Morita, S.
Osaka J. Math.
21 (1984), 545-563

NONTRIVIALITY OF THE GELFAND-FUCHS CHARACTERISTIC CLASSES FOR FLAT S^1-BUNDLES

SHIGEYUKI MORITA[1)]

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

(Received September 12, 1983)

1. Introduction

Motivated by the work of Gelfand and Fuchs [2], Bott and Haefliger (see [4]) have defined homomorphisms

$$\bar{\Phi}: H^*(\mathcal{L}_{S^1}) \to H^*(B\overline{\mathrm{Diff}}_+(S^1); \boldsymbol{R})$$
$$\Phi: H^*(\mathcal{L}_{S^1}, so(2)) \to H^*(B\mathrm{Diff}_+(S^1)^\delta; \boldsymbol{R})$$

where $\mathcal{L}_{S^1}$ is the topological Lie algebra consisting of all C^∞ vector fields on S^1, $H^*(\mathcal{L}_{S^1})$ is its continuous cohomology (=the Gelfand-Fuchs cohomology of S^1) and $H^*(\mathcal{L}_{S^1}, so(2))$ is the continuous cohomology of $\mathcal{L}_{S^1}$ relative to the subalgebra $so(2)\subset\mathcal{L}_{S^1}$. $\mathrm{Diff}_+(S^1)$ is the topological group of all orientation preserving C^∞ diffeomorphisms of S^1 and $B\overline{\mathrm{Diff}}^+(S^1)$ (resp. $B\mathrm{Diff}_+(S^1)^\delta$) is the classifying space for the topological group $\overline{\mathrm{Diff}}_+(S^1)$ (=homotopy theoretical fibre of the forgetful homomorphism $\mathrm{Diff}_+(S^1)^\delta\to\mathrm{Diff}_+(S^1)$, here δ denotes the discrete topology) (resp. $\mathrm{Diff}_+(S^1)^\delta$). $B\overline{\mathrm{Diff}}_+(S^1)$ (resp. $B\mathrm{Diff}_+(S^1)^\delta$) classifies foliated S^1-products (resp. foliated S^1-bundles) (see [17]). Gelfand and Fuchs [2] have proved that $H^*(\mathcal{L}_{S^1})$ is a free graded algebra with two generators α of degree 2 and β of degree 3 and it follows that $H^*(\mathcal{L}_{S^1}, so(2))=\boldsymbol{R}[\alpha, \chi]/(\alpha\chi)$ where χ is the Euler class (see [4]). We may call the images of $\bar{\Phi}$ and Φ the Gelfand-Fuchs characteristic classes for flat S^1-bundles. Thurston [16] has constructed examples of foliated S^1-bundles to show that the classes α and χ (we omit the symbols $\bar{\Phi}$ and Φ for simplicity, thus α stands for $\Phi(\alpha)$ for example) are independent and also that all the classes α^n ($n\in\boldsymbol{N}$) vary continuously, namely there are homology classes $\sigma_t\in H_{2n}(B\mathrm{Diff}_+(S^1)^\delta; \boldsymbol{Z})$ with $\langle\sigma_t, \alpha^n\rangle=t$ for all $t\in\boldsymbol{R}$. In this paper we describe an extension of Thurston's argument which proves the nontriviality of the classes $\alpha^{n-1}\beta$ and χ^n ($n\in\boldsymbol{N}$).

1) Partially supported by the Sonderforschungsbereich Theoretische Mathematik 40, Universität Bonn.

Thus we can conclude

Theorem 1.1. *The homomorphisms* $\bar{\Phi}$ *and* Φ *are injective. Moreover all the classes except* χ^n $(n\in \boldsymbol{N})$ *vary continuously so that the Gelfand-Fuchs characteristic classes define surjective homomorphisms*

$$H_n(B\overline{\mathrm{Diff}}_+(S^1);\boldsymbol{Z})\to \boldsymbol{R}\to 0 \qquad (n\geqq 2)$$
$$H_{2n}(B\mathrm{Diff}_+(S^1)^\delta;\boldsymbol{Z})\to \boldsymbol{R}\oplus\boldsymbol{Z}\to 0 \qquad (n\in\boldsymbol{N}).$$

This answers a question of Haefliger (Problem 4 in [13]) affirmatively. We can also obtain more informations on the homology of $B\overline{\mathrm{Diff}}_+(S^1)$ and $B\mathrm{Diff}_+(S^1)^\delta$ by considering the discontinuous invariants of them. See Remark 4.4 and [12].

2. The Gelfand-Fuchs characteristic classes for flat S^1-bundles

Let us begin by recalling the definition of the homomorphisms $\bar{\Phi}$, Φ very briefly (see [4] for details). We also derive some relations among their images=the Gelfand-Fuchs characteristic classes. Henceforth we write G for the topological group $\mathrm{Diff}_+(S^1)$.

Suppose we are given a foliated S^1-product over a C^∞ manifold M. Thus there is given a codimension one foliation $\mathcal{F}$ on $M\times S^1$ transverse to the fibres. Equivalently we are given an $\mathcal{L}_{S^1}$-valued C^∞ 1-form ω on M satisfying the integrability condition $d\omega+[\omega,\omega]=0$. For a vector field $X\in\mathcal{L}_M$, let us write $\omega(X)=\omega_X(x,t)\dfrac{\partial}{\partial t}$ where $\omega_X(x,t)$ is a C^∞ function on $M\times S^1$. Then the subbundle $T(\omega)=\{X(x)+\omega_X(x,t)\dfrac{\partial}{\partial t}\in T_{(x,t)}(M\times S^1);\ X\in\mathcal{L}_M,\ (x,t)\in M\times S^1\}$ of the tangent bundle of $M\times S^1$ is equal to that of the foliation $\mathcal{F}$. Let $C^*(\mathcal{L}_{S^1})$ be the cochain complex of all continuous alternating forms on $\mathcal{L}_{S^1}$. Then the linear map

$$\bar{\varphi}\colon C^*(\mathcal{L}_{S^1})\to\Omega^*(M)\ (=\text{de Rham complex of } M)$$

defined by $\bar{\varphi}(c)\,(X_1,\cdots,X_q)=c(\omega(X_1),\cdots,\omega(X_q))$ $(c\in C^q(\mathcal{L}_{S^1}),\ X_i\in\mathcal{L}_M)$ commutes with the differentials and we have the induced homomorphism on cohomology

$$\bar{\Phi}\colon H^*(\mathcal{L}_{S^1})\to H^*(M;\boldsymbol{R}).$$

This construction is functorial on the category of foliated S^1-products and hence we have the homomorphism $\bar{\Phi}\colon H^*(\mathcal{L}_{S^1})\to H^*(B\bar{G};\boldsymbol{R})$ at the universal space level. Next let $\pi\colon E\to N$ be an oriented foliated S^1-bundle over a C^∞ manifold N and let U be a coordinate neighborhood in N. Choose a trivialization $\pi^{-1}(U)\cong U\times S^1$ as an $SO(2)$-bundle. This trivialization gives rise to a foliated S^1-product structure on $\pi^{-1}(U)$ and so, as before we obtain a cochain map $\bar{\varphi}$: $C^*(\mathcal{L}_{S^1}) \to \Omega^*(U)$. It turns out that this homomorphism restricted to the subcomplex $C^*(\mathcal{L}_{S^1}, so(2))$ is independent of the choice of the trivialization $\pi^{-1}(U) \cong U \times S^1$. Thus patching together these homomorphisms over each coordinate neighborhoods in N, we obtain a cochain map φ: $C^*(\mathcal{L}_{S^1}, so(2)) \to \Omega^*(N)$ and this construction defines the homomorphism Φ: $H^*(\mathcal{L}_{S^1}, so(2)) \to H^*(BG^\delta; \boldsymbol{R})$. We recall that the classes $\alpha \in H^2(\mathcal{L}_{S^1})$ and $\beta \in H^3(\mathcal{L}_{S^1})$ are represented by the following cocycles $\alpha \in C^2(\mathcal{L}_{S^1})$ and $\beta \in C^3(\mathcal{L}_{S^1})$ respectively [2] (we use the same symbols for the cocycles)

$$\alpha\left(f\frac{\partial}{\partial t}, g\frac{\partial}{\partial t}\right) = \int_{S^1} \begin{vmatrix} f' & f'' \\ g' & g'' \end{vmatrix} dt$$

$$\beta\left(f\frac{\partial}{\partial t}, g\frac{\partial}{\partial t}, h\frac{\partial}{\partial t}\right) = \int_{S^1} \begin{vmatrix} f & f' & f'' \\ g & g' & g'' \\ h & h' & h'' \end{vmatrix} dt \text{ or } \begin{vmatrix} f & f' & f'' \\ g & g' & g'' \\ h & h' & h'' \end{vmatrix}_{t=1}$$

(these two cocycles for β are cohomologous and t is the coordinate for $S^1 = \boldsymbol{R}/\boldsymbol{Z}$, $1 \in S^1$).

Now we consider the fibration

$$B\bar{G} \to BG^\delta \to BG .$$

Since G has the same homotopy type as the rotation group $SO(2)$, the map $B\bar{G} \to BG^\delta$ has the structure of an S^1-fibration. In fact this can be explained more explicitly as follows. Let $\tilde{G}$ be the universal covering group of G. Then it has the expression $\tilde{G} = \{f \in \mathrm{Diff}_+(\boldsymbol{R});\ Tf = fT\}$, where T is the translation by 1 of $\boldsymbol{R}$ and we have a central extension

$$0 \to \boldsymbol{Z} \to \tilde{G} \to G \to 1 .$$

It is easy to see that there is a natural homotopy equivalence $B\tilde{G}^\delta \sim B\bar{G}$ (henceforth we identify these two spaces) and we have the orientable S^1-fibration

$$B\boldsymbol{Z} = S^1 \to B\tilde{G}^\delta \xrightarrow{p} BG^\delta .$$

This gives rise to the following

Proposition 2.1. *We have the Gysin exact sequences*

$$\cdots \to H_m(B\tilde{G}^\delta) \xrightarrow{p_*} H_m(BG^\delta) \xrightarrow{\cap \chi} H_{m-2}(BG^\delta) \xrightarrow{\mu} H_{m-1}(B\tilde{G}^\delta) \to \cdots ,$$

$$\cdots \to H^m(BG^\delta) \xrightarrow{p^*} H^m(B\tilde{G}^\delta) \xrightarrow{\mu_*} H^{m-1}(BG^\delta) \xrightarrow{\cup \chi} H^{m+1}(BG^\delta) \to \cdots$$

(with any coefficient),

where $\chi \in H^2(BG^\delta; \boldsymbol{Z})$ *is the Euler class.*

Proposition 2.2. *The equality* $\mu^*(\alpha^{n-1}\beta) = -\alpha^n$ *holds for all* $n \in \boldsymbol{N}$.

Proof. Since $p^*(\alpha)=\alpha$, we have only to prove the case $n=1$. Let $\zeta = (\pi\colon E \to N)$ be an oriented foliated S^1-bundle over a C^∞ manifold N. Let us fix an $SO(2)$ structure on ζ. Then the pull back bundle $\pi^*\zeta = (\bar{\pi}\colon \pi^*(E) \to E)$, where $\pi^*(E) = \{(e, e') \in E \times E;\ \pi(e) = \pi(e')\}$ and $\bar{\pi}(e, e') = e$, has the structure of a foliated S^1-product because we have an isomorphism of $SO(2)$-bundles $\pi^*E \cong E \times S^1$ defined by $(e, e') \to (e, g)$ where $e' = eg$, $g \in SO(2)$. These two flat S^1-bundles ζ and $\pi^*\zeta$ are classified by the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \boldsymbol{Z} & \to & \pi_1(E) & \to & \pi_1(N) & \to & 1 \\ & & \| & & \downarrow \tilde{\rho} & & \downarrow \rho & & \\ 0 & \to & \boldsymbol{Z} & \to & \tilde{G} & \to & G & \to & 1 \end{array}$$

where ρ (resp. $\tilde{\rho}$) classifies the bundle ζ (resp. $\pi^*\zeta$). Equivalently we have a morphism of S^1-fibrations

$$\begin{array}{ccc} E & \longrightarrow & B\tilde{G}^\delta \\ \pi \downarrow & & \downarrow \\ N & \longrightarrow & BG^\delta . \end{array}$$

By the universality, it is enough to prove the equality $\mu^*(\beta(\pi^*\zeta)) = -\alpha(\zeta)$, where $\alpha(\zeta) \in H^2(N; \boldsymbol{R})$ (resp. $\beta(\pi^*\zeta) \in H^3(E; \boldsymbol{R})$) is the characteristic class of the bundle ζ (resp. $\pi^*\zeta$) corresponding to α (resp. β). Let U be a coordinate neighborhood in N and choose a trivialization $\pi^{-1}(U) \cong U \times S^1$ as an $SO(2)$-bundle. Then we have a natural isomorphism $\bar{\pi}^{-1}(E') \cong (U \times S^1) \times S^1$ as $SO(2)$-bundles, where $E' = \pi^{-1}(U)$. Let ω be the $\mathcal{L}_{S^1}$-valued C^∞ 1-form on U defined by the flat structure ζ restricted to E' and the trivialization $E' \cong U \times S^1$. Similarly let $\bar{\omega}$ be the $\mathcal{L}_{S^1}$-valued C^∞ 1-form on $U \times S^1$ corresponding to the foliated S^1-product structure on $U \times S^1$. Fix a point $x \in U$ and let us write

$$\omega(X) = \omega_X(t)\frac{\partial}{\partial t} \qquad (X \in T_x(U)) \text{ and}$$

$$\bar{\omega}(X) = \bar{\omega}_X(s, t)\frac{\partial}{\partial t} \qquad (X \in T_x(U) \subset T_{(x,s)}(U \times S^1)).$$

Lemma 2.3. *We have the following equations*:

$$\bar{\omega}_X(s, t) = \omega_X(s+t) \qquad (X \in T_x(U)) \text{ and}$$

$$\bar{\omega}\left(\frac{\partial}{\partial s}\right) = -\frac{\partial}{\partial t}.$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \bar{\pi}^{-1}(E') = (U\times S^1)\times S^1 & \xrightarrow{\lambda} & U\times S^1 = \pi^{-1}(U) \\ \bar{\pi}\downarrow & & \downarrow\pi \\ E' = U\times S^1 & \xrightarrow[\pi]{} & U \end{array}$$

where $\lambda(x, s, t)=(x, s+t)$, $(x, s, t)\in(U\times S^1)\times S^1$. Clearly we have $T(\varpi)_{(x,s,t)} =(\lambda_*)^{-1}T(\omega)_{(x,s+t)}$, where λ_* is the differential of the map λ. Since $T(\omega)_{(x,t)} =\{X+\omega_X(t)\frac{\partial}{\partial t};\ X\in T_x(U)\}$, and $\lambda_*\left(\frac{\partial}{\partial s}\right)=\lambda_*\left(\frac{\partial}{\partial t}\right)=\frac{\partial}{\partial t}$, we obtain

$$T(\varpi)_{(x,s,t)} = \{X+u\frac{\partial}{\partial s}+v\frac{\partial}{\partial t};\ X\in T_x(U),\ u+v = \omega_X(s+t)\}\ .$$

Now let $X\in T_x(U)\subset T_{(x,s)}(U\times S^1)$. Then the lifted vector in $T(\varpi)_{(x,s,t)}$ should have the form $X+\varpi_X(s, t)\frac{\partial}{\partial t}$ and this vector goes to $X+\omega_X(s+t)\frac{\partial}{\partial t}$ under the map λ_*. Hence $\varpi_X(s, t)=\omega_X(s+t)$. Similarly the vector $\frac{\partial}{\partial s}\in T_{(x,s)}(U\times S^1)$ lifts to $\frac{\partial}{\partial s}+\varpi\left(\frac{\partial}{\partial s}\right)\in T(\varpi)_{(x,s,t)}$ and this vector goes to 0 under λ_*. Therefore $\varpi\left(\frac{\partial}{\partial s}\right)=-\frac{\partial}{\partial t}$. This completes the proof of Lemma 2.3.

Now we go back to the proof of Prop. 2.2. Let X, $Y\in T_x(U)$. Then the cohomology class $\alpha(\zeta)\in H^2(N; \boldsymbol{R})$ is represented by a closed form $a\in\Omega^2(N)$ which has the value

$$\begin{aligned} a(X, Y) &= \alpha(\omega(X), \omega(Y)) \\ &= \int_{S^1}\begin{vmatrix} \omega_X'(t) & \omega_X''(t) \\ \omega_Y'(t) & \omega_Y''(t)\end{vmatrix}\, dt\ . \end{aligned}$$

Similarly the element $\beta(\pi^*\zeta)\in H^3(E; \boldsymbol{R})$ is represented by a closed form $b \in\Omega^3(E)$ which has the value for X, $Y\in T_{(x,s)}(U\times S^1)$

$$\begin{aligned} b\left(X, Y, \frac{\partial}{\partial s}\right) &= \beta\left(\varpi(X), \varpi(Y), \varpi\left(\frac{\partial}{\partial s}\right)\right) \\ &= \int_{S^1}\begin{vmatrix} \varpi_X(s, t) & \varpi_X'(s, t) & \varpi_X''(s, t) \\ \varpi_Y(s, t) & \varpi_Y'(s, t) & \varpi_Y''(s, t) \\ -1 & 0 & 0 \end{vmatrix}\, dt \\ &= -\int_{S^1}\begin{vmatrix} \omega_X'(s+t) & \omega_X''(s+t) \\ \omega_Y'(s+t) & \omega_Y''(s+t)\end{vmatrix}\, dt \\ &= -a(X, Y)\ . \end{aligned}$$

Hence locally we have $b=-a\wedge ds+$other terms. By the definition of the homomorphism μ^*: $H^3(E; \boldsymbol{R})\to H^2(N; \boldsymbol{R})$, we have $\mu^*(\beta(\pi^*\zeta))=-\alpha(\zeta)$. This proves Prop. 2.2.

Corollary 2.4. *For any $\sigma\in H_{2n}(BG^\delta)$ and $n\in \boldsymbol{N}$, we have $\langle\sigma, \alpha^n\rangle=-\langle\mu(\sigma), \alpha^{n-1}\beta\rangle$.*

3. Relation with the Godbillon-Vey class

In this section we investigate the relation between the Gelfand-Fuchs characteristic classes for flat S^1-bundles and the Godbillon-Vey class for codimension one foliations [3]. The results of this section should be known to many people and also we do not use them in the remaining part of this paper except in the proof of Th. 4.3. We include them here for completeness.

Let $B\overline{\Gamma}_1$ be the classifying space for oriented codimension one Haefliger structures and let $gv\in H^3(B\overline{\Gamma}_1; \boldsymbol{R})$ be the Godbillon-Vey class.

Proposition 3.1. *Let h: $B\tilde{G}^\delta\times S^1\to B\overline{\Gamma}_1$ be the classifying map for the universal codimension one foliation on $B\tilde{G}^\delta\times S^1$. Then we have*

$$h^*(gv) = \beta\times 1-\alpha\times\iota$$

where $\iota\in H^1(S^1; \boldsymbol{R})$ is the generator.

Proof. Let ζ be a foliated S^1-product over a C^∞ manifold M. Thus there is given a codimension one foliation $\mathcal{F}$ on $M\times S^1$ transverse to the fibres defined by an $\mathcal{L}_{S^1}$-valued C^∞ 1-form ω on M satisfying the integrability condition $d\omega+[\omega, \omega]=0$. By the universality, we have only to prove $gv(\mathcal{F})=\beta(\zeta)\times 1-\alpha(\zeta)\times\iota$. As before for a vector field $X\in\mathcal{L}_M$, let us write $\omega(X)=\omega_X(x, t)\frac{\partial}{\partial t}$ so that the tangent bundle of $\mathcal{F}$ is given by

$$T(\mathcal{F}) = \{X(x)+\omega_X(x, t)\frac{\partial}{\partial t}\in T_{(x,t)}(M\times S^1);\ X\in\mathcal{L}_M,\ (x, t)\in M\times S^1\}.$$

Hence if we define a 1-form $\theta\in\Omega^1(M\times S^1)$ by

$$\theta(X) = -\omega_X\ (X\in\mathcal{L}_M\subset\mathcal{L}_{M\times S^1}) \text{ and } \theta\left(\frac{\partial}{\partial t}\right)=1,$$

then clearly $\mathcal{F}$ is defined by θ, namely $T(\mathcal{F})=\{Z\in T(M\times S^1);\ \theta(Z)=0\}$. Now for two vector fields X, $Y\in\mathcal{L}_M$, the integrability condition of ω implies

$$X\omega(Y)-Y\omega(X)-\omega([X, Y]) = -[\omega(X), \omega(Y)].$$

Hence

$$(*)\qquad X\omega_Y-Y\omega_X-\omega_{[X,Y]} = \omega_Y\frac{\partial}{\partial t}\omega_X-\omega_X\frac{\partial}{\partial t}\omega_Y.$$

On the other hand

$$d\theta(X, Y) = -X\omega_Y + Y\omega_X + \omega_{[X,Y]} \quad \text{and}$$
$$d\theta\Big(X, \frac{\partial}{\partial t}\Big) = \frac{\partial}{\partial t}\omega_X .$$

Thus we obtain

$$d\theta(X, Y) = \omega_X \frac{\partial}{\partial t}\omega_Y - \frac{\partial}{\partial t}\omega_X\omega_Y \quad \text{and}$$
$$d\theta\Big(X, \frac{\partial}{\partial t}\Big) = \frac{\partial}{\partial t}\omega_X .$$

This means that if we define a 1-form $\eta \in \Omega^1(M \times S^1)$ by

$$\eta(X) = \frac{\partial}{\partial t}\omega_X \quad \text{and} \quad \eta\Big(\frac{\partial}{\partial t}\Big) = 0 ,$$

then we have $d\theta = \eta \wedge \theta$. By definition, the Godbillon-Vey class $gv(\mathcal{F})$ is represented by the closed 3-form $\eta \wedge d\eta$. Now applying $\frac{\partial}{\partial t}$ to both sides of (*) we obtain

$$X\frac{\partial}{\partial t}\omega_Y - Y\frac{\partial}{\partial t}\omega_X - \frac{\partial}{\partial t}\omega_{[X,Y]} = \omega_Y \frac{\partial^2}{\partial t^2}\omega_X - \omega_X\frac{\partial^2}{\partial t^2}\omega_Y .$$

On the other hand

$$d\eta(X, Y) = X\frac{\partial}{\partial t}\omega_Y - Y\frac{\partial}{\partial t}\omega_X - \frac{\partial}{\partial t}\omega_{[X,Y]} \quad \text{and}$$
$$d\eta\Big(X, \frac{\partial}{\partial t}\Big) = -\frac{\partial^2}{\partial t^2}\omega_X .$$

Hence if we define a 1-form $\nu \in \Omega^1(M \times S^1)$ by

$$\nu(X) = -\frac{\partial^2}{\partial t^2}\omega_X \quad \text{and} \quad \nu\Big(\frac{\partial}{\partial t}\Big) = 0 ,$$

then we have $d\eta = \nu \wedge \theta$. Therefore the Godbillon-Vey form is $\eta \wedge d\eta = \eta \wedge \nu \wedge \theta$. Now $gv(\mathcal{F})$ is an element of $H^3(M \times S^1; \boldsymbol{R}) \cong H^3(M; \boldsymbol{R}) \oplus H^2(M; \boldsymbol{R})$. The first component of $gv(\mathcal{F})$ is represented by the 3-form $i^*(\eta \wedge \nu \wedge \theta)$ where i: $M \to M \times S^1$ is the inclusion $i(x) = (x, 1)$, $x \in M$, $1 \in S^1$. But clearly $i^*(\eta \wedge \nu \wedge \theta)$ coincides with the 3-form representing the class $\beta(\zeta)$ (here we use the latter form of the cocycle $\beta \in C^3(\mathcal{L}_{S^1})$). Next it is easy to see that $\eta \wedge \nu \wedge \theta = \eta \wedge \nu \wedge dt$ $+$other terms. Therefore the second component of $gv(\mathcal{F})$ is represented by the 2-form $\int_{S^1} \eta \wedge \nu\, dt$. But this is equal to the 2-form representing the class $-\alpha(\zeta)$. Thus we have proved $gv(\mathcal{F}) = \beta(\zeta) \times 1 - \alpha(\zeta) \times \iota$.

Corollary 3.2. *Let $\zeta=(\pi; E\rightarrow N)$ be a foliated S^1-bundle over a C^∞ manifold N and let $\mathcal{F}$ be the corresponding codimension one foliation on E. Then $gv(\mathcal{F})=\beta(\pi^*\zeta)$ and therefore $\mu^*(gv(\mathcal{F}))=-\alpha(\zeta)$, where μ^*: $H^3(E; \boldsymbol{R})\rightarrow H^2(N; \boldsymbol{R})$ is the "integration along the fibres".*

Corollary 3.3. *The cohomology classes $\alpha\in H^2(BG^\delta; \boldsymbol{R})$ and $\beta\in H^3(B\tilde{G}^\delta; \boldsymbol{R})$ can be represented by the following Eilenberg-MacLane cocycles $\alpha\in C^2(G; \boldsymbol{R})$ and $\beta\in C^3(\tilde{G}; \boldsymbol{R})$ (again we use the same symbols)*

$$\alpha(u, v) = -\frac{1}{2}\int_{S^1}\begin{vmatrix} \log v' & (\log v')' \\ \log u'(v) & (\log u'(v))' \end{vmatrix} dt \qquad (u, v\in G)$$

$$\beta(f, g, h) = -\frac{1}{6}\int_0^1\begin{vmatrix} h-id & \log h' & (\log h')' \\ gh-id & \log g'(h) & (\log g'(h))' \\ fgh-id & \log f'(gh) & (\log f'(gh))' \end{vmatrix} dt \qquad (f, g, h\in G).$$

Proof. Thurston has obtained the cocycle α for the Godbillon-Vey class integrated along the fibres of foliated S^1-bundles (see [1] for a proof). The same technique as in [1] or the one in [9] yields the cocycle β for the Godbillon-Vey class restricted to the canonical cross-section of foliated S^1-products (see [10, 11]). Therefore Cor. 3.3 follows from Prop. 3.1.

4. Thurston's argument

In this section we summarize Thurston's argument [16] of proving the nontriviality and variability of the Gelfand-Fuchs characteristic classes. The unit tangent sphere bundles of closed orientable surfaces with constant negative curvature can be considered as examples of foliated S^1-bundles with nontrivial α and χ classes. From the viewpoint of group homology, these examples can be constructed by making use of the standard representation ρ: $SL_2(\boldsymbol{R})\rightarrow G$ which is defined by extending the action of $PSL_2(\boldsymbol{R})$ on the unit disc to the boundary. However it can be checked that $\rho^*(\alpha)$ and $\rho^*(\chi)$ are linearly dependent. At this point Thurston has considered the lift of ρ to the 2-fold coverings: $SL_2(\boldsymbol{R})\rightarrow G^{(2)}$=2-fold covering group of G and composed it with the natural inclusion $G^{(2)}\rightarrow G$ (cf. §5) to obtain another representation ρ': $SL_2(\boldsymbol{R})\rightarrow G$. Two representations ρ and ρ' give rise to the third one $\bar{\rho}$: $SL_2(\boldsymbol{R}) \underset{SO(2)}{*} SL_2(\boldsymbol{R})\rightarrow G$ and it turns out that $\bar{\rho}^*(\alpha)$ and $\bar{\rho}^*(\chi)$ are linearly independent. Moreover he has defined a smooth family of representations π_1 (closed surface of genus 2)$\rightarrow SL_2(\boldsymbol{R}) \underset{SO(2)}{*} SL_2(\boldsymbol{R})$ with varying α classes (see [1] for details). All of the above can be done in the real analytic category. Thus we have

Theorem 4.1 (Thurston). *The classes α and χ are linearly independent in $H^2(B\mathrm{Diff}^\omega_+(S^1)^\delta; \boldsymbol{R})$ and α defines a surjective homomorphism*

$$H_2(B\mathrm{Diff}_+^\omega(S^1)^\delta; \boldsymbol{Z}) \to \boldsymbol{R} \to 0\,.$$

Thurston has also constructed examples with varying α^n. One way to show this is to combine his another theorem [16] that the Godbillon-Vey class defines a surjective homomorphism $\pi_3(B\bar{\Gamma}_1)\to\boldsymbol{R}$ with Mather's deep results [6, 7, 8]. Let us recall the latter. Let $\mathrm{Diff}_K(\boldsymbol{R})$ be the topological group of all C^∞ diffeomorphisms of $\boldsymbol{R}$ with compact supports. We have a natural map h: $B\mathrm{Diff}_K(\boldsymbol{R})^\delta\times\boldsymbol{R}\to B\bar{\Gamma}_1$ which classifies the universal codimension one foliation on $B\mathrm{Diff}_K(\boldsymbol{R})^\delta\times\boldsymbol{R}$. Since this foliation is trivial in a neighborhood of the infinity of the $\boldsymbol{R}$-factor, the map h has an adjoint map H: $B\mathrm{Diff}_K(\boldsymbol{R})^\delta\to\Omega B\bar{\Gamma}_1$.

Theorem 4.2 (Mather [7]). *The map H: $B\mathrm{Diff}_K(\boldsymbol{R})^\delta\to\Omega B\bar{\Gamma}_1$ induces an isomorphism on homology.*

From this and also his yet another result [8] that the group $\mathrm{Diff}_K(\boldsymbol{R})$ is perfect, Mather has concluded that the space $B\bar{\Gamma}_1$ is 2-connected. Thus we have an isomorphism $H^2(\Omega B\bar{\Gamma}_1;\boldsymbol{R})=\mathrm{Hom}(\pi_3(B\bar{\Gamma}_1),\boldsymbol{R})$. Let $gv'\in H^2(\Omega B\bar{\Gamma}_1;\boldsymbol{R})$ be the element corresponding to the Godbillon-Vey class gv: $\pi_3(B\bar{\Gamma}_1)\to\boldsymbol{R}$ under the above isomorphism. Then by Cor. 3.2 we have $H^*(gv')=-\alpha$. Here we are considering $\mathrm{Diff}_K(\boldsymbol{R})$ as a subgroup of G by identifying some open interval $\mathring{I}$ in S^1 with $\boldsymbol{R}$. It is easy to see that thus defined cohomology class $\alpha\in H^2(B\mathrm{Diff}_K(\boldsymbol{R})^\delta;\boldsymbol{R})$ is well defined independent of the choice of $\mathring{I}$ and the diffeomorphism $\mathring{I}\approx\boldsymbol{R}$. Now let $\tau_t(t\in\boldsymbol{R})$ be a one-parameter family of elements of $\pi_3(B\bar{\Gamma}_1)$ with $gv(\tau_t)=t$ (Thurston [16]) and let $\tau'_t\in\pi_2(\Omega B\bar{\Gamma}_1)\cong H_2(\Omega B\bar{\Gamma}_1)$ be the corresponding homology classes. If we set $\sigma_t=H_*^{-1}(\tau'_t)\in H_2(B\mathrm{Diff}_K(\boldsymbol{R})^\delta;\boldsymbol{Z})$, then we have $\langle\sigma_t,\alpha\rangle=-t$. Now for any positive integer $n\in\boldsymbol{N}$, choose n mutually disjoint closed intervals I_i in S^1 $(i=1,\cdots,n)$. If we fix orientation preserving diffeomorphism of each open interval $\mathring{I}_i$ with $\boldsymbol{R}$, we obtain a homomorphism ρ_n: $\mathrm{Diff}_K(\boldsymbol{R})\times\cdots\times\mathrm{Diff}_K(\boldsymbol{R})\to\tilde{G}$ and this induces a map ρ_n: $B\mathrm{Diff}_K(\boldsymbol{R})^\delta\times\cdots\times B\mathrm{Diff}_K(\boldsymbol{R})^\delta\to B\tilde{G}^\delta$. It can be easily checked that $\rho_n^*(\alpha)=\sum_{i=1}^n 1\times\cdots\times 1\times\overset{i}{\alpha}\times 1\times\cdots\times 1$. Now let $\bar{\sigma}_t=\sigma_t\times\sigma_1\times\cdots\times\sigma_1\in H_{2n}(B\mathrm{Diff}_K(\boldsymbol{R})^\delta\times\cdots\times B\mathrm{Diff}_K(\boldsymbol{R})^\delta;\boldsymbol{Z})$. Then we have $\langle(\rho_n)_*(\bar{\sigma}_t),\alpha^n\rangle=\langle\bar{\sigma}_t,(\rho_n)^*(\alpha^n)\rangle=(-1)^n n!\,t$. This proves the variability of the classes $\alpha^n\in H^{2n}(B\tilde{G}^\delta;\boldsymbol{R})$ for all n. In view of Cor. 2.4, we can now conclude

Theorem 4.3. *The classes α^n and $\alpha^{n-1}\beta$ in $H^*(B\tilde{G}^\delta;\boldsymbol{R})$ are variable for all $n\in\boldsymbol{N}$.*

REMARK 4.4. Above argument can be modified to obtain more informations on the homology of $B\tilde{G}^\delta$. Namely it can be shown that the discontinuous invariants of $B\tilde{G}^\delta$ arising from the class α are all nontrivial so that we have surjective homomorphisms

$$H_{2n}(B\overline{\mathrm{Diff}}_+(S^1); \boldsymbol{Z}) \to S_{\boldsymbol{Q}}^n(\boldsymbol{R}) \to 0$$
$$H_{2n+1}(B\overline{\mathrm{Diff}}_+(S^1); \boldsymbol{Z}) \to S_{\boldsymbol{Q}}^{n-1}(\boldsymbol{R})\otimes_{\boldsymbol{Z}}\boldsymbol{R} \to 0 \qquad (n\in\boldsymbol{N})$$
$$H_{2n}(B\mathrm{Diff}_+(S^1)^\delta; \boldsymbol{Z}) \to S_{\boldsymbol{Q}}^n(\boldsymbol{R})\oplus\boldsymbol{Z} \to 0$$

where $S_{\boldsymbol{Q}}^n(\boldsymbol{R})$ denotes the n fold symmetric power of $\boldsymbol{R}$ considered as a $\boldsymbol{Q}$-vector space. See [12] for details. Thurston [17] has extensively generalized Mather's results [6, 7, 8]. Here we mention only the following, which will play an essential role in the proof of the nontriviality of χ^n.

Theorem 4.5 (Thurston [17]). *Let* H: $B\tilde{G}^\delta \to \wedge B\overline{\Gamma}_1 = \mathrm{Map}\,(S^1, B\overline{\Gamma}_1)$ *be the adjoint map of the natural one* $B\tilde{G}^\delta \times S^1 \to B\overline{\Gamma}_1$. *Then* H *induces an isomorphism on homology.*

5. The Euler class

Let H be an abstract group. Then it is well known that the set of isomorphism classes of central extensions

$$0 \to \boldsymbol{Z} \to \tilde{H} \xrightarrow{p} H \to 1$$

can be identified with the cohomology group $H^2(H; \boldsymbol{Z})$. The correspondence is given as follows. Choose a splitting (as a set map) s: $H \to \tilde{H}$ of p such that $s(e)=\tilde{e}$, where e, $\tilde{e}$ are the unit elements, and define a cochain $\chi_s \in C^2(H; \boldsymbol{Z})$ by the equality

$$\chi_s(f, g) = s(f)s(g)s(fg)^{-1} \in \boldsymbol{Z}, \qquad f, g\in H\,.$$

It is easy to check that χ_s is a cocycle and different splitting yields a cohomologous one. Thus we have a well defined cohomology class $[\chi_s]\in H^2(H; \boldsymbol{Z})$. Conversely let $\chi\in C^2(H; \boldsymbol{Z})$ be a cocycle. Define a multiplication on the set $\boldsymbol{Z}\times H$ by

$$(m, f)\,(n, g) = (m+n+\chi(f, g), fg), \qquad m, n\in\boldsymbol{Z}, f, g\in H\,.$$

Then this multiplication is associative and defines a group structure on $\boldsymbol{Z}\times H$, which we denote by $(\boldsymbol{Z}\times H)_\chi$. The set $\boldsymbol{Z}=\{(m, e)\in(\boldsymbol{Z}\times H)_\chi; m\in\boldsymbol{Z}\}$ is a subgroup of $(\boldsymbol{Z}\times H)_\chi$ and we have a central extension

$$0 \to \boldsymbol{Z} \to (\boldsymbol{Z}\times H)_\chi \xrightarrow{p} H \to 1\,.$$

Ths isomorphism class of this extension depends only on the cohomology class $[\chi]\in H^2(H; \boldsymbol{Z})$. In particular if $\chi=\chi_s$, then the correspondence $(\boldsymbol{Z}\times H)_\chi \to \tilde{H}$ given by $(m, f)\to ms(f)$ is an isomorphism.

Now let $\zeta=(\rho$: $E\to N)$ be an oriented S^1-bundle with Euler class $\chi(\zeta)\in H^2(N; \boldsymbol{Z})$ and for a positive integer $k\in\boldsymbol{N}$, let $\zeta_k=(\pi_k$: $E_k\to N)$ be the oriented

S^1-bundle with Euler class $\chi(\zeta_k)=k\chi(\zeta)$. Then it is easy to see that there is a fibre preserving map b: $E\to E_k$ such that b restricted to each fibre is a k-fold covering map of S^1. The following diagram is clearly commutative

$$\begin{array}{ccc} H_m(N) & \xrightarrow{\mu} & H_{m+1}(E) \\ \| & & \downarrow b_* \\ H_m(N) & \xrightarrow{k\cdot\mu} & H_{m+1}(E_k) \end{array}$$

where μ are the homomorphisms in the Gysin sequences of the S^1-bundles ζ, ζ_k. Let us consider a similar situation in the group extension context. Thus let

$$0\to \boldsymbol{Z}\to \tilde{H}\xrightarrow{p} H\to 1$$

be a central extension with cocycle χ_s (s: $H\to\tilde{H}$ is a splitting of p). Then the natural map $B\tilde{H}\to BH$ has a structure of an oriented S^1-bundle with Euler class $[\chi_s]\in H^2(H;\boldsymbol{Z})=H^2(BH;\boldsymbol{Z})$. Now let

$$0\to \boldsymbol{Z}\to \tilde{H}'\underset{s'}{\overset{p'}{\rightleftarrows}} H\to 1$$

be another central extension with splitting s': $H\to\tilde{H}'$ and suppose that $[\chi_{s'}]=k[\chi_s]$ for a positive integer k. Thus there is a cochain $\gamma\in C^1(H;\boldsymbol{Z})$ such that $k\chi_s-\chi_{s'}=\delta\gamma$. In these situations we have

Proposition 5.1. *The map* ρ: $\tilde{H}\to\tilde{H}'$ *defined by* $\rho(\tilde{f})=(k(\tilde{f}s(f)^{-1})+\gamma(f))s'(f)$, *where* $\tilde{f}\in\tilde{H}$ *and* $f=p(\tilde{f})$, *is a group homomorphism and the following diagram is commutative*

$$\begin{array}{ccccccccc} 0 & \to & \boldsymbol{Z} & \to & \tilde{H} & \xrightarrow{p} & H & \to & 1 \\ & & \downarrow\times k & & \downarrow\rho & & \| & & \\ 0 & \to & \boldsymbol{Z} & \to & \tilde{H}' & \xrightarrow{p'} & H & \to & 1\,. \end{array}$$

Corollary 5.2. *The following diagram is commutative*

$$\begin{array}{ccc} H_m(BH) & \xrightarrow{\mu} & H_{m+1}(B\tilde{H}) \\ \| & & \downarrow\rho_* \\ H_m(BH) & \xrightarrow{k\cdot\mu} & H_{m+1}(B\tilde{H}')\,. \end{array}$$

The proof of Prop. 5.1 can be carried out by a direct calculation and we omit it. Now recall that G stands for the topological group $\mathrm{Diff}_+(S^1)$. For a positive integer $k\in\boldsymbol{N}$, let $G^{(k)}$ be the k-fold covering group of G. Thus we can write $G^{(k)}=\{f\in G;\, fR(\frac{1}{k})=R(\frac{1}{k})f\}$ where $R(\frac{1}{k})$ is the rotation of S^1 by $\frac{1}{k}$.

Let $p_k\colon G^{(k)}\to G$ and $i_k\colon G^{(k)}\to G$ be the natural projection and the inclusion. Henceforth we consider $G^{(k)}$ as a subgroup of G via i_k. Let us fix a splitting $s\colon G\to\tilde{G}$ to the projection $p\colon\tilde{G}\to G$ and let $\chi_s\in C^2(G;\boldsymbol{Z})$ be the corresponding cocycle representing the central extension $0\to\boldsymbol{Z}\to\tilde{G}\to G\to 1$. If we denote χ for the Euler class $[\chi_s]\in H^2(G;\boldsymbol{Z})$, then we have

Proposition 5.3. $p_k^*(\chi)=k\,i_k^*(\chi)$.

Proof. Let us define a group homomorphism $\varphi_k\colon\tilde{G}\to\tilde{G}$ by $\varphi_k\tilde{f}(x)=\frac{1}{k}\tilde{f}(kx)$, $\tilde{f}\in\tilde{G}$, $x\in\boldsymbol{R}$. Let f be an element of $G^{(k)}$ and set $f'=p_k(f)$. Thus we have $f'(kt)=k\,f(t)$where $t\in S^1$ and $k\colon S^1\to S^1$ is the k-fold covering map. Then it can be seen that there is an integer $d\in\boldsymbol{Z}$ such that $s(f)=\varphi_k s(f')T(\frac{d}{k})$, where $T(\frac{d}{k})$ denotes the translation of $\boldsymbol{R}$ by $\frac{d}{k}$. We define a cochain $\gamma\in C^1(G^{(k)};\boldsymbol{Z})$ by $\gamma(f)=d$. We can write $s(f)=\varphi_k(s(f')+\gamma(f))$. Now we claim that the equation

$$k\,\chi_s(f,g)-\chi_s(f',g')=\gamma(f)-\gamma(fg)+\gamma(g)$$

holds for all $f,g\in G^{(k)}$ ($f'=p_k(f)$ and $g'=p_k(g)$). Clearly this is enough for the proof. Now we calculate

$$\begin{aligned}
&k\,\chi_s(f,g)-\chi_s(f',g') = k(s(f)s(g)s(fg)^{-1})-s(f')s(g')s(f'g')^{-1}\\
&= k(\varphi_k(s(f')+\gamma(f))\varphi_k(s(g')+\gamma(g))\varphi_k(s(f'g')+\gamma(fg))^{-1})-s(f')s(g')s(f'g')^{-1}\\
&= k(\varphi_k(s(f')s(g')s(f'g')^{-1}+\gamma(f)+\gamma(g)-\gamma(fg)))-s(f')s(g')s(f'g')^{-1}\\
&= \gamma(f)-\gamma(fg)+\gamma(g)\,.
\end{aligned}$$

This completes the proof.

REMARK 5.4. Contrast to Prop. 5.3, it is easy to show that $i_k^*(\alpha)=k\,p_k^*(\alpha)$ even on the cocycle level.

Let $0\to\boldsymbol{Z}\to\tilde{H}\underset{s}{\overset{p}{\rightleftarrows}}H\to 1$ be a central extension with cocycle χ_s. Suppose that we have a group homomorphism $\rho\colon K\to H$. Then the cocycle $\rho^*\chi_s$ defines a central extension

$$0\to\boldsymbol{Z}\to(\boldsymbol{Z}\times K)_{\rho^*\chi_s}\to K\to 1$$

which may be called the pull back extension by ρ. If $s'\colon K\to(\boldsymbol{Z}\times K)_{\rho^*\chi_s}$ is the canonical splitting defined by $s'(k)=(0,k)$, then we have $\chi_{s'}=\rho^*\chi_s$. As for the central extension $0\to Z\to\tilde{G}\underset{s}{\rightleftarrows}G\to 1$ with the splitting s and group homomorphisms i_k and $p_k\colon G^{(k)}\to G$, we have

Lemma 5.5. (i) *The pull back extension by i_k is isomorphic to*

$$0 \to \boldsymbol{Z} \to \tilde{G}_k \underset{s}{\overset{p}{\rightleftarrows}} G^{(k)} \to 1$$

where $\tilde{G}_k = \{\tilde{f} \in \tilde{G};\ \tilde{f}(x+\frac{1}{k}) = \tilde{f}(x)+\frac{1}{k}\}$, *$p$ and s are the restrictions of the original ones.*

(ii) *The pull back extension by p_k is isomorphic to*

$$0 \to \boldsymbol{Z} \to \tilde{G} \times \boldsymbol{Z}/k \underset{s'}{\overset{p'}{\rightleftarrows}} G^{(k)} \to 1$$

where p' restricted to $\tilde{G}$ is defined to be $p' = p \circ \varphi_k$ and it sends the generator κ of the cyclic group $\boldsymbol{Z}/k$ to $R(-\frac{1}{k}) \in G^{(k)}$. $\boldsymbol{Z}$ is generated by (T, κ) and finally the splitting s' is given by $s'(f) = (s(p_k(f)), -\gamma(f) \bmod k)$.

Proof. Direct calculation.

In view of Prop. 5.1, 5.3 and Lemma 5.5 there is defined a homomorphism $\rho: \tilde{G}_k \to \tilde{G} \times \boldsymbol{Z}/k$ so that the following diagram is commutative

$$\begin{array}{ccccccc} 0 \to & \boldsymbol{Z} & \longrightarrow & \tilde{G}_k & \longrightarrow & G^{(k)} & \to 1 \\ & \downarrow \times k & & \downarrow \rho & & \| & \\ 0 \to & \boldsymbol{Z} & \to & \tilde{G} \times \boldsymbol{Z}/k & \to & G^{(k)} & \to 1 . \end{array}$$

We would like to identify this homomorphism.

Proposition 5.6. *The homomorphism $\rho: \tilde{G}_k \to \tilde{G} \times \boldsymbol{Z}/k$ is given by*

$$\rho(\tilde{f}) = (\varphi_k^{-1}(\tilde{f}), 0), \ (\tilde{f} \in \tilde{G}_k) .$$

(*Note that the homomorphism φ_k: $\tilde{G} \to \tilde{G}_k$ is an isomorphism.*)

Proof. Let $\tilde{f} \in \tilde{G}_k$ and we write $f = p(\tilde{f})$, $f' = p_k(f)$. By the definition of ρ (see Prop. 5.1), we have

$$\begin{aligned} \rho(\tilde{f}) &= \{k(\tilde{f} s(f)^{-1}) + \gamma(f)\} s'(f) \\ &= (\{k(\tilde{f} s(f)^{-1}) + \gamma(f)\} s(f'), \{k(\tilde{f} s(f)^{-1}) + \gamma(f)\} - \gamma(f) \bmod k \\ &= (\{\varphi_k^{-1}(\tilde{f} \varphi_k(s(f')^{-1} - \gamma(f))) + \gamma(f)\} s(f'), 0) \\ &= (\varphi_k^{-1}(f), 0) . \end{aligned}$$

This completes the proof.

Let us summarize the above results as

5.7. Diagram. The following diagram is commutative

$$\begin{array}{ccccccc} \boldsymbol{Z} & = & \boldsymbol{Z} & \xleftarrow{\times k} & \boldsymbol{Z} & = & \boldsymbol{Z} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{G} & \leftarrow & \tilde{G} \times \boldsymbol{Z}/k & \xleftarrow{(\varphi_k^{-1}, 0)} & \tilde{G}_k & \to & \tilde{G} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G & \xleftarrow{p_k} & G^{(k)} & = & G^{(k)} & \xrightarrow{i_k} & G . \end{array}$$

Now we have the following main result of this section.

Proposition 5.8. *For each element* $\sigma\in H_m(BG^\delta; \boldsymbol{Q})$, *let* $\sigma'(k)=\frac{1}{k}(i_k)_*(p_k)_*^{-1}(\sigma)$. *Then we have*

$$(\varphi_k)_*\mu(\sigma)=\mu(\sigma'(k)).$$

Moreover if $m=2n$, then

$$\langle\sigma'(k), \chi^n\rangle=k^{-(n+1)}\langle\sigma, \chi^n\rangle \quad and$$
$$\langle\sigma'(k), \alpha^n\rangle=k^{n-1}\langle\sigma, \alpha^n\rangle.$$

Proof. The first statement follows from Diagram 5.7 and Cor. 5.2. The latter can be proved by Prop. 5.3 and Remark 5.4.

6. Space of closed curves

Let X be a simply connected topological space and let ΩX (resp. $\Lambda X=\mathrm{Map}(S^1, X)$) be its loop space (resp. space of closed curves) with the compact open topology. The natural sequence $\Omega X\to\Lambda X\to X$ is a Hurewicz fibration. For a positive integer $k\in\boldsymbol{N}$, let ψ_k: $\Lambda X\to\Lambda X$ be the continuous map defined by $\psi_k(l)(t)=l(kt)$ $(l\in\Lambda X, t\in S^1)$. Then clearly ψ_k Spreserves ΩX and the following diagram is commutative

$$\begin{array}{ccccc}\Omega X & \to & \Lambda X & \to & X\\ \psi_k\downarrow & & \psi_k\downarrow & & \|\\ \Omega X & \to & \Lambda X & \to & X.\end{array}$$

Note that the homotopy group $\pi_m(\Lambda X)$ is naturally isomorphic to $\pi_{m+1}(X)\oplus\pi_m(X)$ and the action of ψ_k is given by multiplication by k on the first factor and the identity on the second. Here we would like to know the property of the linear map $(\psi_k)_*$: $H_*(\Lambda X; \boldsymbol{Q})\to H_*(\Lambda X; \boldsymbol{Q})$. For that we use the theory of minimal models due to Sullivan [14] which describes the rational homotopy theory of spaces in terms of differential graded algebras and maps between them. Thus for each simply connected space K of finite type (i.e. the homology group of K is finitely generated in each degree), there is associated a free differential graded algebra (abbreviated by d.g.a.) $\mathcal{K}$ over $\boldsymbol{Q}$, called the minimal model of K, which is essentially equivalent to the rational Postnikov tower of K. Also for each homotopy class of continuous map f: $K\to L$, there is associated a unique (up to homotopy) d.g.a. map $\hat{f}$: $\mathcal{L}\to\mathcal{K}$. Now let $\mathcal{K}=\Lambda(x)$ (free d.g.a. generated by the generators x) be the minimal model of K. Consider a free algebra $\Lambda\mathcal{K}=\Lambda(x, y)$, where degree y=degree $x-1$. Let s be the derivation of degree -1 on $\Lambda\mathcal{K}$ defined by $s(x)=y$ and $s(y)=0$. We define a differential d on $\Lambda\mathcal{K}$ inductively by the condition $sd+ds=0$. Thus dx is the same as in $\mathcal{K}$ and $dy=-sdx$. It is easy to check that d is a differential, i.e. $d^2=0$.

Theorem 6.1 (Sullivan [14]). *The d.g.a.* $\Lambda\mathcal{K}$ *is the minimal model of* ΛK.

The proof of this theorem is given roughly as follows (see [15] for details). Let e: $\Lambda K\times S^1\to K$ be the evaluation map. Then for any topological space C and a continuous map f: $C\times S^1\to K$, there is defined the adjoint map F: $C\to\Lambda K$ making the following diagram commutative

$$\begin{array}{ccc} \Lambda K\times S^1 & \xrightarrow{e} & K \\ F\times id\nwarrow & & \nearrow f \\ & C\times S^1 & . \end{array}$$

Conversely the space ΛK is characterized by this universal property. Now consider $\Lambda\mathcal{K}(\xi)$, where $d\xi=0$ with degree $\xi=1$, and define a d.g.a. map $\hat{e}$: $\mathcal{K}\to\Lambda\mathcal{K}(\xi)$ by $\hat{e}(x)=x+\xi y$. Then given any d.g.a. C and a map $\hat{f}$: $\mathcal{K}\to C(\xi)$, there is the adjoint map $\hat{F}$: $\Lambda\mathcal{K}\to C$ making the following diagram commutative

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\hat{e}} & \Lambda\mathcal{K}(\xi) \\ \hat{f}\nwarrow & & \nearrow\hat{F}\otimes id \\ & C(\xi) & . \end{array}$$

By the universality, this proves the assertion. Now we have

Proposition 6.2. *Let* K *be a simply connected space of finite type and let* $\mathcal{K}=\Lambda(x)$ *be its minimal model. Then the d.g.a. map* $\hat{\psi}_k$: $\Lambda\mathcal{K}\to\Lambda\mathcal{K}$ *corresponding to the map* ψ_k: $\Lambda K\to\Lambda K$ *is given by* $\hat{\psi}_k(x)=x$ *and* $\hat{\psi}_k(y)=ky$.

Proof. The map ψ_k: $\Lambda K\to\Lambda K$ is characterized by the following universal property. Let C be a topological space and let f: $C\times S^1\to K$ be a continuous map with the adjoint F: $C\to\Lambda K$. Consider the map f_k: $C\times S^1\to K$ given by $f_k(c, t)=f(c, kt)$ ($c\in C$, $t\in S^1$). If F_k: $C\to\Lambda K$ is the adjoint map of f_k, then the following diagram is commutative

$$\begin{array}{ccc} \Lambda K & \xrightarrow{\psi_k} & \Lambda K \\ F\nwarrow & & \nearrow F_k \\ & C & . \end{array}$$

Now let $\hat{f}$: $\mathcal{K}\to C(\xi)$ be the d.g.a. map corresponding to f. Then the d.g.a. map $\hat{f}_k$: $\mathcal{K}\to C(\xi)$ corresponding to f_k is clearly given by the composition $\mathcal{K}\xrightarrow{\hat{f}}C(\xi)$ $\xrightarrow{id\otimes k} C(\xi)$, where $id\otimes k=id$ on C and $(id\otimes k)(\xi)=k\xi$. Then it is easy to check the commutativity of the following diagram

$$\begin{array}{ccc} \Lambda\mathcal{K} & \xrightarrow{\hat{\psi}_k} & \Lambda\mathcal{K} \\ & \hat{F}\nwarrow \quad \nearrow\hat{F}_k & \\ & C & \end{array} .$$

By the universality this completes the proof.

Corollary 6.3. *For all non-negative integer m, the linear map* $(\psi_k)_*$: $H_m(\Lambda K; \boldsymbol{Q}) \to H_m(\Lambda K; \boldsymbol{Q})$ *is diagonalizable with eigen values of the form* k^p *where* p *is a non-negative integer. Namely* $H_m(\Lambda K; \boldsymbol{Q}) = \bigoplus_{p\geqq 0} V_m(p)$ *where* $V_m(p) = \{\sigma \in H_m(\Lambda K; \boldsymbol{Q}); (\psi_k)_*(\sigma) = k^p\sigma$ *for all* $k \geqq 0\}$.

Proof. Since we have assumed that the space K is of finite type, it is enough to prove the assertion for the cohomology group instead of homology. By Prop. 6.2 the linear map $(\psi_k)^*$: $H^*(\Lambda K; \boldsymbol{Q}) \to H^*(\Lambda K; \boldsymbol{Q})$ can be computed from the d.g.a. map $\hat{\psi}_k$: $\Lambda\mathcal{K} \to \Lambda\mathcal{K}$. Now for each non-negative integer p, let $(\Lambda\mathcal{K})_p$ be the subspace of $\Lambda\mathcal{K}$ generated by monomials on x and y whose number of the generators y are exactly p. By the definition of the differential in $\Lambda\mathcal{K}$, it is easy to see that $(\Lambda\mathcal{K})_p$ is a subcomplex of $\Lambda\mathcal{K}$ and $\Lambda\mathcal{K} = \bigoplus_{p\geqq 0}(\Lambda\mathcal{K})_p$. Moreover the map $\hat{\psi}_k$ preserves these subcomplexes. Clearly $\hat{\psi}_k$=multiplication by k^p on $(\Lambda\mathcal{K})_p$. Hence

$$H^*(\Lambda K; \boldsymbol{Q}) \cong H^*(\Lambda\mathcal{K}) = \bigoplus_{p\geqq 0} H^*((\Lambda\mathcal{K})_p)$$

and $(\hat{\psi}_k)^*$=multiplication by k^p on $H^*((\Lambda\mathcal{K})_p)$.

We generalize this to arbitrary space.

Proposition 6.4. *Let X be a simply connected topological space. Then for any non-negative integer m, we have a direct sum decomposition* $H_m(\Lambda X; \boldsymbol{Q}) = \bigoplus_{p\geqq 0} V_m(p)$, *where* $V_m(p) = \{\sigma \in H_m(\Lambda X; \boldsymbol{Q}); (\psi_k)_*(\sigma) = k^p\sigma$ *for all* $k \geqq 1\}$.

Proof. First by considering the geometric realization of the singular chain complex of X, we may assume that X is a CW complex. Now let $\sigma \in H_m(\Lambda X; \boldsymbol{Q})$. Then there is a finite complex L and a continuous map F: $L \to \Lambda X$ such that $\sigma = F_*(\tau)$ for some $\tau \in H_m(L; \boldsymbol{Q})$. Let f: $L \times S^1 \to X$ be the adjoint of F. Since X is simply connected and $L \times S^1$ is compact there is a simply connected finite subcomplex K of X such that $f(L \times S^1) \subset K$. It follows that $F(L) \subset \Lambda K$. Thus the element σ comes from $H_m(\Lambda K; \boldsymbol{Q})$. Since the map ψ_k: $\Lambda X \to \Lambda X$ preserves the subspace ΛK we can apply Cor. 6.3 and we are done.

7. Proof of Theorem 1.1

In this section we prove Theorem 1.1. In view of the results of §4 (Th.

4.1 and Th. 4.3), we have only to prove the nontriviality of the classes χ^n ($n \in \boldsymbol{N}$). For this we use Mather-Thurston's result Th. 4.5 that the map H: $B\tilde{G}^\delta \to \Lambda B\overline{\Gamma}_1$ induces an isomorphism on homology. For a positive integer k let φ_k: $B\tilde{G}^\delta \to B\tilde{G}^\delta$ be the map corresponding to the homomorphism φ_k: $\tilde{G} \to \tilde{G}$ (we use the same symbol) and let ψ_k: $\Lambda B\overline{\Gamma}_1 \to \Lambda B\overline{\Gamma}_1$ be the continuous map defined as before. Then we have

Proposition 7.1. *The following diagram is homotopy commutative*

$$\begin{array}{ccc} B\tilde{G}^\delta & \xrightarrow{H} & \Lambda B\overline{\Gamma}_1 \\ \varphi_k\downarrow & & \downarrow\psi_k \\ B\tilde{G}^\delta & \xrightarrow{H} & \Lambda B\overline{\Gamma}_1 \end{array}$$

Proof. It is easy to see that the following diagram is homotopy commutative

$$\begin{array}{ccc} B\tilde{G}^\delta \times S^1 & \xrightarrow{\varphi_k \times id} & B\tilde{G}^\delta \times S^1 \\ id \times k\downarrow & & \downarrow \\ B\tilde{G}^\delta \times S^1 & \longrightarrow & B\overline{\Gamma}_1 \ . \end{array}$$

The adjoint of this diagram is the desired one.

7.2. Proof of the nontriviality of χ^n. It is enough to prove the existence of elements $\sigma_n \in H_{2n}(BG^\delta; \boldsymbol{Z})$ with $\langle \sigma_n, \chi^n \rangle \neq 0$ for all n, and by the universal coefficient theorem we may use the rational homology. We use the induction on n. For $n=1$ we already know the existence of σ_1. Thus suppose that σ_n is an element of $H_{2n}(BG^\delta; \boldsymbol{Q})$ with $\langle \sigma_n, \chi^n \rangle \neq 0$. By Prop. 6.4 and Prop. 7.1, there is a direct sum decomposition $H_{2n+1}(B\tilde{G}^\delta; \boldsymbol{Q}) = \bigoplus_{p \geq 0} V_{2n+1}(p)$, where $V_{2n+1}(p) = \{\tau \in H_{2n+1}(B\tilde{G}^\delta; \boldsymbol{Q}); (\varphi_k)_*(\tau) = k^p \tau$ for all $k \geqq 0\}$. Now let

$$\mu(\sigma_n) = \sum_{p \geq 0} \tau_p \qquad (\tau_p \in V_{2n+1}(p))$$

be the corresponding decomposition. Of course only finitely many τ_p are non-zero. By Prop. 5.8 we have $(\varphi_k)_* \mu(\sigma_n) = \mu(\sigma'_n(k))$ for some $\sigma'_n(k)$ with $\langle \sigma'_n(k), \chi^n \rangle = k^{-(n+1)} \langle \sigma_n, \chi^n \rangle$. On the other hand we have $(\varphi_k)_* \mu(\sigma_n) = \sum_p k^p \tau_p$ and hence

$$\mu(\sigma'_n(k)) = \sum_{p \geq 0} k^p \tau_p \,.$$

Then it is easy to see that some linear combination of σ_n and $\sigma'_n(k)$'s, for example

$$\bar{\sigma}_n = \sigma_n + \sum_{j=1}^{r} r_j \, \sigma'_n(k^j) \qquad (r_j \in \boldsymbol{Q})$$

where r=number of non-zero τ_p's and $k > 1$, should satisfy the equations $\mu(\bar{\sigma}_n)$

$=0$ and $\langle\bar{\sigma}_n, \chi^n\rangle \neq 0$. Then from the Gysin sequence (Prop. 2.1), there is an element $\sigma_{n+1} \in H_{2n+2}(BG^\delta; \boldsymbol{Q})$ such that $\sigma_{n+1} \cap \chi = \bar{\sigma}_n$ and hence $\langle\sigma_{n+1}, \chi^{n+1}\rangle \neq 0$. This completes the proof.

REMARK 7.3. The above proof of the nontriviality of χ^n is not constructive. Also it does not work in the real analytic context. From this point of view, it would be interesting to try to prove the diagonalizability of the linear map $(\varphi_k)_*: H_*(B\tilde{G}^\delta; \boldsymbol{Q}) \rightarrow H_*(B\tilde{G}^\delta: \boldsymbol{Q})$ directly.

REMARK 7.4. Haefliger has communicated to the author that he has modified the argument (7.2) to obtain a more direct and simple proof. He first notes that the map H in Th. 4.5 is equivariant with respect to the natural G-actions on both sides and then uses the Sullivan model for the corresponding Borel fibrations. From this argument we can conclude, in particular, that there is an element $\sigma \in H_{2m}(BG^\delta; \boldsymbol{Z})$ with $\langle\sigma, \chi^n\rangle = 1$. This follows from the fact that the action of G on the space $\Lambda B\bar{\Gamma}_1$ has fixed points. However it seems to the author that the argument (7.2) remains to be useful because of its explicit nature.

References

[1] R. Bott: *On some formulas for the characteristic classes of group actions*, Differential Topology, Foliations and Gelfand-Fuks Cohomology, Lecture Notes in Math. 652, Berlin-Heidelberg-New York, Springer 1978.

[2] I.M. Gelfand and D.B. Fuchs: *Cohomologies of the Lie algebra of tangential vector fields of a smooth manifold*, I and II, Funkcional. Anal. i Priložen. **3** (1969), no. 3, 32–52 and **4** (1970), no. 2, 23–31.

[3] C. Godbillon and J. Vey: *Un invariant des feuilletages de codimension* 1, C.R. Acad. Sci. Paris **273** (1971), 92–95.

[4] A. Haefliger: *Sur les classes caractéristiques des feuilletages*, Séminaire Bourbaki, No. 412, June 1972.

[5] M. Herman: *Simplicité du groupe des difféomorphismes de class C^∞, isotopes à l'identité, du tore de dimension n*, C.R. Acad. Sci. Paris **273** (1971), 232–234.

[6] J. Mather: *On Haefliger's classifying space* I, Bull. Amer. Math. Soc. **77** (1971), 1111–1115.

[7] J. Mather: *Integrability in codimension* 1, Comment. Math. Helv. **48** (1973), 195–233.

[8] J. Mather: *Commutators of diffeomorphisms*, Comment. Math. Helv. **49** (1974), 512–528.

[9] T. Mizutani, S. Morita and T. Tsuboi: *Foliated J-bundles and the Godbillon-Vey class of codimension one foliations*, preprint.

[10] T. Mizutani and T. Tsuboi: *Foliations without holonomy and foliated bundles*, Sci. Rep. Saitama Univ. **9** (1979), 45–55.

[11] S. Morita and T. Tsuboi: *The Godbillon-Vey class of codimension one foliations*

without holonomy, Topology **19** (1980), 43–49.

[12] S. Morita: *Discontinuous invariants of foliations*, to appear.

[13] P.A. Schweitzer: *Problems* in: Differential Topology, Foliations and Gelfand-Fuks Cohomology, Lecture Notes in Math. 652, Berlin-Heidelberg-New York, Springer, 1978.

[14] D. Sullivan: *Infinitesimal computations in topology*, Inst. Hautes Édudes Sci. Publ. Math. **47** (1977), 269–331.

[15] D. Sullivan and M. Vigué: *The homology theory of the closed geodesic problem*, J. Differential Geom. **11** (1976), 633–644.

[16] W. Thurston: *Noncobordant foliations of S^3*, Bull. Amer. Math. Soc. **78** (1972), 511–514.

[17] W. Thurston: *Foliations and groups of diffeomorphisms*, Bull. Amer. Math. Soc. **80** (1974), 304–307.

Department of Mathematics
College of Arts and Sciences
University of Tokyo
Komaba, Tokyo 153
Japan