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COMPLEX HYPERPOLAR ACTIONS WITH A TOTALLY GEODESIC ORBIT

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Abstract

We first show that homogeneous submanifolds with abelian normal bundle in a symmetric space of non-compact type occur as principal orbits of complex hyperpolar actions on the symmetric space. Next we show that all complex hyperpolar actions with a reflective orbit are orbit equivalent to Hermann type actions. Furthermore, we classify complex hyperpolar actions with a totally geodesic orbit in the case where the ambient symmetric space is irreducible. Also, we list up the cohomogeneities of Hermann type actions on irreducible symmetric spaces.

1. Introduction

A proper isometric action H (which is automatically compact) on a symmetric space of compact type is called a *hyperpolar action* if there exists a properly embedded complete flat submanifold Σ of the symmetric space meeting all H-orbits orthogonally. The submanifold Σ is automatically totally geodesic and it is called a *section* of the action. Hyperpolar actions have necessarily singular orbits, which are interpreted as the polar set of the action. Principal orbits of hyperpolar actions are equifocal submanifolds in the sense of [17]. A. Kollross [14] classified hyperprolar actions on irreducible symmetric spaces of compact type. According to the classification, a hyperpolar action on the symmetric space is a Hermann action or a cohomogeneity one action, where a Hermann action implies the action of a symmetric subgroup of the isometry group of the symmetric space. Recently, the author [11] has introduced the notions of a complex equifocal submanifold in a symmetric space of non-compact type and a complex hyperpolar action on the symmetric space. These notions are defined as follows. Let G/K be a symmetric space of non-compact type and M be an immersed submanifold in G/K. The submanifold M is called a *complex equifocal submanifold* if the following conditions (i)–(iii) hold:

(i) M has abelian normal bundle, that is, the sectional curvature of any 2-plane in the normal space of M is equal to zero,

(ii) the normal holonomy group of M is trivial,

(iii) for any parallel normal vector field v of M, the complex focal radii along the normal geodesic γ_{v_x} with $\gamma'_{v_x}(0) = v_x$ are independent of the choice of $x \in M$, where

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$\gamma'_{v_x}(0)$ is the velocity vector of γ_{v_x} at 0.
In the case where M is complete and real analytic, the complex focal radii along the normal geodesic γ of M surjectively correspond to focal points along the complexified normal geodesic $\gamma^{\mathbf{c}}$ of the complexified submanifold $M^{\mathbf{c}}$ (which is a submanifold in the anti-Kaehlerian symmetric space $G^{\mathbf{c}}/K^{\mathbf{c}}$). See [11] about the definition of the complex focal radius. See [11, 13] about the study of a complex equifocal submanifold. Let H be a closed subgroup of G. If there exists a properly embedded complete flat submanifold Σ meeting all H-orbits orthogonally, then the H-action on G/K is called a *complex hyperpolar action*. The submanifold Σ is automatically totally geodesic and it is called a *section* of the action. Note that complex hyperpolar actions are not necessarily compact group actions. It is known that principal orbits of complex hyperpolar actions are complex equifocal (see Theorem 12 of [11]). In this paper, we first show the following fact.

Theorem A. *All homogeneous submanifolds with abelian normal bundle in a symmetric space of non-compact type occur as principal orbits of complex hyperpolar actions on the symmetric space and hence they are complex equifocal.*

Thus the study of homogeneous submanifolds with abelian normal bundle in a symmetirc space of non-compact type is reduced to that of complex hyperpolar actions. Complex hyperprolar actions do not necessarily have singular orbits (i.e., the polar set). However, for almost all complex hyperpolar actions H on G/K, the complexified actions $H^{\mathbf{c}}$ (which are actions on the anti-Kaehlerian symmetric space $G^{\mathbf{c}}/K^{\mathbf{c}}$) have singular orbits (i.e., the polar set). The polar set should be named complex polar set of the original action. From this reason, the original actions were named complex hyperpolar actions. It is expected that the study of a complex hyperpolar action will be useful to that of harmonic analysis on a symmetric space of non-compact type. If H is a symmetric subgroup of G, then the H-action on G/K is called a *Hermann type action*. It is known that a Hermann type action is a complex hyperpolar action admitting a reflective orbit (see [12]), where the reflectivity implies that the geodesic reflection in the orbit is a globally well-defined isometry of G/K. Conversely we can show the following statement.

Theorem B. *Let G/K be a symmetric space of non-compact type, where G is assumed to be simply connected. Then all complex hyperpolar actions on G/K admitting a reflective orbit are orbit equivalent to Hermann type actions.*

Reflective submanifolds are totally geodesic. Hence we naturally think if the statement of the above theorem holds even if the part of "reflective orbit" is replaced by "totally geodesic orbit". However, it is shown in [5] that there exist cohomogeneity one (hence complex hyperpolar) actions on G/K admitting a totally geodesic orbit which is not orbit equivalent to a Hermann type action. Complex hyperpolar actions on irreducible symmetric space of non-compact type admitting a totally geodesic orbit are classified as follows.

Theorem C. *Complex hyperpolar actions on an irreducible symmetric space of non-compact type admitting a totally geodesic orbit are orbit equivalent to a Hermann type action, the G_2^2-action on $SO_0(3,4)/SO(3)\times SO(4)$, the $SU(1,2)$-action on $G_2^2/SO(4)$, the $SL(3,\mathbf{R})$-action on $G_2^2/SO(4)$, the $G_2^{\mathfrak{c}}$-action on $SO(7,\mathbf{C})/SO(7)$ or the $SL(3,\mathbf{C})$-action on $G_2^{\mathfrak{c}}/G_2$.*

REMARK 1.1. Five actions other than a Hermann type action in this statement are of cohomogeneity one.

From Theorems A, B and C, the following facts for homogeneous submanifolds with abelian normal bundle follow directly.

Corollary D. *Let M be a homogeneous submanifold with abelian normal bundle in a symmetric space G/K of non-compact type.*
(i) *If M admits a reflective focal submanifold and G is simply connected, then M occurs as a principal orbit of a Hermann type action.*
(ii) *If M admits a totally geodesic focal submanifold and G/K is irreducible, then M occurs as a principal orbit of a Hermann type action or one of five non-Hermann type actions in the statement of* Theorem C.

For complex hyperpolar actions admitting a totally geodesic principal orbit, we can show the following fact.

Theorem E. *Complex hyperpolar actions on an irreducible symmetric space of non-compact type admitting a totally geodesic principal orbit are orbit equivalent to the $SO_0(1,m-1)$-action on the hyperbolic space $SO_0(1,m)/SO(m)$, where $m\geq 2$.*

2. Proofs of Theorems A, B, C and E

In this section, we shall prove Theorems A, B, C and E. First, we prepare the following lemma.

Lemma 2.1. *Let G be a semi-simple Lie group equipped with a bi-invariant pseudo-Riemannian metric $\langle\ ,\ \rangle$, H be a closed subgroup of $G\times G$ (where H acts on G by the adjoint representation) and $\mathfrak{a}$ be an abelian subspace of the normal space $T_e^{\perp}(He)$ of the orbit He at e, where e is the identity element of G. Then $\Sigma=\exp_G(\mathfrak{a})$ meets all H-orbits through Σ orthogonally, where $\exp_G$ is the exponential map of G.*

Proof. Denote by $\mathfrak{h}$ the Lie algebra of H. Let $X_0 \in \mathfrak{a}$. Easily we can show

$$T_e(He) = \{Y - Z \mid (Y, Z) \in \mathfrak{h}\}$$

and

$$(\exp_G X_0)_*^{-1}(T_{\exp_G X_0} H(\exp_G X_0)) = \{\mathrm{Ad}(\exp_G X_0)^{-1}(Y) - Z \mid (Y, Z) \in \mathfrak{h}\}.$$

For any $(Y, Z) \in \mathfrak{h}$ and $W \in \mathfrak{a}$, we have

$$\langle \mathrm{Ad}(\exp_G X_0)^{-1}(Y) - Z, W\rangle_e = \langle Y - Z, W\rangle_e = 0.$$

Hence it follows that $(\exp_G X_0)_*^{-1}(T_{\exp_G X_0} H(\exp_G X_0))$ is orthogonal to $\mathfrak{a}$, that is, $T_{\exp_G X_0} H(\exp_G X_0)$ is orthogonal to $(\exp_G X_0)_*(\mathfrak{a}) = T_{\exp_G X_0}\Sigma$. This completes the proof. □

By using this lemma, we can show the following fact.

Lemma 2.2. *Let G/K be a symmetric space of non-compact type, H be a closed subgroup of G and $\mathfrak{a}$ be an abelian subspace of the normal space $T^{\perp}_{eK}H(eK)$ of the orbit $H(eK)$ at eK. Then $\Sigma := \exp(\mathfrak{a})$ meets all H-orbits through Σ orthogonally, where* exp *is the exponential map of G/K.*

Proof. Let $\pi\colon G \to G/K$ be the natural projection. The space $\mathfrak{a}$ is identified with the horizontal lift of $\mathfrak{a}$ to e with respect to π. Since orbits of the $H \times K$-action on G are the inverse images of orbits of the H-action on G/K by π, the subspace $\mathfrak{a}$ is contained in the normal space $T^{\perp}_e(H \times K)e$ of the orbit $(H \times K)e$ at e. Hence, according to the previous lemma, $\tilde{\Sigma} := \exp_G(\mathfrak{a})$ meets all $(H \times K)$-orbits through $\tilde{\Sigma}$ orthogonally. Therefore $\Sigma := \exp(\mathfrak{a})$ meets all H-orbits through Σ orthogonally. □

By using this lemma, we prove Theorem A.

Proof of Theorem A. Let M be a homogeneous submanifold with abelian normal bundle in a symmetric space G/K of non-compact type. Since M is homogeneous, there exists a closed subgroup H of G having M as an H-orbit. We shall show that the H-action is complex hyperpolar. Without loss of generality, we may assume that $M = H(eK)$. By the assumption, the normal space $T^{\perp}_{eK}M$ is an abelian subspace of $T_{eK}(G/K)$. According to Lemma 2.2, it is shown that the complete flat totally geodesic submanifold $\Sigma := \exp(T^{\perp}_{eK}M)$ meets all H-orbits through Σ orthogonally. Take any $gK \in G/K$ and a piecewise smooth curve $\mu\colon [0, 1] \to G/K$ with $\mu(0) = eK$ and $\mu(1) = gK$. Since H-orbits give a singular Reimannian foliation on G/K and G/K is complete, by imitating the proof of Lemma 2.1 of [3] (even if the foliation has singular leaves), we can construct a rectangle $\delta\colon [0, 1] \times [0, 1] \to G/K$ such that $\delta(t, \cdot)$ lies in an H-orbit for $t \in [0,1]$, $\delta(\,\cdot\,, s)$ meets H-orbits orthogonally and that $\delta(t,t) = \mu(t)$ ($t \in [0,1]$). Clearly we have $\delta(1,0) \in H(gK) \cap \Sigma$. Thus the orbit $H(gK)$ meets Σ. Therefore we see that all H-orbits meet Σ orthogonally. This implies that the H-action is complex hyperpolar and that $M = H(eK)$ is a principal orbit of the H-action. This completes the proof. □

Next, by using Lemma 2.2, we prove Theorem B.

Proof of Theorem B. Let G/K be as in the statement of Theorem B and H be a closed subgroup of G such that the H-action on G/K is a complex hyperpolar action admitting a reflective orbit. By replacing H by its suitable conjugate group if necessary, we may assume that $H(eK)$ is a reflective orbit. For simplicity, set $M := H(eK)$. Since M is reflective, so is also $M^\perp := \exp(T^\perp_{eK}M)$. Denote by $\mathfrak{g}$ (resp. $\mathfrak{k}$) the Lie algebra of G (resp. K). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Let B be the $\mathrm{Ad}(G)$-invariant non-degenerate inner product of $\mathfrak{g}$ such that $B|_{\mathfrak{k}\times\mathfrak{k}}$ is negative definite, $B|_{\mathfrak{k}\times\mathfrak{p}} = 0$ and $B|_{\mathfrak{p}\times\mathfrak{p}} = \langle\ ,\ \rangle_{eK}$, where $\langle\ ,\ \rangle$ is the Riemannian metric of G/K and $\mathfrak{p}$ is identified with $T_{eK}(G/K)$. Set $\mathfrak{h}' := \mathfrak{n}_{\mathfrak{k}}(T_{eK}M) \oplus T_{eK}M$ and $\mathfrak{m}' := (\mathfrak{k} \ominus \mathfrak{n}_{\mathfrak{k}}(T_{eK}M)) \oplus T^\perp_{eK}M$. Clearly we have $\mathfrak{g} = \mathfrak{h}' \oplus \mathfrak{m}'$ (orthogonal direct sum) and $\mathfrak{n}_{\mathfrak{k}}(T_{eK}M) = \mathfrak{n}_{\mathfrak{k}}(T^\perp_{eK}M)$. Since M is reflective, both $T_{eK}M$ and $T^\perp_{eK}M$ are Lie triple systems. By using these facts, we can show

$$(2.1)\qquad [\mathfrak{h}', \mathfrak{h}'] \subset \mathfrak{h}', \quad [\mathfrak{h}', \mathfrak{m}'] \subset \mathfrak{m}' \quad \text{and} \quad [\mathfrak{m}', \mathfrak{m}'] \subset \mathfrak{h}'.$$

Let H' be the connected subgroup of G having $\mathfrak{h}'$ as its Lie algebra. It follows from (2.1) and the simply connectedness of G that H' is a symmetric subgroup of G. That is, the H'-action on G/K is a Hermann type action. We shall show that the H'-action and the H-action have the same orbits. Denote by $\mathrm{pr}_{\mathfrak{p}}$ (resp. $\mathrm{pr}_{\mathfrak{k}}$) the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{p}$ (resp. $\mathfrak{k}$). Also, let $\mathfrak{h}$ be the Lie algebra of H. We have $T_e((H\times K)e) = \mathrm{pr}_{\mathfrak{p}}(\mathfrak{h}) + \mathfrak{k}$ and $T_e((H'\times K)e) = \mathrm{pr}_{\mathfrak{p}}(\mathfrak{h}') + \mathfrak{k} = T_{eK}M + \mathfrak{k}$. On the other hand, it follows from $\pi^{-1}(M) = (H\times K)e$ that $T_{eK}M = \mathrm{pr}_{\mathfrak{p}}(T_e((H\times K)e))$. So we have $T_{eK}M = \mathrm{pr}_{\mathfrak{p}}(\mathfrak{h})$ and hence $T_e((H'\times K)e) = T_e((H\times K)e)$. This implies $(H'\times K)e = (H\times K)e$, which furthermore implies $H'(eK) = H(eK)$. Let Σ be a section of the H-action through eK. Set $\mathfrak{a} := T_{eK}\Sigma$, which is abelian. Since $\mathfrak{a} \subset T^\perp_{eK}(H'(eK))$, it follows from Lemma 2.2 that H'-orbits through Σ meet Σ orthogonally. On the other hand, we have $[\mathrm{pr}_{\mathfrak{k}}(\mathfrak{h}), T_{eK}M] = \mathrm{pr}_{\mathfrak{p}}([\mathfrak{h}, T_{eK}M]) \subset T_{eK}M$, that is, $\mathrm{pr}_{\mathfrak{k}}(\mathfrak{h}) \subset \mathfrak{n}_{\mathfrak{k}}(T_{eK}M)$, which together with $\mathrm{pr}_{\mathfrak{p}}(\mathfrak{h}) = T_{eK}M$ implies that $\mathfrak{h} \subset \mathfrak{h}'$, that is, $H \subset H'$. Hence H'-principal orbits through Σ coincides with H-principal orbits through Σ. From this fact, it follows that the H'-action and the H-action have the same orbits. This completes the proof. □

Next we prove Theorem C.

Proof of Theorem C. Let G/K be as in the statement of Theorem C and H be a closed subgroup of G such that the H-action on G/K is a complex hyperpolar action admitting a totally geodesic orbit. According to the classification of cohomogeneity one actions admitting a totally geodesic orbit by Berndt-Tamaru [5], if the H-action is of cohomogeneity one, then the action is orbit equivalent to a Hermann type action or one of five non-Hermann type actions in the statement of Theorem C. Assume that the H-action is of cohomogeneity greater than one. By replacing H by its suitable conjugate group if necessary, we may assume that $H(eK)$ is a totally geodesic orbit. For simplicity, set $M := H(eK)$. Let H' be a connected subgroup of G defined as in the proof of Theorem B. By the same argument as in the proof of Theorem B, we can show that the H'-action and the H-action have the same orbits. Set $\mathfrak{h}'^* := \mathfrak{n}_{\mathfrak{f}}(T_{eK}M) \oplus \sqrt{-1}T_{eK}M$ $(\subset \mathfrak{g}^* := \mathfrak{f} + \sqrt{-1}\mathfrak{p} \subset \mathfrak{g}^{\mathbf{c}})$. Let H'^* be the connected subgroup of G^* $(:= \exp \mathfrak{g}^*)$ having $\mathfrak{h}'^*$ as its Lie algebra. Since the H'-action is a complex hyperpolar action of cohomogeneity greater than one, it follows that the H'^*-action is a hyperpolar action of cohomogeneity greater than one on the irreducible symmetric space G^*/K of compact type. According to the classification by A. Kollross [14] of hyperpolar actions on irreducible symmetric spaces of compact type, the H'^*-action is orbit equivalent to a Hermann action. We denote this Hermann action by $(H'^*)'$. By replacing $(H'^*)'$ by its suitable conjugate group, we may assume that $(H'^*)'(eK) = (H'^*)(eK)$. Then the dual action $(H'^*)'^*$ of $(H'^*)'$ is defined. The $(H'^*)'^*$-action is a Hermann type action and it is orbit equivalent to the H'-action. Therefore, the H-action is orbit equivalent to the Hermann type $(H'^*)'^*$-action. This completes the proof. □

Next we prove Corollary D.

Proof of Corollary D. According to Theorem A, M occurs as a principal orbit of a complex hyperpolar action on G/K. If M admits a reflective (resp. totally geodesic) focal submanifold, then the focal submanifold is a reflective (resp. totally geodesic) singular orbit of the action. Hence the statements (i) and (ii) of Corollary D follow from Theorems B and C, respectively. □

Next we prove Theorem E.

Proof of Theorem E. Let G/K be an irreducible symmetric space of non-compact type and H be a closed subgroup of G whose action on G/K is a complex hyperpolar action admitting totally geodesic principal orbit. By replacing H by its suitable conjugate group if necessary, we may assume that the orbit $H(eK)$ is a totally geodesic principal orbit. Let H' be a connected subgroup of G defined as in the proof of Theorem B. As in the proof of Theorem B, we can show that the H'-action and the H-action have the same orbits. Let $H'^* \times G^*/K \to G^*/K$ be the dual action of the H'-action. Then it is shown that $H'^*(eK)$ is totally geodesic principal orbit. Hence, according to the result in [6], the H'^*-action is conjugate to the $SO(m)$-action on the sphere $SO(m+1)/SO(m)$ $(m \geq 2)$ and hence it is also conjugate to the $SO'(m)$-action on the sphere, where $SO(m)$ denotes the subgroup $\left\{\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \,\middle|\, A \in SO(m)\right\}$ of $SO(m+1)$ and $SO'(m)$ denotes the conjugate group $\left\{\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, A \in SO(m)\right\}$. It is easy to show that the tangent spaces $T_{eSO(m)}H'^{*}(eSO(m))$ and $T_{eSO(m)}SO'(m)(eSO(m))$ are Lie triple systems of $T_{eSO(m)}(SO(m+1)/SO(m))$ $(\subset \mathfrak{so}(m+1))$ and that they map to each other by an element of the $\mathrm{Ad}(SO(m))$-action. From this fact, it follows that the H'-action is orbit equivalent to the dual action of the $SO'(m)$-action, that is, the $SO_0(1, m-1)$-action on the hyperbolic space $SO_0(1, m)/SO(m)$. After all, the H-action is orbit equivalent to the $SO_0(1, m-1)$-action. □

3. Hermann type actions and their cohomogeneities

In this section, we shall list up Hermann type actions on irreducible symmetric spaces of non-compact type and their cohomogeneities. Let G/K be an irreducible symmetirc space of non-compact type and H be a symmetric subgroup of G. Let θ (resp. σ) be an involution of G with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$ (resp. $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$). By replacing H by its suitable conjugate group if necessary, we may assume that $\theta \circ \sigma = \sigma \circ \theta$. Set $L := \mathrm{Fix}(\sigma \circ \theta)$. The orbit $H(eK)$ and $\exp(T^{\perp}_{eK}(H(eK)))$ are reflective submanifolds, where exp is the exponential map of G/K. It is shown that the submanifold $\exp(T^{\perp}_{eK}(H(eK)))$ is isometric to the symmetric space $L/H \cap K$ and that the cohomogeneity of the H-action is equal to the rank of $L/H \cap K$. By using this fact, the cohomogeneities of Hermann type actions on irreducible symmetric spaces of non-compact type are listed up as in Tables 1–5. In Tables 1–5, $\widetilde{A \cdot B}$ denotes $A \times B/\Pi$, where Π is a discrete center of $A \times B$. The symbol $\widetilde{SO_0(1, 8)}$ in the line of Type FII-II$'$ of Table 5 denotes the universal covering of $SO_0(1, 8)$ and the symbol α in the line of Type G'' of Table 5 denotes an outer automorphism of G_2^2.

Table 1.

type	H	G/K	$\cong L/H\cap K$	cohom
AI-I	$SO(n)$	$SL(n,\mathbf{R})/SO(n)$	$SL(n,\mathbf{R})/SO(n)$	$n-1$
AI-I$'$	$SO_0(p,n-p)$	$SL(n,\mathbf{R})/SO(n)$	$SL(p,\mathbf{R})/SO(p)$ $\times SL(n-p,\mathbf{R})/SO(n-p)\times\mathbf{R}$	$n-1$
AI-II	$Sp(n,\mathbf{R})$	$SL(2n,\mathbf{R})/SO(2n)$	$SL(n,\mathbf{C})/SU(n)$	$n-1$
AI-III	$(SL(p,\mathbf{R})\times SL(n-p,\mathbf{R}))$ $\cdot\mathbf{R}_*$	$SL(n,\mathbf{R})/SO(n)$	$SO_0(p,n-p)/SO(p)\times SO(n-p)$	$\min\{p,n-p\}$
AI-III$'$	$SL(n,\mathbf{C})\cdot U(1)$	$SL(2n,\mathbf{R})/SO(2n)$	$Sp(n,\mathbf{R})/U(n)$	n
AII-I	$SO^*(2n)$	$SU^*(2n)/Sp(n)$	$SL(n,\mathbf{C})/SU(n)$	$n-1$
AII-II	$Sp(n)$	$SU^*(2n)/Sp(n)$	$SU^*(2n)/Sp(n)$	$n-1$
AII-II$'$	$Sp(p,n-p)$	$SU^*(2n)/Sp(n)$	$SU^*(2p)/Sp(p)$ $\times SU^*(2n-2p)/Sp(n-p)\times U(1)$	$n-1$
AII-III	$SU^*(2p)\times SU^*(2n-2p)$ $\times U(1)$	$SU^*(2n)/Sp(n)$	$Sp(p,n-p)/Sp(p)\times Sp(n-p)$	$\min\{p,n-p\}$
AII-III$'$	$SL(n,\mathbf{C})\cdot U(1)$	$SU^*(2n)/Sp(n)$	$SO^*(2n)/U(n)$	$[n/2]$
AIII-I	$SO_0(p,q)$	$SU(p,q)/S(U(p)\times U(q))$	$SO_0(p,q)/SO(p)\times SO(q)$	$\min\{p,q\}$
AIII-I$'$	$SO^*(2p)$	$SU(p,p)/S(U(p)\times U(p))$	$Sp(p,\mathbf{R})/U(p)$	p
AIII-II	$Sp(p,q)$	$SU(2p,2q)/S(U(2p)\times U(2q))$	$Sp(p,q)/Sp(p)\times Sp(q)$	$\min\{p,q\}$
AIII-II$'$	$Sp(p,\mathbf{R})$	$SU(p,p)/S(U(p)\times U(p))$	$SO^*(2p)/U(p)$	$[p/2]$
AIII-III	$S(U(i,j)\times U(p-i,q-j))$	$SU(p,q)/S(U(p)\times U(q))$	$SU(p-i,j)/S(U(p-i)\times U(j))$ $\times SU(i,q-j)/S(U(i)\times U(q-j))$	$\min\{p-i,j\}$ $+\min\{i,q-j\}$
AIII-III$'$	$SL(p,\mathbf{C})\cdot U(1)$	$SU(p,p)/S(U(p)\times U(p))$	$SL(p,\mathbf{C})/SU(p)$	p
IV-A1	$SO(n,\mathbf{C})$	$SL(n,\mathbf{C})/SU(n)$	$SL(n,\mathbf{R})/SO(n)$	$n-1$
IV-A2	$SL(n,\mathbf{R})$	$SL(n,\mathbf{C})/SU(n)$	$SO(n,\mathbf{C})/SO(n)$	$[n/2]$
IV-A3	$SL(i,\mathbf{C})\times SL(n-i,\mathbf{C})$ $\times U(1)$	$SL(n,\mathbf{C})/SU(n)$	$SU(i,n-i)/S(U(i)\times U(n-i))$	$\min\{i,n-i\}$
IV-A4	$SU(i,n-i)$	$SL(n,\mathbf{C})/SU(n)$	$SL(i,\mathbf{C})/SU(i)$ $\times SL(n-i,\mathbf{C})/SU(n-i)$	$n-2$
IV-A5	$Sp(n,\mathbf{C})$	$SL(2n,\mathbf{C})/SU(2n)$	$SU^*(2n)/Sp(n)$	$n-1$
IV-A6	$SU^*(2n)$	$SL(2n,\mathbf{C})/SU(2n)$	$Sp(n,\mathbf{C})/Sp(n)$	n

Table 2.

type	H	G/K	$\cong L/H\cap K$	cohom
BDI-I	$SO_0(i,j)$ $\times SO_0(p-i,q-j)$	$SO_0(p,q)/SO(p)\times SO(q)$	$(SO_0(p-i,j)/SO(p-i)\times SO(j))$ $\times(SO_0(i,q-j)/SO(i)\times SO(q-j))$	$\min\{p-i,j\}$ $+\min\{i,q-j\}$
DI-I′	$SO(p,\mathbf{C})$	$SO_0(p,p)/SO(p)\times SO(p)$	$SL(p,\mathbf{R})\cdot U(1)/SO(p)$	p
DI-III	$SU(p,q)\cdot U(1)$	$SO_0(2p,2q)/SO(2p)\times SO(2q)$	$SU(p,q)/SU(p)\times SU(q)$	$\min\{p,q\}$
DI-III′	$SL(p,\mathbf{R})\cdot U(1)$	$SO_0(p,p)/SO(p)\times SO(p)$	$SO(p,\mathbf{C})/SO(p)$	$[p/2]$
DIII-I	$SO^*(2i)\times SO^*(2n-2i)$	$SO^*(2n)/U(n)$	$SU(i,n-i)/SU(i)\times SU(n-i)$	$\min\{i,n-i\}$
DIII-I′	$SO(n,\mathbf{C})$	$SO^*(2n)/U(n)$	$SO(n,\mathbf{C})/SO(n)$	$[n/2]$
DIII-III	$U(n)$	$SO^*(2n)/U(n)$	$SO^*(2n)/U(n)$	$[n/2]$
DIII-III′	$SU(i,n-i)\cdot U(1)$	$SO^*(2n)/U(n)$	$SO^*(2i)/U(i)$ $\times SO^*(2n-2i)/U(n-i)$	$[i/2]+[(n-i)/2]$
DIII-III″	$SU^*(2n)\cdot U(1)$	$SO^*(4n)/U(2n)$	$SU^*(2n)/Sp(n)$	$n-1$
IV-BD1	$SO(i,\mathbf{C})\times SO(n-i,\mathbf{C})$	$SO(n,\mathbf{C})/SO(n)$	$SO_0(i,n-i)/SO(i)\times SO(n-i)$	$\min\{i,n-i\}$
IV-BD2	$SO_0(i,n-i)$	$SO(n,\mathbf{C})/SO(n)$	$SO(i,\mathbf{C})/SO(i)$ $\times SO(n-i,\mathbf{C})/SO(n-i)$	$[i/2]+[(n-i)/2]$
IV-BD3	$SL(n,\mathbf{C})\cdot SO(2,\mathbf{C})$	$SO(2n,\mathbf{C})/SO(2n)$	$SO^*(2n)/U(n)$	$[n/2]$
IV-BD4	$SO^*(2n)$	$SO(2n,\mathbf{C})/SO(2n)$	$SL(n,\mathbf{C})/SU(n)\times SO(2,\mathbf{C})/SO(2)$	n
CI-I	$U(n)$	$Sp(n,\mathbf{R})/U(n)$	$Sp(n,\mathbf{R})/U(n)$	n
CI-I′	$SU(i,n-i)\cdot U(1)$	$Sp(n,\mathbf{R})/U(n)$	$Sp(i,\mathbf{R})/U(i)$ $\times Sp(n-i,\mathbf{R})/U(n-i)$	n
CI-I″	$SL(n,\mathbf{R})\cdot U(1)$	$Sp(n,\mathbf{R})/U(n)$	$SL(n,\mathbf{R})/SO(n)$	$n-1$
CI-II	$Sp(i,\mathbf{R})\times Sp(n-i,\mathbf{R})$	$Sp(n,\mathbf{R})/U(n)$	$SU(i,n-i)/S(U(i)\times U(n-i))$	$\min\{i,n-i\}$
CI-II′	$Sp(n,\mathbf{C})$	$Sp(2n,\mathbf{R})/U(2n)$	$Sp(n,\mathbf{C})/Sp(n)$	n
CII-I	$SU(p,q)\cdot U(1)$	$Sp(p,q)/Sp(p)\times Sp(q)$	$SU(p,q)/S(U(p)\times U(q))$	$\min\{p,q\}$
CII-I′	$SU^*(2p)\cdot U(1)$	$Sp(p,p)/Sp(p)\times Sp(p)$	$Sp(p,\mathbf{C})/Sp(p)$	p
CII-II	$Sp(p)\times Sp(q)$	$Sp(p,q)/Sp(p)\times Sp(q)$	$Sp(p,q)/Sp(p)\times Sp(q)$	$\min\{p,q\}$
CII-II′	$Sp(i,j)$ $\times Sp(p-i,q-j)$	$Sp(p,q)/Sp(p)\times Sp(q)$	$(Sp(p-i,j)/Sp(p-i)\times Sp(j))$ $\times(Sp(i,q-j)/Sp(i)\times Sp(q-j))$	$\min\{p-i,j\}$ $+\min\{i,q-j\}$
CII-II″	$Sp(p,\mathbf{C})$	$Sp(p,p)/Sp(p)\times Sp(p)$	$SU^*(2p)/Sp(p)$	$p-1$
IV-C1	$SL(n,\mathbf{C})\cdot SO(2,\mathbf{C})$	$Sp(n,\mathbf{C})/Sp(n)$	$Sp(n,\mathbf{R})/U(n)$	n
IV-C2	$Sp(n,\mathbf{R})$	$Sp(n,\mathbf{C})/Sp(n)$	$SL(n,\mathbf{C})/SU(n)\times SO(2,\mathbf{C})/SO(2)$	n
IV-C3	$Sp(i,\mathbf{C})\times Sp(n-i,\mathbf{C})$	$Sp(n,\mathbf{C})/Sp(n)$	$Sp(i,n-i)/Sp(i)\times Sp(n-i)$	$\min\{i,n-i\}$
IV-C4	$Sp(i,n-i)$	$Sp(n,\mathbf{C})/Sp(n)$	$Sp(i,\mathbf{C})/Sp(i)$ $\times Sp(n-i,\mathbf{C})/Sp(n-i)$	n

Table 3.

type	H	G/K	$\cong L/H\cap K$	cohom
EI-I	$Sp(4)/\{\pm 1\}$	$E_6^6/(Sp(4)/\{\pm 1\})$	$E_6^6/(Sp(4)/\{\pm 1\})$	6
EI-I′	$Sp(4,\mathbf{R})$	$E_6^6/(Sp(4)/\{\pm 1\})$	$SL(6,\mathbf{R})/SO(6)\times SL(2,\mathbf{R})/SO(2)$	6
EI-I″	$Sp(2,2)$	$E_6^6/(Sp(4)/\{\pm 1\})$	$SO_0(5,5)\cdot\mathbf{R}/SO(5)\times SO(5)$	6
EI-II	$SU^*(6)\cdot SU(2)$	$E_6^6/(Sp(4)/\{\pm 1\})$	$F_4^4/Sp(3)\cdot Sp(1)$	4
EI-II′	$SL(6,\mathbf{R})\times SL(2,\mathbf{R})$	$E_6^6/(Sp(4)/\{\pm 1\})$	$Sp(4,\mathbf{R})/(U(4)/\{\pm 1\})$	4
EI-III	$SO_0(5,5)\cdot\mathbf{R}$	$E_6^6/(Sp(4)/\{\pm 1\})$	$Sp(2,2)/((Sp(2)\times Sp(2))/\{\pm 1\})$	2
EI-IV	F_4^4	$E_6^6/(Sp(4)/\{\pm 1\})$	$SU^*(6)\cdot SU(2)/Sp(3)\cdot Sp(1)$	2
EII-I	$Sp(1,3)$	$E_6^2/SU(6)\cdot SU(2)$	$F_4^4/Sp(3)\cdot Sp(1)$	4
EII-I′	$Sp(4,\mathbf{R})$	$E_6^2/SU(6)\cdot SU(2)$	$Sp(4,\mathbf{R})/U(4)$	4
EII-II	$SU(6)\cdot SU(2)$	$E_6^2/SU(6)\cdot SU(2)$	$E_6^2/SU(6)\cdot SU(2)$	4
EII-II′	$SU(2,4)\cdot SU(2)$	$E_6^2/SU(6)\cdot SU(2)$	$SO_0(4,6)/SO(4)\times SO(6)$	4
EII-II″	$SU(3,3)\cdot SL(2,\mathbf{R})$	$E_6^2/SU(6)\cdot SU(2)$	$SU(3,3)/S(U(3)\times U(3))\times SL(2,\mathbf{R})/SO(2)$	4
EII-III	$SO^*(10)\cdot U(1)$	$E_6^2/SU(6)\cdot SU(2)$	$SO^*(10)/U(5)$	2
EII-III′	$SO_0(4,6)\cdot U(1)$	$E_6^2/SU(6)\cdot SU(2)$	$SU(2,4)/S(U(2)\times U(4))$	2
EII-IV	F_4^4	$E_6^2/SU(6)\cdot SU(2)$	$Sp(1,3)/Sp(1)\times Sp(3)$	1
EIII-I	$Sp(2,2)$	$E_6^{-14}/Spin(10)\cdot U(1)$	$Sp(2,2)/Sp(2)\times Sp(2)$	2
EIII-II	$SU(2,4)\cdot SU(2)$	$E_6^{-14}/Spin(10)\cdot U(1)$	$SU(2,4)/S(U(2)\times U(4))$	2
EIII-II′	$SU(1,5)\cdot SL(2,\mathbf{R})$	$E_6^{-14}/Spin(10)\cdot U(1)$	$SO^*(10)/U(5)$	2
EIII-III	$Spin(10)\cdot U(1)$	$E_6^{-14}/Spin(10)\cdot U(1)$	$E_6^{-14}/Spin(10)\cdot U(1)$	2
EIII-III′	$SO^*(10)\cdot U(1)$	$E_6^{-14}/Spin(10)\cdot U(1)$	$SU(1,5)\cdot SL(2,\mathbf{R})/S(U(1)\times U(5))\cdot SO(2)$	2
EIII-III″	$SO_0(2,8)\cdot U(1)$	$E_6^{-14}/Spin(10)\cdot U(1)$	$SO_0(2,8)/SO(2)\times SO(8)$	2
EIII-IV	F_4^{-20}	$E_6^{-14}/Spin(10)\cdot U(1)$	$F_4^{-20}/Spin(9)$	1
EIV-I	$Sp(1,3)$	E_6^{-26}/F_4	$SU^*(6)\cdot SU(2)/Sp(3)\cdot Sp(1)$	2
EIV-II	$SU^*(6)\cdot SU(2)$	E_6^{-26}/F_4	$Sp(1,3)/Sp(1)\cdot Sp(3)$	1
EIV-III	$SO_0(1,9)\cdot U(1)$	E_6^{-26}/F_4	$F_4^{-20}/Spin(9)$	1
EIV-IV	F_4	E_6^{-26}/F_4	E_6^{-26}/F_4	2
EIV-IV′	F_4^{-20}	E_6^{-26}/F_4	$SO_0(1,9)\cdot U(1)/SO(1)\times SO(9)$	2
IV-$E_6$1	E_6	$E_6^{\mathbf{c}}/E_6$	$E_6^{\mathbf{c}}/E_6$	6
IV-$E_6$2	E_6^6	$E_6^{\mathbf{c}}/E_6$	$Sp(4,\mathbf{C})/Sp(4)$	4
IV-$E_6$3	E_6^2	$E_6^{\mathbf{c}}/E_6$	$SL(6,\mathbf{C})\cdot SL(2,\mathbf{C})/SU(6)\cdot SU(2)$	6
IV-$E_6$4	E_6^{-14}	$E_6^{\mathbf{c}}/E_6$	$SO(10,\mathbf{C})\cdot Sp(1)/Spin(10)\cdot U(1)$	6
IV-$E_6$5	$Sp(4,\mathbf{C})$	$E_6^{\mathbf{c}}/E_6$	$E_6^6/Sp(4)$	6
IV-$E_6$6	$SL(6,\mathbf{C})\cdot SL(2,\mathbf{C})$	$E_6^{\mathbf{c}}/E_6$	$E_6^2/SU(6)\cdot SU(2)$	4
IV-$E_6$7	$SO(10,\mathbf{C})\cdot Sp(1)$	$E_6^{\mathbf{c}}/E_6$	$E_6^{-14}/Spin(10)\cdot U(1)$	2
IV-$E_6$8	$F_4^{\mathbf{C}}$	$E_6^{\mathbf{c}}/E_6$	E_6^{-26}/F_4	2
IV-$E_6$9	E_6^{-26}	$E_6^{\mathbf{c}}/E_6$	$F_4^{\mathbf{c}}/F_4$	4

Table 4.

type	H	G/K	$\cong L/H\cap K$	cohom
EV-V	$SU(8)/\{\pm1\}$	$E_7^7/(SU(8)/\{\pm1\})$	$E_7^7/(SU(8)/\{\pm1\})$	7
EV-V′	$SL(8,\mathbf{R})$	$E_7^7/(SU(8)/\{\pm1\})$	$SL(8,\mathbf{R})/SO(8)$	7
EV-V″	$SU^*(8)$	$E_7^7/(SU(8)/\{\pm1\})$	$E_6^6\cdot U(1)/Sp(4)$	7
EV-V‴	$SU(4,4)$	$E_7^7/(SU(8)/\{\pm1\})$	$SO_0(6,6)\cdot SL(2,\mathbf{R})/(SO(6)\times SO(6))\cdot SO(2)$	7
EV-VI	$SO^*(12)\cdot SU(2)$	$E_7^7/(SU(8)/\{\pm1\})$	$E_6^2\cdot U(1)/SU(6)\cdot SU(2)\cdot U(1)$	4
EV-VI′	$SO_0(6,6)\cdot SL(2,\mathbf{R})$	$E_7^7/(SU(8)/\{\pm1\})$	$SU(4,4)/S(U(4)\times U(4))$	4
EV-VII	$E_6^6\cdot U(1)$	$E_7^7/(SU(8)/\{\pm1\})$	$SU^*(8)/Sp(4)$	3
EV-VII′	$E_6^2\cdot U(1)$	$E_7^7/(SU(8)/\{\pm1\})$	$SO^*(12)/U(6)$	3
EVI-V	$SU(4,4)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	$SU(4,4)/S(U(4)\times U(4))$	4
EVI-V′	$SU(2,6)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	$E_6^2/SU(6)\cdot SU(2)$	4
EVI-VI	$SO'(12)\cdot SU(2)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	4
EVI-VI′	$SO^*(12)\cdot SL(2,\mathbf{R})$	$E_7^{-5}/SO'(12)\cdot SU(2)$	$SO^*(12)\cdot SL(2,\mathbf{R})/U(6)\cdot SO(2)$	4
EVI-VI″	$SO_0(4,8)\cdot SU(2)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	$SO_0(4,8)/SO(4)\times SO(8)$	4
EVI-VII	$E_6^2\cdot U(1)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	$SU(2,6)/S(U(2)\times U(6))$	2
EVI-VII′	$E_6^{-14}\cdot U(1)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	$E_6^{-14}/Spin(10)\cdot U(1)$	2
EVII-V	$SU^*(8)$	$E_7^{-25}/E_6\cdot U(1)$	$SU^*(8)/Sp(4)$	3
EVII-V′	$SU(2,6)$	$E_7^{-25}/E_6\cdot U(1)$	$SO^*(12)/SU(6)$	3
EVII-VI	$SO^*(12)\cdot SU(2)$	$E_7^{-25}/E_6\cdot U(1)$	$SU(2,6)/S(U(2)\times U(6))$	2
EVII-VI′	$SO_0(2,10)\cdot SL(2,\mathbf{R})$	$E_7^{-25}/E_6\cdot U(1)$	$E_6^{-14}/Spin(10)\cdot U(1)$	2
EVII-VII	$E_6\cdot U(1)$	$E_7^{-25}/E_6\cdot U(1)$	$E_7^{-25}/E_6\cdot U(1)$	3
EVII-VII′	$E_6^{-14}\cdot U(1)$	$E_7^{-25}/E_6\cdot U(1)$	$SO_0(2,10)\cdot SL(2,\mathbf{R})/(SO(2)\times SO(10))\cdot SO(2)$	3
EVII-VII″	$E_6^{-26}\cdot U(1)$	$E_7^{-25}/E_6\cdot U(1)$	$E_6^{-26}\cdot U(1)/F_4$	2
IV-$E_7$1	E_7	$E_7^{\mathbf{c}}/E_7$	$E_7^{\mathbf{c}}/E_7$	7
IV-$E_7$2	E_7^7	$E_7^{\mathbf{c}}/E_7$	$SL(8,\mathbf{C})/SU(8)$	7
IV-$E_7$3	E_7^{-5}	$E_7^{\mathbf{c}}/E_7$	$SO(12,\mathbf{C})\cdot SL(2,\mathbf{C})/SO'(12)\cdot SU(2)$	7
IV-$E_7$4	E_7^{-25}	$E_7^{\mathbf{c}}/E_7$	$E_6^{\mathbf{c}}\cdot\mathbf{C}^*/E_6\cdot U(1)$	7
IV-$E_7$5	$SL(8,\mathbf{C})$	$E_7^{\mathbf{c}}/E_7$	$E_7^7/SU(8)$	7
IV-$E_7$6	$SO(12,\mathbf{C})\cdot SL(2,\mathbf{C})$	$E_7^{\mathbf{c}}/E_7$	$E_7^{-5}/SO'(12)\cdot SU(2)$	4
IV-$E_7$7	$E_6^{\mathbf{c}}\cdot\mathbf{C}^*$	$E_7^{\mathbf{c}}/E_7$	$E_7^{-25}/E_6\cdot U(1)$	3

Table 5.

type	H	G/K	$\cong L/H\cap K$	cohom
EVIII-VIII	$SO'(16)$	$E_8^8/SO'(16)$	$E_8^8/SO'(16)$	8
EVIII-VIII′	$SO^*(16)$	$E_8^8/SO'(16)$	$E_7^7\cdot SL(2,\mathbf{R})/SU(8)\cdot SO(2)$	4
EVIII-VIII″	$SO_0(8,8)$	$E_8^8/SO'(16)$	$SO_0(8,8)/SO(8)\times SO(8)$	8
EVIII-IX	$E_7^{-5}\cdot Sp(1)$	$E_8^8/SO'(16)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	4
EVIII-IX′	$E_7^7\cdot SL(2,\mathbf{R})$	$E_8^8/SO'(16)$	$SO^*(16)/U(8)$	4
EIX-VIII	$SO^*(16)$	$E_8^{-24}/E_7\cdot Sp(1)$	$SO^*(16)/U(8)$	4
EIX-VIII′	$SO_0(4,12)$	$E_8^{-24}/E_7\cdot Sp(1)$	$E_7^{-5}/SO'(12)\cdot SU(2)$	4
EIX-IX	$E_7\cdot Sp(1)$	$E_8^{-24}/E_7\cdot Sp(1)$	$E_8^{-24}/E_7\cdot Sp(1)$	4
EIX-IX′	$E_7^{-5}\cdot Sp(1)$	$E_8^{-24}/E_7\cdot Sp(1)$	$SO_0(4,12)/SO(4)\times SO(12)$	4
EIX-IX″	$E_7^{-25}\cdot SL(2,\mathbf{R})$	$E_8^{-24}/E_7\cdot Sp(1)$	$E_7^{-25}\cdot SL(2,\mathbf{R})/E_6\cdot U(1)\cdot SO(2)$	4
IV$E_8$1	E_8	$E_8^{\mathbf{c}}/E_8$	$E_8^{\mathbf{c}}/E_8$	8
IV$E_8$2	E_8^8	$E_8^{\mathbf{c}}/E_8$	$SO(16,\mathbf{C})/SO(16)$	8
IV$E_8$3	E_8^{-24}	$E_8^{\mathbf{c}}/E_8$	$E_7^{\mathbf{c}}\times SL(2,\mathbf{C})/E_7\cdot SU(2)$	8
IV$E_8$4	$SO(16,\mathbf{C})$	$E_8^{\mathbf{c}}/E_8$	$E_8^8/SO(16)$	8
IV$E_8$5	$E_7^{\mathbf{c}}\times SL(2,\mathbf{C})$	$E_8^{\mathbf{c}}/E_8$	$E_8^{-24}/E_7\cdot SU(2)$	4
FI-I	$Sp(3)\cdot Sp(1)$	$F_4^4/Sp(3)\cdot Sp(1)$	$F_4^4/Sp(3)\cdot Sp(1)$	4
FI-I′	$Sp(1,2)\cdot Sp(1)$	$F_4^4/Sp(3)\cdot Sp(1)$	$SO_0(4,5)/SO(4)\times SO(5)$	4
FI-I″	$Sp(3,\mathbf{R})\cdot SL(2,\mathbf{R})$	$F_4^4/Sp(3)\cdot Sp(1)$	$Sp(3,\mathbf{R})/U(3)\times SL(2,\mathbf{R})/SO(2)$	4
FI-II	$SO_0(4,5)$	$F_4^4/Sp(3)\cdot Sp(1)$	$Sp(1,2)/Sp(1)\times Sp(2)$	1
FII-I	$Sp(1,2)\cdot Sp(1)$	$F_4^{-20}/Spin(9)$	$SO_0(1,8)/SO(1)\times SO(8)$	1
FII-II	$Spin(9)$	$F_4^{-20}/Spin(9)$	$F_4^{-20}/Spin(9)$	1
FII-II′	$\widetilde{SO_0(1,8)}$	$F_4^{-20}/Spin(9)$	$Sp(1,2)/Sp(1)\times Sp(2)$	1
IV-F1	F_4^4	$F_4^{\mathbf{C}}/F_4$	$Sp(3,\mathbf{C})/Sp(3)\times SL(2,\mathbf{C})/SU(2)$	4
IV-F2	F_4^{-20}	$F_4^{\mathbf{C}}/F_4$	$SO(9,\mathbf{C})/SO(9)$	4
IV-F3	$SO(9,\mathbf{C})$	$F_4^{\mathbf{C}}/F_4$	$F_4^{-20}/Spin(9)$	1
IV-F4	$Sp(3,\mathbf{C})\cdot SL(2,\mathbf{C})$	$F_4^{\mathbf{C}}/F_4$	$F_4^4/Sp(3)\cdot Sp(1)$	4
G	$SO(4)$	$G_2^2/SO(4)$	$G_2^2/SO(4)$	2
G′	$SL(2,\mathbf{R})\times SL(2,\mathbf{R})$	$G_2^2/SO(4)$	$SO(4)/SO(2)\times SO(2)$	2
G″	$\alpha(SO(4))$	$G_2^2/SO(4)$	$SL(2,\mathbf{R})/SO(2)\times SL(2,\mathbf{R})/SO(2)$	2
IV-G1	G_2^2	$G_2^{\mathbf{c}}/G_2$	$SL(2,\mathbf{C})/SU(2)\times SL(2,\mathbf{C})/SU(2)$	2
IV-G2	$SL(2,\mathbf{C})\times SL(2,\mathbf{C})$	$G_2^{\mathbf{c}}/G_2$	$G_2^2/SO(4)$	2

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