1)/\text{Spin}(2n)) \cong \wedge(\tilde{\Delta}_{2n}^+)$ for $n \geq 1$ where $\tilde{\Delta}_{2n}^+ = \Delta_{2n}^+ - 2^{n-1}$.

4. DIII

We regard $U(n)$ as a subgroup of $SO(2n)$ by the map

$$A = ((a_{ij})) \rightarrow A' = \left(\begin{pmatrix} x_{2i-1, 2j-1} & -x_{2i, 2j} \\ x_{2i, 2j} & x_{2i-1, 2j-1} \end{pmatrix} \right)$$

where $a_{ij} = x_{2i-1, 2j-1} + \sqrt{-1} x_{2i, 2j}$ ($1 \leq i, j \leq n$).

We see that the canonical inclusion map of $SO(2n-1)$ to $SO(2n)$ induces a homeomorphism

$$(4.1) \quad SO(2n-1)/U(n-1) \approx SO(2n)/U(n)$$

because of $SO(2n-1) \cap U(n) = U(n-1)$ and $SO(2n) = U(n) \cdot SO(2n-1)$. Let $\pi: \text{Spin}(2n) \rightarrow SO(2n)$ denote the two fold covering map of $SO(2n)$ and define $\tilde{U}(n)$ (resp. $\tilde{U}(n-1)$) to be the inverse image of $U(n)$ (resp. $U(n-1)$) by π . By (4.1) we have homeomorphisms

$$\begin{aligned}
 (4.2) \quad & \text{Spin}(2n-1)/\tilde{U}(n-1) \approx \text{Spin}(2n)/\tilde{U}(n) \\
 \text{and} \quad & SO(2n)/U(n) \approx \text{Spin}(2n)/\tilde{U}(n).
 \end{aligned}$$

Next we shall consider the complex representation ring of $\tilde{U}(n)$. Let T be the standard maximal torus of $U(n)$ and put $\tilde{T} = \pi^{-1}(T)$, which becomes a maximal torus of $\tilde{U}(n)$. Here, using the notation of [10], §13 we define the

homomorphism

$$f: R(T)[u_n]/(u_n^2 - (\alpha_1 \cdots \alpha_n)^{-1}) \rightarrow R(\tilde{T})$$

by $f(x + yu_n) = \pi^*(x) + \pi^*(y)(\alpha_1 \cdots \alpha_n)^{-1/2}$ $x, y \in R(T)$. Then we can easily check that f is isomorphic and compatible with the actions of the Weyl groups of $U(n)$ and $\tilde{U}(n)$, and so we have

(4.3) $R(\tilde{U}(n))$ is isomorphic to the algebra

$$Z[\lambda^1 \rho_n, \dots, \lambda^n \rho_n, (\lambda^n \rho_n)^{-1}, u_n]/I$$

where I is the ideal generated by the elements

$$(\lambda^n \rho_n)(\lambda^n \rho_n)^{-1} - 1 \text{ and } u_n^2 - (\lambda^n \rho_n)^{-1}.$$

Theorem 4.1. *With the above notation*

$$K^*(Spin(2n)/\tilde{U}(n)) \cong \bigoplus_{\substack{e_k=0,1 \\ 0 \leq k \leq n-2}} Z(u_n^{e_0} g_1^{e_1} \cdots g_{n-2}^{e_{n-2}})$$

for $n \geq 2$ where

$$g_k = u_n \left\{ \sum_{s_1 \geq 1, \dots, s_k \geq 1} \sum_{t=0}^k (-1)^t \binom{k}{t} g(n, 2s_1 + \dots + 2s_k - k + t + 1) \right\}$$

for $k=1, \dots, n-2$ and

$$g(n, i) = \lambda^{n-i} \rho_n + \lambda^{n-i-2} \rho_n + \dots$$

for $i=0, \dots, n$.

Proof. Denote by $i_n: Spin(2n-1)/\tilde{U}(n-1) \rightarrow Spin(2n)/\tilde{U}(n)$ the homeomorphism of (4.2) and put

$$R(\tilde{T}) = Z[\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}, (\alpha_1 \cdots \alpha_n)^{-1/2}]$$

using the notation of [10], §13, Proposition 8.3. We proceed by induction on n .

The homomorphism $i_2^*: K^*(Spin(4)/\tilde{U}(2)) \rightarrow K^*(Spin(3)/\tilde{U}(1))$ is isomorphic, and we have

$$R(U(1)) \otimes_{R(Spin(3))} Z = Z[\alpha^{-1/2}]/((\alpha^{-1/2} - 1)^2),$$

$$i_2^*(u_2) = \alpha^{-1/2}$$

when we put $R(U(1)) = Z[\alpha^{1/2}, \alpha^{-1/2}]$. Therefore we get the statement when $n=2$.

Put $E = Spin(2n+1)/\tilde{U}(n)$, $F = Spin(2n)/\tilde{U}(n)$ and denote the inclusions $(F, \phi) \rightarrow (E, \phi) \rightarrow (E, F)$ by i and j respectively. Then there is a short exact sequence

$$0 \rightarrow K^*(E, F) \xrightarrow{j^*} K^*(E) \xrightarrow{i^*} K^*(F) \rightarrow 0.$$

Moreover we denote the projection $E \rightarrow Spin(2n+1)/Spin(2n)$ by p . Then we have an isomorphism

$$\varphi: K^*(F) \otimes \tilde{K}^*(Spin(2n+1)/Spin(2n)) \rightarrow K^*(E, F)$$

defined by $j^*\varphi(x \otimes \tilde{\Delta}_{2n}^+) = y p^*(\tilde{\Delta}_{2n}^+)$ $x \in K(F)$ where y is an element of $K^*(E)$ such that $i^*(y) = x$.

Here suppose that the assertion for $K^*(Spin(2n)/\tilde{U}(n))$ is true. By Theorem 3.6 we may assume that $K^*(Spin(2n+1)/Spin(2n)) = \wedge(\Delta_{2n}^- - 2^{n-1})$. Consider the element $i_{n+1}^{*-1} p^*(\Delta_{2n}^- - 2^{n-1})$ of $K^*(Spin(2n+2)/\tilde{U}(n+1))$. By the definition of Δ_{2n}^-

$$p^*(\Delta_{2n}^- - 2^{n-1}) = u_n(\lambda^{n-1} \rho_n + \lambda^{n-3} \rho_n + \dots) - 2^{n-1}.$$

Hence

$$i_{n+1}^{*-1} p^*(\Delta_{2n}^- - 2^{n-1}) = u_{n+1} \{ \sum_{s \geq 1} (g(n+1, 2s) - g(n+1, 2s+1)) \} - 2^{n-1}$$

because of $i_{n+1}^*(g(n+1, i) - g(n+1, i+1)) = \lambda^{n-i+1} \rho_n$.

For the completion of the induction it is sufficient to prove that

$$(i_{n+1} i)^*(u_{n+1} \{ \sum_{s_1 \geq 1, \dots, s_{k+1} \geq 1} \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} g(n+1, 2s_1 + \dots + 2s_{k+1} - k + i) \}) = g_k$$

for $k=2, \dots, n-1$. This follows from the following equalities:

$$\begin{aligned} & u_{n+1} \left(\sum_{s_1 \geq 1, \dots, s_{k+1} \geq 1} \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} g(n+1, 2s_1 + \dots + 2s_{k+1} - k + i) \right) \\ &= u_{n+1} \left\{ \sum_{s_1 \geq 1, \dots, s_{k+1} \geq 1} (g(n+1, 2s_1 + \dots + 2s_{k+1} - k) + \sum_{i=1}^k (-1)^i \binom{k}{i} + \binom{k}{i-1}) g(n+1, 2s_1 \right. \\ & \quad \left. + \dots + 2s_{k+1} - k + i) + (-1)^{k+1} g(n+1, 2s_1 + \dots + 2s_{k+1} + 1) \right\} \\ &= u_{n+1} \left\{ \sum_{s_1 \geq 1, \dots, s_k \geq 1} \sum_{i=0}^k (-1)^i \binom{k}{i} (\sum_{s_{k+1} \geq 1} (g(n+1, 2s_1 + \dots + 2s_k + 2s_{k+1} - k + i) \right. \\ & \quad \left. - g(n+1, 2s_1 + \dots + 2s_k + 2s_{k+1} - k + i + 1))) \right\} \end{aligned}$$

and $g(n, j) = (i_{n+1} i)^*(\sum_{k \geq 1} (g(n+1, j+2k-1) - g(n+1, j+2k)))$ for $j \geq 0$.

5. EI and FI (1)

In this section we discuss the symmetric spaces $E_6/PSp(4)$ and $F_4/Sp(3) \times_{\mathbb{Z}_2} SU(2) (= F_4/Sp(3) \cdot SU(2))$ ([11], p. 131).

We reproduce the Dynkin diagram of F_4 in [I] added the maximal root $\tilde{\alpha}$ and the simple roots $\alpha_1, \dots, \alpha_4$ corresponding to the vertexes.

$$(5.1) \quad \begin{array}{cccccc} \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & -\tilde{\alpha} & \\ \circ & \circ & \longleftarrow \circ & \circ & \circ & \\ \rho' & \overline{\lambda^2 \rho'} & \overline{\lambda^3 \rho'} & Ad_{F_4} & & \\ 26 & 273 & 1274 & 52 & & \end{array}$$

Then the Dynkin diagram of $Sp(3) \times SU(2)$ is obtained by omitting the vertex with the symbol α_1 .

$$(5.2) \quad \begin{array}{cccc} \beta_1 & \beta_2 & \beta_3 & \beta \\ \circ & \text{---} \circ & \longleftarrow \circ & \circ \\ \rho_3 & \lambda^2 \rho_3 & \lambda^3 \rho_3 & \rho_2 \\ 6 & 14 & 14 & 2 \end{array}$$

where the explanation of the symbols and the numbers is quite similar to that of the above diagram.

According to [16], Tables I, III and VIII, the fundamental weights of F_4 and $Sp(3) \cdot SU(2)$ determined by the above fundamental root systems are as follows:

$$(5.3) \quad \begin{aligned} w_1 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \tilde{\alpha}, \\ w_2 &= 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4, \\ w_3 &= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4, \\ w_4 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \\ \bar{w}_1 &= \beta_1 + \beta_2 + \frac{1}{2}\beta_3, \\ \bar{w}_2 &= \beta_1 + 2\beta_2 + \beta_3, \\ \bar{w}_3 &= \beta_1 + 2\beta_2 + \frac{3}{2}\beta_3, \\ \bar{w} &= \frac{1}{2}\beta. \end{aligned}$$

Hereafter, for simplicity we denote the weights $m_1\alpha_1 + \dots + m_4\alpha_4$, $n_1\beta_1 + \dots + n_3\beta_3$, $n_1\beta_1 + \dots + n_3\beta_3 + n\beta$ by $(m_1 \dots m_4)$, $(n_1 \dots n_3)$ and $(n_1 \dots n_3, n)$ respectively.

Since ρ_3 is the irreducible representation of $Sp(3)$ with $(11 \frac{1}{2})$ as the highest weight, by acting the elements of the Weyl group on it we get the all weights of ρ_3 :

$$(5.4) \quad \left(11 \frac{1}{2}\right) \left(01 \frac{1}{2}\right) \left(00 \frac{1}{2}\right) \left(00 - \frac{1}{2}\right) \left(0 - 1 - \frac{1}{2}\right) \left(-1 - 1 - \frac{1}{2}\right).$$

Let $i: Sp(3) \cdot SU(2) \rightarrow F_4$ be the inclusion of $Sp(3) \cdot SU(2)$ and $i^*(w)$ denote the reduction of a weight w of F_4 to $Sp(3) \cdot SU(2)$. Then we have

$$i^*(-\tilde{\alpha}) = \beta, i^*(\alpha_2) = \beta_3, i^*(\alpha_3) = \beta_2 \text{ and } i^*(\alpha_4) = \beta_1$$

and so using the first formula of (5.3)

$$(5.5) \quad \begin{aligned} i^*(1000) &= \left(-1 - 2 - \frac{3}{2}, -\frac{1}{2}\right), \\ i^*(0100) &= (001, 0), \\ i^*(0010) &= (010, 0), \\ i^*(0001) &= (100, 0). \end{aligned}$$

Proposition 5.1. *With the notations of (5.1) and (5.2) we have in $R(Sp(3) \cdot SU(2))$*

- (i) $i^*(\rho') = \lambda^2 \rho_3 + \rho_2 \rho_3 - 1$,
- (ii) $i^*(Ad_{F_4}) = \rho_2 \lambda^3 \rho_3 - \lambda^2 \rho_3 + \rho_2^2 + \rho_3^2 - \rho_2 \rho_3 - 1$.

Proof. By restricting all the weights of the adjoint representation of E_6 to F_4 we obtain those of ρ' , which are listed at the end of this section, since we know all the roots of F_4 ([16], Table VIII). It follows obviously that the weights of ρ_2 are $\frac{1}{2}\beta$ and $-\frac{1}{2}\beta$.

(i) When we observe the restrictions of the weights of ρ' to $Sp(3) \cdot SU(2)$ making use of (5.5) we get (i).

(ii) Considering that

$$Ad_{Sp(3)} = \rho_3^2 - \lambda^2 \rho_3 \quad \text{and} \quad Ad_{SU(2)} = \rho_2^2 - 1$$

we get (ii) similarly. q.e.d.

Lemma 5.2. *In $R(Sp(3) \cdot SU(2)) \otimes_{R(F_4)} Z$ we have*

- (i) $\lambda^2 \rho_3 = -\rho_2 \rho_3 + 27$,
- (ii) $\rho_2 \lambda^3 \rho_3 = -\rho_2^2 - \rho_3^2 + 80$,
- (iii) $\rho_3 \lambda^3 \rho_3 = \rho_2^2 \rho_3^2 + \rho_2^3 \rho_3 - \rho_3^2 - 27 \rho_2^2 - 30 \rho_2 \rho_3 + 432$,
- (iv) $(\lambda^3 \rho_3)^2 = \rho_2^4 + \rho_3^4 - \rho_2^4 \rho_3^2 + 54 \rho_2^3 \rho_3 + 2 \rho_2 \rho_3^3 + 6 \rho_2^2 \rho_3^2 - 216 \rho_2 \rho_3 - 136 \rho_2^2 - 812 \rho_2^2 + 6080$.

Proof. (i) and (ii) These are immediate results of Proposition 5.1.

(iii) From (i) of Proposition 5.1 we get

$$i^*(\lambda^2 \rho' + \rho') = \lambda^2(\rho_2 \rho_3) + (\rho_2 \rho_3) \lambda^2 \rho_3 + \lambda^2(\lambda^2 \rho_3)$$

and by the direct calculation we have

$$\begin{cases} \lambda^2(\rho_2 \rho_3) = (\rho_2^2 - 2) \lambda^2 \rho_3 + \rho_3^2 \\ \lambda^2(\lambda^2 \rho_3) = \rho_3 \lambda^3 \rho_3 - \lambda^2 \rho_3. \end{cases}$$

Therefore,

$$i^*(\lambda^2 \rho') = \rho_3 \lambda^3 \rho_3 + (\rho_2^2 + \rho_2 \rho_3 - 4) \lambda^2 \rho_3 + (\rho_3^2 - \rho_2 \rho_3 + 1)$$

and so from (i), (iii) follows.

(iv) By the direct calculation we get

$$\begin{cases} \lambda^2(\rho_2^2) = 2\rho_2^2 - 2 \\ \lambda^2(\rho_3^2) = -2(\lambda^2 \rho_3)^2 + 2\rho_3^2 \lambda^2 \rho_3 \\ \lambda^2(\lambda^3 \rho_3) = (\lambda^2 \rho_3)^2 - \rho_3^2 + 1 \end{cases}$$

and from (ii) we have

$$\begin{aligned} &(\lambda^3 \rho_3)^2 + (\rho_2^2 - 2)\lambda^2(\lambda^3 \rho_3) + \lambda^2(\rho_2^2) + \lambda^2(\rho_3^2) \\ &+ \rho_2^3 \lambda^3 \rho_3 + \rho_2 \rho_3^2 \lambda^3 \rho_3 + \rho_2^2 \rho_3^2 = 3160. \end{aligned}$$

Therefore, making use of the above formulas, (i) and (ii) we have (iv). q.e.d.

Theorem 5.3. *With the notation of [I], Proposition 7.3*

$$K^*(E_6/PSp(4)) \cong \wedge(\beta(\rho_1 - \rho_2), \beta(\lambda^2 \rho_1 - \lambda^2 \rho_2)) \otimes Z[\rho_4^2]/((\rho_4^2 - 64)^3).$$

Proof. Let $j: Sp(3) \cdot SU(2) \rightarrow PSp(4)$ be the inclusion map of $Sp(3) \cdot SU(2)$. Then we have

$$\begin{cases} j^*(\lambda^2 \rho_4) = \lambda^2 \rho_3 + \rho_2 \rho_3 + 1 \\ j^*(\lambda^4 \rho_4) = \rho_2 \lambda^3 \rho_3 + 2\lambda^2 \rho_3 \\ j^*(\rho_4^2) = (\rho_2 + \rho_3)^2 \\ j^*((\lambda^3 \rho_4)^2) = (\lambda^3 \rho_3 + \rho_2 \lambda^2 \rho_3 + \rho_3)^2 \\ j^*(\rho_4(\lambda^3 \rho_4)) = (\rho_2 + \rho_3)(\lambda^3 \rho_3 + \rho_2 \lambda^2 \rho_3 + \rho_3) \end{cases}$$

and from Lemma 5.2 we have in $R(PSp(4)) \otimes_{R(\mathbb{Z}_6)} Z$

$$\begin{cases} \lambda^2 \rho_4 = 28 \\ \lambda^4 \rho_4 = -\rho_4^2 + 134 \\ \rho_4 \lambda^3 \rho_4 = -\rho_4^2 + 512 \\ (\lambda^3 \rho_4)^2 = \rho_4^4 - 191\rho_4^2 + 11264. \end{cases}$$

This and (2.2) show that

$$R(PSp(4)) \otimes_{R(\mathbb{Z}_6)} Z = Z[\rho_4^2]/((\rho_4^2 - 64)^3)$$

and so Theorem 5.3 follows from [I], Proposition 7.3. q.e.d.

(5.6) The weights of ρ' and the positive roots of F_4 are as follows respectively:

1 2 3 2		2 3 4 2			
1 2 3 1		1 3 4 2			
1 2 2 1		1 2 4 2			
1 1 2 1		1 2 3 2			
1 1 1 1	0 1 2 1	1 2 3 1	1 2 2 2		
1 1 1 0	0 1 1 1	1 2 2 1	1 1 2 2		
0 1 1 0	0 0 1 1	1 2 2 0	0 1 2 2	1 1 2 1	
0 0 1 0	0 0 0 1	1 1 2 0	0 1 2 1	1 1 1 1	
0 0 0 0	0 0 0 0	1 1 1 0	0 1 2 0	0 1 1 1	

$$\begin{array}{ll|llll}
0 & 0-1 & 0 & 0 & 0 & 0-1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0-1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & -1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & -1 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & -2 & -1 & & & & & & & & & & & & & & & & & & \\
-1 & -2 & -2 & -1 & & & & & & & & & & & & & & & & & & \\
-1 & -2 & -3 & -1 & & & & & & & & & & & & & & & & & & \\
-1 & -2 & -3 & -2 & & & & & & & & & & & & & & & & & &
\end{array}$$

where the sequence of integers $m_1 \cdots m_4$ indicates a weight $m_1 \alpha_1 + \cdots + m_4 \alpha_4$.

6. EI and FI (2)

This section is a continuation of the section 5.

Put

$$(6.1) \quad x = \rho_2^2, y = (\rho_2 + \rho_3)^2 \text{ and } w = \rho_2 \rho_3 + x.$$

Then

$$(6.2) \quad w^2 = xy$$

obviously. We obtain from Lemma 5.2

$$\begin{aligned}
& \text{(i)} \quad \rho_3^2 = x + y - 2w, \\
& \text{(ii)} \quad \lambda^2 \rho_3 = x - w + 27, \\
(6.3) \quad & \text{(iii)} \quad \rho_2 \lambda^3 \rho_3 = -2x + (2w - y + 80), \\
& \text{(iv)} \quad \rho_3 \lambda^3 \rho_3 = (2 - w)x + (w^2 - 28w - y + 432), \\
& \text{(v)} \quad (\lambda^3 \rho_3)^2 = -x^3 + (2w - 48)x^2 + (-w^2 + 44w - 732)x \\
& \quad \quad \quad + (6w^2 + 56w - 2yw + y^2 - 136y + 6080)
\end{aligned}$$

and from Theorem 5.3

$$(6.4) \quad (y - 64)^3 = 0.$$

From (6.2), (iii) and (iv) of (6.3) we have

$$\begin{aligned}
(6.5) \quad & \text{(i)} \quad wx^2 + (-w^2 + 24w - 512)x + (4w^2 - yw + 80w) = 0, \\
& \text{(ii)} \quad -wx^2 + (2w^2 - 24w + 512)x + (-w^3 + 20w^2 + 5yw - 592w - y^2 \\
& \quad \quad \quad + 80y) = 0.
\end{aligned}$$

From (6.2) and (6.5) we have

$$(6.6) \quad -w^2x + w^3 - 24w^2 - 4yw + 512w + y^2 - 80y = 0.$$

From (6.2), (6.4) and (6.6) we have

$$(6.7) \quad w^4 - yw^3 + 24yw^2 + 4y^2w - 512yw - 112y^2 + 12288y - 262144 = 0.$$

From (6.2), (iii) and (v) of (6.3) we have

$$(6.8) \quad x^4 + (48 - 2w)x^3 + (w^2 - 44w + 736)x^2 + (-6w^2 - 64w - 6400)x + (2w^3 + 144w^2 + 320w - yw^2 - 4yw + y^2 - 160y + 6400) = 0.$$

(2.2), (6.1) and (6.3) show that $R(Sp(3) \cdot SU(2)) \otimes_{R(\mathbb{F}_4)} Z$ is generated by the elements x, y and w as an algebra and moreover (6.4), (6.7) and (6.8) imply

Lemma 6.1. $R(Sp(3) \cdot SU(2)) \otimes_{R(\mathbb{F}_4)} Z$ is generated by the elements $x^a y^b w^c$ for $a, c=0, 1, 2, 3$ and $b=0, 1, 2$, as a module.

Let M denote the submodule of $R(Sp(3) \cdot SU(2)) \otimes_{R(\mathbb{F}_4)} Z$ generated by the elements:

$$1, x, x^2, x^3, y, y^2, w, w^2, w^3, xw, yw, y^2w, yw^2, y^2w^2, yw^3, y^2w^3.$$

From (6.4) and (6.7) we have

$$(6.9) \quad y^i w^j \in M \quad \text{for } i, j \geq 0.$$

Hence, from (6.6) we have

$$(6.10) \quad xw^{j+2} \in M \quad \text{for } j \geq 0.$$

From (i) of (6.5), (6.9) and (6.10) we have

$$(6.11) \quad x^2 w^j \in M \quad \text{for } j \geq 0.$$

From (i) of (6.5) and (6.6) we get

$$(6.12) \quad x^2 w = w^3 - 28w^2 + 432w - 3yw - 24xw + 512x + y^2 - 80y$$

and so we see that x^2w, x^3w, x^3w^2 and x^3w^3 are contained in M from (6.9), (6.10) and (6.11). Thus we obtain

Lemma 6.2. *With the above notation*

$$R(Sp(3) \cdot SU(2)) \otimes_{R(\mathbb{F}_4)} Z = M.$$

Theorem 6.3. *With the notation of (6.1) $K^*(F_4/Sp(3) \cdot SU(2))$ is a free module generated by the elements*

$$1, x, x^2, x^3, y, y^2, w, w^2, w^3, xw, yw, yw^2.$$

Proof. Let N be the submodule of $K^*(F_4/Sp(3) \cdot SU(2))$ generated by the elements mentioned in the theorem.

From (iii), (iv) and (v) of (6.3) we have

$$\begin{aligned} & -x^4 + (3w - 48)x^3 + (-3w^2 + 94w - 736)x^2 \\ & + (w^3 - 42w^2 + 768w + 5376)x + (-5w^3 - 168w^2 - 7456w + 2yw^2 \\ & - y^2w + 162yw + y^2 - 512y + 34560) = 0. \end{aligned}$$

From this equality and (6.8) we have

$$\begin{aligned} & wx^3 + (-2w^2 + 50w)x^2 + (w^3 - 48w^2 + 704w - 1024)x \\ & + (-3w^3 - 24w^2 - 7136w + yw^2 - y^2w + 158yw + 2y^2 - 672y + 40960) = 0. \end{aligned}$$

Moreover, from this equality, (6.5) and (6.6) we have

$$(6.13) \quad y^2w = 512x^2 - 512xw + 12288x - 10240w + 192yw - 512y + 40960.$$

This shows

$$(6.14) \quad y^2w \in N.$$

From (6.4) we have

$$(6.15) \quad y^2w^2 = 192yw^2 - 12288w^2 + 262144x$$

using (6.2) and so

$$(6.16) \quad y^2w^2 \in N.$$

From (6.2) and (6.13)

$$\begin{aligned} yw^3 &= x(y^2w) \\ &= 512x^3 - 512x^2w + 12288x^2 - 10240xw + 192w^3 - 512w^2 + 40960x. \end{aligned}$$

and so we have

$$(6.17) \quad yw^3 \in N$$

since $x^2w \in N$ by (6.12). From (6.15) and (6.17) we have

$$(6.18) \quad y^2w^3 \in N.$$

(6.14), (6.16), (6.17) and (6.18) imply Theorem 6.3 since $K^*(F_4/Sp(3) \cdot SU(2))$ is a free module of rank 12. q.e.d.

7. FII and G

Type FII($F_4/Spin(9)$). According to [15], Theorem 15.1 we have in $R(Spin(9)) \otimes_{R(F_4)} Z$

$$\begin{cases} \lambda^1 \rho_9 = -\Delta_9 + 25 \\ \lambda^2 \rho_9 = -\Delta_9 + 52 \\ \lambda^3 \rho_9 = \Delta_9^2 - 23\Delta_9 + 196 \\ (\Delta_9 - 16)^3 = 0. \end{cases}$$

This proves

Theorem 7.1. $K^*(F_4/Spin(9)) \cong Z[\Delta_9]/((\Delta_9 - 16)^3)$.

Type $G(G_2/SU(2) \cdot SU(2))$. We observe the extended Dynkin diagram of G_2 with the irreducible representations corresponding to the vertices and their dimensions written next to vertices ([16], Table 30):

$$(7.1) \quad \begin{array}{ccc} \alpha_1 & \alpha_2 & -\tilde{\alpha} \\ \circ \longleftarrow \circ & \text{---} & \circ \\ \rho & Ad_{G_2} & \\ 7 & 14 & \end{array}$$

where α_1, α_2 are the simple roots and $\tilde{\alpha}$ is the maximal root.

Let us denote by σ the involutive automorphism of G_2 for the symmetric space of type G ([11], Theorem 3.1). Then we see that the subgroup consisting of fixed points of σ is $SU(2) \times_{Z_2} SU(2) (= SU(2) \cdot SU(2))$ where Z_2 is the intersection of the centers of the two groups $SU(2)$, and its Dynkin diagram is obtained by omitting the vertex with the symbol α_2 .

$$(7.2) \quad \begin{array}{cc} \beta_1 & \beta_2 \\ \circ & \circ \\ \rho_2 & \rho_2' \\ 2 & 2 \end{array}$$

in which the explanation of the symbols and the numbers are as in the diagram of G_2 .

If we denote the fundamental weights of G_2 and $SU(2) \cdot SU(2)$ by w_k and \bar{w}_k for $k=1, 2$ respectively, then we have from [16], Tables I and IX

$$(7.3) \quad \begin{aligned} w_1 &= 2\alpha_1 + \alpha_2, \\ w_2 &= 3\alpha_1 + 2\alpha_2 = \tilde{\alpha}, \\ \bar{w}_1 &= \frac{1}{2}\beta_1, \\ \bar{w}_2 &= \frac{1}{2}\beta_2. \end{aligned}$$

Let $i: SU(2) \cdot SU(2) \rightarrow G_2$ be the inclusion of $SU(2) \cdot SU(2)$ and $i^*(w)$ be the reduction of a weight w of G_2 to $SU(2) \cdot SU(2)$. Since

$$i^*(\alpha_1) = \beta_1 \text{ and } i^*(-\tilde{\alpha}) = \beta_2,$$

we have by (7.3)

$$(7.4) \quad \begin{aligned} i^*(\alpha_1) &= \beta_1, \\ i^*(\alpha_2) &= -\frac{3}{2}\beta_1 - \frac{1}{2}\beta_2. \end{aligned}$$

Proposition 7.2. *With the notation of (7.1)*

- (i) $i^*(\rho) = \rho_2^2 + \rho_2\rho_2' - 1,$
- (ii) $i^*(Ad_{G_2}) = \rho_2^2 + \rho_2'^2 + \rho_2^3\rho_2' - 2\rho_2\rho_2' - 2$

where $i^*: R(G_2) \rightarrow R(SU(2) \cdot SU(2))$ is the restriction.

Proof. Denote the weights $m_1\alpha_1 + m_2\alpha_2$ and $n_1\beta_1 + n_2\beta_2$ by (m_1, m_2) and (n_1, n_2) respectively.

(i) Since ρ is the irreducible representation of G_2 with $(2, 1)$ as the highest weight, by operating the elements of the Weyl group on it we see that the weights of ρ is as follows:

$$(2, 1) \ (1, 1) \ (1, 0) \ (0, 0) \ (-1, 0) \ (-1, -1) \ (-2, -1).$$

Consider the restrictions of the weights of ρ to $SU(2) \cdot SU(2)$ using (7.4) then (i) follows because the weights of ρ_2 and ρ_2' are

$$\left(\frac{1}{2}, 0\right) \ \left(-\frac{1}{2}, 0\right) \text{ and } \left(0, \frac{1}{2}\right) \ \left(0, -\frac{1}{2}\right)$$

respectively.

(ii) From [16], Table IX the weights of Ad_{G_2} are as follows:

$$\begin{aligned} (3, 2) \ (3, 1) \ (2, 1) \ (1, 1) \ (1, 0) \ (0, 1) \ (0, 0) \ (-3, -2) \\ (-3, -1) \ (-2, -1) \ (-1, -1) \ (-1, 0) \ (0, -1) \ (0, 0). \end{aligned}$$

By observing the reduction of these weights to $SU(2) \cdot SU(2)$ we obtain (ii) analogously. q.e.d.

Theorem 7.3. $K^*(G_2/SU(2) \times_{Z_2} SU(2)) \cong Z[\rho_2^2]/((\rho_2^2 - 4)^3)$

with the notation of (7.2).

Proof. By Proposition 7.2 we get in $R(SU(2) \cdot SU(2)) \otimes_{R(G_2)} Z$

$$\rho_2^2 + \rho_2\rho_2' = 8 \text{ and } \rho_2^2 + \rho_2'^2 + \rho_2^3\rho_2' - 2\rho_2\rho_2' = 16.$$

From these equalities we have

$$\begin{cases} \rho_2 \rho_2' = 8 - \rho_2^2 \\ \rho_2'^2 = \rho_2^4 - 11\rho_2^2 + 32 \\ (\rho_2^2 - 4)^3 = 0. \end{cases}$$

Therefore the theorem is proved because $R(SU(2) \cdot SU(2))$ equals the ring $Z[\rho_2^2, \rho_2 \rho_2', \rho_2'^2]$. q.e.d.

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