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ON CHARACTER CORRESPONDENCES IN π-SEPARABLE GROUPS

Dedicated to Professor Tuyosi Oyama on his 60th birthday

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1. Introduction

Let A and G be finite groups and suppose A acts on G by automorphisms. We denote by $\mathrm{Irr}(G)$ the set of ordinary (complex) irreducible characters of G. For a prime p, $\mathrm{IBr}_p(G)$ denotes the set of all irreducible Brauer characters of G with respect to p. If φ is a class function of G and $a\in A$, φ^a, defined by $\varphi^a(g^a)=\varphi(g)$ for $g\in G$, is again a class function. For a set S of class functions of G which is stable under the action of A, we write S_A to denote the set of all A-invariant elements of S. Let π be a set of prime numbers and let π' be the set of primes complementary to π. For $\chi\in\mathrm{Irr}(G)$, we denote by $\hat{\chi}$ the restriction of χ to the set $\hat{G}$ of all π-elements of G. If $\hat{\chi}$ can not be written in the form $\hat{\chi}=\hat{\zeta}+\hat{\psi}$ with ordinary characters ζ, ψ of G, then we say that χ is π-irreducible and that $\hat{\chi}$ is a π-irreducible character of G. We denote the set of all π-irreducible characters of G by $\mathrm{I}_\pi(G)$. We say that G is π-separable if every composition factor of G is either a π-group or a π'-group.

For a π-separable group G, Isaacs [8] considered the vector space c.f. $(\hat{G})$ of all complex-valued class functions defined on $\hat{G}$ and showed that $\mathrm{I}_\pi(G)$ is a basis of c.f. $(\hat{G})$ which has the following properties.

(1) If $\chi\in\mathrm{Irr}(G)$, then $\hat{\chi}$ is a nonnegative integer linear combination of elements of $\mathrm{I}_\pi(G)$.

(2) If $\varphi\in\mathrm{I}_\pi(G)$, then $\varphi=\hat{\chi}$ for some $\chi\in\mathrm{Irr}(G)$. These imply that $\mathrm{I}_\pi(G)$ behaves as a π-generalization of Brauer characters.

Now assume that A acts on G by automorphisms and $(|A|, |G|)=1$. Under the assumption that A is solvable, Glauberman [2] established a natural bijection from $\mathrm{Irr}(G)_A$ onto $\mathrm{Irr}(C_G(A))$. If A is non-solvable, then $|A|$ is even by the Odd-order Theorem and hence $|G|$ is odd. In that case, Isaacs [4] showed that there also exists a similar bijection from $\mathrm{Irr}(G)_A$ onto $\mathrm{Irr}(C_G(A))$.

On the other hand, Uno [10] studied a character correspondence between Brauer characters. He proved that if G is p-solvable, then there exists a bijection from $\mathrm{IBr}_p(G)_A$ onto $\mathrm{IBr}_p(C_G(A))$ and this has similar properties as those of

Glauberman and Isaacs.

The purpose of this paper is to generalize this result to π-separable groups by applying Isaacs's π-generalization of Brauer characters. Namely, we have the following theorem.

Theorem. *Let A act on G such that $(|G|, |A|)=1$. Suppose G is π-separable. Then there exists a natural bijection*

$$\tilde{\Pi}(G, A): \mathrm{I}_{\pi}(G)_A \to I_{\pi}(C_G(A))$$

and the following hold.

(1) *If $B \trianglelefteq A$, then $\tilde{\Pi}(G, A)=\tilde{\Pi}(G, B)\tilde{\Pi}(C_G(B), A/B)$.*

(2) *If A is a q-group for a prime q and $\psi \in I_{\pi}(G)_A$, then $(\psi)\tilde{\Pi}(G, A)$ is a unique π-irreducible constituent of $\psi_{C_G(A)}$ with multiplicity prime to q.*

A p-solvable group is just a $\{p\}'$-separable group, and Uno's methods work mostly in our case, with slight modifications. However we shall reproduce them for the completeness. Also, we shall start with a review of the character correspondences of Glauberman and Isaacs, and collect, in section 3, some facts on characters of π-separable groups ([8], [9]) that will be needed to prove the above theorem. In section 4, we shall consider a correspondence of π-irreducible characters of π-separable groups and prove the theorem.

Concerning our terminologies and notations, we refer to Gorenstein [3] and Isaacs [6].

2. Character Correspondences of Glauberman and Isaacs

Here we summalize some properties of the character correspondences of Glauberman and Isaacs.

HYPOTHESIS 2.1. A acts on G such that $(|A|, |G|)=1$. Put $C=C_G(A)$ and let $\Gamma=GA$ be the semidirect product of G by A.

Theorem 2.2. *Assume Hypothesis* 2.1. *Then there is a natural bijection*

$$\Pi(G, A): \mathrm{Irr}(G)_A \to \mathrm{Irr}(C)$$

and the following hold.

(1) *If $B \trianglelefteq A$ and $D=C_G(B)$, then $\Pi(G, B)$ maps $\mathrm{Irr}(G)_A$ onto $\mathrm{Irr}(D)_A$.*

(2) *In the situation of* (1), *$\Pi(G, A)=\Pi(G, B)\Pi(D, A/B)$.*

(3) *If A is a p-group and $\chi \in \mathrm{Irr}(G)_A$, then $(\chi)\Pi(G, A)$ is a unique irreducible constituent of χ_C such that*
$p \nmid [\chi_C, (\chi)\Pi(G, A)]$.

(4) *If α is an automorphism of Γ which leaves G and A invarint, then C is α-invariant and $(\chi^{\alpha})\Pi(G, A)=\{(\chi)\Pi(G, A)\}^{\alpha}$ for every $\chi \in \mathrm{Irr}(G)_A$.*

(5) *Let* $\chi\in\mathrm{Irr}(G)_A$ *and* $\xi=(\chi)\Pi(G, A)$. *Then* $\boldsymbol{Q}(\chi)=\boldsymbol{Q}(\xi)$, *where* $\boldsymbol{Q}(\chi)$ *is the field obtained by adjoining all the values of* χ *to the rational field* $\boldsymbol{Q}$.

Proof. See Corollary 5.2 of [11], Theorem 2.2 and Lemma 4.2 of [10].

Theorem 2.3. *Assume Hypothesis* 2.1 *and let* N *be an* A*-invariant normal subgroup of* G. *Let* $\chi\in\mathrm{Irr}(G)_A$, $\theta\in\mathrm{Irr}(N)_A$, $T=I_G(\theta)$, $\psi=(\chi)\Pi(G, A)$ *and* $\phi=(\theta)\Pi(N, A)$ *where* $I_G(\theta)$ *denotes the inertia group of* θ *in* G. *Then the following hold.*

(1) $[\chi_N, \theta]\neq 0$ *if and only if* $[\psi_{N\cap C}, \phi]\neq 0$.

(2) $T\cap C=I_C(\phi)$ *and* $(\varphi^G)\Pi(G, A)=\{(\varphi)\Pi(T, A)\}^C$ *for every* $\varphi\in\mathrm{Irr}(T\mid\theta)$.

Proof. See Lemma 2.5 of [12].

Assume Hypothesis 2.1. Then for $\chi\in\mathrm{Irr}(G)_A$, there exists a unique extension χ' of χ to Γ such that $A\leqq\ker(\det\chi')$. We call χ' the canonical extension of χ (cf. [6], Chapter 13).

Theorem 2.4. *Assume Hypothesis* 2.1 *and let* A *is cyclic. Let* $\chi\in\mathrm{Irr}(G)_A$ *and* χ' *be the canonical extension of* χ *to* Γ. *Then there exists* $\varepsilon=\pm 1$ *such that* $\chi'(ca)=\varepsilon(\chi)\Pi(G, A)(c)$ *for all* $c\in C$ *and all generators* a *of* A.

Proof. See Theorem 3.3 of [2].

3. Results of Isaacs on $B_\pi(G)$ and $I_\pi(G)$

Suppose that G is a p-solvable group. By the Fong-Swan Theorem (cf. [1], Theorem 72.1), each $\varphi\in\mathrm{IBr}_p(G)$ is the form $\varphi=\hat{\chi}$ for some $\chi\in\mathrm{Irr}(G)$, where $\hat{\chi}$ denotes the restriction of χ to the set of p'-elements of G. From this point of view, Isaacs [7] defined a characteristic subset ${}^q\mathcal{J}(G)\subseteqq\mathrm{Irr}(G)$ such that the restriction map $\chi\to\hat{\chi}$ defines a bijection from ${}^q\mathcal{J}(G)$ onto $\mathrm{IBr}_p(G)$. Later he generalized its construction to π-separable groups and constructed a subset $B_\pi(G)\subseteqq\mathrm{Irr}(G)$ such that the restriction to the set of π-elements induces a bijection from $B_\pi(G)$ onto $I_\pi(G)$. Moreover $B_\pi(G)$ is stable under the natural action of $\mathrm{Aut}(G)$ on $\mathrm{Irr}(G)$.

We do not mention here the definition of $B_\pi(G)$, but the main result of Isaacs [8] can be stated as follows:

Theorem 3.1. *Let* G *be* π*-separable. Then the following hold.*

(1) *The restriction map* $\psi\to\hat{\psi}$ *defines a bijection from* $B_\pi(G)$ *onto* $I_\pi(G)$. *In partiqular* $I_\pi(G)$ *is a basis of c.f.* $(\hat{G})$.

(2) *There exist nonnegative integers* $d_{\chi\varphi}$ *for* $\chi\in\mathrm{Irr}(G)$ *and* $\varphi\in B_\pi(G)$ *such that* $\hat{\chi}=\sum_{\varphi\in B_\pi(G)} d_{\chi\varphi}\hat{\varphi}$.

Proof. See Corollary 10.1 (a) and Corollary 10.2 of [8].

Let us write $\boldsymbol{Q}_\pi$ to denote the field obtained by adjoining all the complex n-th roots of unity to $\boldsymbol{Q}$, for all π-numbers n.

The following two results are concerned with the characterizations of characters in $B_\pi(G)$.

Theorem 3.2. *Let G be π-separable and let $\chi \in \mathrm{Irr}(G)$.*

(1) *If $\chi \in B_\pi(G)$, then $\chi(g) \in \boldsymbol{Q}_\pi$ for all $g \in G$.*

(2) *Assume $2 \in \pi$ or $2 \nmid |G|$. Let χ have values in $\boldsymbol{Q}_\pi$ and suppose that $\hat{\chi} \in I_\pi(G)$. Then $\chi \in B_\pi(G)$.*

Proof. See Corollary 12.1 and Theorem 12.3. of [8].

Theorem 3.3. *Let G be π-separable and let $N \trianglelefteq G$.*

(1) *If $\chi \in B_\pi(G)$, then every irreducible constituent of χ_N belongs to $B_\pi(N)$.*

(2) *If G/N is a π-group and $\psi \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G \mid \psi)$, then $\psi \in B_\pi(N)$ if and only if $\chi \in B_\pi(G)$.*

(3) *If G/N is a π'-group, then for $\psi \in B_\pi(N)$, there exists a unique irreducible constituent χ of ψ^G which belongs to $B_\pi(G)$. If ψ is G-invariant, then χ is an extension of ψ.*

Proof. See Theorem 6.2, Corollary 6.3 and Theorem 7.1 of [8].

In the rest of this section, we give facts about π-irreducible characters. The next result is an analog of Clifford's character correspondence.

Theorem 3.4. *Let G be π-separable and $N \trianglelefteq G$. Let $\theta \in I_\pi(N)$ and $T = I_G(\theta)$. Then induction defines a bijection from $I_\pi(T \mid \theta)$ onto $I_\pi(G \mid \theta)$.*

Proof. See Proposition 3.2 of [9].

Now we show the following character correspondence which can be considered as a π-generalization of Theorem 3.1 of [5], and the proof there also works in our case. However we give here a rather short proof.

Theorem 3.5. *Let G be π-separable and N be a normal subrgoup of G such that G/N is a π-group. Let $\psi \in \mathrm{Irr}(N)$ and assume*

(1) $\hat{\psi} \in I_\pi(N)$

(2) $\psi^g = \psi$ *for those $g \in G$ with $\hat{\psi}^g = \hat{\psi}$.*

Then $\wedge$ defines a one to one correspondence between $\mathrm{Irr}(G \mid \psi)$ and $I_\pi(G \mid \hat{\psi})$.

Proof. Let $\psi' \in B_\pi(N)$ be such that $\hat{\psi}' = \hat{\psi}$, $S = \mathrm{Irr}(G \mid \psi)$, $T = I_\pi(G \mid \hat{\psi})$ and $U = \{\varphi \in B_\pi(G);\ [\varphi_N, \psi'] \neq 0\}$. Note that the map $\wedge$ gives a one to one correspondence between T and U. Write $\psi^G = \sum_{\chi \in S} a_\chi \chi$, $\psi'^G = \sum_{\varphi \in U} b_\varphi \varphi$ with

positive integers a_χ, b_φ and $\hat{\chi}=\sum_{\hat{\varphi}\in T} d_{\chi\varphi}\hat{\varphi}$ with $d_{\chi\varphi}\geq 0$. Since $\hat{\psi}^G=\hat{\psi}'^G$, we have $b_\varphi=\sum_{\chi\in S} a_\chi d_{\chi\varphi}$. On the other hand, we get $I_G(\psi)=I_G(\hat{\psi})=I_G(\hat{\psi}')$ from the assumption. If we denote this common group by I, then we have

$$\sum_{\chi\in S} a_\chi^2 = [\psi^G, \psi^G] = |I: N| = [\psi'^G, \psi'^G] = \sum_{\varphi\in U} b_\varphi^2 .$$

So

$$\sum_{\varphi\in U}(\sum_{\chi\in S} a_\chi d_{\chi\varphi})^2 = \sum_{\chi\in S} a_\chi^2 .$$

Expanding the left-hand side of the above equation, we find easily

$$\sum_{\varphi\in U} d_{\chi\varphi}^2 = 1 \quad \text{and} \quad d_{\chi\varphi} d_{\eta\varphi} = 0 \qquad \text{for } \eta\neq\chi \text{ in } S .$$

Thus $d_{\chi\varphi}=1$ for some $\varphi\in U$, while $d_{\chi\varphi'}=0$ if $\varphi'\neq\varphi$. Therefore $\hat{\chi}=\hat{\varphi}\in T$. If, for $\eta\in S$ different from χ, $\hat{\eta}=\hat{\varphi}'$ with $\varphi'\in U$, then we get $\varphi\neq\varphi'$ from the above. Hence $\hat{\eta}\neq\hat{\chi}$ and thus $\wedge$ defines an injection from S into T. If $\hat{\varphi}\in T$ with $\varphi\in U$, then $b_\varphi\neq 0$ and hence $d_{\chi\varphi}\neq 0$ for some $\chi\in S$. This yields that $\hat{\chi}=\hat{\varphi}$. Therefore $\wedge$ is a bijection.

4. Correspondences of π-Irreducible Characters

Let Ω be a subset of $\mathrm{Irr}(G)$. For a subgroup H of G, we set $\Omega(H)=\{\psi\in\mathrm{Irr}(H);\ [\chi_H, \psi]\neq 0 \text{ for some } \chi\in\Omega\}$.

DEFINITION. Let G be π-separable and assume that A acts on G. Let $G=G_0\supseteq G_1\supseteq\cdots\supseteq G_n=\{1\}$ be a normal series of G and let Ω be a subset of $\mathrm{Irr}(G)$. If Ω satisfies the following two conditions:

(1) Ω is A-invariant

(2) the map $\wedge$ is a bijection from $\Omega(G_i)$ onto $I_\pi(G_i)$ for each i, $0\leq i\leq n$,

then we say, following Uno [10], that Ω has the π-lifting property with respect to A and $\{G_i\}_{i=0}^n$. If Ω has the π-lifting property with respect to A and every normal series of G, then we simply say that Ω has the π-lifting property with respect to A. If Ω has π-lifting property with respect to A, then Ω has π-lifting property with respect to B for any subgroup B of A. Furthermore, we note that $B_\pi(G)$ has the π-lifting property with respect to A.

The next lemma is a generalization of Lemma 3.4 of Uno [10].

Lemma 4.1. *Assume A acts on G. Let $G=G_0\supseteq G_1\supseteq\cdots\supseteq G_n=\{1\}$ be a normal series of G. If $\Omega\subseteq\mathrm{Irr}(G)$ has the π-lifting property with respect to A and $\{G_i\}_{i=0}^n$, then the following hold.*

(1) *If G_i/G_{i+1} is a π-group, then $\mathrm{Irr}(G_i|\theta)\subseteq\Omega(G_i)$ for every $\theta\in\Omega(G_{i+1})$.*

(2) *If G_i/G_{i+1} is a π'-group, then $\mathrm{Irr}(G_i|\theta)\cap\Omega(G_i)$ consists of a single element for every $\theta\in\Omega(G_{i+1})$. If θ is G_i-invariant, then the element of*

$\mathrm{Irr}(G_i|\theta)\cap\Omega(G_i)$ *is an extension of* θ.

Proof. (1) Let $\theta\in\Omega(G_{i+1})$ and $\chi\in\mathrm{Irr}(G_i|\theta)$. By the π-lifting property of Ω, it follows that $\hat{\theta}\in I_\pi(G_{i+1})$ and $I_{G_i}(\theta)=I_{G_i}(\hat{\theta})$. So by Theorem 3.5. $\hat{\chi}\in I_\pi(G_i)$. Again by the π-lifting property of Ω, there exists $\psi\in\Omega(G_i)$ such that $\hat{\psi}=\hat{\chi}$. Set $\psi_{G_{i+1}}=\sum_{j=1}^{t}\eta_j$ where each $\eta_j\in\Omega(G_{i+1})$. Then every $\hat{\eta}_j\in I_\pi(G_{i+1})$ and $\hat{\theta}$ is π-irreducible constituent of $\hat{\psi}_{G_{i+1}}$. By Theorem 3.1 (1), $\hat{\theta}=\hat{\eta}_j$ for some j. Since $\wedge$ is a bijection from $\Omega(G_{i+1})$ onto $I_\pi(G_{i+1})$, it follows that $\theta=\eta_j$ and $\psi\in\mathrm{Irr}(G_i|\theta)$. Thus by Theorem 3.5, we conclude that $\chi=\psi\in\Omega(G_i)$.

(2) Let $\theta\in\Omega(G_{i+1})$ and $\chi\in\Omega(G_i)\cap\mathrm{Irr}(G_i|\theta)$. By the π-lifting property of Ω, $\hat{\theta}\in I_\pi(G_{i+1})$ and $\hat{\chi}\in I_\pi(G_i|\hat{\theta})$. Let χ' (resp. θ') be an element of $B_\pi(G_i)$ (resp. $B_\pi(G_{i+1})$) such that $\hat{\chi}'=\hat{\chi}$ (resp. $\hat{\theta}'=\hat{\theta}$). By Theorem 3.3(1), we can write ${\chi'}_{G_{i+1}}=\sum_j\eta_j$ where $\eta_j\in B_\pi(G_{i+1})$. Thus By Theorem 3.1 (1), ${\hat{\chi}'}_{G_{i+1}}=\sum_j\hat{\eta}_j$ where $\hat{\eta}_j\in I_\pi(G_{i+1})$. Thus $\hat{\eta}_j=\hat{\theta}'$ for some j and so $\eta_j=\theta'$. Applying Theorem 3.3 (3), we see that $B_\pi(G_i)\cap\mathrm{Irr}(G_i|\theta')$ consists of the single element χ'. Therefore, we obtain from Theorem 3.1 (1) that $I_\pi(G_i|\hat{\theta})=\{\hat{\chi}\}$. If $\zeta\in\Omega(G_i)\cap\mathrm{Irr}(G_i|\theta)$, then $\zeta\in I_\pi(G_i|\hat{\theta})$ and so $\hat{\zeta}=\hat{\chi}$. Thus by the π-lifting property of Ω, $\zeta=\chi$. Therefore $\Omega(G_i)\cap\mathrm{Irr}(G_i|\theta)$ consists of a single element χ. If θ is G_i-invariant, then θ' is also G_i-invariant. Thus by Theorem 3.3(3), $\chi(1)=\theta(1)$. this implies that χ is an extension of θ.

The next lemma corresponds to Proposition 3.6 of Uno [10]. However the proof needs some elabolations in part.

Lemma 4.2. *Assume Hypothesis* 2.1 *and that* G *is* π*-separable. Let* $G=G_0\rhd G_1\rhd\cdots\rhd G_n=\{1\}$ *be an* A*-composition series of* G *and let* $\Omega\subseteqq\mathrm{Irr}(G)$ *have the* π*-lifting property with respect to* A *and* $\{G_i\}_{i=0}^n$. *Then the image of* Ω_A *by* $\Pi(G, A)$ *has the* π*-lifting property with respect to* $\{1_{Aut(G)}\}$ *and* $\{G_i\cap C\}_{i=0}^n$.

Proof. We proceed by induction on $|G|$.

Set $C_i=G_i\cap C$, $0\leq i\leq n$ and let $\Lambda=\{\Omega_A\}\Pi(G, A)$. First we claim that $\Lambda(C_i)=\{\Omega(G_i)_A\}\Pi(G_i, A)$ for each i, $0\leq i\leq n$. Let η be any element of $\Omega(G_i)_A$. If G_{i-1}/G_i is a π-group, then by Lemma 4.1 (1), $\mathrm{Irr}(G_{i-1}|\eta)_A\subseteqq\Omega(G_{i-1})_A$ and $\mathrm{Irr}(G_{i-1}|\eta)_A$ is nonempty (cf. [6], Chapter 13). If G_{i-1}/G_i is a π'-group, then by Lemma 4.1 (2), $\Omega(G_{i-1})\cap\mathrm{Irr}(G_{i-1}|\eta)$ has just one element. Since both $\Omega(G_{i-1})$ and $\mathrm{Irr}(G_{i-1}|\eta)$ are A-invariant, $(\Omega(G_{i-1})\cap\mathrm{Irr}(G_{i-1}|\eta))_A=\Omega(G_{i-1})\cap\mathrm{Irr}(G_{i-1}|\eta)$. Therefore $\Omega(G_{i-1})_A\cap\mathrm{Irr}(G_{i-1}|\eta)$ is nonempty in any case. Applying Theorem 2.3 (1) repeatedly, we can find an element $\chi\in\Omega_A\cap\mathrm{Irr}(G|\eta)$ such that $(\chi)\Pi(G, A)\in\mathrm{Irr}(C|(\eta)\Pi(G_i, A))$. This implies that $\{\Omega(G_i)_A\}\Pi(G, A)\subseteqq\Lambda(C_i)$. Conversely, let ξ be any element of $\Lambda(C_i)$ and let $\eta=(\xi)\Pi^{-1}(G_i, A)$. From the definitions of Λ and $\Lambda(C_i)$, there exists $\chi\in\Omega(G)_A$ such that ξ is a constituent of $(\chi)\Pi(G, A)_{C_i}$. By Theorem 2.3 (1), we see $\chi\in\mathrm{Irr}(G|\eta)$. Thus

$\eta \in \Omega(G_i)$. This implies that $\Lambda(C_i) \subseteqq \{\Omega(G_i)_A\}\Pi(G, A)$ and the claim is proved.

Now, if we set $\Omega_1 = \Omega(G_1)$, then Ω_1 is A-invariant and $\Omega_1(G_i) = \Omega(G_i)$ for all $i \geq 1$. So, by the inductive hypothesis applied to G_1, we may assume that the restriction map $\wedge$ from $\Lambda(C_i)$ to $I_\pi(C_i)$ is a bijection for each i, $1 \leq i \leq n$. So, it suffices to show that $\wedge$ gives a bijection from Λ onto $I_\pi(C)$.

If $C \subseteqq G_1$, then $\Lambda = \Lambda(C_1)$ and the assertion holds by the inductive hypothesis. Now we assume $C \nsubseteqq G_1$. Let $\theta_1, \cdots, \theta_k$ be representatives of C-orbits on $\Omega(G_1)_A$ and set $\phi_i = (\theta_i)\Pi(G_1, A)$, $1 \leq i \leq k$. By Theorem 2.2 (4),

$$\phi_i^c = \{(\theta_i)\Pi(G_1, A)\}^c = (\theta_i^c)\Pi(G_1, A) \qquad \text{for all} \quad c \in C .$$

Thus we find that $\phi_1, \cdots, \phi_k$ are representatives of C-orbits on $\Lambda(C_1)$. Furthermore, since $\Omega(G_1)$ is G-stable, it is easy to see that $I_G(\theta_i) = I_G(\hat{\theta}_i)$ for all i. Also, we have $I_C(\phi_i) = I_C(\hat{\phi}_i)$ for all i, because we have assumed that $\{(\Omega_1)_A\}\Pi(G_1, A)$ has the π-lifting property. In particular, it follows that $\hat{\phi}_1, \cdots, \hat{\phi}_k$ are representatives of C-orbits on $I_\pi(C_1)$.

We divide the proof into two cases.

Case 1. G/G_1 is a π-group.

If $\chi \in \Omega_A$, then $\chi \in \mathrm{Irr}(G \mid \theta_i)$ for some i. So, by Theorem 2.3(1), we have $(\chi)\Pi(G, A) \in \mathrm{Irr}(C \mid \phi_i)$ and thus $\Lambda \subseteqq \bigcup_{i=1}^{k} \mathrm{Irr}(C \mid \phi_i)$. Conversely if $\psi \in \mathrm{Irr}(C \mid \phi_i)$, then from Theorem 2.3 (1) and Lemma 4.1 (1), $(\psi)\Pi^{-1}(G, A) \in \mathrm{Irr}(G \mid \theta_i)_A \subseteqq \Omega_A$. Thus $\psi \in \Lambda$ and we conclude that

$$\Lambda = \bigcup_{i=1}^{k} \mathrm{Irr}(C \mid \phi_i) .$$

On the other hand, we have clearly $I_\pi(C) = \bigcup_{i=1}^{k} I_\pi(C \mid \hat{\phi}_i)$. So, by Theorem 3.5, $\wedge$ defines a bijection from Λ onto $I_\pi(C)$.

Case 2. G/G_1 is a π'-group.

If $\chi \in \Omega_A$, then $\chi \in \mathrm{Irr}(G \mid \theta_i) \cap \Omega_A$ for some i. Thus $\Omega_A = \bigcup_{i=1}^{k} \mathrm{Irr}(G \mid \theta_i) \cap \Omega_A$. By Lemma 4.1 (2), each $\mathrm{Irr}(G \mid \theta_i) \cap \Omega$ consists of a single element. Since both $\mathrm{Irr}(G \mid \theta_i)$ and Ω are A-invariant, it follows that $\mathrm{Irr}(G \mid \theta_i) \cap \Omega = \mathrm{Irr}(G \mid \theta_i) \cap \Omega_A$, $1 \leq i \leq k$. Set $\mathrm{Irr}(G \mid \theta_i) \cap \Omega_A = \{\chi_i\}$, $1 \leq i \leq k$. By Lemma 4.1 (2), $\chi_i \neq \chi_j$ for $i \neq j$. Thus $\Omega_A = \{\chi_1, \cdots, \chi_k\}$ and $\Lambda = \{(\chi_i)\Pi(G, A);\ 1 \leq i \leq k\}$. Furthermore, we obtain from Theorem 2.3 (1) that each $\hat{\phi}_i$ is a π-irreducible constituent of $\widehat{(\chi_i)\Pi(G, A)_{C_1}}$. Let $\hat{\psi}$ be any element of $I_\pi(C \mid \hat{\phi}_i)$. Let ϕ_i' be the element of $B_\pi(C_1)$ which corresponds to $\hat{\phi}_i$. Since $B_\pi(C)$ has the π-lifting property (with respect to the trivial action of A on C), we see from Lemma 4.1(2) that $B_\pi(C \mid \phi_i')$ consists of a single element. Set $B_\pi(C \mid \phi_i') = \{\psi_i\}$, $1 \leq i \leq k$, so we have $I_\pi(C \mid \hat{\phi}_i) = \{\hat{\psi}_i\}$, $1 \leq i \leq k$ and $I_\pi(C) = \{\hat{\psi}_1, \cdots, \hat{\psi}_k\}$. Set $T_i = I_G(\theta_i)$, $1 \leq i \leq k$. For each χ_i, $1 \leq i \leq k$, there exists a unique irreducible character $\xi_i \in \mathrm{Irr}(T_i \mid \theta_i)$ such that $\xi_i^G = \chi_i$. Then $\xi_i \in \mathrm{Irr}(T_i \mid \theta_i) \cap \Omega(T_i)$ and it follows

from Lemma 4.1 (2) that ξ_i is an extension of θ_i. Since χ_i and θ_i are A-invariant, ξ_i is also A-invariant and $(\xi_i)\Pi(T_i, A)^C=(\chi_i)\Pi(G, A)$ by Theorem 2.3 (2).

If each $(\xi_i)\Pi(T_i, A)$ is π-irreducible, then $(\chi_i)\Pi(G, A)$ is also π-irreducible by Theorem 3.4 and thus $\widehat{(\chi_i)\Pi(G, A)}=\hat{\psi}_i$, $1\leq i\leq k$. This implies that $\wedge$ is a bijection from Λ onto $I_\pi(C)$.

In order to prove the π-irreducibility of $(\xi_i)\Pi(T_i, A)$, we consider two cases.

(a) $|A|$ is even.

From Hypothesis 2.1 and the Odd-Order Theorem, G is solvable. Thus G/G_1 is abelian and so $G=G_1C$. Now, we know that $\mathrm{Irr}(T_i|\theta_i)=\{\xi_i\lambda;\ \lambda\in \mathrm{Irr}(T_i/G_1)\}$ (cf. [6], Corollary 6.17). For any $\xi_i\lambda\in\mathrm{Irr}(T_i|\theta_i)$, we have

$$(\xi_i\lambda)^a(t)=\xi_i^a\lambda^a(t)=\xi_i(t)\lambda(\bar{t}^{a^{-1}})=\xi_i\lambda(t) \qquad \text{for all} \quad a\in A \text{ and } t\in T_i\,.$$

Therefore by Theorem 2.3(1), we have

$$|\mathrm{Irr}(T_i\cap C|\phi_i)|=|\mathrm{Irr}(T_i|\theta_i)|=|\mathrm{Irr}(T_i/G_1)|=|T_i\cap C\colon C_1|\,.$$

This implies that every element of $\mathrm{Irr}(T_i\cap C|\phi_i)$ is an extension of ϕ_i. In particular, we see that $(\xi_i)\Pi(T_i, A)$ is π-irreducible.

(b) $|A|$ is odd.

Then A is a solvable group. Let $A=A_0\rhd A_1\rhd\cdots\rhd A_m=\{1\}$ be a composition series of A and let $\{\bar{A}_i=A_{i-1}/A_i\}_{i=1}^m$ be composition factors of A. Let $X_j=C_{G_1}(A_j)$, $Y_j=C_{T_i}(A_j)$, $\sigma_j=(\theta_i)\Pi(G_1, A_j)$ and $\tau_j=(\xi_i)\Pi(T_i, A_j)$ for $j=0, \cdots, m$. Then

$$X_j=C_{X_{j+1}}(\bar{A}_{j+1}),\ Y_j=C_{Y_{j+1}}(\bar{A}_{j+1}),\ X_j\trianglelefteq Y_j\,,$$
$$\sigma_j\in\mathrm{Irr}(X_j)_{A_{j-1}} \text{ and } \tau_j\in\mathrm{Irr}(Y_j)_{A_{j-1}} \qquad \text{for} \quad j=1, \cdots, m\,.$$

$\tau_m=\xi_i$ is an extension of $\sigma_m=\theta_i$. Now, we assume that τ_j is an extension of σ_j for some $j\geq 1$. Let us denote by Δ (resp. Δ') the semidirect product $X_j\bar{A}_j$ (resp. $Y_j\bar{A}_j$) and let $\sigma\in\mathrm{Irr}(\Delta)$ (resp. $\tau\in\mathrm{Irr}(\Delta')$) be the canonical extension of σ_j (resp. τ_j). Then τ_Δ is irreducible and it satisfies $\ker(\det\tau_\Delta)\geqq\bar{A}_j$. So, by the uniqueness of the canonical extension of σ_j, we obtain that $\tau_\Delta=\sigma$. Since $\bar{A}_j$ is cyclic, there exist, by Theorem 2.4, $\varepsilon=\pm 1$ and $\varepsilon'=\pm 1$ such that

$$\tau(x\bar{a})=\varepsilon\tau_{j-1}(x),\ \sigma(x\bar{a})=\varepsilon'\sigma_{j-1}(x) \qquad \text{for all} \quad x\in X_{j-1} \text{ and } \bar{a}\in\bar{A}_j-\{\bar{1}\}\,.$$

So,

$$\tau_{j-1}(1)=\varepsilon\tau_\Delta(\bar{a})=\varepsilon\sigma(\bar{a})=\varepsilon\varepsilon'\sigma_{j-1}(1)\,.$$

This implies $\varepsilon=\varepsilon'$ and hence τ_{j-1} is an extension of σ_{j-1}. Repeating the same argument, we conclude that τ_0 is an extension of σ_0 and the proof is complete.

From Lemma 4.2, we immediately obtain the following corollary.

Corollary 4.3. *Assume Hypothesis* 2.1 *and that* G *is* π*-separable. Let* $\Omega \subseteq \mathrm{Irr}(G)$. *If* Ω *has the* π*-lifting property with respect to* A*, then the restriction map* $\wedge$ *gives a bijection from* $(\Omega_A)\Pi(G, A)$ *onto* $I_\pi(C)$.

Assume Hypothesis 2.1 and that G is π-separable. Suppose $\Omega \subseteq \mathrm{Irr}(G)$ has the π-lifting property with respect to A. Then $\wedge : \chi \to \hat{\chi}$ gives a bijection from Ω_A onto $I_\pi(G)_A$ and we have the sequence of bijections:

$$I_\pi(G)_A \xrightarrow{\wedge^{-1}} \Omega_A \xrightarrow{\Pi(G, A)} (\Omega_A)\Pi(G, A) \xrightarrow{\wedge} I_\pi(C)$$

where $\wedge^{-1}$ is the inverse map of $\wedge$. Thus the composite map $\tilde{\Pi}(G, A) = \wedge^{-1}\Pi(G, A)\wedge$ gives a bijection from $I_\pi(G)_A$ onto $I_\pi(C)$. From its construction, it seems likely that $\tilde{\Pi}(G, A)$ depends on the choice of Ω. However, as we shall see below, if A be solvable, then $\tilde{\Pi}(G, A)$ does not depend on the choice of Ω. On the other hand, if A is non-solvable, then $|G|$ is odd by the Odd-Order Theorem. When $|G|$ is odd, we shall show that $\Pi(G, A)$ actually gives a bijection from $B_\pi(G)_A$ onto $B_\pi(C)$. So the bijection map $\tilde{\Pi}(G, A)$ is naturally defined anyway.

The following Lemmas 4.4 and 4.5 correspond to Lemma 3.9 and Proposition 3.8 of Uno [10] respectively, and his proof also works in our case.

Lemma 4.4. *Assume Hypothesis* 2.1 *and that* A *is cyclic of prime order. Let,* $\chi, \psi \in \mathrm{Irr}(G)_A$ *be such that* $\hat{\chi} = \hat{\psi} \in I_\pi(G)$. *Then* $\widehat{(\chi)\Pi(G, A)} = \widehat{(\psi)\Pi(G, A)}$.

Proof. Let $|A| = q$.

Case 1. $q \in \pi$.

In this case, Γ/G is cyclic q-group and Γ is π-separable. Since χ and ψ are A-invariant, it follows from Theorem 3.5 that the restriction maps

$$\mathrm{Irr}(\Gamma \mid \chi) \to I_\pi(\Gamma \mid \hat{\chi}) \quad \text{and} \quad \mathrm{Irr}(\Gamma \mid \psi) \to I_\pi(\Gamma \mid \hat{\psi})$$

are bijections.

Let χ' (resp. ψ') be the canonical extension of χ (resp. ψ) to Γ. Then $\mathrm{Irr}(\Gamma \mid \chi) = \{\mu\chi' ; \mu \in \mathrm{Irr}(A)\}$. So $\hat{\psi}' = \widehat{\mu\chi'}$ for some $\mu \in \mathrm{Irr}(A)$. By Theorem 2.4, there exist $\varepsilon = \pm 1$ and $\varepsilon' = \pm 1$ such that

$$(\chi)\Pi(G, A)(c) = \varepsilon\chi'(ca) \quad \text{and}$$
$$(\psi)\Pi(G, A)(c) = \varepsilon'\psi'(ca) \qquad \text{for all} \quad c \in C \text{ and } a \in A - \{1\}.$$

Thus for every π-element $c \in C$ and $a \in A - \{1\}$,

$$(\psi)\Pi(G, A)(c) = \varepsilon'\psi'(ca) = \varepsilon'\mu(a)\chi'(ca) = \varepsilon'\varepsilon\mu(a)(\chi)\Pi(G, A)(c),$$

and in particular, $(\psi)\Pi(G, A)(1)=\varepsilon'\varepsilon\mu(a)(\chi)\Pi(G, A)(1)$. Since $\mu(a)$ is a root of unity, we obtain $\varepsilon\varepsilon'\mu(a)=1$. Thus $\widehat{(\chi)\Pi(G, A)}=\widehat{(\psi)\Pi(G, A)}$ as required.

Case 2. $q\in\pi'$.

Set $\pi_0=\pi\cup\{q\}$. Then G is π_0-separable. Since q does not divide $|G|$, $I_{\pi_0}(G)=I_\pi(G)$ and $\hat{\chi}=\hat{\psi}$ is a π_0-irreducible character. So the assertion is clear from the case 1.

Lemma 4.5. *Assume Hypothesis* 2.1 *and that G is π-separable and A is solvable. Let $B\trianglelefteq A$, $D=C_G(B)$ and assume that $\Omega\subseteq\mathrm{Irr}(G)$ and $\Lambda\subseteq\mathrm{Irr}(D)$ both have the π-lifting property with respect to A. Let $\chi\in\Omega_A$ and let ϕ be the unique element of $\Lambda_{A/B}$ such that $\hat{\phi}=\widehat{(\chi)\Pi(G, B)}$. (Note that $(\chi)\Pi(G, B)$ is π-irreducible by Corollary* 4.3). *Then $\widehat{(\chi)\Pi(G, A)}=\widehat{(\phi)\Pi(D, A/B)}$.*

Proof. We proceed by induction on $|A|$.

We may assume $A\neq B$. Let H be a maximal normal subgroup of A containing B. By Corollary 4.3, $\widehat{(\chi)\Pi(G, H)}$ and $\widehat{(\phi)\Pi(D, H/B)}$ are both contained in $I_\pi(C_G(H))$. From the inductive hypothesis, we may assume that $\widehat{(\chi)\Pi(G, H)}=\widehat{(\phi)\Pi(D, H/B)}$. Since A/H is a cyclic group of prime order, it follows from Lemma 4.4 that

$$\widehat{\{(\chi)\Pi(G, H)\}\Pi(C_G(H), A/H)}=\widehat{\{(\phi)\Pi(D, H/B)\}\Pi(C_G(H), A/H)}.$$

Thus the assertion follows immediately from Theorem 2.2 (2).

We are now ready to prove the following theorem.

Theorem 4.6. *Assume Hypothesis* 2.1 *and that G is π-separable and A is solvable. Then there exists a bijection*

$$\tilde{\Pi}(G, A): I_\pi(G)_A\to I_\pi(C),$$

which is independent of the choice of Ω which satisfies the π-lifting property with respect to A. And the following hold.

(1) *If $B\trianglelefteq A$, then $\tilde{\Pi}(G, A)=\tilde{\Pi}(G, B)\tilde{\Pi}(C_G(B), A/B)$.*

(2) *If A is a q-group for a prime q and $\psi\in I_\pi(G)_A$, then $(\psi)\tilde{\Pi}(G, A)$ is a unique π-irreducible constituent of ψ_C with multiplicity prime to q.*

Proof. In Lemma 4.5, let $B=\{1\}$. Then we see that $\tilde{\Pi}(G, A)$ is independent of the choice of Ω which satisfies the π-lifting property with respect to A. If $B\trianglelefteq A$, it is easily seen by Lemma 4.5 that $\tilde{\Pi}(G, A)=\tilde{\Pi}(G, B)\tilde{\Pi}(C_G(B), A/B)$. Now fix Ω which satisfies π-lifting property with respect to A. For $\psi\in I_\pi(G)_A$, there exists $\chi\in\Omega$ such that $\hat{\chi}=\psi$. If A is a q-group, then by Theorem 2.2 (3), $\chi_C=m(\chi)\Pi(G, A)+q\zeta$ where m is a positive integer prime to

q and ζ is zero or a character of C. Therefore by the definition of $\tilde{\Pi}(G, A)$, $\psi_C = m(\psi)\tilde{\Pi}(G, A) + q\hat{\zeta}$. This completes the proof.

Finally we note the following fact which asserts that $\Pi(G, A)$ induces a bijection between B_π-characters under certain circumstances.

Theorem 4.7. *Assume Hypothesis* 2.1 *and that* G *is* π*-separable. If either* $2 \in \pi$ *or* $2 \nmid |G|$, *then* $\Pi(G, A)$ *gives a bijection from* $B_\pi(G)_A$ *onto* $B_\pi(C)$.

Proof. It suffices to show that if $\chi \in B_\pi(G)_A$, then $\xi = (\chi)\Pi(G, A) \in B_\pi(C)$. We know that $\hat{\xi} \in I_\pi(C)$ by Corollary 4.3 (applied to $\Omega = B_\pi(G)$) and ξ has values in $\boldsymbol{Q}_\pi$ by Theorem 2.2 (5) and Theorem 3.2 (1). Thus we conclude from Theorem 3.2 (2) that $\xi \in B_\pi(C)$.

Under the assumption of the above theorem, we get the natural bijection $\tilde{\Pi}(G, A)$: $I_\pi(G) \to I_\pi(C)$. And clearly this coincides with the bijection $\tilde{\Pi}(G, A)$ given in Theorem 4.6, provided A is solvable. Thus the proof of our main theorem is completed.

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Shortly after this work was completed, Wolf [13] was published, where he showed the same result as our main theorem (cf. Theorem 4.4 in it). However his argument is slightly different from ours. The author wishes to express his gratitude to Professor Y. Tsushima for his helpful advice and encouragement.

References

[1] L. Dornhoff: Group Representation Theory, Marcel Dekker, New York, 1972.
[2] G. Glauberman: *Correspondences of characters for relatively prime operator groups,* Canad. J. Math. **20** (1968), 1465–1488.
[3] D. Gorenstein: Finite Groups, Harper & Row, New York, 1968; 2nd., Chelsea, New York, 1980.
[4] I.M. Isaacs: *Characters of solvable and symplectic groups,* Amer. J. Math. **95** (1973), 594–635.
[5] I.M. Isaacs: *Lifting Brauer characters of p-solvable groups,* Pacific J. Math. **53** (1974), 171–188.
[6] I.M. Isaacs: Character Theory of Finite Groups, Academic Press, New York, 1976.
[7] I.M. Isaacs: *Lifting Brauer characters of p-solvable groups* II, J. Algebra **51** (1978), 476–490.
[8] I.M. Isaacs: *Characters of π-separable groups,* J. Algebra **86** (1984), 98–128.
[9] I.M. Isaacs: *Fong characters in π-separable groups,* J. Algebra **99** (1986), 89–107.
[10] K. Uno: *Character correspondences in p-solvable groups,* Osaka J. Math. **20** (1983), 713–725.

[11] T.R. Wolf: *Character correspondences in solvable groups* Illinois J. Math. **22** (1978), 327–340.

[12] T.R. Wolf: *Character correspondences induced by subgroups of operator groups,* J. Algebra **57** (1979), 502–521.

[13] T.R. Wolf: *Character correspondences and π-special characters in π-separable groups,* Canad. J. Math. **39**, No. 4 (1987), 920–937.

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