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GREENBERG'S THEOREM AND EQUIVALENCE PROBLEM ON COMPACT RIEMANN SURFACES

Dedicated to Professor Masatake Kuranishi on his 80th birthday

SATORU MIZUTA and MAKOTO NAMBA

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Abstract

Another proof of Greenberg's theorem on automorphism groups of compact Riemann surfaces is given. Using the idea of the proof, the equivalence problem for finite Galois coverings of the compact projective line is answered affirmatively, except special type of coverings.

1. Introduction

Greenberg [5] showed the following theorem on automorphism groups of connected compact Riemann surfaces in 1963:

Theorem 1.1 ([5], Theorem 4). *Let G be a non-trivial, finite group. Then there exists a connected compact Riemann surface S whose automorphism group* $\mathrm{Aut}(S)$ *is isomorphic to G.*

By using Fenchel-Nielsen's theory, Greenberg showed a list which states that for some Fuchsian group Γ there exists a Fuchsian group Γ' containing Γ as a proper subgroup with finite index. Using the list, he proved the above theorem. But he didn't give enough explanation for his list, nor for the proof of the above theorem.

In this paper, we give a proof of the above theorem, using only elementary knowledge on branched Galois coverings and hyperbolic geometry, without using Fenchel-Nielsen's theory.

As an application of our proof of Greenberg's theorem, we give an answer to the equivalence problem (see §5 for detail and terminology):

Theorem 1.2 (c.f. Theorem 5.8). *Let $f = \{f_u\}_{u\in N}\colon X = \{X_u\}_{u\in N} \to Y$ be a non-degenerate family of finite Galois coverings of the complex projective line $\mathbb{P}^1$ with a $\mathbb{P}^1$-bundle $\rho\colon Y \to N$. Assume $g \geq 2$, where g is the genus of X_u $(u \in N)$. Assume*

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that the number s of the branch points and the set $\{e_1, e_2, \ldots, e_s\}$ of ramification indices of f_u $(u \in N)$ are either
(1) $s \neq 4$, *or*
(2) $s = 4$ *and* $\{e_1, e_2, e_3, e_4\}$ *does not satisfy* $e_1 = e_2 \leq e_3 = e_4$.
Then, for any two points u and u' in N, X_u and $X_{u'}$ are biholomorphic if and only if f_u and $f_{u'}$ are holomorphically equivalent.

We also have

Theorem 1.3 (c.f. Theorem 5.7). *Under the same conditions to* Theorem 1.2 *on g, s and $\{e_1, e_2, \ldots, e_s\}$, the canonical holomorphic map of the moduli space of holomorphic equivalence classes of finite Galois coverings of $\mathbb{P}^1$ to the moduli space $\mathbb{M}_g$ of compact Riemann surfaces of genus g is injective.*

2. General Klein tiles

Let s be an integer with $s \geq 3$ and $e_1, e_2, \ldots, e_s$ integers with $e_j \geq 2$ $(j = 1, 2, \ldots, s)$ which satisfy the inequality

$$\sum_{j=1}^{s} \frac{1}{e_j} < s - 2. \tag{2.1}$$

We call a hyperbolic $2(s-1)$-gonal polygon T in the upper half plane $\mathcal{U}$ a *general Klein tile* if T satisfies the following conditions (see Fig. 1 for the case $s = 4$):
(1) If we label vertices of T as

$$V_1, V_2, \ldots, V_{2s-2}$$

counterclockwisely, then we have $\rho(V_j, V_{j+1}) = \rho(V_{2s-j}, V_{2s-j-1})$ with the hyperbolic metric ρ for each $j = 1, 2, \ldots, s-1$, where $V_{2s-1} = V_1$.
(2) Let $\angle V_j$ be the inner angle at V_j for each $j = 1, 2, \ldots, s$. For $j = 2, 3, \ldots, s-1$, the equality $\angle V_j + \angle V_{2s-j} = 2\pi/e_j$ holds.
(3) $\angle V_1 = 2\pi/e_1$ and $\angle V_s = 2\pi/e_s$.

A general Klein tile T is called a *Klein tile* if the s-polygon $V_1V_2 \cdots V_s$ (a half of T) is congruent to $V_{2s-1}V_{2s-2} \cdots V_s$ (another-half of T) by the reflection with respect to the hyperbolic line $\overline{V_1V_s}$. (If $s = 3$, then a general Klein tile is necessarily a Klein tile.) Klein tiles and tessellation by them appeared in Klein [9].

Now let T be a general Klein tile. Let Γ be the subgroup of $PSL(2, \mathbb{R})$ generated by the hyperbolic rotations φ_j with the center V_j and the angle $2\pi/e_j$ for $j = 1, 2, \ldots, 2s-2$. (We put $e_{2s-j} = e_j$ for $j = 1, 2, \ldots, s$.) Then Γ is a Fuchsian group of the first kind and has T as a fundamental domain. Γ is, in fact, generated by φ_j

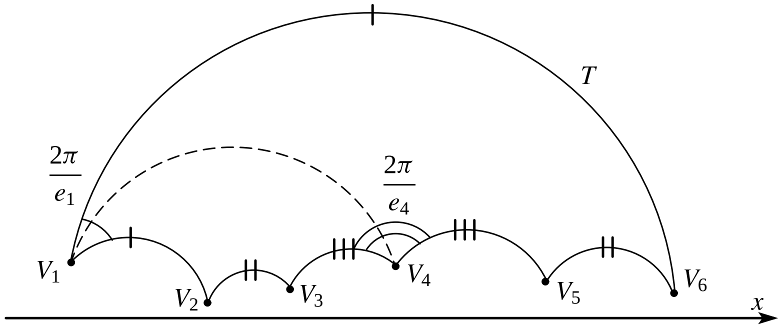

Fig. 1. A general Klein tile T for $s = 4$

$(j = 1, 2, \dots, s)$ and is presented as follows:

$$\Gamma = \left\langle \varphi_1, \varphi_2, \dots, \varphi_s \;\middle|\; \varphi_1^{e_1} = \varphi_2^{e_2} = \cdots = \varphi_s^{e_s} = \varphi_1\varphi_2\cdots\varphi_s = 1 \right\rangle .$$

Γ is said to have the signature $(0; e_1, e_2, \dots, e_s)$.

For each $j = 1, 2, \dots, 2s-2$, let (x_j, y_j) be the coordinates of V_j in $\mathcal{U}$, a_j the Euclidean center of the hyperbolic line C_j through V_j and V_{j+1} which is a circle in Euclidean geometry, and r_j the radius of C_j.

Note that a_j (a point on the real axis) and r_j can be determined algebraically by x_j and y_j for $j = 1, 2, \dots, 2s-2$. In fact, for example, by easy calculations, we have

$$\begin{aligned} a_1 &= \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2}{2(x_1 - x_2)}, \\ r_1^2 &= \frac{\left\{(x_1 - x_2)^2 + (y_1 - y_2)^2\right\}\left\{(x_1 - x_2)^2 + (y_1 + y_2)^2\right\}}{4(x_1 - x_2)^2}. \end{aligned} \tag{2.2}$$

Conversely, x_j and y_j can be determined algebraically by a_j and r_j for $j = 1, 2, \dots, 2s-2$. In fact, for example, we have

$$\begin{aligned} x_2 &= \frac{a_2^2 - a_1^2 + r_1^2 - r_2^2}{2(a_2 - a_1)}, \\ y_2^2 &= \frac{\left\{(a_1 - a_2)^2 - (r_1 - r_2)^2\right\}\left\{(r_1 + r_2)^2 - (a_1 - a_2)^2\right\}}{4(a_1 - a_2)^2}. \end{aligned} \tag{2.3}$$

Consider the fields attaching these numerical data on the rational number field $\mathbb{Q}$:

$$\begin{aligned} K(T) &= \mathbb{Q}(x_1, y_1, x_2, y_2, \dots, x_{2s-2}, y_{2s-2}), \\ K'(T) &= \mathbb{Q}(a_1, r_1, a_2, r_2, \dots, a_{2s-2}, r_{2s-2}), \\ K''(T) &= K(T)K'(T) \quad \text{(the composite field)}. \end{aligned} \tag{2.4}$$

Then, by the above consideration, $K''(T)/K(T)$ and $K''(T)/K'(T)$ are finite algebraic extensions. In particular, the transcendence degree over $\mathbb{Q}$ of these fields are the same:

$$\mathrm{Tr.\,deg}_{\mathbb{Q}}\, K(T) = \mathrm{Tr.\,deg}_{\mathbb{Q}}\, K'(T) = \mathrm{Tr.\,deg}_{\mathbb{Q}}\, K''(T). \tag{2.5}$$

The numerical data

$$x_1, y_1, x_2, y_2, \dots, x_{2s-2}, y_{2s-2},$$
$$a_1, r_1, a_2, r_2, \dots, a_{2s-2}, r_{2s-2}$$

determine the general Klein tile T.

We show that the $2s-3$ data

$$x_2, y_2, \dots, x_{s-2}, y_{s-2}, x_{2s-2}, y_{2s-2} \quad \text{and} \quad a_{s-2}$$

determine T and can be chosen algebraically independent, and the other data can be determined algebraically by these data.

We use the following formula (see Jones and Singerman [7]): for $z, w \in \mathcal{U}$

$$\sinh^2\left(\frac{1}{2}\rho(z,w)\right) = \frac{|z-w|^2}{4\,\mathrm{Im}(z)\,\mathrm{Im}(w)}. \tag{2.6}$$

The triangle $\triangle V_1V_2V_{2s-2}$ is a hyperbolic isosceles triangle such that the top angle $\angle V_1$ is equal to $2\pi/e_1$. Hence, by the sine rule, the cosine rule for hyperbolic geometry (see Beardon [1]) and (2.6), the point V_1 can be determined algebraically by V_2 and V_{2s-2}.

If $s \geq 5$, then V_{2s-3} can be determined algebraically by V_2, V_3 and V_{2s-2}. In fact, since

$$\angle V_{2s-2} = \frac{2\pi}{e_2} - \angle V_2,$$

the direction $\overrightarrow{V_{2s-2}V_{2s-3}}$ can be determined. Since

$$\rho(V_2, V_3) = \rho(V_{2s-2}, V_{2s-3}),$$

the vertex V_{2s-3} can be determined algebraically by (2.6).

In a similar way, $V_{2s-4}, \dots, V_{s+2}$ can be determined algebraically by $V_2, V_3, \dots, V_{s-2}$ if $s \geq 6$.

Since a_{s-2} is given, r_{s-2} can be determined algebraically. Hence $\angle V_{s-2}$ can be determined. $\cos\angle V_{s-2}$ can be determined algebraically. Since

$$\angle V_{s+2} = \frac{2\pi}{e_{s-2}} - \angle V_{s-2},$$

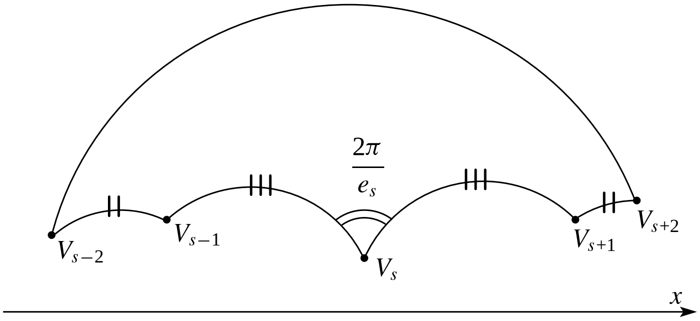

Fig. 2. The pentagon $V_{s-2}V_{s-1}V_sV_{s+1}V_{s+2}$

$\cos\angle V_{s+2}$ can be determined algebraically. Hence the direction $\overrightarrow{V_{s+2}V_{s+1}}$ can be determined. In particular, a_{s+1} and r_{s+1} can be determined algebraically.

Finally, we show that the pentagon $V_{s-2}V_{s-1}V_sV_{s+1}V_{s+2}$ (see Fig. 2) can be determined algebraically.

By the elementary geometry, we have the following 6 equations for the pentagon:

$$r_{s-2}^2 + r_{s-1}^2 - 2r_{s-2}r_{s-1}\cos V_{s-1} = (a_{s-2} - a_{s-1})^2,$$
$$r_{s+1}^2 + r_s^2 - 2r_{s+1}r_s\cos V_{s+1} = (a_{s+1} - a_s)^2,$$
$$r_{s-1}^2 + r_s^2 - 2r_{s-1}r_s\cos\frac{2\pi}{e_s} = (a_s - a_{s-1})^2,$$
$$\cos\frac{2\pi}{e_{s-1}} = \cos\angle V_{s-1}\cos\angle V_{s+1} - \sqrt{(1-\cos^2\angle V_{s-1})(1-\cos^2\angle V_{s+1})},$$
$$\rho(V_{s-2}, V_{s-1}) = \rho(V_{s+2}, V_{s+1}),$$
$$\rho(V_{s-1}, V_s) = \rho(V_{s+1}, V_s).$$

By (2.6), the last two equations give algebraic relations among $x_{s-1}, y_{s-1}, x_s, y_s, x_{s+1}, y_{s+1}$. By (2.2), we can express them as algebraic relations among $a_{s-1}, r_{s-1}, a_s, r_s$.

Then these 6 equations with the 6 unknowns

$$a_{s-1},\ r_{s-1},\ a_s,\ r_s,\ \cos\angle V_{s-1},\ \cos\angle V_{s+1}$$

are algebraically independent. (In fact, for example, from these equations except the 4-th equation, we have a family of pentagons $V_{s-2}V_{s-1}V_sV_{s+1}V_{s+2}$ such that $\cos(\angle V_{s-1} + \angle V_{s+1})$ is not constant.) Hence, from these 6 equations, the above 6 unknowns can be determined algebraically. Hence, by (2.3),

$$V_{s-1} = (x_{s-1}, y_{s-1}),\quad V_s = (x_s, y_s),\quad V_{s+1} = (x_{s+1}, y_{s+1})$$

can be determined algebraically.

In the above discussion, it is noted that the given data

$$x_2, y_2, \ldots, x_{s-2}, y_{s-2}, x_{2s-2}, y_{2s-2} \quad \text{and} \quad a_{s-2}$$

can be chosen algebraically independent.

Thus we conclude, by (2.2) and (2.3),

Proposition 2.1. *Suppose that* $s \geq 3$ *and integers* $e_1, \ldots, e_s$ *with* $e_j \geq 2$ $(j = 1, 2, \ldots, s)$ *satisfy the inequality* (2.1). *Let* T *be a general Klein tile with these data defined as above. Then*
(1) *some* $2s-3$ *coordinates among* $4s-4$ *coordinates of the* $2s-2$ *vertices of* T *can be taken algebraically independent and other coordinates can be determined algebraically from these* $2s-3$ *coordinates.*
(2) $\mathrm{Tr.\,deg}_{\mathbb{Q}}\, K(T) \leq 2s-3$. *Here equality holds for a general* T.

3. Proof of Greenberg's theorem

We first explain our terminology (c.f., Namba [14]). Let $f\colon X \to Y$ be a surjective holomorphic map between connected compact Riemann surfaces X and Y. Then f can be regarded as a *branched covering*: For any point p of X, there are local coordinate systems z and w around p and $q = f(p)$, respectively, with $z(p) = w(q) = 0$ such that f is locally expressed as

$$f\colon z \longmapsto w = z^e.$$

The positive integer e is called the *ramification index of* f at p. If $e \geq 2$, then p and q are called a *ramification point* and a *branch point of* f, respectively. We denote by R_f (resp. B_f) the set of all ramification points (resp. branch points) and call it the *ramification locus* (resp. *branch locus*) of f. They are finite sets. Note that $R_f \subset f^{-1}(B_f)$ and

$$f\colon X - f^{-1}(B_f) \longrightarrow Y - B_f$$

is a usual finite covering map (i.e., finite unbranched covering map). Its mapping degree is called the *degree of* f and is denoted by $\deg(f)$. f is called a *Galois covering* if the *automorphism group*

$$\mathrm{Aut}(f) = \{\psi \in \mathrm{Aut}(X) \mid f \circ \psi = f\}$$

of f acts transitively on each fiber of f. In this case, $\deg(f)$ is equal to the order of $\mathrm{Aut}(f)$ and Y is canonically biholomorphic to the quotient space $X/\mathrm{Aut}(f)$. A Galois covering is called an *abelian* (resp. *cyclic*) *covering* if $\mathrm{Aut}(f)$ is abelian (resp. cyclic). If $f\colon X \to Y$ is a Galois covering, then for any point $q \in B_f$, the ramification index

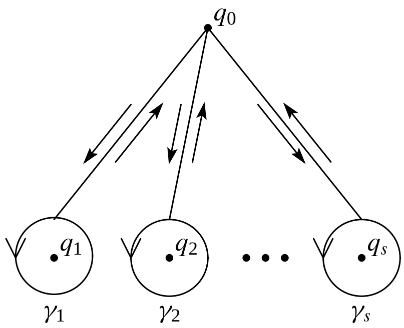

Fig. 3. The meridians

of f at a point $p \in f^{-1}(q)$ is independent of p and depends only on q. Hence we may call it the *ramification index of* f *at* q, which is a divisor of $\deg(f)$. Let f be Galois, $B_f = \{q_1, q_2, \dots, q_s\}$ and e_j (≥ 2) the ramification index of f at q_j $(j = 1, 2, \dots, s)$. Then the positive divisor

$$e_1 q_1 + e_2 q_2 + \cdots + e_s q_s$$

on Y is called the *branch divisor of* f.

Now let G be any non-trivial finite group. G can be presented as follows:

$$G = \left\langle g_1, g_2, \dots, g_s \;\middle|\; g_1^{e_1} = g_2^{e_2} = \cdots = g_s^{e_s} = g_1 g_2 \cdots g_s = 1, \;\; *, \dots, * \right\rangle, \tag{3.1}$$

where each g_j is not the identity 1 and e_j (≥ 2) is the order of g_j. ($*$ are other relations.) We allow $g_1, g_2, \dots, g_s$ to overlap a number of times; for example, we can select such as $g_1 = g_2 = \cdots = g_s$ if G is a cyclic group. So we can enlarge the value of s even if the order $\#G$ of G is much smaller.

We assume that $s \geq 3$ and $e_1, e_2, \dots, e_s$ satisfy the inequality (2.1). (Note that (2.1) is automatically satisfied if $s \geq 5$.)

Take distinct s points

$$q_1, q_2, \dots, q_s$$

in $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$, the complex projective line. Put $B = \{q_1, q_2, \dots, q_s\}$. Take a point $q_0 \in \mathbb{P}^1$ as a base point which is not contained in B. Then the fundamental group $\pi_1(\mathbb{P}^1 - B, q_0)$ has the presentation as follows:

$$\pi_1\left(\mathbb{P}^1 - B, q_0\right) = \langle \gamma_1, \gamma_2, \dots, \gamma_s \mid \gamma_1 \gamma_2 \cdots \gamma_s = 1 \rangle,$$

where γ_j $(j = 1, 2, \dots, s)$ are (the homotopy classes of) the meridians around q_j as in Fig. 3. Consider an epimorphism

$$\xi : \pi_1\left(\mathbb{P}^1 - B, q_0\right) \longrightarrow G, \quad \gamma_j \longmapsto g_j \quad (j = 1, 2, \dots, s) \tag{3.2}$$

and a finite unbranched Galois covering

$$f'\colon X' \longrightarrow \mathbb{P}^1 - B$$

which corresponds to the kernel $\operatorname{Ker}\xi$ of ξ. f' can be extended to a finite branched covering

$$f\colon X \longrightarrow \mathbb{P}^1,$$

where X is a compact Riemann surface. (The extension is unique up to isomorphisms. See Theorem 4.3.) f is a finite Galois covering whose automorphism group $\operatorname{Aut}(f)$ is isomorphic to G. The branch divisor of f is

$$e_1q_1 + e_2q_2 + \cdots + e_sq_s.$$

The genus g of X can be calculated by the Riemann-Hurwitz formula:

$$2g - 2 = d\left\{\sum_{j=1}^{s}\left(1 - \frac{1}{e_j}\right) - 2\right\}, \tag{3.3}$$

where d is the order of the group G. Hence, by the assumption of the inequality (2.1), we have $g \geq 2$. In particular $\operatorname{Aut}(X)$ is a finite group.

$\operatorname{Aut}(f)$ is a subgroup of $\operatorname{Aut}(X)$ and is isomorphic to G. The quotient space $X/\operatorname{Aut}(X)$ is also biholomorphic to $\mathbb{P}^1$ and the projection map

$$\hat{f}\colon X \longrightarrow X/\operatorname{Aut}(X) \simeq \mathbb{P}^1$$

is a finite Galois covering with the Galois group $\operatorname{Aut}(X)$. Let

$$\hat{e}_1\hat{q}_1 + \hat{e}_2\hat{q}_2 + \cdots + \hat{e}_{\hat{s}}\hat{q}_{\hat{s}}$$

be its branch divisor. There exists a surjective holomorphic map (i.e., a rational function)

$$h\colon X/\operatorname{Aut}(f) \simeq \mathbb{P}^1 \longrightarrow X/\operatorname{Aut}(X) \simeq \mathbb{P}^1$$

such that $h \circ f = \hat{f}$. Let m be the degree of the map h. m is then the index of $\operatorname{Aut}(f)$ in $\operatorname{Aut}(X)$ and md is the order of $\operatorname{Aut}(X)$. Comparing (3.3) with the Riemann-Hurwitz formula with respect to $\hat{f}$, we have

$$m\left\{\sum_{k=1}^{\hat{s}}\left(1 - \frac{1}{\hat{e}_k}\right) - 2\right\} = \sum_{j=1}^{s}\left(1 - \frac{1}{e_j}\right) - 2. \tag{3.4}$$

The following lemma is obvious from the definition of the ramification index.

Lemma 3.1. *For a point $p \in X$, let c and c' be the ramification indices of f at p and of h at $f(p)$, respectively. Then the ramification index of $\hat{f} = h \circ f$ at p is cc'.*

The following lemma is also obvious from Lemma 3.1 and the definition of Galois coverings.

Lemma 3.2. *Let f, h and $\hat{f} = h \circ f$ be as above. Then the following* (i) *to* (v) *hold*:

(i) $B_f \subset h^{-1}(B_{\hat{f}})$ *and* $B_h \subset B_{\hat{f}}$.

(ii) *For $r \in B_{\hat{f}}$, the ramification index of $\hat{f}$ at r can be divisible by l, where l is the least common multiple of the ramification indices of h at points of $h^{-1}(r)$.*

(iii) *Let $r \in B_{\hat{f}}$. If the ramification index of h at a point q in $h^{-1}(r)$ is less than l in* (ii), *then $q \in B_f$.*

(iv) *Let $r \in B_{\hat{f}}$. If $q \in h^{-1}(r)$ is not a ramification point of h, then $q \in B_f$.*

(v) *Let $r \in B_h$. Assume that*

(1) *there is $q \in h^{-1}(r)$ such that q is not a ramification point of h, and*

(2) $h^{-1}(r) \not\subset B_f$.

Then the following (a) *and* (b) *hold*:

(a) $h^{-1}(r) - B_f = \{q' \in h^{-1}(r) \mid$ *the ramification index of h at q' is l in* (ii)$\}$,

(b) *the ramification index of f at q is l in* (ii).

Now we prove the following key proposition for the proof of Greenberg's theorem.

Proposition 3.3. *Let $f, h, \hat{f} = h \circ f, s, \hat{s}$ and m be as above. Assume $m \geq 2$, i.e.,* $\mathrm{Aut}(f) \neq \mathrm{Aut}(X)$. *If either*

(1) $s \geq 5$, *or*

(2) $s = 4$ *and* $\{e_1, e_2, e_3, e_4\}$ *does not satisfy* $e_1 = e_2 \leq e_3 = e_4$,

then $\hat{s} < s$.

Proof. We divide the proof into several steps.

(i) Assume first that $m = 2$. Then h is a double covering with 2 branch points. If $q \in B_f$ and $q' \in h^{-1}(h(q))$, then $q' \in B_f$ by (iv) of Lemma 3.2. This implies easily that

(i-1) if $s \geq 5$, then $s > \hat{s}$, and

(i-2) if $s = 4$, then $\hat{s} = 3$ or 4.

In (i-2), $\hat{s} = 4$ holds if and only if $\{q_1, q_2, q_3, q_4\}$ and $\{\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_4\}$ are as in Fig. 4. In this case,

$$e_1 = e_2 (= \hat{e}_2) \quad \text{and} \quad e_3 = e_4 (= \hat{e}_3)$$

by Lemma 3.1. (Since the genus g of X satisfies $g \geq 2$, e_1 and e_3 must satisfy $(1/e_1)+$

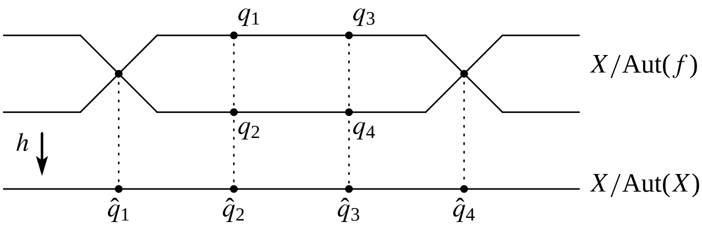

Fig. 4. $m = 2$, $s = 4$, $\hat{s} = 4$

$(1/e_3) < 1$.)

(ii) In the following, we assume that $m \geq 3$.

The left hand side of the equality in (3.4) is greater than or equal to $m((\hat{s}/2) - 2)$. Hence

$$m\left(\frac{\hat{s}}{2} - 2\right) \leq \sum_{j=1}^{s}\left(1 - \frac{1}{e_j}\right) - 2 = s - A - 2, \quad \text{where} \quad A = \sum_{j=1}^{s}\frac{1}{e_j}.$$

Hence

$$\hat{s} \leq \frac{2s}{m} - \frac{2A}{m} - \frac{4}{m} + 4.$$

Hence $s > \hat{s}$ holds if

$$\frac{2s}{m} - \frac{2A}{m} - \frac{4}{m} + 4 < s.$$

This inequality holds if and only if

$$4m - 2A - 4 < (m-2)s.$$

Since $m \geq 3$, this inequality holds if and only if

$$4 + \frac{4 - 2A}{m - 2} < s.$$

If $m \geq 6$, then

$$4 + \frac{4 - 2A}{m - 2} < 4 + \frac{4}{m - 2} \leq 5.$$

Hence if $m \geq 6$ and $s \geq 5$, then $\hat{s} < s$.

If $m = 5$, then

$$4 + \frac{4 - 2A}{m - 2} < 4 + \frac{4}{5 - 2} = \frac{16}{3} < 6.$$

Hence if $m = 5$ and $s \geq 6$, then $\hat{s} < s$.

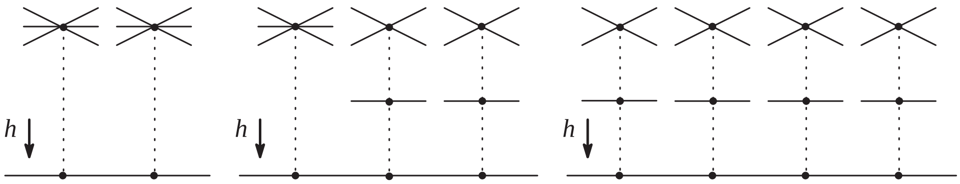

Fig. 5. $m = 3$

If $m = 4$, then

$$4 + \frac{4 - 2A}{m - 2} < 4 + \frac{4}{4 - 2} = 6.$$

Hence if $m = 4$ and $s \geq 6$, then $\hat{s} < s$.

If $m = 3$, then

$$4 + \frac{4 - 2A}{m - 2} < 4 + \frac{4}{3 - 2} = 8.$$

Hence if $m = 3$ and $s \geq 8$, then $\hat{s} < s$.

(iii) Hence if $m \geq 3$ and $s \geq 5$, then it is enough to check the cases (iii-1) $m = 3$ and $s = 5, 6, 7$ and (iii-2) $m = 4$ or 5 and $s = 5$:

(iii-1) Assume $m = 3$. The picture of ramifications for h with $\deg(h) = 3$ are 3 pictures in Fig. 5.

Using Lemma 3.2, we can check that the case $5 \leq s \leq \hat{s}$ can not occur for each picture in Fig. 5. For example, in the last picture in Fig. 5, the 4 unramified points of h in $h^{-1}(B_h)$ must be points in B_f by (iii) of Lemma 3.2. If $\hat{s} \geq 5$, then there is a point $r \in (\mathbb{P}^1 - B_h) \cap B_{\hat{f}}$. Then the 3 points in $h^{-1}(r)$ must be points in B_f by (iv) of Lemma 3.2. Hence

$$s \geq 4 + 3(\hat{s} - 4) > \hat{s}.$$

Thus if $m = 3$ and $s \geq 5$, then $\hat{s} < s$.

(iii-2) Assume $m = 4$ or $m = 5$. We draw all possible pictures of ramifications for h with $m = 4$ and $m = 5$. (There are 14 pictures for $m = 4$ and 36 pictures for $m = 5$.) Using Lemma 3.2, we can check as in the case (iii-1) that the case $\hat{s} \geq s = 5$ can not occur for each picture for $m = 4$ and $m = 5$.

Thus if $m = 4$ or 5 and $s = 5$, then $\hat{s} < s$.

Hence we conclude that if $s \geq 5$, then $\hat{s} < s$.

(iv) Finally we consider the case $m \geq 3$ and $s = 4$. We look for the case $\hat{s} \geq s = 4$.

(iv-1) If $m = 3$, then by Lemma 3.2, the case $\hat{s} \geq s = 4$ occurs only if $\hat{s} = s = 4$ and the picture of ramifications is as in Fig. 6.

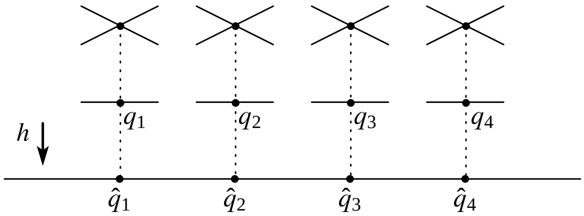

Fig. 6. $m = 3$, $s = 4$, $\hat{s} = 4$

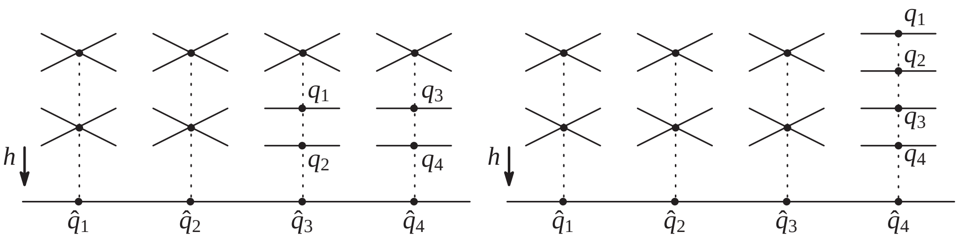

Fig. 7. $m = 4$, $s = 4$, $\hat{s} = 4$

But in this case, every ramification index of f at q_j $(j = 1, 2, 3, 4)$ is 2 by (v) of Lemma 3.2. Hence the genus of X is 1, a contradiction. Hence this case does not occur.

(iv-2) If $m = 4$, then by Lemma 3.2, the case $\hat{s} \geq s = 4$ occurs only if $\hat{s} = s = 4$ and the picture of ramifications is one of 2 pictures in Fig. 7.

In the left picture in Fig. 7, every ramification index of f at q_j $(j = 1, 2, 3, 4)$ must be 2 by (v) of Lemma 3.2. Hence the genus of X is 1, a contradiction. Hence this case does not occur.

The right picture in Fig. 7 may occur. In this case, by (v) of Lemma 3.2,

$$e_1 = e_2 = e_3 = e_4 \ (= \hat{e}_4).$$

(Since $g \geq 2$, e_1 must satisfy $e_1 \geq 3$.)

(iv-3) Finally we assume $m \geq 5$ and $\hat{s} \geq s = 4$. We show that this case does not occur.

If $\#B_h \leq 3$, then there is a point $r \in B_{\hat{f}} - B_h$. By (iv) of Lemma 3.2, $h^{-1}(r) \subset B_f$. Hence

$$5 \leq m = \#h^{-1}(r) \leq s = 4,$$

a contradiction. Hence $\#B_h \geq 4$.

Moreover by a similar reason, we must have $B_{\hat{f}} = B_h$. Hence $\hat{s} = \#B_h$ (≥ 4). Put

$$B_h = B_{\hat{f}} = \{r_1, r_2, \ldots, r_{\hat{s}}\}$$

and

$$h^{-1}(r_j) = \left\{q_j^1, q_j^2, \dots, q_j^{t_j}\right\} \quad (j = 1, 2, \dots, \hat{s}).$$

Let m_j^k be the ramification index of h at q_j^k. Then

$$\begin{aligned} m_1^1 + m_1^2 + \cdots + m_1^{t_1} &= m, \\ m_2^1 + m_2^2 + \cdots + m_2^{t_2} &= m, \\ &\cdots \\ m_{\hat{s}}^1 + m_{\hat{s}}^2 + \cdots + m_{\hat{s}}^{t_{\hat{s}}} &= m. \end{aligned}$$

Adding them, we have

$$\sum_{j=1}^{\hat{s}} \sum_{k=1}^{t_j} m_j^k = \hat{s}m.$$

On the other hand,

$$\sum_{j=1}^{\hat{s}} \sum_{k=1}^{t_j} (m_j^k - 1) = 2m - 2 \tag{3.5}$$

by the Riemann-Hurwitz formula with respect to h. Hence

$$\hat{s}m - (t_1 + t_2 + \cdots + t_{\hat{s}}) = 2m - 2.$$

Hence

$$t_1 + t_2 + \cdots + t_{\hat{s}} = (\hat{s} - 2)m + 2. \tag{3.6}$$

Assume that

$$1 = m_j^1 = \cdots = m_j^{a_j} < m_j^{a_{j+1}} \leq \cdots \leq m_j^{t_j}.$$

($a_j = 0$ if $m_j^k \geq 2$ for all k.) Now (3.5) can be rewritten as

$$\sum_{j=1}^{\hat{s}} \sum_{m_j^k \geq 2} (m_j^k - 1) = 2m - 2.$$

Every term in the left hand side satisfies

$$m_j^k - 1 \geq 1.$$

Hence

$$(t_1 - a_1) + (t_2 - a_2) + \cdots + (t_{\hat{s}} - a_{\hat{s}}) \leq 2m - 2.$$

By (3.6),

$$(\hat{s} - 2)m + 2 - (a_1 + a_2 + \cdots + a_{\hat{s}}) \leq 2m - 2.$$

Hence

$$(a_1 + a_2 + \cdots + a_{\hat{s}}) \geq (\hat{s} - 4)m + 4.$$

By (iv) of Lemma 3.2,

$$4 = s \geq a_1 + a_2 + \cdots + a_{\hat{s}}.$$

Hence

$$4 = s \geq a_1 + a_2 + \cdots + a_{\hat{s}} \geq (\hat{s} - 4)m + 4.$$

Hence

$$\hat{s} = 4, \quad a_1 + a_2 + a_3 + a_4 = 4, \quad m_j^{a_j+1} = \cdots = m_j^{t_j} = 2.$$

Moreover, if m is odd, then

$$a_1 = a_2 = a_3 = a_4 = 1.$$

If m is even, then either

$$a_1 = a_2 = 0, \quad a_3 = a_4 = 2$$

or

$$a_1 = a_2 = a_3 = 0, \quad a_4 = 4.$$

Since $m \geq 5$, by (v) of Lemma 3.2, we have

$$e_1 = e_2 = e_3 = e_4 = 2.$$

Hence the genus of X is 1, a contradiction. Hence this case does not occur.

We conclude that if $\hat{s} \geq s = 4$, then either
(1) $m = 2$, $\hat{s} = s = 4$ and $e_1 = e_2 \leq e_3 = e_4$ $((1/e_1) + (1/e_3) < 1)$, or
(2) $m = 4$, $\hat{s} = s = 4$ and $e_1 = e_2 = e_3 = e_4$ (≥ 3) □

There exist examples which satisfy the conditions stated at the end of the proof of Proposition 3.3:

EXAMPLE 3.4. (1) Let λ be a complex number with $\lambda \neq 0, 1$ and X_λ be the Riemann surface of the algebraic function

$$X_\lambda : y^4 - (x-1)^2(x-\lambda)^2 x = 0.$$

The mapping

$$f_\lambda : (x, y) \in X_\lambda \longmapsto x \in \mathbb{P}^1$$

is a cyclic covering of degree 4, branching at $x = 1, \lambda, \infty$ and 0 with the ramification indices $2, 2, 4$ and 4, respectively. Hence, in this case,

$$s = 4, \quad e_1 = e_2 = 2, \quad e_3 = e_4 = 4 \quad \text{and} \quad g = 2.$$

The mappings

$$\alpha : (x, y) \in X_\lambda \longmapsto \left(x, \sqrt{-1}y\right) \in X_\lambda,$$
$$\beta : (x, y) \in X_\lambda \longmapsto \left(\lambda/x, \lambda^{3/4}(x-1)(x-\lambda)/xy\right) \in X_\lambda$$

are automorphisms of X_λ. α generates $\mathrm{Aut}(f_\lambda)$.

The subgroup D_λ of $\mathrm{Aut}(X_\lambda)$ generated by α and β is isomorphic to the dihedral group of order 8.

D_λ coincides with $\mathrm{Aut}(X_\lambda)$ for general λ. In fact, if we set

$$u = \frac{y^2}{(x-1)(x-\lambda)},$$

then

$$u^2 = x \quad \text{on} \quad X_\lambda.$$

Hence X_λ is expressed by the equation

$$X_\lambda : y^2 = (u^2 - 1)(u^2 - \lambda)u.$$

This is an equation of hyperelliptic Riemann surfaces and the mapping

$$(u, y) \in X_\lambda \longmapsto u \in \mathbb{P}^1$$

is a double covering. The linear pencil of degree 2 on a hyperelliptic Riemann surface is unique. Using this fact, we can determine the group $\mathrm{Aut}(X_\lambda)$ of automorphisms of X_λ. The result can be stated as follows:

(1-i) If $\lambda \neq 0, 1, -1, 9, 1/9$, then $\mathrm{Aut}(X_\lambda) = D_\lambda$, which is isomorphic to the dihedral group of degree 8. Every element of D_λ gives a holomorphic equivalence of f_λ to itself. Hence, in this case,

$$m = [\mathrm{Aut}(X_\lambda) : \mathrm{Aut}(f_\lambda)] = 2, \quad \hat{s} = s = 4.$$

(1-ii) If $\lambda = -1$, then the order of $\mathrm{Aut}(X_{-1})$ is 48 and there is the following exact sequence:

$$1 \longrightarrow (\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathrm{Aut}(X_{-1}) \longrightarrow S_4 \longrightarrow 1.$$

(1-iii) If $\lambda = 9$ or $1/9$, then the order of $\mathrm{Aut}(X_\lambda)$ is 24 and there is the following exact sequence:

$$1 \longrightarrow (\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathrm{Aut}(X_\lambda) \longrightarrow D(12) \longrightarrow 1,$$

where $D(12)$ is the dihedral group of order 12.

(2) Let λ is a complex number with $\lambda \neq 0, 1$ and X_λ be the Riemann surface of the algebraic function

$$X_\lambda : y^4 - (x-1)(x-\lambda)x^3 = 0.$$

The mapping

$$f_\lambda : (x, y) \in X_\lambda \longmapsto x \in \mathbb{P}^1$$

is a cyclic covering of degree 4, branching at $x = 1, \lambda, \infty$ and 0 with the ramification index 4 equally. Hence, in this case,

$$s = 4, \quad e_1 = e_2 = e_3 = e_4 = 4 \quad \text{and} \quad g = 3.$$

The mappings

$$\begin{aligned}
\alpha &: (x, y) \in X_\lambda \longmapsto \left(x, \sqrt{-1}y\right) \in X_\lambda, \\
\beta &: (x, y) \in X_\lambda \longmapsto \left(\lambda/x, \lambda y/x^2\right) \in X_\lambda, \\
\gamma &: (x, y) \in X_\lambda \longmapsto \left((\lambda x - \lambda)/(x - \lambda), \lambda\sqrt{\lambda - 1}x(x-1)/(xy - \lambda y)\right) \in X_\lambda
\end{aligned}$$

are automorphisms of X_λ. α generates $\mathrm{Aut}(f_\lambda)$.

The subgroup E_λ of $\mathrm{Aut}(X_\lambda)$ generated by α, β and γ is isomorphic to

$$D(8) \times (\mathbb{Z}/2\mathbb{Z}),$$

where $D(8)$ is the dihedral group of order 8.

If we put

$$\hat{x} = y/x \quad \text{and} \quad \hat{y} = 2x - 1 - \lambda - (y/x)^4,$$

then X_λ can be expressed by the equation

$$\hat{y}^2 = \hat{x}^8 + 2(\lambda + 1)\hat{x}^4 + (\lambda - 1)^2,$$

which is an equation of hyperelliptic Riemann surfaces. From this, we can determine the group $\mathrm{Aut}(X_\lambda)$ of automorphisms of X_λ. The result can be stated as follows:

$$
\begin{tikzcd}
\mathcal{U} \arrow[d, "\mu"'] \arrow[dr, bend left, "\pi"] \arrow[dd, bend right=60, "\hat{\pi}"'] \arrow[dr, phantom, "\circlearrowright" near start] & \\
X \arrow[r, "f"] \arrow[d, "\hat{f}"'] \arrow[dr, phantom, "\circlearrowright" near start] & X/\mathrm{Aut}(f) \arrow[dl, "h"] \\
X/\mathrm{Aut}(X) &
\end{tikzcd}
$$

Fig. 8. Commutative diagram (i)

(2-i) If $\lambda \neq 0, 1, -1, 3/4, 4/3$, then $\mathrm{Aut}(X_\lambda) = E_\lambda$, which is isomorphic to $D(8) \times (\mathbb{Z}/2\mathbb{Z})$. Every element of E_λ gives a holomorphic equivalence of f_λ to itself. Hence, in this case,

$$m = [\mathrm{Aut}(X_\lambda) : \ \mathrm{Aut}(f_\lambda)] = 4, \quad \hat{s} = s = 4.$$

(2-ii) If $\lambda = -1$, then $\mathrm{Aut}(X_{-1})$ is isomorphic to $D(16) \times (\mathbb{Z}/2\mathbb{Z})$. The order is 32.
(2-iii) If $\lambda = 3/4$ or $4/3$, then the order of $\mathrm{Aut}(X_\lambda)$ is 48 and there is the following exact sequence:

$$1 \longrightarrow (\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathrm{Aut}(X_\lambda) \longrightarrow S_4 \longrightarrow 1.$$

Now let

$$\mu : \mathcal{U} \longrightarrow X$$

be the universal covering map of X. Put

$$\pi = f \circ \mu \quad \text{and} \quad \hat{\pi} = \hat{f} \circ \mu.$$

Then μ, π and $\hat{\pi}$ are infinite Galois branched coverings. We have the commutative diagram in Fig. 8.

Put

$$\Lambda = \mathrm{Aut}(\mu), \quad \Gamma = \mathrm{Aut}(\pi) \quad \text{and} \quad \hat{\Gamma} = \mathrm{Aut}(\hat{\pi}).$$

Then they are Fuchsian groups of the first kind and $\Lambda \subset \Gamma \subset \hat{\Gamma}$. Λ is a normal subgroup of finite index of both Γ and $\hat{\Gamma}$. Note that Λ has no elliptic element and $\hat{\Gamma}$ is the normalizer of Λ in $PSL(2, \mathbb{R})$ (see Jones and Singerman [7]).

The Galois correspondence of the commutative diagram in Fig. 8 asserts that

$$\mathrm{Aut}(f) \simeq \Gamma/\Lambda \quad \text{and} \quad \mathrm{Aut}(X) \simeq \hat{\Gamma}/\Lambda.$$

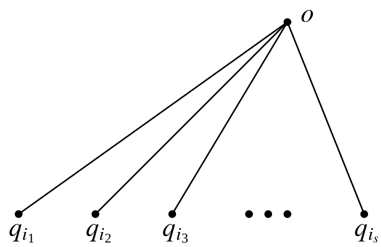

Fig. 9. A point o and geodesics

Moreover we have

$$X \simeq \mathcal{U}/\Lambda,$$
$$\mathbb{P}^1 \simeq X/\operatorname{Aut}(f) \simeq \mathcal{U}/\Gamma,$$
$$X/\operatorname{Aut}(X) \simeq \mathcal{U}/\hat{\Gamma}\ (\simeq \mathbb{P}^1).$$

Let F and $\hat{F}$ be the sets of fixed points in $\mathcal{U}$ of Γ and $\hat{\Gamma}$, respectively. Then $F \subset \hat{F}$, $F = \pi^{-1}(B)$ and $\hat{F} = \hat{\pi}^{-1}(\hat{B})$, where $\hat{B} = \{\hat{q}_1, \hat{q}_2, \dots, \hat{q}_{\hat{s}}\}$, the branch locus of $\hat{f}$.

Lemma 3.5. (1) *There exists a finite subset V (resp. $\hat{V}$) in F (resp. $\hat{F}$) such that V (resp. $\hat{V}$) forms the set of vertices of a general Klein tile T (resp. $\hat{T}$), a $2(s-1)$-gonal (resp. $2(\hat{s}-1)$-gonal) polygon and such that T (resp. $\hat{T}$) is a fundamental domain of Γ (resp. $\hat{\Gamma}$).*
(2) *A fundamental domain of Λ can be obtained as a union of finite number of consecutive T's (resp. $\hat{T}$'s).*

Proof. (2) is an easy consequence of (1), for Λ is a normal subgroup of finite index of both Γ and $\hat{\Gamma}$.

We prove the assertion (1) with respect to Γ. The assertion (1) with respect to $\hat{\Gamma}$ can be proved in a similar way.

First, note that $PSL(2,\mathbb{R})$ is the group of all orientation preserving isometries of $\mathcal{U}$ with the standard Riemannian metric. Hence Γ is a group of orientation preserving isometries. We introduce a metric on $\mathbb{P}^1 \simeq \mathcal{U}/\Gamma$ from that of $\mathcal{U}$ through π. Then $\mathbb{P}^1 \simeq \mathcal{U}/\Gamma$ is a 2-dimensional Riemannian manifold with the thorns $q_1, q_2, \dots, q_s$. Take a point $o \in \mathbb{P}^1 - B$, $B = \{q_1, q_2, \dots, q_s\}$, and a positive number δ such that

(1) B is contained in a non-Euclidean ball $\Delta(o,\delta)$ with the center o and the radius δ, and
(2) every thorn can be jointed in $\Delta(o,\delta)$ to o by a unique, mutually distinct, geodesic (see Fig. 9).

In Fig. 9, $i_1 i_2 \cdots i_s$ is a permutation of $12 \cdots s$. (Note that even if (2) is not satisfied for o, (2) will be satisfied for o' very near from o. In fact, we can find o' from where we can watch every thorn.)

We pull back the graph in Fig. 9 to $\mathcal{U}$ over the map π. Then we get a tiling of $\mathcal{U}$ by a tile S of $2s$-polygon with vertices

$$O_1, P_1, O_2, P_2, \ldots, O_s, P_s$$

ordered counterclockwisely such that

(1) $\pi(O_\nu) = o$ $(\nu = 1, 2, \ldots, s)$, and

(2) $\pi(P_\nu) = q_{i_\nu}$ $(\nu = 1, 2, \ldots, s)$.

We draw new geodesics

$$P_1P_2,\ P_2P_3, \ldots, P_sP_1$$

at every tile S so that we get new s-polygons

$$T' = P_1P_2\cdots P_s$$

with the vertices ordered counterclockwisely.

Now we throw away (i.e., forget) old sides of S. Then we get a new tiling of $\mathcal{U}$ with a tile T which consists of the union of T' and T'' which is also an s-polygon and adjoins T' with a side in common, and the vertices of which are ordered clockwisely.

The tile T is a general Klein tile which is a $2(s-1)$-polygon. □

REMARK 3.6. The tiling of $\mathcal{U}$ by T is a kind of dual to the tiling by S.

Let D and $\hat{D}$ be fundamental domains of Λ in (2) of Lemma 3.5, which are unions of finite numbers of consecutive T's and $\hat{T}$'s, respectively. Then D and $\hat{D}$ can be taken so that they are almost same. This means that D can be obtained by cutting off from $\hat{D}$ some small polygons, some sides L of which are on the boundary of $\hat{D}$, and pasting them to other parts of the boundary of $\hat{D}$ which are equivalent to L under $\hat{\Gamma}$.

We express this that D is equal to $\hat{D}$ with its boundary modified.

Lemma 3.7. *Let D and $\hat{D}$ be fundamental domains of Λ which can be obtained as unions of finite numbers of consecutive T's and $\hat{T}$'s, respectively, as in* (2) *of* Lemma 3.5. *Then D can be taken to be equal to $\hat{D}$ with its boundary modified*.

Proof. As in the proof of Lemma 3.5, we obtain a tiling Σ of X by pulling back the graph in Fig. 9 over f. We also obtain a tiling $\hat{\Sigma}$ of X by pulling-back over $\hat{f}$ a similar graph in $X/\operatorname{Aut}(X)$ to that in Fig. 9. We then consider the dual tiling Σ^* and $\hat{\Sigma}^*$ of Σ and $\hat{\Sigma}$, respectively (see Remark 3.6).

Note that

(1) the union of all tiles in Σ^* (resp. $\hat{\Sigma}^*$) is X itself, and

(2) the vertices of the tiles of Σ^* are contained in those of $\hat{\Sigma}^*$.

Now we pull back over μ both tilings of Σ^* and $\hat{\Sigma}^*$ of X to $\mathcal{U}$ and obtain two tilings of $\mathcal{U}$ by general Klein tiles T and $\hat{T}$, respectively. Then by (1) and (2) above, we conclude that D can be taken to be equal to $\hat{D}$ with its boundary modified. □

The Fuchsian group Γ is generated by the rotations of $\mathcal{U}$ with the angle $2\pi/e$ with the center the vertices P of the general Klein tile T in Lemma 3.5. (e is the ramification index of π at P.)

Now, conversely we start from a general Klein tile T in $\mathcal{U}$. If we are given a general Klein tile

$$T = V_1 V_2 \cdots V_{2s-2}$$

of $2(s-1)$-polygon in $\mathcal{U}$, then we get a Fuchsian group Γ of the first kind which is generated by rotations of $\mathcal{U}$ at the centers the vertices of T and a fundamental domain of which is T. We also get a normal subgroup Λ of Γ such that

(1) $\Gamma/\Lambda \simeq G$, and

(2) Λ has no elliptic element.

This is because Γ has the following presentation as an abstract group:

$$\Gamma = \left\langle \varphi_1, \varphi_2, \ldots, \varphi_s \;\middle|\; \varphi_1^{e_1} = \varphi_2^{e_2} = \cdots = \varphi_s^{e_s} = \varphi_1\varphi_2\cdots\varphi_s = 1 \right\rangle.$$

$X = \mathcal{U}/\Lambda$ is a compact Riemann surface and

$$f\colon X = \mathcal{U}/\Lambda \longrightarrow \mathcal{U}/\Gamma \simeq \mathbb{P}^1$$

is a finite Galois branched covering with $\mathrm{Aut}(f)$ isomorphic to G. Let $\hat{\Gamma}$ be the normalizer of Λ in $PSL(2,\mathbb{R})$. Then $\hat{\Gamma}$ is a Fuchsian group of the first kind and $\hat{\Gamma}/\Lambda$ is isomorphic to $\mathrm{Aut}(X)$. Moreover we get a commutative diagram as in Fig. 8.

Let

$$\hat{T} = \hat{V}_1 \hat{V}_2 \cdots \hat{V}_{2\hat{s}-2}$$

be a general Klein tile of $2(\hat{s}-1)$-polygon in Lemma 3.5.

Let (x_j, y_j) $(j = 1, 2, \ldots, 2s-2)$ (resp. $(\hat{x}_k, \hat{y}_k)$ $(k = 1, 2, \ldots, 2\hat{s}-2)$) be the coordinate of V_j (resp. $\hat{V}_k$) in $\mathcal{U}$. As in §1, consider the field

$$K(T) = \mathbb{Q}(x_1, y_1, x_2, y_2, \ldots, x_{2s-2}, y_{2s-2})$$
$$(\text{resp. } K(\hat{T}) = \mathbb{Q}(\hat{x}_1, \hat{y}_1, \hat{x}_2, \hat{y}_2, \ldots, \hat{x}_{2\hat{s}-2}, \hat{y}_{2\hat{s}-2})).$$

By Proposition 2.1, we may choose the Klein tile T such that $\mathrm{Tr.deg}_{\mathbb{Q}} K(T) = 2s-3$. By Proposition 2.1 again, we have

$$\mathrm{Tr.deg}_{\mathbb{Q}} K(\hat{T}) \leq 2\hat{s} - 3.$$

By Lemma 3.5, a fundamental domain $\hat{D}$ of Λ is a union of finite consecutive tiles of $\hat{T}$'s.

By Lemma 3.7, a fundamental domain D of Λ, which is equal to $\hat{D}$ with its boundary modified, is a union of finite consecutive tiles of T's.

Now let K (resp. $\hat{K}$) be the field over $\mathbb{Q}$ attaching coordinates of all vertices in $D \cup \hat{D}$ of the tiles of the tiling by T (resp. $\hat{T}$). Every tile of the tiling by T is congruent to the adjoining tile. Hence the coordinates of every vertex of the tiles of the tiling by T in $\mathcal{U}$ depend algebraically on the coordinates of those of T. Hence K is a finite extension of $K(T)$. In particular

$$\mathrm{Tr.\,deg}_{\mathbb{Q}}\, K = \mathrm{Tr.\,deg}_{\mathbb{Q}}\, K(T).$$

In a similar way,

$$\mathrm{Tr.\,deg}_{\mathbb{Q}}\, \hat{K} = \mathrm{Tr.\,deg}_{\mathbb{Q}}\, K\left(\hat{T}\right).$$

On the other hand, as noted above,

$$F = \pi^{-1}(\{q_1, q_2, \ldots, q_s\}) \subset \hat{F} = \hat{\pi}^{-1}(\{\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{\hat{s}}\}).$$

Hence $K \subset \hat{K}$. Hence

$$\begin{aligned} 2s - 3 &= \mathrm{Tr.\,deg}_{\mathbb{Q}}\, K(T) = \mathrm{Tr.\,deg}_{\mathbb{Q}}\, K \\ &\leq \mathrm{Tr.\,deg}_{\mathbb{Q}}\, \hat{K} = \mathrm{Tr.\,deg}_{\mathbb{Q}}\, K\left(\hat{T}\right) \leq 2\hat{s} - 3. \end{aligned}$$

Hence

$$s \leq \hat{s}.$$

By Proposition 3.3, if either

(1) $s \geq 5$, or

(2) $s = 4$ and $\{e_1, e_2, e_3, e_4\}$ does not satisfy $e_1 = e_2 \leq e_3 = e_4$,

then

$$\mathrm{Aut}(X) = \mathrm{Aut}(f) \simeq G.$$

Thus Greenberg's theorem is proved. □

4. Moduli spaces of Galois coverings

In this section, we discuss moduli spaces of Galois coverings. We first prepare some terminologies (c.f., Namba [14]).

Let Y be a connected complex manifold. A *finite branched covering* $f : X \to Y$ of Y is by definition a finite proper holomorphic map of an irreducible normal complex

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowright"] & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

Fig. 10. Commutative diagram (ii)

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "\varphi"'] & Y' \end{tikzcd}$$

Fig. 11. Commutative diagram (iii)

space X onto Y. As in the case of Riemann surfaces,

$$R_f = \{p \in X \mid f \text{ is not biholomorphic around } p\}$$

and $B_f = f(R_f)$ are called the *ramification locus* and the *branch locus of* f, respectively. They are hypersurfaces of X and Y, respectively. Two such coverings $f\colon X \to Y$ and $f'\colon X' \to Y$ are said to be *isomorphic* (denoted by $f \simeq f'$) if there is a biholomorphic map $\psi\colon X \to X'$ such that the diagram in Fig. 10 is commutative.

Two finite branched coverings $f\colon X \to Y$ and $f'\colon X' \to Y'$ are said to be *holomorphically equivalent* (resp. *topologically equivalent*) if there are biholomorphic maps (resp. orientation preserving homeomorphisms) $\psi\colon X \to X'$ and $\varphi\colon Y \to Y'$ such that the diagram in Fig. 11 is commutative.

A finite branched covering $f\colon X \to Y$ is called a *Galois covering* if the automorphism group

$$\mathrm{Aut}(f) = \{\psi \in \mathrm{Aut}(X) \mid f \circ \psi = f\}$$

of f acts transitively on each fiber of f. (Moreover if $\mathrm{Aut}(f)$ is abelian or cyclic, then f is said to be *abelian* or *cyclic*, respectively.) In this case Y is canonically biholomorphic to the quotient space $X/\mathrm{Aut}(f)$.

For a non-trivial finite group G, a finite branched Galois covering $f\colon X \to Y$ of Y is called a *G-covering* if $\mathrm{Aut}(f)$ is isomorphic to G. A finite (not necessarily Galois) branched covering $f\colon X \to Y$ is called a *family of G-coverings of* $\mathbb{P}^1$ if

(1) $\rho\colon Y \to N$ is a holomorphic $\mathbb{P}^1$-bundle over a connected complex manifold N,

(2) $\rho^{-1}(t)$ is not contained in B_f for every $t \in N$,

(3) there is an open covering $\{U_\alpha\}_\alpha$ of N such that the restriction

$$f_\alpha = f\colon X_\alpha = f^{-1}(Y_\alpha) \longrightarrow Y_\alpha = \rho^{-1}(U_\alpha)$$

is a G-covering for every α, and

(4) the restriction

$$f_t = f : X_t = f^{-1}(Y_t) \longrightarrow Y_t = \rho^{-1}(t) \ \left(\simeq \mathbb{P}^1\right)$$

is a G-covering of $\mathbb{P}^1$ for every point $t \in N$.

In this case, f and X are written as $f = \{f_t\}_{t\in N}$ and $X = \{X_t\}_{t\in N}$, respectively. A family $\{f_t\}_{t\in N}$ of G-coverings of $\mathbb{P}^1$ is said to be *non-degenerate* if the number s of the branch points of f_t is constant for $t \in N$.

In the rest of this paper, we assume $s \geq 3$.

Theorem 4.1. *Members of a non-degenerate family of G-coverings of $\mathbb{P}^1$ are mutually topologically equivalent. Conversely, any two topologically equivalent G-coverings of $\mathbb{P}^1$ belong (up to isomorphisms) to the same non-degenerate family of G-coverings of $\mathbb{P}^1$.*

In order to prove the first half of the theorem, we need some preparations. Let $\rho : Y \to N$ be a holomorphic $\mathbb{P}^1$-bundle and $f = \{f_t\}_{t\in N} : X = \{X_t\}_{t\in N} \to Y$ be a non-degenerate family of G-coverings of $\mathbb{P}^1$. It suffices to show the first half of Theorem 4.1 locally. That is, it suffices to show that, for an arbitrary point $o \in N$, there exists a connected open neighborhood W of o in N such that f_t is topologically equivalent to f_o for all $t \in W$.

Let $\{q_1(o), q_2(o), \ldots, q_s(o)\}$ be the branch locus of f_o. Let U_j $(j = 1, 2, \ldots, s)$ be mutually disjoint small balls with the center $q_j(o)$ with respect to a Riemannian metric on $\mathbb{P}^1$. Let W be a small ball in N with the center o with respect to a Riemannian metric on N. For a point $t \in W$, let $\{q_1(t), q_2(t), \ldots, q_s(t)\}$ be the branch locus of f_t. By the assumption, s is independent of $t \in W$. Each $q_j(t)$ $(j = 1, 2, \ldots, s)$ depends holomorphically on $t \in W$. Taking W small enough, we may assume that each $q_j(t)$ $(j = 1, 2, \ldots, s)$ belongs to U_j. Note that

$$Z = \bigcup_{j=1}^{s} \{q_j(t) \mid t \in W\}$$

is a non-singular hypersurface of $\rho^{-1}(W)$.

Now we recall the following known lemma whose proof can be found in e.g., Matsuno [10]:

Lemma 4.2. *There exists an orientation preserving homeomorphism $\hat{\varphi} : W \times \mathbb{P}^1 \to \rho^{-1}(W)$ such that*

(1) $\rho(\hat{\varphi}(t, q)) = t$ *for* $(t, q) \in W \times \mathbb{P}^1$,

(2) $\varphi_t = \hat{\varphi} : t \times \mathbb{P}^1 \to \rho^{-1}(t)$ *maps* $q_j(o)$ *to* $q_j(t)$ *for* $t \in W$ *and* $j = 1, 2, \ldots, s$, *and*

$$\begin{tikzcd}
W \times \left(X_o - f_o^{-1}(B)\right) \arrow[r, "\hat{\psi}"] \arrow[d, "id \times f_o"'] \arrow[dr, phantom, "\circlearrowleft"] & X'' \arrow[d, "f''"] \\
W \times (\mathbb{P}^1 - B) \arrow[r, "\hat{\varphi}"'] & \rho^{-1}(W) - Z
\end{tikzcd}$$

Fig. 12. Commutative diagram (iv)

(3) *φ_t maps U_j to U_j for $t \in W$ and $j = 1, 2, \dots, s$, and φ_t is equal to the identity map on $\mathbb{P}^1 - U_1 \cup U_2 \cup \cdots \cup U_s$.*

Lemma 4.2 implies that $\rho^{-1}(W) - Z$ is homeomorphic to $W \times (\mathbb{P}^1 - B)$, where $B = \{q_1(o), q_2(o), \dots, q_s(o)\}$. Since W is a ball, $\pi_1(\rho^{-1}(W) - Z, (o, q_o))$ is isomorphic through $\hat{\varphi}$ to $\pi_1(\mathbb{P}^1 - B, q_o)$:

$$\hat{\varphi}_* : \pi_1\left(\mathbb{P}^1 - B, q_o\right) \simeq \pi_1\left(\rho^{-1}(W) - Z, (o, q_o)\right), \tag{4.1}$$

where q_o is a point in $\mathbb{P}^1 - U_1 \cup U_2 \cup \cdots \cup U_s$.

Let ξ be a homomorphism of $\pi_1(\mathbb{P}^1 - B, q_o)$ onto G whose kernel $\mathrm{Ker}(\xi)$ corresponds to $f_o : X_o \to \mathbb{P}^1$ (see (3.2)). Let

$$f'' : X'' \longrightarrow \rho^{-1}(W) - Z$$

be a finite unbranched G-covering corresponding to $\mathrm{Ker}(\xi)$ under the isomorphism φ_* in (4.1). We extend f'' to a finite branched G-covering $f' : X' \to \rho^{-1}(W)$ by the following theorem of Grauert and Remmert:

Theorem 4.3 (Grauert and Remmert [4]). *Let M and S be a connected complex manifold and its hypersurface, respectively. Let $f : X \to M - S$ be a finite unbranched covering. Then there exists a unique* (*up to isomorphisms*) *finite* (*branched*) *covering $\hat{f} : \hat{X} \to M$ which extends f.*

The extended X' in our case is non-singular, for Z is non-singular (see Namba [14]). Note that

$$f' = \{f'_t\}_{t \in W} : X' = \{X'_t\}_{t \in W} \longrightarrow \rho^{-1}(W) \tag{4.2}$$

is a non-degenerate G-covering.

By the isomorphism in (4.1), the homeomorphism $\hat{\varphi}$ induces the topological equivalence in Fig. 12. In Fig. 12, *id* is the identity map. Since the extension in Theorem 4.3 is topologically equivalent to the Fox completion (see Fox [3]), the above

$$\begin{tikzcd} W \times X_o \arrow[r, "\hat{\psi}"] \arrow[d, "id \times f_o"'] \arrow[dr, phantom, "\circlearrowright"] & X' \arrow[d, "f'"] \\ W \times \mathbb{P}^1 \arrow[r, "\hat{\varphi}"'] & \rho^{-1}(W) \end{tikzcd}$$

Fig. 13. Commutative diagram (v)

topological equivalence can be naturally extended to the topological equivalence in Fig. 13.

Now, by (4.1) again, $\pi_1(\rho^{-1}(W) - Z, (o, q_o))$ is isomorphic to $\pi_1(\mathbb{P}^1 - B, q_o)$. Hence the unbranched covering

$$f: f^{-1}\left(\rho^{-1}(W) - Z\right) \longrightarrow \rho^{-1}(W) - Z$$

and its extension by Theorem 4.3

$$f_{|W} = \{f_t\}_{t \in W} : X_{|W} = f^{-1}\left(\rho^{-1}(W)\right) \longrightarrow \rho^{-1}(W)$$

also corresponds to Ker(ξ). (In particular, they are G-coverings.)

Hence $f_{|W} = \{f_t\}_{t \in W}$ and $f' = \{f'_t\}_{t \in W}$ in (4.2) are isomorphic.

Thus the trivial family of G-coverings

$$W \times X_o \longrightarrow W \times \mathbb{P}^1$$

and

$$f_{|W} = \{f_t\}_{t \in W}$$

are topologically equivalent. In particular, f_t is topologically equivalent to f_o for every $t \in W$. This completes the proof of the first half of Theorem 4.1.

The second half of Theorem 4.1 follows from the assertion that the set of all (isomorphism classes of) G-coverings of $\mathbb{P}^1$, which are topologically equivalent to a given G-covering $f_o: X_o \to \mathbb{P}^1$, forms a non-degenerate family

$$\hat{f} = \{f_m\}_{m \in M} : \hat{X} = \{X_m\}_{m \in M} \longrightarrow M \times \mathbb{P}^1 \tag{4.3}$$

of G-coverings of $\mathbb{P}^1$. (The projection $M \times \mathbb{P}^1 \to M$ is the product $\mathbb{P}^1$-bundle over a connected complex manifold M.)

We call this family a *complete non-degenerate family with respect to* f_o. Such a family was constructed in Völklein [18]. Thus the proof of Theorem 4.1 has completed. (In Appendix, we give a sketch of a construction of the family in (4.3), which is apparently different from that in Völklein [18].) □

We need, however, some informations about the complete non-degenerate family with respect to f_o in (4.3).

First of all, note that two topological equivalent G-coverings of $\mathbb{P}^1$ have the same number s of branch points and the same set of ramification indices $\{e_1, e_2, \dots, e_s\}$. Hence s and $\{e_1, e_2, \dots, e_s\}$ are constant for G-coverings $f_m\colon X_m \to \mathbb{P}^1$, $m \in M$, of the family in (4.3).

Next, we need to observe how a neighborhood W of a given G-covering f_o in the family with respect to f_o in (4.3) can be constructed. In fact, it can be constructed in a similar way to the proof of the first part of Theorem 4.1 as follows: The s-times symmetric product $S^s\mathbb{P}^1$ of $\mathbb{P}^1$ can be naturally identified with the s-dimensional complex projective space $\mathbb{P}^s$: $S^s\mathbb{P}^1 = \mathbb{P}^s$. By the projection

$$\nu\colon \left(\mathbb{P}^1\right)^s \longrightarrow S^s\mathbb{P}^1 = \mathbb{P}^s,$$

the diagonal is mapped to the discriminant locus Δ, which is an irreducible hypersurface of degree $2s - 2$ in $\mathbb{P}^s$. Let

$$B^o = \left\{q_1^o, q_2^o, \dots, q_s^o\right\} \quad \left(\subset \mathbb{P}^1\right)$$

be the branch locus of f_o. Let U_j $(j = 1, 2, \dots, s)$ be small open balls with the center q_j^o with respect to a Riemannian metric on $\mathbb{P}^1$ which are disjoint each other. Put

$$W = \nu(U_1 \times U_2 \times \cdots \times U_s).$$

Then W is a simply connected open neighborhood of the divisor (on $\mathbb{P}^1$)

$$m_o = q_1^o + q_2^o + \cdots + q_s^o$$

in $S^s\mathbb{P}^1 = \mathbb{P}^s$ and is disjoint from Δ.

Put

$$Z = \Big\{(m, q) \in W \times \mathbb{P}^1 \;\Big|\; m = q_1 + q_2 + \cdots + q_s, \\ q_j = q \quad \text{for some} \quad j = 1, 2, \dots, s\Big\}.$$

Then Z is a non-singular hypersurface in $W \times \mathbb{P}^1$.

Lemma 4.2 can be applied to this case. Thus there exists an orientation preserving homeomorphism

$$\hat{\varphi}\colon W \times \mathbb{P}^1 \longrightarrow W \times \mathbb{P}^1 \tag{4.4}$$

such that

(1) $\rho(\hat{\varphi}(m, q)) = m$ for $(m, q) \in W \times \mathbb{P}^1$, where $\rho\colon W \times \mathbb{P}^1 \to W$ is the projection,
(2) $\varphi_m = \hat{\varphi}\colon m \times \mathbb{P}^1 \to \rho^{-1}(m)$ maps q_j^o to q_j for $m \in W$ and $j = 1, 2, \dots, s$, where $q_1 + q_2 + \cdots + q_s = m$, and

(3) φ_m for $m \in W$ maps U_j to U_j for $m \in W$ and $j = 1, 2, \dots, s$, and φ_m is equal to the identity map on $\mathbb{P}^1 - U_1 \cup U_2 \cup \cdots \cup U_s$.

In particular,

$$\hat{\varphi}\colon W \times \left(\mathbb{P}^1 - \left\{q_1^o, q_2^o, \dots, q_s^o\right\}\right) \longrightarrow \left(W \times \mathbb{P}^1\right) - Z$$

is an orientation preserving homeomorphism. Hence (as in the proof of the first part of Theorem 4.1) there exists a non-degenerate family

$$f_W = \{f_m\}_{m\in W}\colon X_W = \{X_m\}_{m\in W} \longrightarrow W \times \mathbb{P}^1 \tag{4.5}$$

of G-coverings (with X_W a connected complex manifold) and an orientation preserving homeomorphism

$$\hat{\psi}\colon W \times X_o \longrightarrow X_W \tag{4.6}$$

which, together with $\hat{\varphi}$, gives a topological equivalence between the trivial family

$$id \times f_o\colon W \times X_o \longrightarrow W \times \mathbb{P}^1$$

(id is the identity map) and f_W.

We thus obtained a local chart f_W in (4.5). Patching up these local charts, we get the complete non-degenerate family $\hat{f} = \{f_m\}_{m\in M}$ of G-coverings in (4.3). (In Appendix, we give a global construction of the family.)

The family $\hat{f} = \{f_m\}_{m\in M}$ in (4.3) consists of the set of all (isomorphism classes of) G-coverings which are topologically equivalent to a given $f_o\colon X_o \to \mathbb{P}^1$ (see (4.6)).

By the construction (see Appendix), there exists a finite unbranched covering

$$\eta\colon M \longrightarrow \mathbb{P}^s - \Delta. \tag{4.7}$$

Hence M is a quasi-projective manifold.

$\mathrm{Aut}(\mathbb{P}^1)$ acts on M. The action is defined by

$$\varphi \circ f_m \simeq f_{\varphi(m)}$$

for $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ and $m \in M$.

Note that f_m and $f_{\varphi(m)}$ are holomorphically equivalent. Conversely, if f_m and $f_{m'}$, for $m, m' \in M$, are holomorphically equivalent, then there exists $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ such that $m' = \varphi(m)$.

$\mathrm{Aut}(\mathbb{P}^1)$ also acts on $\mathbb{P}^s - \Delta$. The actions are equivariant with respect to η. Every point of $\mathbb{P}^s - \Delta$ is stable under the action of $\mathrm{Aut}(\mathbb{P}^1)$ (see Mumford [11]). Hence every point of M is also stable, for η is a finite unbranched covering. Hence the quotient space

$$M / \mathrm{Aut}(\mathbb{P}^1) \tag{4.8}$$

$$\begin{tikzcd}
W \times \mathcal{U} \arrow[r, "\tilde{\psi}"] \arrow[d, "id \times \mu_o"'] \arrow[dr, phantom, "\circlearrowright"] & \tilde{X}_W \arrow[d, "\mu"] \\
W \times X_o \arrow[r, "\hat{\psi}"] \arrow[d, "id \times f_o"'] \arrow[dr, phantom, "\circlearrowright"] & X_W \arrow[d, "f_W"] \\
W \times \mathbb{P}^1 \arrow[r, "\hat{\varphi}"'] & W \times \mathbb{P}^1
\end{tikzcd}$$

Fig. 14. Commutative diagram (vi)

is an irreducible normal quasi-projective variety of dimension $s-3$, which we call *the moduli space of G-coverings* (for a given G-covering $f_o \colon X_o \to \mathbb{P}^1$).

We now assume the inequality (2.1) for a given $f_o \colon X_o \to \mathbb{P}^1$ which branches at $e_1q_1(o) + e_2q_2(o) + \cdots + e_sq_s(o)$. We return back to discussions on the homeomorphism $\hat{\psi}$ in (4.6).

Let

$$\mu \colon \tilde{X}_W \longrightarrow X_W$$

be the universal covering of X_W. Then $\hat{\psi}$ induces an orientation preserving homeomorphism

$$\tilde{\psi} \colon W \times \mathcal{U} \longrightarrow \tilde{X}_W \quad (\mathcal{U}\text{: the upper half plane})$$

which, together with $\hat{\psi}$, makes the diagram in Fig. 14 commutative. ($\mu_o \colon \mathcal{U} \to X_o$ is the universal covering of X_o.)

Note that every fiber $\mu^{-1}(X_m)$, $m \in W$, is biholomorphic to $\mathcal{U}$, so can be identified with $\mathcal{U}$. $\mu_m = \mu \colon \mathcal{U} = \mu^{-1}(X_m) \to X_m$ is the universal covering of X_m. Note also that $\hat{\varphi}$, $\hat{\psi}$ and $\tilde{\psi}$ induce orientation preserving homeomorphisms

$$\begin{aligned}
&\varphi_m \colon o \times \mathbb{P}^1 \longrightarrow m \times \mathbb{P}^1,\\
&\psi_m \colon X_o \longrightarrow X_m, \quad \text{and}\\
&\tilde{\psi}_m \colon \mathcal{U} = \mu^{-1}(X_o) \longrightarrow \mathcal{U} = \mu^{-1}(X_m),
\end{aligned}$$

respectively.

Let $T(m)$, for $m \in W$, be the general Klein tile constructed in the proof of Lemma 3.5 with the respect to the G-covering $f_m \colon X_m \to \mathbb{P}^1$. Then the commutative diagram in Fig. 14 implies that

(1) $T(m)$ is an image of $T(o)$ by the orientation preserving homeomorphism $\tilde{\psi}_m$ which is in a small neighborhood of the identity map of the group $\mathrm{Homeo}(\mathcal{U}, \mathcal{U})$ of all homeomorphisms of $\mathcal{U}$ onto itself, and

(2) $T(m)$ is a small deformation of $T(o)$, that is, the coordinates of vertices of $T(m)$ are near from those of vertices of $T(o)$.

Note that the real dimension of the moduli space $M/\operatorname{Aut}(\mathbb{P}^1)$ is $2s-6$. On the other hand, Proposition 2.1 implies that the coordinates of vertices of general Klein tiles, modulo congruences of tiles, depend $(2s-3)-3=2s-6$ real parameters, for $\dim PSL(2,\mathbb{R})=3$.

Hence we conclude that every small deformation of $T(o)$ is $T(m)$ for some $m \in W$. Thus, by the proof of Greenberg's theorem in §3, we have

Proposition 4.4. *Let $\hat{f}=\{f_m\}_{m\in M}\colon \hat{X}=\{X_m\}_{m\in M} \to M\times\mathbb{P}^1$ be the complete non-degenerate family of G-coverings with respect to $f_o\colon X_o\to\mathbb{P}^1$ in* (4.3). *Suppose that either*
(1) $s\geq 5$, *or*
(2) $s=4$ *and* $\{e_1,e_2,e_3,e_4\}$ *does not satisfy* $e_1=e_2\leq e_3=e_4$.
Then there exists a dense set U in M such that $\operatorname{Aut}(f_m)=\operatorname{Aut}(X_m)$ *for all $m\in U$.*

In the next section, we show that U can be taken so that U is a Zariski open set in M.

5. Equivalence problem

In the section, let s be an integer with $s\geq 3$ and $e_1,e_2,\ldots,e_s$ integers with $e_j\geq 2$ $(j=1,2,\ldots,s)$ which satisfy the inequality (2.1). Let

$$f=\{f_u\}_{u\in N}\colon X=\{X_u\}_{u\in N}\longrightarrow Y \tag{5.1}$$

be a non-degenerate family of G-coverings of $\mathbb{P}^1$ with a $\mathbb{P}^1$-bundle $\rho\colon Y\to N$ such that each $f_u\colon X_u\to\mathbb{P}^1$ branches at the divisor

$$e_1q_1(u)+e_2q_2(u)+\cdots+e_sq_s(u). \tag{5.2}$$

Theorem 5.1. *Under the above notations, the set*

$$S=\{u\in N \mid \operatorname{Aut}(f_u)\neq\operatorname{Aut}(X_u)\}$$

is a closed complex subset of N.

In order to prove Theorem 5.1, we need some preparations. The disjoint union

$$A=\bigcup_{u\in N}\operatorname{Aut}(X_u)$$

forms a complex space such that the projection

$$\kappa\colon A\longrightarrow N$$

$$\begin{tikzcd} X_U \arrow[r, "\hat{h}"] \arrow[d, "\rho \circ f"'] \arrow[dr, phantom, "\circlearrowright"] & \hat{X}_g \arrow[d, "\hat{f}_g"] \\ U \arrow[r, "h"'] & T_g \end{tikzcd}$$

Fig. 15. Commutative diagram (vii)

is holomorphic (see Schuster [17] and Namba [13]). This is a relative Douady space (see Pourcin [15]).

In the present case, every fiber

$$\kappa^{-1}(u) = \mathrm{Aut}(X_u)$$

is a finite group of order $\leq 84(g-1)$, where g is the genus of X_u $(u \in N)$. We first prove

Lemma 5.2. *κ is a proper map.*

Proof. Let $\{u_j\}_{j=1,2,\dots}$ be a sequence of points in N which converges to a point $o \in N$. Let $\{\psi_j\}_{j=1,2,\dots}$ be a sequence of points in A such that $\kappa(\psi_j) = u_j$ $(j = 1, 2, \dots)$. It suffices to show that we can choose a subsequence from $\{\psi_j\}_{j=1,2,\dots}$ which converges to an element of $\mathrm{Aut}(X_o)$.

Let T_g and $\hat{X}_g = \{\hat{X}_t\}_{t \in T_g}$ be the Teichmüller space and the Teichmüller family of compact Riemann surfaces of genus g, where g is defined by (3.3).

Let $\hat{f}_g : \hat{X}_g \to T_g$ be the projection. By the completeness of the family $\{\hat{X}_t\}_{t \in T_g}$, there are a connected open neighborhood U of o in N and a holomorphic map

$$h : U \to T_g$$

such that the family $X_U = \{X_u\}_{u \in U}$ of compact Riemann surfaces is isomorphic to the family induced over h of the Teichmüller family $\{\hat{X}_t\}_{t \in T_g}$. Thus there is a holomorphic map

$$\hat{h} : X_U \longrightarrow \hat{X}_g$$

such that

(1) $\hat{h}$ makes the diagram in Fig. 15 commutative and

(2) the restriction $\hat{h}_u : X_u \to \hat{X}_{h(u)}$ of $\hat{h}$ on X_u is a biholomorphic map of X_u onto $\hat{X}_{h(u)}$.

Let Γ_g be the Teichmüller modular group. Γ_g acts properly discontinuously on both T_g and $\hat{X}_g$, and equivariantly with respect to $\hat{f}_g$. The quotient space

$$\mathbb{M}_g = T_g / \Gamma_g$$

is the moduli space of compact Riemann surfaces of genus g. Recall that, for every point $t \in T_g$, the isotropy subgroup

$$\Gamma_g(t) = \{\gamma \in \Gamma_g \mid \gamma(t) = t\}$$

is a finite subgroup of Γ_g which can be identified with $\mathrm{Aut}(\hat{X}_t)$ by the action of the group $\Gamma_g(t)$ on $\hat{X}_t : \Gamma_g(t) = \mathrm{Aut}(\hat{X}_t)$. (See, e.g., Imayoshi and Taniguchi [6].)

Now we may assume that every point u_j $(j = 1, 2, \dots)$ belongs to U. The sequence $\{h(u_j)\}_{j=1,2,\dots}$ of points in T_g converges to $h(o)$. Note that

$$\lambda_j = \hat{h}_{u_j} \circ \psi_j \circ \hat{h}_{u_j}^{-1}$$

is an element of $\mathrm{Aut}(\hat{X}_{h(u_j)}) = \Gamma_g(h(u_j))$. Hence

$$\lambda_j(h(u_j)) = h(u_j) \quad (j = 1, 2, \dots).$$

Since the action of Γ_g on T_g is properly discontinuous, (taking a subsequence if necessary) we may assume that

$$\lambda_1 = \lambda_2 = \cdots = \lambda \quad \text{(a constant element of } \Gamma_g).$$

Then $\lambda(h(o)) = h(o)$, so $\lambda \in \Gamma_g(h(o))$. Moreover

$$\psi_j = \hat{h}_{u_j}^{-1} \circ \lambda \circ \hat{h}_{u_j} \quad (j = 1, 2, \dots).$$

Hence the sequence $\{\psi_j\}_{j=1,2,\dots}$ converges to

$$\psi_o = \hat{h}_o^{-1} \circ \lambda \circ \hat{h}_o \in \mathrm{Aut}(X_o). \qquad \square$$

Next we must observe the local structure of the complex space $A = \bigcup_{u \in N} \mathrm{Aut}(X_u)$. Take a point $o \in N$ and an automorphism ψ_o in $\mathrm{Aut}(X_o)$. Then an open neighborhood W of ψ_o in A can be identified with the analytic subset

$$\{(u, v) \in U \times V \mid \alpha(u, v) = 0\}$$

of $U \times V$, where U is a connected open neighborhood of o in N, V is a connected open neighborhood of 0 in $\mathrm{H}^0(X_o, \mathcal{O}(\psi_o^* T X_o))$ and α is a holomorphic map

$$\alpha : U \times V \longrightarrow \mathrm{H}^1(X_o, \mathcal{O}(\psi_o^* T X_o))$$

such that

(1) $\alpha(o, 0) = 0$ and

(2) $(d\alpha)_{(o,0)} = (\tau, 0)$

(see Namba [13], §3.2). Here TX_o is the tangent bundle of X_o, $\mathrm{H}^k(X_o, \mathcal{O}(\psi_o^* TX_o))$ are the cohomology groups of the sheaf of sections of the vector bundle $\psi_o^* TX_o$ and

$$\tau = \psi_o^* \rho_o - \psi_{o*} \rho_o.$$

Here

$$\rho_o\colon T_o N \longrightarrow \mathrm{H}^1(X_o, \mathcal{O}(TX_o))$$

is the Kodaira-Spencer map for the family $\{X_u\}_{u \in N}$ and

$$\psi_o^*\colon \mathrm{H}^1(X_o, \mathcal{O}(TX_o)) \longrightarrow \mathrm{H}^1(X_o, \mathcal{O}(\psi_o^* TX_o)),$$
$$\psi_{o*}\colon \mathrm{H}^1(X_o, \mathcal{O}(TX_o)) \longrightarrow \mathrm{H}^1(X_o, \mathcal{O}(\psi_o^* TX_o))$$

are the linear map induced by ψ_o.

In the present case, $\psi_o^* TX_o$ is a line bundle such that

$$\deg \psi_o^* TX_o = \deg TX_o = 2 - 2g < 0.$$

Hence

$$\mathrm{H}^0(X_o, \mathcal{O}(\psi_o^* TX_o)) = 0.$$

This means that there are a closed complex subspace

$$R = \{u \in U \mid \alpha(u, 0) = 0\}$$

of U and a holomorphic section

$$\zeta\colon R \longrightarrow A$$

of $\kappa\colon A \to N$ on R such that the image $W = \zeta(R)$ is a connected open neighborhood of ψ_o in A. Hence $\kappa(W) = R$.

If ψ_o is in $\mathrm{Aut}(f_o)$, then $\kappa(W) = U$. In fact, take a point $q \in \mathbb{P}^1 - B_{f_o}$, where B_{f_o} is the branch locus of f_o. We may assume that $q \in \mathbb{P}^1 - B_{f_u}$ for all $u \in U$. Consider the finite subset

$$f_u^{-1}(q) = \{p_1(u), p_2(u), \ldots, p_d(u)\}$$

of X_u. Each $p_k(u)$ $(k = 1, 2, \ldots, d)$ depends holomorphically on $u \in U$. Let $\psi_k(u)$ $(k = 1, 2, \ldots, d)$ be the automorphism of X_u such that

$$\psi_k(u)\colon p_1(u) \longmapsto p_k(u).$$

Then

$$\zeta_k\colon u \in U \longmapsto \psi_k(u) \in A$$

gives a holomorphic section of κ such that $W_k = \zeta_k(U)$ is an open set of A.

Thus we conclude that

$$A' = \bigcup_{u \in N} \mathrm{Aut}(f_u)$$

is an open subset of A such that $\kappa(A') = N$.

Note that A' is also closed in A. In fact, if $\{\psi_j\}_{j=1,2,\dots}$ is a sequence of points in A' which converges to $\psi_o \in A$, then

$$f_{u_j} \circ \psi_j = f_{u_j} \quad (j = 1, 2, \dots),$$

where $u_j = \kappa(\psi_j)$. Hence $f_o \circ \psi_o = f_o$, where $o = \kappa(\psi_o)$. Hence $\psi_o \in \mathrm{Aut}(f_o)$.

Thus A' is open and closed in A. Hence $A'' = A - A'$ is also open and closed in A.

Thus, by Lemma 5.2,

$$S = \kappa(A'') = \{u \in N \mid \mathrm{Aut}(f_u) \neq \mathrm{Aut}(X_u)\}$$

is a closed complex subspace of N. This proves Theorem 5.1.

By Theorem 5.1 and Proposition 4.4, we have

Theorem 5.3. *Let $\hat{f} = \{f_m\}_{m \in M} : \hat{X} = \{X_m\}_{m \in M} \to M \times \mathbb{P}^1$ be the complete non-degenerate family of G-coverings with respect to $f_o : X_o \to \mathbb{P}^1$ in* (4.3). *Suppose that either*

(1) $s \geq 5$, *or*

(2) $s = 4$ *and* $\{e_1, e_2, e_3, e_4\}$ *does not satisfy* $e_1 = e_2 \leq e_3 = e_4$.

Then

$$S = \{m \in M \mid \mathrm{Aut}(f_m) \neq \mathrm{Aut}(X_m)\}$$

is a closed complex subspace of M such that $S \neq M$.

Now, *an equivalence problem* asks the following problem: For two G-coverings $f_1 : X_1 \to \mathbb{P}^1$ and $f_2 : X_2 \to \mathbb{P}^1$, is it true that X_1 and X_2 are biholomorphic if and only if f_1 and f_2 are holomorphically equivalent (under suitable conditions)?

The 'if' part of the problem is trivial. The difficult part is the 'only if' part. As for cyclic coverings or Kummer coverings, answers (under various conditions) are known (see Namba [12], Kato [8] and Sakurai-Suzuki [16]).

Theorem 5.4. *Let $f = \{f_u\}_{u \in N} : X = \{X_u\}_{u \in N} \to Y$ be a non-degenerate family of G-coverings of $\mathbb{P}^1$ in* (5.1) *with a $\mathbb{P}^1$-bundle $\rho : Y \to N$ and with the branch divisors* (5.2) *which satisfy the inequality* (2.1). *Assume that there is a point $o \in N$ such that* $\mathrm{Aut}(f_o) = \mathrm{Aut}(X_o)$. *Then, for any two points u and u' in N, X_u and $X_{u'}$ are biholomorphic if and only if f_u and $f_{u'}$ are holomorphically equivalent.*

In order to prove Theorem 5.4, we need some preparations. By the assumption and Theorem 5.1,

$$S = \{u \in N \mid \mathrm{Aut}(f_u) \neq \mathrm{Aut}(X_u)\}$$

is a closed complex subspace of N such that $S \neq N$. Hence $N - S$ is a Zariski open subset of N.

Lemma 5.5. *If u and v belong to $N - S$ and if X_u and X_v are biholomorphic, then f_u and f_v are holomorphically equivalent. More precisely, every biholomorphic map $\psi: X_u \to X_v$ induces an automorphism $\varphi: \mathbb{P}^1 \to \mathbb{P}^1$ such that ψ and φ give the holomorphic equivalence of f_u and f_v, that is, $f_v \circ \psi = \varphi \circ f_u$.*

Proof. The quotient spaces

$$X_u / \mathrm{Aut}(f_u) = X_u / \mathrm{Aut}(X_u) \quad \text{and} \quad X_v / \mathrm{Aut}(f_v) = X_v / \mathrm{Aut}(X_v)$$

can be identified with $\mathbb{P}^1$. Note that

$$\psi \circ \mathrm{Aut}(X_u) \circ \psi^{-1} = \mathrm{Aut}(X_v).$$

Hence ψ induces a biholomorphic map φ of the quotient spaces such that $f_v \circ \psi = \varphi \circ f_u$. $\square$

For $u \in N - S$ and $v \in S$, X_u and X_v cannot be biholomorphic, for

$$\deg \mathrm{Aut}(X_u) = d < \deg \mathrm{Aut}(X_v).$$

For $u, v \in S$, suppose that X_u and X_v are biholomorphic. We will show that f_u and f_v are holomorphically equivalent.

As in the proof of Lemma 5.2, there are a connected open neighborhood U (resp. V) of u (resp. v) and holomorphic maps

$$h: U \longrightarrow T_g \quad (\text{resp.} \quad h': V \longrightarrow T_g)$$

and

$$\hat{h}: X_U \longrightarrow \hat{X}_g \quad (\text{resp.} \quad \hat{h}': X_V \longrightarrow \hat{X}_g)$$

such that $\hat{f}_g \circ \hat{h} = h \circ \rho \circ f$ (resp. $\hat{f}_g \circ \hat{h}' = h' \circ \rho \circ f$) and the restriction $\hat{h}_{u_1}$ (resp. $\hat{h}'_{u_2}$) on X_{u_1} (resp. X_{u_2}) of $\hat{h}$ (resp. $\hat{h}'$) is a biholomorphic map of X_{u_1} (resp. X_{u_2}) onto $\hat{X}_{h(u_1)}$ (resp. $\hat{X}_{h'(u_2)}$).

By the assumption,

$$\pi_g(h(u)) = \pi_g(h'(v)),$$

where

$$\pi_g : T_g \longrightarrow T_g/\Gamma_g = \mathbb{M}_g$$

is the projection.

Hence there are sequences $\{u_j\}_{j=1,2,\ldots}$ of points in $U \cap (N - S)$ and $\{v_j\}_{j=1,2,\ldots}$ of points in $V \cap (N - S)$ such that

(1) $\{u_j\}_{j=1,2,\ldots}$ (resp. $\{v_j\}_{j=1,2,\ldots}$) converges to u (resp. v) and

(2) $\pi_g(h(u_j)) = \pi_g(h'(u_j))$ $(j = 1, 2, \ldots)$.

(2) implies that there is a sequence $\{\lambda_j\}_{j=1,2,\ldots}$ in the Teichmüller modular group Γ_g such that

$$\lambda_j(h(u_j)) = h'(v_j) \quad (j = 1, 2, \ldots).$$

Since Γ_g acts properly discontinuously on T_g, (taking a subsequence if necessarily) we may assume that

$$\lambda_1 = \lambda_2 = \cdots = \lambda \quad (\text{a constant element of } \Gamma_g).$$

Put

$$\psi_j = \hat{h}'^{-1}_{v_j} \circ \lambda \circ \hat{h}_{u_j} \quad (j = 1, 2, \ldots).$$

Then ψ_j is a biholomorphic map of X_{u_j} onto X_{v_j}. The sequence $\{\psi_j\}_{j=1,2,\ldots}$ converges to $\psi = \hat{h}'^{-1}_v \circ \lambda \circ \hat{h}_u$, a biholomorphic map of X_u onto X_v, in the relative Douady space

$$\bigcup_{(p,q)\in N\times N} \mathrm{Hol}(X_p, X_q)$$

of holomorphic maps.

By Lemma 5.5, each ψ_j $(j = 1, 2, \ldots)$ induces an automorphism φ_j of $\mathbb{P}^1$ such that

$$\varphi_j \circ f_{u_j} = f_{v_j} \circ \psi_j \quad (j = 1, 2, \ldots).$$

Let p^1 and p^2 be distinct points in X_u such that $f_u(p^1) = f_u(p^2)$. Let $\{p^1_j\}_{j=1,2,\ldots}$ and $\{p^2_j\}_{j=1,2,\ldots}$ be sequences of points in X_U such that

(1) $p^1_j, p^2_j \in X_{u_j}$ $\quad (j = 1, 2, \ldots)$,

(2) $\{p^1_j\}_{j=1,2,\ldots}$ (resp. $\{p^2_j\}_{j=1,2,\ldots}$) converges to p^1 (resp. p^2) and

(3) $f_{u_j}(p^1_j) = f_{u_j}(p^2_j)$ $(j = 1, 2, \ldots)$.

The sequences $\{\psi_j(p^1_j)\}_{j=1,2,\ldots}$ and $\{\psi_j(p^2_j)\}_{j=1,2,\ldots}$ of points in X_V converges to the points $\psi(p^1)$ and $\psi(p^2)$ in X_V, respectively.

Now we have

$$f_{v_j}\left(\psi_j\left(p^1_j\right)\right) = \varphi_j\left(f_{u_j}\left(p^1_j\right)\right) = \varphi_j\left(f_{u_j}\left(p^2_j\right)\right) = f_{v_j}\left(\psi_j\left(p^2_j\right)\right) \quad (j = 1, 2, \ldots).$$

Hence

$$f_v\left(\psi\left(p^1\right)\right) = f_v\left(\psi\left(p^2\right)\right).$$

This shows that ψ induces a holomorphic map

$$\varphi\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

such that

(1) $\{\varphi_j\}_{j=1,2,\dots}$ converges to φ in $\mathrm{Hol}(\mathbb{P}^1, \mathbb{P}^1)$ and

(2) $\varphi \circ f_u = f_v \circ \psi$.

A similar argument can be applied to ψ^{-1}, and so we conclude that φ is an automorphism of $\mathbb{P}^1$. This completes the proof of Theorem 5.4.

REMARK 5.6. We have given a proof of Theorem 5.4 without using special properties of $\mathbb{P}^1$. So it will work for G-coverings of a higher dimensional projective manifold, under suitable conditions.

By Theorem 5.4 and Theorem 5.3, we have

Theorem 5.7. *Let $\hat{f} = \{f_m\}_{m\in M} \colon \hat{X} = \{X_m\}_{m\in M} \to M \times \mathbb{P}^1$ be the complete non-degenerate family of G-coverings with respect to $f_o\colon X_o \to \mathbb{P}^1$ in* (4.3) *with $g \geq 2$, where g is the genus of X_o. Assume that the number s of the branch points and the set $\{e_1, e_2, \dots, e_s\}$ of ramification indices of f_o are either*

(1) *$s \neq 4$, or*

(2) *$s = 4$ and $\{e_1, e_2, e_3, e_4\}$ does not satisfy $e_1 = e_2 \leq e_3 = e_4$.*

Then

(1) *for any two points m and m' in M, X_m and $X_{m'}$ are biholomorphic if and only if f_m and $f_{m'}$ are holomorphically equivalent, and*

(2) *the holomorphic map*

$$\iota\colon M/\operatorname{Aut}\left(\mathbb{P}^1\right) \longrightarrow \mathbb{M}_g, \quad m \ \left(\operatorname{mod}\operatorname{Aut}\left(\mathbb{P}^1\right)\right) \longmapsto [X_m]$$

is injective, where $\mathbb{M}_g$ is the moduli space of compact Riemann surfaces of genus g.

Note that (1) and (2) of Theorem 5.7 is trivial for $s = 3$, because the moduli space $M/\operatorname{Aut}(\mathbb{P}^1)$ is one point in the case.

QUESTION. Is the map ι in (2) of Theorem 5.7 a holomorphic injection?

Any two G-coverings of $\mathbb{P}^1$ in a non-degenerate family are topologically equivalent by Theorem 4.1. Hence they belongs (up to isomorphisms) to the complete non-degenerate family in (4.3) as members. Hence, by Theorem 5.7, we finally have as an answer to the equivalence problem:

Theorem 5.8. *Let* $f = \{f_u\}_{u\in N} \colon X = \{X_u\}_{u\in N} \to Y$ *be a non-degenerate family of* G*-coverings of* $\mathbb{P}^1$ *with a* $\mathbb{P}^1$*-bundle* $\rho \colon Y \to N$. *Assume* $g \geq 2$, *where* g *is the genus of* X_u $(u \in N)$. *Assume that the number* s *of the branch points and the set* $\{e_1, e_2, \dots, e_s\}$ *of ramification indices of* f_u $(u \in N)$ *are either*

(1) $s \neq 4$, *or*

(2) $s = 4$ *and* $\{e_1, e_2, e_3, e_4\}$ *does not satisfy* $e_1 = e_2 \leq e_3 = e_4$.

Then, for any two points u *and* u' *in* N, X_u *and* $X_{u'}$ *are biholomorphic if and only if* f_u *and* $f_{u'}$ *are holomorphically equivalent.*

REMARK 5.9. We do not know if, in the exceptional case $s = 4$ and $e_1 = e_2 \leq e_3 = e_4$, the affirmative answer to the equivalence problem still holds or not. The affirmative answer to the equivalence problem still holds for the cyclic coverings in Example 3.4. In fact, in both (1) and (2) of Example 3.4, using the uniqueness of linear pencils of degree 2 on hyperelliptic Riemann surfaces, we can show that X_λ and X_μ are biholomorphic if and only if $\mu = \lambda$ or $\mu = 1/\lambda$. When $\mu = 1/\lambda$, a biholomorphic mapping of X_λ onto $X_{1/\lambda}$ is given by

$$(x, y) \in X_\lambda \longmapsto \left(x/\lambda,\, y\big/\left(\lambda^{5/4}\right)\right) \in X_{1/\lambda},$$

which gives clearly a holomorphic equivalence of f_λ to $f_{1/\lambda}$.

Appendex: Construction of complete non-degenerate families

We give a sketch of a construction of the complete non-degenerate family of G-coverings of $\mathbb{P}^1$ with respect to a given f_o in (4.3). Our construction is apparently different from that in Völklein [18].

We identify the symmetric product $S^s\mathbb{P}^1$ of $\mathbb{P}^1$ with $\mathbb{P}^s$: $S^s\mathbb{P}^1 = \mathbb{P}^s$. The set of divisors which contain the point ∞ is then identified with the hyperplane H_∞ at infinity. Hence $\mathbb{P}^s - \Delta - H_\infty = \mathbb{C}^s - \Delta$, where Δ is the discriminant locus. Let $\{q_1^o, q_2^o, \dots, q_s^o\}$ be the branch locus of f_o. We assume that

$$q_j^o \neq \infty \quad (j = 1, 2, \dots, s).$$

The divisor $D^o = q_1^o + q_2^o + \cdots + q_s^o$ can be regarded as a point of $\mathbb{C}^s - \Delta$. The fundamental group $\pi_1(\mathbb{C}^s - \Delta, D^o)$ can be identified with the Artin braid group B_s of s-strings: $\pi_1(\mathbb{C}^s - \Delta, D^o) = B_s$.

Note also that $\pi_1(\mathbb{P}^s - \Delta, D^o)$ can be identified with $B_s(\mathbb{P}^1)$ the braid group of s-strings in $\mathbb{P}^1$: $\pi_1(\mathbb{P}^s - \Delta, D^o) = B_s(\mathbb{P}^1)$.

There are natural surjective homomorphisms

$$\alpha \colon B_s \longrightarrow B_s\left(\mathbb{P}^1\right),$$
$$\beta \colon B_s\left(\mathbb{P}^1\right) \longrightarrow \mathrm{Map}\left(\mathbb{P}^1, D^o\right),$$

where $\mathrm{Map}(\mathbb{P}^1, D^o)$ is the mapping class group. ($\mathrm{Map}(\mathbb{P}^1, D^o) = M(0, s)$ by the notation in Birman [2].) It is known (see Birman [2]) that
(1) $\mathrm{Ker}(\alpha)$ is the smallest normal subgroup in B_s which contains

$$\sigma_1\sigma_2 \cdots \sigma_{s-1}\sigma_{s-1} \cdots \sigma_2\sigma_1,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_{s-1}$ are the standard generators of B_s and
(2) $\mathrm{Ker}(\beta)$ is the center of $B_s(\mathbb{P}^1)$, which is the smallest normal subgroup of $B_s(\mathbb{P}^1)$ which contains $(\sigma_1\sigma_2 \cdots \sigma_{s-1})^s$.

Consider the subgroup

$$H = \{\sigma \in B_s = \pi_1(\mathbb{C}^s - \Delta, D^o) \mid (\beta\alpha)(\sigma)_*(\mathrm{Ker}(\xi)) = \mathrm{Ker}(\xi)\}$$

of $\pi_1(\mathbb{C}^s - \Delta, D^o)$, where ξ is the homomorphism in (3.2) with $B = B^o = \{q_1^o, q_2^o, \ldots, q_s^o\}$ and $q_o = \infty$, and

$$(\beta\alpha)(\sigma)_* \colon \pi_1\left(\mathbb{P}^1 - B^o, \infty\right) \longrightarrow \pi_1\left(\mathbb{P}^1 - B^o, \infty\right)$$

is the isomorphism induced by the mapping class $(\beta\alpha)(\sigma)$.

We can rewrite H as follows:

$$\begin{aligned} H = \{\sigma \in B_s = \pi_1(\mathbb{C}^s - \Delta, D^o) \mid \\ \text{there is an } A \in \mathrm{Aut}(G) \text{ such that } \xi \circ (\beta\alpha)(\sigma)_* = A \circ \xi\}. \end{aligned}$$

Let

$$\eta' \colon (M', o) \longrightarrow (\mathbb{C}^s - \Delta, D^o)$$

($o \in M'$) be the finite unbranched covering of the pair $(\mathbb{C}^s - \Delta, D^o)$ which corresponds to H. Then η' induces an isomorphism

$$\eta'_* \colon \pi_1(M', o) \longrightarrow H.$$

By the theorem of Grauert and Remmert (Theorem 4.3), η' can be uniquely (up to isomorphisms) extended to a finite covering

$$\eta \colon M \longrightarrow \mathbb{P}^s - \Delta.$$

η is again a finite unbranched covering, for the meridian around H_∞ is (a conjugate of) $\sigma_1\sigma_2 \cdots \sigma_{s-1}\sigma_{s-1} \cdots \sigma_2\sigma_1$ and belongs to H.

Next put

$$Z = \left\{(m, q) \in M \times \mathbb{P}^1 \;\middle|\; q \text{ is contained in the divisor } \eta(m)\right\}.$$

Then Z is a non-singular hypersurface of $M \times \mathbb{P}^1$. The map

$$h\colon m \in M' \longmapsto (m, \infty) \in M' \times \mathbb{P}^1 - Z$$

is a holomorphic section of the projection

$$\rho\colon M' \times \mathbb{P}^1 - Z \longrightarrow M'.$$

Lemma 4.2 can be applied to ρ, so ρ is a topological fiber bundle. Hence there are the following exact sequence:

$$\begin{aligned}&\xrightarrow{i_*} \pi_2\left(M' \times \mathbb{P}^1 - Z\right) \xrightarrow{\rho_*} \pi_2\left(M'\right)\\ &\longrightarrow \pi_1\left(\mathbb{P}^1 - D^o, \infty\right) \xrightarrow{i_*} \pi_1\left(M' \times \mathbb{P}^1 - Z, (o, \infty)\right) \xrightarrow{\rho_*} \pi_1\left(M', o\right) \longrightarrow 1\end{aligned}$$

and the homomorphisms

$$\begin{aligned}h_*&\colon \pi_2\left(M'\right) \longrightarrow \pi_2\left(M' \times \mathbb{P}^1 - Z\right),\\ h_*&\colon \pi_1\left(M', o\right) \longrightarrow \pi_1\left(M' \times \mathbb{P}^1 - Z, (o, \infty)\right)\end{aligned}$$

such that $\rho_* h_* = 1$. Hence h_* is injective and ρ_* is surjective. Hence

$$1 \longrightarrow \pi_1\left(\mathbb{P}^1 - D^o, \infty\right) \xrightarrow{i_*} \pi_1\left(M' \times \mathbb{P}^1 - Z, (o, \infty)\right) \xrightarrow{\rho_*} \pi_1\left(M', o\right) \longrightarrow 1$$

is exact and

$$\pi_1\left(M' \times \mathbb{P}^1 - Z, (o, \infty)\right) \simeq \pi_1\left(M', o\right) \ltimes \pi_1\left(\mathbb{P}^1 - D^o, \infty\right)$$

(semi-direct product). We identify $h_*(\pi_1(M', o))$ with $\pi_1(M', o)$. Then

$$\pi_1\left(M' \times \mathbb{P}^1 - Z, (o, \infty)\right) = \pi_1\left(M', o\right) \bullet \pi_1\left(\mathbb{P}^1 - D^o, \infty\right)$$

(the product set).

Ker(ξ) is not only a normal subgroup of $\pi_1(\mathbb{P}^1 - D^o, \infty)$, but also a normal subgroup of $\pi_1(M' \times \mathbb{P}^1 - Z, (o, \infty))$. Hence the product set $\pi_1(M', o) \bullet \mathrm{Ker}(\xi)$ is a subgroup of $\pi_1(M' \times \mathbb{P}^1 - Z, (o, \infty))$.

Let

$$\hat{f}'\colon \hat{X}' \longrightarrow M' \times \mathbb{P}^1 - Z$$

be the unbranched covering which corresponds to $\pi_1(M', o) \bullet \mathrm{Ker}(\xi)$.

By Theorem 4.3, $\hat{f}'$ can be uniquely (up to isomorphisms) extended to

$$\hat{f}''\colon \hat{X}'' \longrightarrow M \times \mathbb{P}^1 - Z.$$

In a similar reason to η', $\hat{f}''$ is an unbranched covering.

By Theorem 4.3 again, $\hat{f}''$ can be uniquely (up to isomorphisms) extended to a branched covering

$$\hat{f}: \hat{X} \longrightarrow M \times \mathbb{P}^1.$$

$\hat{X}$ is non-singular, for Z is non-singular. The map $\hat{f}$ gives a non-degenerate family of G-coverings of $\mathbb{P}^1$:

$$\hat{f} = \{f_m\}_{m \in M}: \hat{X} = \{X_m\}_{m \in M} \longrightarrow M \times \mathbb{P}^1.$$

This is the complete non-degenerate family of G-coverings of $\mathbb{P}^1$ in (4.3). (f_o is equal to the given f_o.) M is the set of all (isomorphism classes of) G-coverings of $\mathbb{P}^1$ which are topologically equivalent to f_o.

EXAMPLE A.1. Let G be the 3rd symmetric group S_3 and $e_1 = e_2 = \cdots = e_s = 2$ with even s (≥ 4). Then M is independent of the choice of ξ and the mapping degree of

$$\eta: M \longrightarrow \mathbb{P}^s - \Delta$$

is $(3^{s-2} - 1)/2$. In fact, the number of the orderings of $s - 1$ iterated elements from the transpositions $(1\,2)$, $(2\,3)$, $(1\,3)$ is 3^{s-1}. We must delete 3 orderings

$$(1\,2), (1\,2), \ldots, (1\,2),$$
$$(2\,3), (2\,3), \ldots, (2\,3),$$
$$(1\,3), (1\,3), \ldots, (1\,3)$$

from them, because they do not generate S_3. Hence there are $3^{s-1} - 3$ surjective homomorphisms from $\pi_1(\mathbb{P}^1 - D^o, \infty)$ to S_3. $\mathrm{Aut}(S_3)$ is isomorphic to the 3rd symmetric group acting on $\{(1\,2), (2\,3), (1\,3)\}$ as the permutation group. Hence the number of surjective homomorphisms from $\pi_1(\mathbb{P}^1 - D^o, \infty)$ to S_3 up to $\mathrm{Aut}(S_3)$ is

$$\frac{3^{s-1} - 3}{6} = \frac{3^{s-2} - 1}{2}.$$

Moreover we can directly check that for any two such homomorphisms ξ and ξ', there is $\sigma \in B_s$ such that

$$\xi' = \xi \circ (\beta\alpha)(\sigma)_*.$$

We can also know the degree of η by another argument: Every such S_3-covering f of $\mathbb{P}^1$ can be decomposed as

$$f: X \xrightarrow{g} Y \xrightarrow{h} \mathbb{P}^1,$$

where $h: Y \to \mathbb{P}^1$ is a double covering of $\mathbb{P}^1$ with s branch points and $g: X \to Y$ is an unbranched covering of degree 3. The set of isomorphism classes of such g's is in one-to-one correspondence to the set of subgroups of order 3 of the Jacobi variety $J(Y)$ of Y (see, e.g., Namba [14]). There are $(3^{s-2} - 1)/2$ such subgroups.

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