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FREDHOLM DETERMINANT FOR PIECEWISE LINEAR TRANSFORMATIONS

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1. Introduction

Let F be a piecewise linear transformaion from a finite union of bounded intervals I into itself and P be the Perron-Frobenius operator assoicated with it. Since F is piecewise smooth, P can be expressed as

$$Pf(x) = \sum_{y\,:\,F(y)=x} f(y)\,|F'(y)|^{-1} \qquad \text{a.e.x}$$

for $f \in L^1$, the set of all Lebesgue integrable functions on I. In this paper, we assume that

$$(1) \qquad \xi = \operatorname*{essinf}_{x\in I} \liminf_{n\to\infty} n^{-1} \log |F^{n\prime}(x)| > 0\,.$$

We call the number ξ the lower Lyapunov number. We will study Spec(F), the spectrum of $P|_{BV}$, the restriction of P to the subspace BV of functions with bounded variation. The generating function of P is determined by the orbits of the division points of the partition, and the orbits are characterized by a finite dimensional matrix $\Phi(z)$ which is defined by a renewal equation (§ 3). Hence, we can show that $D(z)=\det(I-\Phi(z))$, which we call a Fredholm determinant, is the determinant of $I-zP=\sum_{n=0}^{\infty} z^n P^n$ in the following sense:

Theorem A. *Let $\lambda \in C$ and assume that $|\lambda| > e^{-\xi}$. Then λ belongs to* Spec(F) *if and only if $z=\lambda^{-1}$ is a zero of $D(z)$:*

$$D(\lambda^{-1}) = 0\,.$$

Furthermore, such λ is an eigenvalue of $P|_{BV}$.

We can calculate the eigenvalues of $P|_{BV}$ concretely by this theorem. Some examples are shown in § 6. Also we can prove the following intrinsic characterization of the power series $D(z)$ from a detailed re-examination of the renewal equation. Let us denote the following Ruelle-Artin-Mazur zeta function by $\zeta(z)$:

$$\zeta(z)=\exp\sum_{n=1}^{\infty}n^{-1}z^{n}\sum_{p\,:\,F^{n}(p)=p}|F^{n\prime}(p)|^{-1}.$$

Theorem B. *If a mapping F satisfies the endpoint condition stated in* § 2, *then*

$$\zeta(z)=1/D(z). \tag{2}$$

In particular, $\zeta(z)^{-1}$ is analytic in the domain $|z|<e^{\xi}$.

As it will be shown later as Theorem C of § 5, the assertion (2) remains true with a slight modification even if the endpoint condition is violated.

A heuristic argument for Theorems A and B was given in [16]. Some rigorous results were given in [18] for unimodal linear transformations. It is shown in [4] that for a piecewise linear transformation singularities of the zeta function coincide with reciprocals of eigenvalues of P (cf. also [10]). We get the another proof of this by combining Theorems A and C. However, it is difficult to calculate the zeta function, because, in general, it is almost impossible to calculate all the periodic orbits. Some related topics can also be found in [20].

If a piecewise linear transformation F is Markov, one can immediately find a subspace on which the restriction of P is the dual of the transition matrix T of the finite Markov chain associated with F. Then $D(z)=\det(I-zT)$ is well-defined and the dual of $I-zT$ can be identified with the operator $I-zP$ restricted to the subspace. On the other hand, it follows from a direct computation for a Markov map that $\zeta(z)$ is essentially equal to $\{\det(I-zP)\}^{-1}$. Hence we obtain a result similar to Theorem B (which is a corollary of Theorem C). The absence of Markov property is the first crusial difficulty. Another difficulty lies in the non-compactness of P.

The Perron-Frobenius operator is originally a nonnegative contraction operator on the space L^1 which is defined as a dual of F-action on bounded measurable functions. On one hand, it is known that the eigenvalues of modulus one of P on L^1 determine the ergodic properties of F. For instance (cf. [11], [12], [15]):

a. The dimension of the eigenspace corresponding to eigenvalue one equals the number of the ergodic components of the dynamical system.

b. On each ergodic component, if one is the unique eigenvalue on the unit circle of P restricted to the ergodic component, then the restricted dynamical system is mixing.

On the other hand, it is found in [18] that the spectrum of P on L^1 has strange property such that any $\lambda\in\boldsymbol{C}$ with $|\lambda|<1$ is an eigenvalue of P with infinite multiplicity. Even if we restrict P to BV, as it was shown in [9],

$$\{z:\ |z|<e^{-h(F)}\}\subset\mathrm{Spec}(F),$$

where $h(F)$ is the topological entropy of the dynamical system. Thus neither P on L^1 nor $P|_{BV}$ is compact. Hence, we cannot define the determinant in the usual sense ([2]). Nevertheless, the eigenfunctions corresponding to the eigenvalues of modulus one belong to BV and the ergodic theorem of Ionescu-Tulcea type works well for the pair (L^1, BV) ([11], [12], [15]).

The first idea to overcome these difficulties is to construct a renewal equation for time correlation functions. The renewal equation is well-known concept in the theory of Markov chains and it describes the time evolution of certain functionals on the path space of Markov chain which may not have Markov properties ([3]). Mappings on intervals are not so far from Markov property as is suggested for instance by [1]: a weakly mixing mapping of an interval is isomorphic to Bernoulli. It is also supported by [13], [14] and [19]. Hence it is not so surprizing to expect that a renewal equation governs the time evolution of good test functions under piecewise linear transformations. The resultant renewal equation enables us to define our Fredholm determinant although F does not generally satisfy the Markov property. To deduce the renewal equation, we use a symbolic dynamics.

The second idea is to introduce the signed symbolic dynamics, since the ordinary symbolic dynamics is almost useless (at least to the author) to deduce a renewal equation. We prepare signed alphabets, signed words and signed sentences. The signed alphabets a^+, a^- will be regarded as two endpoints of the subinterval (a) which corresponds to an ordinary alphabet a. Then we have to manage double copies of ordinary words, but this method makes the structure of the dynamical system transparent. We can treat it as if it has Markov property and construct a renewal equation on the signed symbolic dynamics. Due to this renewal equation, our Fredholm determinant is defined and it can be shown to determine the spectrum of the Perron-Frobenius operator (Theorem A). Furthermore, this renewal equation inherits some of dynamical structure of F. Particularly, it carries the information of periodic orbits. Hence, appealing to the intermediate value theorem, it enables us to calculate the zeta function (Theorem C) (after rather annoying enumeration).

Finally we remark that we can treat some of more general Ruelle-Artin-Mazur zeta functions

$$\zeta(z) = \exp\Big[\sum_{n-1}^{\infty} z^n n^{-1} \sum_{p\,:\,F^n(p)=p} \exp \sum_{i=0}^{n-1} U(F^i(p))\Big]$$

with weight function U. We can similarly deduce the Fredholm determinant to prove the analogues of Theorems A and B. In addition to our weight function $U(p)=-\log|F'(p)|$, the special cases of our interest are as follows:

a. $U(p)=0$: there follows the Artin-Mazur zeta function

$$\zeta(z) = \exp\Big[\sum_{n=1}^{\infty} z^n n^{-1} \# \{p\colon F^n(p)=p\}\Big],$$

b. $U(p)=i\theta f(p)-\log|F'(p)$: this is the case of imaginary perturbed Perron-Frobenius operator

$$P(\theta:f)g(x)=P(e^{i\theta f}g(x)=\sum_{y:F(y)=x} g(y)e^{i\theta f(y)}|F'(y)|^{-1},$$

which is shown to be powerfull tool to prove the central limit theorem (cf. [6], [17]).

In the next § 2, we will state the endpoint and the other conditions and define the signed symbolic dynamics. In § 3, we will introduce the renewal equation on the signed symbolic dynamics and the Fredholm determinant. In § 4, we prove Theorem A, by dividing it into two cases according as the mapping satisfies the endpoint condition or not. In § 5, we prove Theorem C and then Theorem B follows as its corollary. In the final 6, we will calculate the Fredholm determinant for several examples.

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2. Preliminaries

2.1 Endpoint Condition

Let I be a finite union of bounded intervals and F be a piecewise linear transformation on I. Then we can take a finite partition of I into subintervals $\{I_a\}_{a\in A}$ so that F is linear on each subinterval I_a. We denote by F^n the n-th iterate of F:

$$F^n(x)=\begin{cases} x & n=0\,, \\ F(F^{n-1}(x)) & n\geqq 1\,. \end{cases}$$

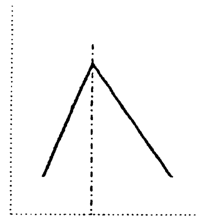
For a set J, we denote by $\mathrm{int}J$, $\mathrm{cl}J$ and ∂J the interior, the closure and the boundary of J, respectively. We denote the set of division points of the partition $\{I_a\}_{a\in A}$ by

$$\Delta I=\bigcup_{a\in A}\partial I_a\,.$$

Now we will state the endpoint condition. We will divide the proof of Theorem A into two cases according as the mapping F satisfies this condition or not (see § 3, § 4). Note that Markov mappings do not satisfy this endpoint condition.

Endpoint condition: If for $x, y\in\Delta I$, there exist nonnegative integers m, n and sequences $\{x_k\}$, $\{y_k\}\subset I$ such that $\lim_{k\to\infty} x_k=x$ and $\lim_{k\to\infty} y_k=y$ and $\lim_{k\to\infty} F^m(x_k)=\lim_{k\to\infty} F^n(y_k)$. Then $m=n$ and $x=y$.

The endpoint condition means that the orbits of different endpoints do not meet except the case of fig. (I) (here we consider that the point p expresses



P

Fig. I

two different endpoints: one belongs to the left hand side and the other belongs to the right hand side).

We assume until § 4 that the partition $\{I_a\}_{a\in A}$ of I is minimal in the following sense:

MINIMALITY CONDITION: If $x\in \Delta I\cap \mathrm{int}(\mathrm{cl}I)$, then there holds either $\lim_{y\uparrow x} F(y)\neq \lim_{y\downarrow x} F(y)$ or $\lim_{y\uparrow x} F'(y)\neq \lim_{y\downarrow x} F'(y)$.

This minimality condition is assumed in the proof of Theorem A. Afterwards in § 5, we can remove this technical condition by using a relation between the Fredholm determinant and the zeta function which we prove in Theorem C.

2.2 Alphabets, Words and Sentences

Hereafter, to make the notation simple, we denote the set I_a by (a). We call each element $a\in A$ an alphabet. For an alphabet $a\in A$, we set

$$F'(a) = F'(x) \qquad x\in \mathrm{int}\,(a)\,,$$
$$\mathrm{sgn}\,a = \mathrm{sgn}\,F'(a) = \begin{cases} + & \text{if } F'(a)>0\,, \\ - & \text{if } F'(a)<0\,. \end{cases}$$

We call a finite sequence of alphabets a word and for a word $w=a_1\cdots a_n$ $(a_i\in A)$, we denote

$$|w| = n \text{ (the length of word } w)\,,\quad w[k] = a_k\,,$$
$$w[k, m]=a_k\cdots a_m \ (1\leqq k\leqq m\leqq n)\,,\quad \theta^k w = w[k+1,\ |w|] = a_{k+1}\cdots a_n\,,$$
$$(w) = \bigcap_{i=1}^{n} F^{-i+1}((a_i)) \text{ (the subinterval corresponding to } w)\,,$$
$$F(w) = F^{|w|}|_{(w)}\,,\quad F'(w) = \prod_{i=1}^{n} F'((a_i)) = F^{|w|\prime}(x)\ \ (x\in \mathrm{int}(w))\,,$$
$$\mathrm{sgn}\,w = \mathrm{sgn}\,F'(w) = \prod_{i=1}^{n} \mathrm{sgn}\,a_i\,,\ (+\cdot+=-\cdot-=+,\ +-\cdot=-\cdot+=-)\,.$$

For given two words $w_1=a_1\cdots a_n$ and $w_2=b_1\cdots b_m$, we denote $w_1\cdot w_2=a_1\cdots a_n b_1\cdots b_m$. We call a word w admissible if $(w)\neq\phi$. We denote the set of all admissible words with length n by W_n and we denote $W=\bigcup_{n=0}^{\infty} W_n$ where W_0

is the set which consists only of the empty word e. The empty word e is characeterized as follows: for any word w, $e\cdot w=e\cdot w=w$. We set $\operatorname{sgn} e=+$, $(e)=I$ and $w[m, n]=e$ for any word w if $m>n$. We call an infinite sequence of alphabets $\alpha=a_1a_2\cdots$ a setence and similarly we denote

$$\alpha[k] = a_k\,, \quad \alpha[k, m] = a_k\cdots a_m \ (k\leqq m)\,,$$
$$\alpha[k, \infty) = a_ka_{k+1}\cdots\,, \quad \{\alpha\} = \bigcap_{n=1}^{\infty} \mathrm{cl}(\alpha[1, n])\,.$$

We call a sentence α admissible if $\{\alpha\}\neq\phi$. We denote the set of all admissible sentences by S and the set of sentences $\alpha\in S$ for which $\bigcap_{n=1}^{\infty}(\alpha[1, n])\neq\phi$ by S^0.

Lemma 2.1. *For any $\varepsilon>0$, there exists an integer $N=N(\varepsilon)$ such that if* $u\in\bigcup_{n\geqq N} W_n$

$$(3)\qquad |F'(u)|<e^{(\xi-\varepsilon)|u|}\,,$$

and

$$(3')\qquad \text{Lebes}\,(u)<e^{-(\xi-\varepsilon)|u|}\,\text{Lebes}\,I\,,$$

where Lebes J *is the Lebesgue measure of a set* J.

The proof follows directly from the definition (1) of the lower Lyapunov number.

Hence, the assumption $\xi>0$ implies that the set $\{\alpha\}$ consists of exactly one point if α is admissible. We adopt the following notations: the expression $x<_\sigma y$ ($\sigma\in\{+, -\}$) means $x<y$ if $\sigma=+$ and $x>y$ if $\sigma=-$, and

$$\delta[L] = \begin{cases} 1 & \text{if a statement } L \text{ is true}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We sometimes denote $\delta[x<y\colon \sigma]=\delta[x<_\sigma y]$.

We define orders on the alphabet set, the word set and the sentence set in the following way.

1. For alphabets $a_1, a_2\in A$;

$$a_1<a_2 \text{ if } x_1<x_2 \text{ for } x_i\in(a_i) \quad (i = 1, 2)\,.$$

2. For words w_1, w_2;
$w_1<w_2$ if there exists an integer i such that $w_1[1, i]=w_2[1, i]$ and $w_1[i+1]<_\sigma w_2[i+1]$ where $\sigma=\operatorname{sgn} w_1[1, i]$.

3. For sentences α_1, α_2;
$\alpha_1<\alpha_2$ if for some integer n, the inequality $\alpha_1[1, n]<\alpha_2/[1, n]$ holds.

Lemma 2.2. *a) Suppose that both words w_1, w_2 are admissible (i.e. $(w_i)\neq\phi$*

$i=1, 2$), *then* $w_1<w_2$ *if and oaly if* $x_1<x_2$ *for* $x\in(w_i)$ ($i=1, 2$).

b) *Suppose that both sentences* α_1, α_2 *are admissible* ($\alpha_i\in S$ $i=1, 2$) *and* $\{\alpha_1\}\neq\{\alpha_2\}$, *then* $\alpha_1<\alpha_2$ *if and only if* $\{\alpha_1\}<\{\alpha_2\}$.

The proof is trivial, because for a word w and $x, y\in(w)$, if $x<y$ then the inequality $F^{|w|}(x)<_{\mathrm{sgn}\,w}F^{|w|}(y)$ holds.

2.3. Plus and Minus Expansions

For $x\in I$, we define a sentence $\alpha^x=a_1^x a_2^x\cdots\in S$, called the expansion of x, by the conditions $F^{i-1}(x)\in(a_i^x)$ for all i. Then, $x=\{\alpha^x\}$ since $\xi>0$. We sometimes identify the point x with its expansion α^x as notational convention. For instance for a word $w=a_1\cdots a_n$ we denote by $w\cdot x$ the sentence $a_1\cdots a_n a_1^x a_2^x\cdots$. Thus for $\alpha\in S^0$ there exists a point in I whose expansion equals α.

We will define signed expansions. For a word w, the plus expansion $w^+\in S$ and the minus expansion $w^-\in S$ are defined by $w^+=\sup_{y\in\mathrm{int}(w)}\alpha^y$ and $w^-=\inf_{y\in\mathrm{int}(w)}\alpha^y$. Hence $\{w^+\}$ (respectively $\{w^-\}$) is the limit from $\mathrm{int}(w)$ to $\sup\{x\in(w)\}$ (respectively $\inf\{x\in(w)\}$). Now set

$$\tilde{A}=\{a^+, a^-: a\in A\}\ \text{(the signed alphabet set)},$$

$$\tilde{W}_n=\{w^+, w^-: w\in W_n\},\qquad (\tilde{W}_1=\tilde{A}),$$

$$\tilde{W}=\bigcup_{n=0}^{\infty}\tilde{W}_n\ \text{(the singed word set)},$$

$$\tilde{B}=\{w^\sigma[n, \infty): w^\sigma\in\tilde{W},\ n\geqq 1,\ \sigma\in\{+, -\}\}.$$

We regard that $\tilde{A}\subset\tilde{W}\subset\tilde{B}\subset S$. We denote the sign of the word w^σ by $\mathcal{E}(w^\sigma)$ ($\sigma\in\{+, -\}$):

$$\mathcal{E}(w^\sigma)=\sigma$$

and we use the conventions that $\mathcal{E}(w^\sigma[n, \infty)))=\sigma$ for $n\geqq 1$ when such an expression appears in below. Similarly we define an order on $\tilde{W}$ by

a. $w^+>w^-$,

b. $w_1^\sigma>w_2^\tau$ if $w_1>w_2$ ($\sigma, \tau\in\{+, -\}$).

Moreover, we denote $x<a^\sigma$ if $\alpha^x<a^\sigma$.

Let

$$(\sup I)^+=\sup_{x\in I}\alpha^x,$$

and

$$(\inf I)^-=\inf_{x\in I}\alpha^x.$$

For $\tilde{a}\in\tilde{A}$, we set

$$\tilde{a}^\wedge = \begin{cases} \inf\{\tilde{b}\in\tilde{A}\colon \tilde{b}>\tilde{a}\} & \text{if } \varepsilon(\tilde{a})=+ \text{ and } \tilde{a}\neq(\sup I)^+, \\ \sup\{\tilde{b}\in\tilde{A}\colon \tilde{b}<\tilde{a}\} & \text{if } \varepsilon(\tilde{a})=- \text{ and } a\neq(\inf I)^-, \\ (\inf I)^- & \text{if } \tilde{a}=(\sup I)^+, \\ (\sup I)^+ & \text{if } \tilde{a}=(\inf I)^-. \end{cases}$$

If $\{\tilde{a}\}\notin\Delta I\cap\operatorname{int}(\operatorname{cl} I)$, then $\tilde{a}^\wedge$ is its alternative, that is, $\tilde{a}\neq\tilde{a}^\wedge$ and $\{\tilde{a}\}=\tilde{a}^\wedge\}$. Since the restriction of the mapping F on (a) $(a\in A)$ can be extended to $\operatorname{cl}(a)$ by continuity, we can define naturally the values $F(\tilde{a})$ and $F'(\tilde{a})$. In this way the mapping F is identified with the shift operator on S, that is, $F(\alpha)=\{\alpha[2,\infty)\}$. Moreover, the endpoint condition and the minimality condition can be expressed by singed alphabets as follows.

1. Endpoint condition: Assume that $\{\tilde{a}[n,\infty)\}=\{\tilde{b}[m,\infty)\}$ for some integers m and n and $\tilde{a},\tilde{b}\in\tilde{A}$, then $m=n$ and $\{\tilde{a}\}=\{\tilde{b}\}$, i.e. there exist two cases (1) $\tilde{a}=\tilde{b}$ and (2) $\tilde{a}=\tilde{b}^\wedge$ and $F(\tilde{a})=F(\tilde{b})$ (see fig. (I)).

2. Minimality condition: For $\tilde{a}\in\tilde{A}$ ($\tilde{a}\neq(\inf I)^-$ or $(\sup I)^+$), either $F(\tilde{a})\neq F(\tilde{a}^\wedge)$ or $F'(\tilde{a})\neq F'(\tilde{a}^\wedge)$ holds.

2.4. Perron-Frobenius Operator

Definition 2.1. The Perron-Frobenius operator P associated with a mapping F is defined by the formula

$$\int f(x)g(F(x))dx = \int Pf(x)\cdot g(x)dx$$

for $f\in L^1$ and $g\in L^\infty$.

It is well-known that P is a nonnegative operator on L^1 with operator norm 1 and since F is piecewise smoth, it can be expressed in the form:

$$Pf(x) = \sum_{y\,:\,F(y)=x} f(y)\,|F'(y)|^{-1} \qquad \text{a.e.x,}$$

and

$$P^n f(x) = \sum_{y\,:\,F^n(y)=x} f(y)\,|F^{n\prime}(y)|^{-1} \qquad \text{a.e.x.}$$

The further basic properties of the Perron-Frobenius operator can be found in [15].

Definition 2.2. We define a formal series

$$\begin{aligned}(f,g)(z) &= \sum_{n=0}^{\infty} z^n \int f(x)g(F^n(x))dx \\ &= \sum_{n=0}^{\infty} z^n \int P^n f(x)\cdot g(x)dx\,.\end{aligned}$$

Note that $(f,g)(z)(f\in L^1, g\in L^\infty)$ converges in $|z|<1$.

Lemma 2.3. *For a set J, set*

$$s^J(z: x) = \sum_{n=0}^{\infty} z^n \sum_y |F^{n\prime}(y)|^{-1},$$

where the sum $\sum_y$ is taken over those points $y \in J$ such that $F^n(y)=x$. Then we get for any $g \in L^\infty$

$$(1_J, g)(z) = \int s^J(z: x) g(x) dx,$$

where 1_J is the indicator function of a set J.

The proof is trivial.

For a aord $w \in W$, we write $s^w(z: x)$ and 1_w instead of $s^{(w)}(z: x)$ and $1_{(w)}$, omitting brakets.

3. Renewal Equation and Fredholm Determinant

In this section, we assume that the mapping F satisfies the endpoint condition. We are going to construct a renewal equation for

$$s^a(z: x) = s^{(a)}(z: x) = \sum_{n=0}^{\infty} z^n \sum_y |F^{n\prime}(y)|^{-1},$$

where the sum $\sum_y$ is taken over those points $y \in (a)$ such that $F^n(y)=x$. Now let us divide the series into four parts:

$$s^a(z: x) = 1_a(x) + z|F'(a)|^{-1} \sum_{b: F((a)) \supset (b)} s^b(z: x)$$
$$+ z|F'(a)|^{-1} s^{J_+}(z: x) + z|F'(a)|^{-1} s^{J_-}(z: x).$$

The second term is the main term. The third and the fourth terms are residual terms which correspond to the intervals J_σ with endpoints $\{a^\sigma[2, \infty)\}$ and $\{a^\sigma[2]^{-\sigma}\}$ $(\sigma \in \{+, -\})$. By repeating to renew the residual terms, we will reach to an expression which depends only on the orbits of the end points of (a). But it is too difficult to carry out this idea directly for $s^a(z: x)$. Thus our idea here is to divide $s^a(z: x)$ into two terms:

$$s^a z: x) = s^{a^+}(z: x) + s^{a^-}(z: x),$$

where the functions $s^{a^\sigma}(z: x)$ are to depend only on a^σ.

Now we need several notations to define s^{a^+} and s^{a^-}. For $\tilde{\alpha} \in \tilde{B}$ and $\beta \in S$, we denote

$$\chi(\alpha, \beta) = \delta[\beta < \tilde{\alpha}: \varepsilon(\tilde{\alpha})] - 1/2 = \begin{cases} +1/2 & \text{if } \beta <_{\varepsilon(\tilde{\alpha})} \tilde{\alpha}, \\ -1/2 & \text{otherwise}. \end{cases}$$

For $\tilde{\alpha} \in \tilde{B}$ and $a \in A$, we define

$$\psi(\tilde{\alpha}, a) = \delta[a < \tilde{\alpha}[2]: \varepsilon(\tilde{\alpha})\, sgn\, \tilde{\alpha}[1]] + \delta[a = \tilde{\alpha}[2], sgn\, \tilde{\alpha}[1] = -] - 1/2.$$

Hence $\psi(\tilde{\alpha}, a)$ takes values $\pm 1/2$ and the equality $\psi(\tilde{\alpha}, a)=+1/2$ holds if and only if one of the following conditions is satisfied:

1. $\varepsilon(\tilde{\alpha})=\operatorname{sgn}\tilde{\alpha}[1]=+$ and $a<\tilde{\alpha}[2]$,
2. $\varepsilon(\tilde{\alpha})=+$, $\operatorname{sgn}\tilde{\alpha}[1]=-$ and $a\geqq\tilde{\alpha}[2]$,
3. $\varepsilon(\tilde{\alpha})=-$, $\operatorname{sgn}\tilde{\alpha}[1]=+$ and $a>\tilde{\alpha}[2]$,
4. $\varepsilon(\tilde{\alpha})=\operatorname{sgn}\tilde{\alpha}[1]=-$ and $a\leqq\tilde{\alpha}[2]$.

Notice here in the case where $\operatorname{sgn}\tilde{\alpha}[1]=-$, the extra term $\delta[a=\tilde{\alpha}[2], \operatorname{sgn}\tilde{\alpha}[1]=-]$ is added.

Now we introduce for $\tilde{\alpha}\in\tilde{\beta}$ the following functions:

$$s^{\tilde{\alpha}}(z: x)=\chi(\tilde{\alpha}, x)+\sum_{w\in W}\delta[w[1]=\tilde{\alpha}[1], \theta w\cdot x\in S^0]z^{|w|}|F'(w)|^{-1}\chi(\tilde{\alpha}, w\cdot x),$$

Note here we identify x and its expansion α^x. Here it must be emphasized that the sentences $w.x$ appearing in the summation are not necessarliy restricted to S^0. The extra summands will be cancelled out as will be seen by the formula (5) below. On the other hand, they make it possible to deduce the desired renewal equation.

Lemma 3.1. *Suppose that $\tilde{\alpha}$, $\tilde{\beta}\in\tilde{B}$ satisfy the following conditions:*

1. $\{\tilde{\alpha}\}<\{\tilde{\beta}\}$,
2. $\varepsilon(\tilde{\alpha}(=-$ *and* $\varepsilon(\tilde{\beta})=+$,
3. $\tilde{\alpha}[1]=\tilde{\beta}[1]$.

Then we set

$$s^{(\{\tilde{\alpha}\},\{\tilde{\beta}\})}(z: x)=s^{\tilde{\alpha}}(z: x)+s^{\tilde{\beta}}(z: x)\quad \text{a.e.x.} \tag{4}$$

The proof of this Lemma is immediate.

REMARK 3.1. The exceptional set for (4) is at most countable and it consists of the points $\{\tilde{\alpha}[n, \infty)\}$, $\{\tilde{\beta}[n, \infty)\}$, $n>0$.

Lemma 3.2.

$$\begin{aligned} s^{\tilde{\alpha}}(z: x)=\chi(\tilde{\alpha}, x)+|zF'(\tilde{\alpha}[1])|^{-1}\{&\sum_{b\in A}\psi(\tilde{\alpha}, b)s^b(z: x)\\ &+\operatorname{sgn}\tilde{\alpha}[1]s^J(z: x)\}\quad a.e.x, \end{aligned} \tag{5}$$

where $J=(\tilde{\alpha}[2, \infty), \tilde{\alpha}[2]^{-\varepsilon(\tilde{\alpha})})$ *and*

$$(\alpha, \tilde{\beta})=\begin{cases} (\{\alpha\}, \{\tilde{\beta}\}) & if\quad \{\tilde{\alpha}\}<\{\tilde{\beta}\}, \\ (\{\tilde{\beta}\}, \{\alpha\}) & if\quad \{\tilde{\beta}\}<\{\tilde{\alpha}\}. \end{cases}$$

The rough sketch of the proof is the following: in case $\varepsilon(\tilde{\alpha})=\operatorname{sgn}\tilde{\alpha}[1]=+$, for instance,

case 1) for $b\in A$ $b<\tilde{\alpha}[2]$, we add all words w ($w[1, 2]=\tilde{\alpha}[1]b$) with coefficient $+1/2z|/F'(\tilde{\alpha}[1])|^{-1}$, since $\chi(\tilde{\alpha}, w\cdot x)=+1/2$, case 2) for $b\in A$ $b>\tilde{\alpha}[2]$, we add all words w ($w[1, 2]=\tilde{\alpha}[1]b$) with coefficeint $-1/2z|F'(\tilde{\alpha}[1])|^{-1}$, since $\chi(\tilde{\alpha}, w\cdot x)=-1/2$,

case 3) even for $b\in A$ $b=\tilde{\alpha}[2]$, we add all words w ($w[1, 2]=\tilde{\alpha}[1]b$) with coefficient $-1/2z|F'(\tilde{\alpha}[1])|^{-1}$.

The second term is constructed by the above way. But for words w such that $\{w\cdot x\}\in J$, in spite of $\chi(\tilde{\alpha}, w, x)=+1/2$ we add in case 3) with coefficient $-1/2z|F'(\tilde{\alpha}[1])|^{-1}$. Hence we need to adjust it. This term appears in the third term.

Now we proveed to construct the renewal equation. For $\tilde{a}\in\tilde{B}$ and $\tilde{a}, \tilde{b}\in\tilde{A}$; let

$$\chi^{\tilde{a}}(z: x) = \sum_{n=0}^{\infty} z^n(F'(\tilde{\alpha}[1.\ n]))^{-1}\chi(\tilde{\alpha}[n+1, \infty), x) \quad (F'(e) = 1)\,,$$

$$\phi(\tilde{\alpha}, \tilde{a}) = \delta[\tilde{a}\leqq\tilde{\alpha}[2]^-]-1/2 = \begin{cases} +1/2 & \text{if } \tilde{a}\leqq\tilde{\alpha}[2]^-\,, \\ -1/2 & \text{if } \tilde{a}>\tilde{\alpha}[2]^-\,, \end{cases}$$

$$\Phi(z)_{\tilde{a},\tilde{b}} = \sum_{n=1}^{\infty} \varepsilon(\tilde{\alpha})z^n(F'(\tilde{\alpha}[1, n])^{-1}\phi(\tilde{\alpha}([n, \infty), \tilde{b})\,, \tag{6}$$

and let $\Phi(z)$ be a matrix whose $(\tilde{a}, \tilde{b})$-component is $\Phi(z)_{\tilde{a},\tilde{b}}$. We define *two* vectors as

$$\chi(z: x) = (\chi^{a}(z: x))_{\tilde{a}\in\tilde{A}}\,, \quad s(z: x) = (s^{a}(z: x))_{\tilde{a}\in\tilde{A}}\,.$$

Note that $\chi^{\tilde{a}}(z: x)$ and $\Phi(z)_{\tilde{a},\tilde{b}}$ is analytic in the region $|z|<e^{\xi}$ by the definition (1) of the lower Lyapunov number.

Lemma 3.3. *For sufficiently small* $|z|$, *we get for* $\tilde{\alpha}\in\tilde{\beta}$

$$s^{\tilde{a}}(z: x) = \chi^{\tilde{a}}(z: x)+ \sum_{\tilde{b}\in\tilde{A}} \Phi(z)_{\tilde{a},\tilde{b}}\, s^{b}(z: x) \quad a.e.x\,, \tag{7}$$

especially we have

$$s(z: x) = \chi(z: x)+\Phi(z)s(z: x) \quad a.e.x. \tag{7'}$$

Proof. Lemma 3.1 implies for the interval $J=(\tilde{\alpha}[2, \infty), \tilde{\alpha}')$ $(\tilde{\alpha}'=\tilde{\alpha}[2]^{-\varepsilon(\tilde{\alpha})})$

$$s^{J}(z: x)=s^{\tilde{\alpha}[2,\infty)}(z: x)+s^{\tilde{\alpha}'}(z: x) \quad a.e.x,$$

amd for $b\in A$

$$s^{b}(z: x) = s^{b^+}(z: x)+s^{b^-}(z: x)\,.$$

Then by Lemma 3.2, we get

$$\begin{aligned} s^{\tilde{\alpha}}(z: x) = \chi(\tilde{\alpha}, x)&+z|F'(\tilde{\alpha}[1])|^{-1}\sum_{b\in A}\sum_{\sigma} \psi(\tilde{\alpha}, b)s^{b^{\sigma}}(z: x) \\ &+z|F'(\tilde{\alpha}[1])|^{-1}\operatorname{sgn}\tilde{\alpha}[1]\{s^{\tilde{\alpha}'}(z: x)+s^{\tilde{\alpha}[2,\infty)}(z: x) \quad a.e.x \end{aligned} \tag{8}$$

$$
\begin{aligned}
&= \chi(\tilde{\alpha}, x)+z(F'(\tilde{\alpha}[1]))^{-1}\sum_{b\in A}\sum_{\sigma}\{\operatorname{sgn}\tilde{\alpha}[1]\psi(\tilde{\alpha}, b)+\delta[b^{\sigma}=\tilde{\alpha}']\} \\
&\quad \times s^{b^{\sigma}}(z: x)+z(F'(\tilde{\alpha}[1]))^{-1}s^{\tilde{\alpha}[2,\infty)}(z: x) \quad a.e.x. \\
&= \chi(\tilde{\alpha}, x)+z(F'(\tilde{\alpha}[1]))^{-1}\sum_{\tilde{b}\in\tilde{A}}\varepsilon(\tilde{\alpha})\phi(\tilde{\alpha}, \tilde{b})s^{b}(z: x) \\
&\quad +z(F'(\tilde{\alpha}[1]))^{-1}s^{\tilde{\alpha}[2,\infty)}(z: x) \quad a.e.x.
\end{aligned}
$$

Substituting (5) to $s^{\tilde{\alpha}[2,\infty)}(z: x)$ repeatedly, we obtain (7),

The equation (7) and (7′) is the renewal equation which is our main tool to solve the spectral problem of the Perron-Frobenius operator.

Definition 3.1. We call $D(z)=\det(I-\Phi(z))$ the Fredholm determinant of the Perron-Feobenius operator P.

4. The Proof of Theorem A

The proof of Theorem A is reduced to the proof of the following two lemmas.

Lemma 4.1. *For any $f\in BV$ and $g\in L^{\infty}$, the singular points z of the function $(f, g)(z)=\sum_{n=0}^{\infty} z^n \int f(x)g(F^n(x))dx$ satisfy $D(z)=0$.*

Lemma 4.2. *For any z which satisfies $|z|<e^{\xi}$ and $D(z)=0$, there exists functions $f\in BV$ and $g\in L^{\infty}$ such that z is not the removable singularlity of $(f, g)(z)$.*

The proof of Theorem A. First note that for any $\varepsilon>0$ Spec(F) of the Perron-Frobenius operator $P|_{BV}$ consists of only finite number of eigenvalues in the domain $\{\lambda: |\lambda|>e^{-(\xi-\varepsilon)}\}$ ([9]). Hence (f, g) has meromorphic extension to the domain $|z|<e^{\xi}$ and $z^{-1}\in S$pec(F) if and only if there exists $f\in BV$ and $g\in L^{\infty}$ such that (f, g) has singularity at z. By Lemma 4.1, (f, g) is analytic at z if $D(z)\neq 0$ for any $f\in BV$ and $g\in L^{\infty}$. Thereofre, z^{-1} does not belong to Spec(F). On the other hand, by Lemma 4.2, those z^{-1} which satisfy $|z|<e^{\xi}$ and $D(z)=0$ belongs to Spec(F). This proves Theorem A.

4.1. The proof of Lemma 4.1

For a word $w\in W$, we get by (8)

$$
\begin{aligned}
(1_w, g)(z) &= \sum_{n=0}^{\infty} z^n \int 1_w(w)g(F^n(x))dx \\
&= \int s^{w}(z: x)g(x)dx \\
&= \sum_{\sigma}\int s^{w^{\sigma}}(z: x)g(x)dx \\
&= \sum_{\sigma}\int \{\chi^{w^{\sigma}}(z: x)+\sum_{n=1}^{\infty}\sigma z^n(F'(w^{\sigma}[1, n]))^{-1} \\
&\quad \times \sum_{\tilde{b}\in\tilde{A}}\phi(w^{\sigma}[n, \infty), \tilde{b})s^{\tilde{b}}(z: x)\}\, g(x)dx
\end{aligned}
$$

$$= \sum_{\sigma} \int \{\chi^{w^\sigma}(z: x) + \sum_{n=|w|}^{\infty} \sigma z^n (F'(w^\sigma[1, n]))^{-1} \times \sum_{\tilde{b}\in\tilde{A}} \phi(w^\sigma[n, \infty), \tilde{b}) s^{\tilde{b}}(z: x)\} g(x)dx .$$

Now let us introduce row vectors $\Phi^w(z)$:

$$\Phi^w(z)_{\tilde{b}} = \sum_{\sigma} \sum_{n=|w|} \sigma z^n (F'(w^\sigma[1, n]))^{-1} \phi(w^\sigma[n, \infty), \tilde{b}) .$$

We will estimate $\Phi^w(z)$ and $\chi^{w^\sigma}(z: x)$.

Now for any $\varepsilon>0$ $(\xi-2\varepsilon>0)$ let $|z|<e^{\xi-2\varepsilon}$. Then for sufficiently large N we get by (3)

$$\begin{aligned} |\Phi^w(z)_{\tilde{b}}| &\leqq 1/2 |\sum_{\sigma} \sum_{n=|w|}^{\infty} z^n (F'(w^\sigma[1, n]))^{-1}| \\ &\leqq 1/2 |\sum_{\sigma} \sum_{n=|w|}^{N-1} z^n (F'(w^\sigma[1, n]))^{-1} \delta | [|w|<N] \\ &\quad + (|z| e^{-(\xi-\varepsilon)})^{N \vee |w|} (1-(|z| e^{-(\xi-\varepsilon)})^{-1} . \end{aligned}$$

Thus there exists a constant $K_1=K_1(\varepsilon)$ and $K_1'=K_1'(\varepsilon)$ such that for $|z|<e^{\xi-2\varepsilon}$

$$|\Phi^w(z)_{\tilde{b}}| \leqq K_1(|z| e^{-(\xi-\varepsilon)})^{|w|} + K_1'[\delta |w|<N] . \tag{9}$$

On the other hand,

$$\begin{aligned} &|\sum_{\sigma} \int \chi^{w^\sigma}(z: x) g(x) dx| \\ &\leqq \sum_{\sigma} |\int \sum_{n=0}^{\infty} z^n (F'(w^\sigma[1, n]))^{+1} \chi(w^\sigma[n+1, \infty), x) g(x) dx| \\ &\leqq A(w, z) + B(w, z) , \end{aligned}$$

where

$$A(w, z) = |\sum_{n=0}^{|w|} \int z^n 1_w(x) g(F^n(x)) dx| ,$$

and

$$B(w, z) = 1/2 \sum_{\sigma} \sum_{n>|w|} |z^n (F'(w^\sigma[1, n]))^{-1}| \, ||g||_\infty .$$

Then we get by (3′) for $|w| \geqq N$

$$\begin{aligned} A(w, z) &\leqq |w| e^{(\xi-2\varepsilon)} e^{-(\xi-\varepsilon)|w|} ||g||_\infty \text{ Lebes } I \\ &= |w| e^{-\varepsilon|w|} ||g||_\infty \text{ Lebes } I . \end{aligned}$$

Therefore, there exists a constant $K_2=K_2(\varepsilon)$ such that

$$A(w, z) \leqq K_2 e^{-1/2\, \varepsilon |w|} ||g||_\infty . \tag{10}$$

On the other hand, there exists a constant $K_3=K_3(\varepsilon)$ such that if $|z|<e^{\xi-2\varepsilon}$,

$$B(w, z) \leqq K_3(|z| e^{-(\xi-\kappa)})^{|w|} ||g||_\infty . \tag{11}$$

Combining (10) and (11) with (9), we get for $|z|<e^{\xi-2\varepsilon}$

$$(12)\qquad |(1_w, g)(z)| \leqq K_2 e^{-1/2\varepsilon|w|}\|g\|_\infty + K_3(|z|e^{-(\xi-\varepsilon)})^{|w|}\|g\|_\infty$$
$$+\{K_1(|z|e^{-(\xi-\varepsilon)})^{|w|} + K_1'\delta[|w|<N]\}\|g\|_\infty \sup_{x\in I} \sum_{\tilde{b}\in\tilde{A}} |s^{\tilde{b}}(z\colon x)| .$$

Lemma 4.3. *For any $f\in BV$, there exists a decomposition $f(x)=\sum_{w\in W} C_w 1_w(x)$ such that for any $\gamma(0<\gamma<1)$*

$$(13)\qquad |\sum_{w\in W} C_w \gamma^{|w|}| \leqq |C_e| + V(f)\#A/(1-\gamma),$$

where $V(f)$ is the total variation of f when I is the union of disjoint intervals, $V(f)$ is the sum of total variations on each intervals).

Proof. For simplicity, we consider that I is an interval. Any $f\in BV$ can be expressed as $f=f_1+f_2$, where f_1 (f_2) is monotone increasing (decreasing, respectively). Hence it suffices to show (13) for a monotone function. For a monotone function f, we define inductively

$$C_e = \inf_{x\in I} f(x),$$
$$C_w = \inf_{x\in(u)} \{f(x) - \sum_u C_u 1_u(x)\} \qquad (W\neq e,\ w\in W),$$

where the summation is taken over those words u which belong to $\cup_{k=0}^{n-1} w_k$. Then, by the assumption $\xi>0$,

$$|\sum_{w\in W} C_w \gamma^{|w|}| \leqq \sum_{n=0} \gamma^n \sum_{w\in W_n}^{\infty} |C_w| \leqq |C_e| + V(f)\#A/(1-\gamma).$$

By (12) and (13), setting $\gamma=|z|e^{-(\xi-\varepsilon)}$ or $e^{-\varepsilon/2}$, there exists a constant $K=K(\varepsilon)$ and $K'=K'(\varepsilon)$ such that

$$|(f, g)(z)| \leqq K\|g\|_\infty \sup_{x\in I} \sum_{\tilde{b}\in\tilde{A}} |s^{\tilde{b}}(z\colon x)| + K'\|g\|_\infty .$$

This shows that at z which satisfies $|z|<e^\xi$ and $D(z)\neq 0$, $(f, g)(z)$ is analytic. This proves Lemma 4.1.

4.2. The proof of Lemma 4.2 when F satisfies the endpoint condition

In this and the next sections, to simplity the proof, we only treat the case where $I=[0, 1]$ since the other cases can be treated in a similar way. First we prove:

Lemma 4.4. *Let z_0 be any point which satisfies $|z_0|<e^\xi$ and $D(z_0)=0$. Then there exists $\tilde{a}\in\tilde{A}$ and a function $g\in L^\infty$ such that $\int s^{\tilde{a}}(z\colon x)g(x)dx$ has singularity at $z=z_0$.*

Proof. Recall

$$s(z: x) = (I-\Phi(z))^{-1}\chi(z: x),$$

and so

$$(\int s^{\tilde{a}}((z: x)g(x)dx)_{\tilde{a}\in\tilde{A}} = (I-\Phi(z))^{-1}(\int \chi^{\tilde{a}}(z: x)g(x)dx)_{\tilde{a}\in\tilde{A}}.$$

Thus it is sufficient to prove that vectors $(\int \chi^{\tilde{a}}(z: x)g(x)dx)_{\tilde{a}\in\tilde{A}}(g\in L^\infty)$ spans $C^{\sharp\tilde{A}}$ for any z with $|z|<e^{\xi}$. For a function $g\in L^\infty$, put $G(x)=\int_0^x g(y)dy$. Then

$$\int \chi^{\tilde{a}}(z: x)g(x)dx = \\ = \sum_{n=0}^{\infty} z^n(F'(\tilde{a}[1, n]))^{-1}\varepsilon(\tilde{a})G(\{\tilde{a}[n+1, \infty)\})+\chi^{\tilde{a}}(z: 1)G(1).$$

On the other hand, for a fixed z with $|z|<e^{\xi}$ and for any $\varepsilon>0$, there exists an integer M such that

$$\max_{\tilde{a}\in\tilde{A}} |\sum_{n=M}^{\infty} z^n(F'(\tilde{a}[1, n]))^{-1}|<\varepsilon.$$

By the endpoint condition, there eixsts $\varepsilon'>0$ such that the set of subintervals

$$\{[\{\tilde{a}\}-\varepsilon', \{\tilde{a}\}+\varepsilon'], [\{\tilde{a}[2, \infty)\}-\varepsilon', \{\tilde{a}[2, \infty)\}+\varepsilon']\}_{\tilde{a}\in\tilde{A}}$$

are mutually disjoint and for any integer $n(3\leqq n\leqq M)$ and $\tilde{a}\in\tilde{A}$

$$\{\tilde{a}[n, \infty)\}\notin \bigcup_{\tilde{b}\in\tilde{A}} \{[\{\tilde{b}\}-\varepsilon', \{\tilde{b}\}+\varepsilon']\cup[\{\tilde{b}[2, \infty)\}-\varepsilon', \{\tilde{b}[2, \infty)\}+\varepsilon']\}.$$

Now assume that $G(x)=\int_0^x g(y)dy$ $(g\in L^\infty)$ satisfies

(1) $|G|\leqq 1$

(2) $G=0$ on the outside of the set

$$\bigcup_{\tilde{b}\in\tilde{A}} \{[\{\tilde{b}\}-\varepsilon', \{\tilde{b}\}+\varepsilon']\cup[\{\tilde{b}[2, \infty)\}-\varepsilon', \{\tilde{b}[2, \infty)\}+\varepsilon']\}.$$

Then we get

$$|\int \chi^{\tilde{a}}(z: x)g(x)dx-\varepsilon(\tilde{a})\{G(\tilde{a})+G(\tilde{a}[2, \infty))(F'(\tilde{a}[1]))^{-1}\}-\chi^{\tilde{a}}(z: 1)G(1)|<\varepsilon.$$

Therefore, noticing the fact that the partition $\{(a)\}_{a\in A}$ is minimal (cf. fig. (I)), it is trivial taht vectors $(\int \chi^{\tilde{a}}(z: x)g(x)dx)_{\tilde{a}\in\tilde{A}}$ $(g\in L^\infty)$ spans $C^{\sharp\tilde{A}}$. This proves the lemma.

Now we prove Lemma 4.2. Let z_0 be a zero of $D(z)$ and $|z_0|<e^{\xi}$. Assume that for any $b\in A$ and $g\in L^\infty$ $(1_b, g)(z)=\int s^b(z: x)g(x)dx$ has not sigularity at z_0. Then by (5) and Lemma 4.4, there must exists $\tilde{a}\in\tilde{A}$ and a function $g\in L^\infty$ such that $\int s^J(z: x)g(x)dx$ has singularity at $z=z_0$, where $J=(\tilde{a}[2, \infty), \tilde{a}[2]^{-\varepsilon(\tilde{a})})$. Hence for the word $w=\tilde{a}[1, 2]$ the function $(1_w, g)(z)$ has a singularity at $z=z_0$, because

$$(1_w, g)(z) = \int s^w(z: x)g(x)dx$$
$$= \int 1_w(x)g(x)dx + z(F'(\tilde{a}[1])^{-1} \int s^J(z: x)g(x)dx\,.$$

This proves Lemma 4–2.

4.3. The proof of Lemma 4.2 for a mapping which does not satisfy the endpoint condition

If a mapping F does not satisfy the endpoint condition, as we see in 4–2, vectors $(\int \chi^{\tilde{a}}(z: x)g(x)dx)_{\tilde{a}\in\tilde{A}}$ $(g\in L^\infty)$ do not necessarily span $C^{\sharp\tilde{A}}$. Therefore, we will reconstruct symbolic dynamics over new alphabets to show if z_0 satisfies $D(z_0)=0$, then there exist $f\in BV$ and $g\in L^\infty$ such that $(f, g)(z)$ has singularity at z_0. We begin with the definition of several notations. Letting $\tilde{a}, \tilde{b}\in\tilde{A}$.

$$N(\tilde{a}, \tilde{b}) = \inf\{n: \tilde{a}[n, \infty) = \tilde{b}[m, \infty] \quad \text{for some } m\geqq 1\} \ (\inf\phi=\infty)\,,$$
$$N(\tilde{a}) = \max\{N(\tilde{a}, \tilde{b}): \tilde{a}\in\tilde{A}, N(\tilde{a}, \tilde{b})<\infty\}\,.$$

As the new division points, we take

$$\Delta I_* = \{\{\tilde{a}[n, \infty)\} : 1\leqq n\leqq N(\tilde{a}), \tilde{a}\in\tilde{A}\}\,.$$

We denote by $\{(a_*)\}_{a^*\in A^*}$ the new partition of I into subintervals which is defined by ΔI_*. As in the previous sections, we can similarly define the sets A_*, W_*, S_*, $\tilde{A}_*$, $\tilde{W}_*$, $\tilde{B}_*$ and so on. Now we will construct a new renewal equation for $s^{\tilde{a}_*}(z: x)$. If $\tilde{a}_*[2, \infty)\notin\tilde{A}_*$, then we define $\Phi_*(z)_{\tilde{a}_*,\tilde{b}_*}$ as (6):

$$\Phi_*(z)_{\tilde{a}_*,\tilde{b}_*} = \sum_{n=1}^{\infty} \varepsilon(\tilde{a}_*)z^n(F'(\tilde{a}_*[1, n])^{-1}\phi_*(\tilde{a}_*[n+1, \infty), \tilde{b}_*)\,,$$
$$\phi_*(\tilde{\alpha}_*, \tilde{b}_*) = \delta[\tilde{b}_*\leqq\alpha_*[2]^-]-1/2\,,$$
$$\chi_*^{\tilde{a}_\circ}(z: x) = \sum_{n=0}^{\infty} z^n(F'(\tilde{a}_*[1, n])^{-1}\chi_*(\tilde{a}_*[n+1, \infty), x)\,,$$
$$\chi_*(\tilde{\alpha}_*, x) = \delta[x<\tilde{\alpha}_*: \varepsilon(\tilde{\alpha}_*)]-1/2\,.$$

If $\tilde{a}_*$, $\tilde{a}_*[2, \infty)\in\tilde{A}_*$, then the right hand side of (5) can be expressd only by $s^{\tilde{b}_*}(z: x)(\tilde{b}_*\in\tilde{A}_*)$, hence noticing the convention $\varepsilon(a_*^\sigma[2, \infty))=\sigma$, especially taking into account of the case when sgn $a_*=-$, we define

$$\Phi_*(z)_{\tilde{a}_*,\tilde{b}_*} = z|F'(\tilde{a}_*)|^{-1}\phi_*(\tilde{a}_*, \tilde{b}_*)\,,$$
$$\phi_*(\tilde{a}_*, \tilde{b}_*) = \delta[\tilde{b}_*\leqq\tilde{a}_*[2]: \varepsilon(\tilde{a}_*)\,\mathrm{sgn}\,\tilde{a}_*[1]]-1/2\,,$$
$$\chi_*^{\tilde{a}_*}(z: x) = \delta[x<\tilde{a}_*: \varepsilon(\tilde{a}_*)]-1/2\,.$$

We also put

$$\chi_*(z: x) = \chi_*^{\tilde{a}_*}(z: x))_{\tilde{a}_*\in\tilde{A}_*}\,,$$
$$s_*(z: x) = (s^{\tilde{a}_*}(z: x))_{\tilde{a}_*\in\tilde{A}_*}\,,$$

and

$$D_*(z) = \det(I - \Phi_*(z)) .$$

Then we have still

$$s_*(z: x) = \chi_*(z: x) + \Phi_*(z) s_*(z: x) .$$

From now on we omit the asterique $*$ in this section to make notations simple. Now let

$$V = \{(x_{\tilde{a}})_{\tilde{a}\in\tilde{A}} : x_{\tilde{a}} = -x_{\tilde{b}} \text{ if } \tilde{a} \neq \tilde{b},\ \{\tilde{a}\} = \{\tilde{b}\} \text{ and } \{\tilde{a}[2, \infty)\},\ \{\tilde{b}[2, \infty)\} \in \Delta I\} .$$

Lemma 4.5. *For every* z $(|z| < e^{\xi})$,

$$V = \{\textstyle\int \chi(z: x) g(x) dx : g \in L^\infty\} .$$

The proof is almost the same as the proof of Lemma 4.4. Hence we omit it. We only need to take care of $\tilde{a}$ and $\tilde{b}$ components for such $\tilde{a}$, $\tilde{b}$ as $\tilde{a} \neq \tilde{b}$, $\{\tilde{a}\} = \{\tilde{b}\}$ and $\{\tilde{a}[2, \infty)\}$, $\{\tilde{b}[2, \infty)\} \in \Delta I$.

Lemma 4.6. *The kernel of* $I - \Phi(z)$ *is contained in* $V + (I - \Phi(z))V$.

Proof. We denote by V_* the subspace of all the vectors $(x_{\tilde{a}})_{\tilde{a}\in\tilde{A}}$ which satisfy the following conditions:

1. $\varepsilon(\tilde{a}) F'(\tilde{a}) x_{\tilde{a}} = \varepsilon(\tilde{b}) F'(\tilde{b}) x_{\tilde{b}}$

if $\tilde{a}$, $\tilde{b} \in \tilde{A}$ satisfy the condition $\{\tilde{a}[2, \infty)\}$, $\{\tilde{b}[2, \infty)\} \in \Delta I$ and one of the following:

a. $\tilde{a}[2, \infty) = \tilde{b}[2, \infty)$ as a sentence,

b. $\tilde{a}[2, \infty) \neq (\inf I)^-, (\sup I)^+$ and $\tilde{a}[2, \infty)^\wedge = \tilde{b}[2, \infty)$ as a sentence,

2. $\varepsilon(\tilde{a}) F'(\tilde{a}) x_{\tilde{a}} = \varepsilon(\tilde{b}) F'(\tilde{b}) x_{\tilde{b}}$, if $\{\tilde{a}[2, \infty)\} = \inf I$ and $\{\tilde{b}[2, \infty)\} = \sup I$.

It suffices to show that

a. the kernel of $I - \Phi(z)$ is contained in V_*,

and

b. the subspace V_* is contained in $V + (I - \Phi(z))V$.

Let $\{\tilde{a}[2, \infty)\}$, $\{\tilde{b}[2, \infty)\} \in \Delta I$. If a vector $\vec{x} = (x_{\tilde{a}})_{\tilde{a}\in\tilde{A}}$ belongs to the kernel $I - \Phi(z)$, then

$$((I - \Phi(z)\vec{x})_{\tilde{a}} = x_{\tilde{a}} - z|F'(\tilde{a})|^{-1} \sum_{\tilde{c}\in\tilde{A}} \{\delta[\tilde{c} \geq \tilde{a}][2] : \varepsilon(\tilde{a}) \operatorname{sgn} \tilde{a}[1]] - 1/2\} x_{\tilde{c}} = 0 ,$$

and so

$$F'(a) x_{\tilde{a}} = z \sum_{\tilde{c}\in\tilde{A}} \{\delta[\tilde{c} \leqq a[2] : \varepsilon(\tilde{a})] - 1/2\} x_{\tilde{c}} ,$$

and the similar equality holds for $\tilde{b}$. Hence $\vec{x} \in V_*$.

Now we define the singed indicator $\vec{e}(a)$ and the ordinary indicator $\vec{f}(\tilde{a})$ as:

$$\vec{e}(\tilde{a})_{\tilde{c}} = \begin{cases} +1 & \text{if } \tilde{c} = \tilde{a}, \text{ or if } \tilde{c} = \tilde{a}^\wedge = (\inf I)^-, (\sup I)^+ \text{ as a sntence} , \\ -1 & \text{if } \tilde{c} = \tilde{a}^\wedge \neq (\inf I)^-, (\sup I)^+ \text{ as a sentence} , \\ 0 & \text{otherwise} , \end{cases}$$

and

$$\vec{f}(\tilde{a})_{\tilde{c}} = \begin{cases} +1 & \text{if } \tilde{c} = \tilde{a} \text{ as a sentence ,} \\ 0 & \text{otherwise .} \end{cases}$$

Then $\vec{e}(\tilde{a}) \in V$ for any $\tilde{a} \in \tilde{A}$ and $\vec{f}(\tilde{a}) \in V$ if $\{\tilde{a}[2, \infty)\} \notin \Delta I$ or $\{\tilde{a}^\wedge[2, \infty)\} \notin \Delta I$. Conversely, V is spanned by the vectors $\vec{e}(\tilde{a})$ and $\vec{f}(\tilde{a})$ ($\{\tilde{a}[2, \infty)\}$ or $\{\tilde{a}^\wedge[2, \infty)\}$ $\notin \Delta]I$).

Now we will show that $\Phi(z)\vec{e}(\tilde{a})$ and $\vec{f}(\tilde{a})$ span V_*. If $\tilde{c} \in \tilde{A}$ satisfies $\{\tilde{c}[2, \infty)\} \in \Delta I$, then

$$\Phi(z)\vec{e}(\tilde{a})_{\tilde{c}} = \begin{cases} z(F'(\tilde{c}))^{-1}\varepsilon(\tilde{a})\varepsilon(\tilde{c}) & \text{if } \tilde{c}[2, \infty) = \tilde{a} \text{ or if} \\ & \quad \tilde{c}[2, \infty) = \tilde{a}^\wedge \mp (\inf I)^-, (\sup I)^+ , \\ & \quad \text{as sentence ,} \\ -z(F'(\tilde{c}))^{-1}\varepsilon(\tilde{a})\varepsilon(\tilde{c}) & \text{if } \tilde{c}[2, \infty) = \tilde{a}^\wedge = (\inf I)^-, (\sup I)^+ \\ & \quad \text{as a sentence ,} \\ 0 & \text{otherwise ,} \end{cases}$$

because if $\varepsilon(\tilde{c}) \operatorname{sgn} \tilde{c}[1] = +$(or $-$), then $\{\tilde{c}[2, \infty)\} = \{\tilde{c}[2]^+\}$ (or $\{\tilde{c}[2]^-\}$ respectively). Hence the proof of Lemma 4–6 is completed.

Now we will, prove Lemma 4.2. Assume $D(z_0)=0$. Let $\vec{x}$ belongs to the kernel of $(I-\Phi(z_0))$. We get the decomposition $\vec{x}=\vec{x}_1+(I-\Phi(z_0))\vec{x}_2$ ($\vec{x}_1, \vec{x}_2 \in V$) by Lemma 4–6. Then we have two possibilities:

case 1. $(I-\Phi(z))^{-1}\vec{x}_1$ is unbounded at $z=z_0$,

and

case 2. $(I-\Phi(z))^{-1}\vec{x}_1$ is bounded at $z=z_0$.

In case 1, we have.

$$\int s(z: x)g(x)dx = (I-\Phi(z))^{-1} \int \chi(z: x)g(x()dx, \ g \in L^\infty ,$$

and by Lemma 4.5 there is $g \in L^\infty$ with $\vec{x}_1 = \int \chi(z: x)g(x)dx$. Hence $\int s(z: x)g(x)dx$ has a not removable singularity at $z=z_0$. Next recall

$$\mathrm{proj}_E\, \vec{y} = (2\pi i)^{-1} \int_{|z-z_0|=r} (I-\Phi(z))^{-1}\vec{y}dz ,$$

for sufficiently small $r>0$, where proj_E is the projection to the eigenspace E corresponding to eigenvalue 1 of $\Phi(z_0)$. In case 2, $\mathrm{proj}_E \vec{x}_1=0$ and so

$$\vec{x} = \mathrm{proj}_E\, \vec{x} = \mathrm{proj}_E\, (I-\Phi(z_0))\vec{x}_2 ,$$

Hence $(I-\Phi(z_0))^2\vec{x}_2=0$, namely, $\vec{x}_2$ has non-zero component in the generalized eigenspace corresponding to eigenvalue 1 of $\Phi(z_0)$. Therefore $(I-\Phi(z))^{-1}\vec{x}_2$ diverges as $z \to z_0$. This completes the proof of Lemma 4.2.

4.4. Corollaries to Theorem A

As we mentioned in § 1, we can express the ergodic properties of the dynamical system in terms of the eigenvalues of the Perron-Frobenius operator P. Now we will express them in terms of our matrix $\Phi(z)$.

Corollary 4.7. *Assume, as before, that* $\xi > 0$.

a) The number of ergodic components is equal to the dimension of the eigenspace of $I - \Phi(1)$ *corresponding to eigenvalue* 0 *and it is also equal to the order of the zero* $z=1$ *of* $\det(I-\Phi(z))$.

b) In particular, if 0 *is a simple eigenvalue of* $I-\Phi(1)$, *then the dynamical system is ergodic. Moreover, if* $\{|z|=1\} \cap \mathrm{Spec}(F) = \{1\}$, *then the dynamical system is mixing.*

REMARK 4.1. The assumption of the statement b (0 is a simple eigenvalue of $I-\Phi(1)$) is always satisfied if we restrict F to an ergodic component, which is still a finite union of subintervals. Moreover, since the eigenvalues of $P|_{BV}$ on the unit circle are the roots of unity, the dynamical systme is mixing for a suitable induced map of F.

Now we will consider the density functions of absolutely continuous invariant measures.

Corollary 4.8. *a) There exists the limit* $\lim_{z\uparrow 1}(1-z)(I-\Phi(z))^{-1}$ *and it does not vanish. We denote it by* $(\Phi'(1))^{-1}$.

b) A function $f \in BV$ *is an invariant function under* F *if and only if there exists a row vector* $\vec{a}$ *such that* $f(x) = \vec{a}(\Phi'(1))^{-1}\chi(1:x)$. *Particularly,* $(1, \cdot, \cdot, \cdot, 1)(\Phi'(1))^{-1}\chi(1:x)$ *is the density function of an absolutely continuous invariant probability measure.*

c) Let μ *be an absolutely continuous invariant probability measure. Suppose that the dynamical system* (F, μ) *is mixing. Then the residue of* $(f,g)(z)$ *at* $z=1$ $(f \in BV, g \in L^\infty)$ *equals* $\int f dx \int g d\mu$.

Proof. Since $(f,g)(z) = \int (I-zP)^{-1}f(x)\cdot g(x)dx$ has singularity of order one at $z=1$ for ceratin functions $f \in BV$ and $g \in L^\infty$, $\lim_{z\uparrow 1}(1-z)s(z:x)$ exists and does not vanish. This proves the assertion a). Let $f \in BV$ satisfy $Pf=f$. Then we get

$$\sum_{n=0}^{\infty} z^n \int P^n f(x)\cdot g(x)dx = (1-z)^{-1}\int f(x)\cdot g(x)dx$$

Expressing $f = \sum_{w\in W} C_w 1_w$, we get

$$\lim_{z\uparrow 1}(1-z)\sum_{n=0}^{\infty} z^n \int P^n f(x)\cdot\cdot g(x)dx$$
$$= \lim_{z\uparrow 1}(1-z)\sum_{w\in W} C_w \int s^w(z:x)g(x)dx$$

$$= \lim_{z \uparrow 1} (1-z) \sum_{w \in W} C_w \{ \int \Phi^w(z) s(z: x) g(x) dx + \sum_{\sigma} \chi^{w^\sigma}(z: x) g(x) dx \}$$
$$= \sum_{w \in W} C_w \int \Phi^w(1) (\Phi'(1))^{-1} \chi(1: x) g(x) dx .$$

This proves $f(x) = \sum_{w \in W} C_w \Phi^w(1) (\Phi'(1))^{-1} \chi(1: x)$. On the other hand,

$$\lim_{z \uparrow 1} (1-z) \sum_{b \in A} s^{\tilde{a}}(z: b \cdot x) z |F'(b)|^{-1} = \lim_{z \uparrow 1} (1-z) s^{\tilde{a}}(z: x) \cdot$$

Hence,

$$P\{(\Phi'(1))^{-1} \chi(1: x)\}_{\tilde{a}} = \sum_{b \in A} \{(\Phi'(1))^{-1} \chi(1: b \cdot x) |F'(b)|^{-1}\}_{\tilde{a}}$$
$$= \{(\Phi'(1))^{-1} \chi(1: x)\}_{\tilde{a}} .$$

This proves the first part of the assertion b). The second part follows from the facts (1) $s^a(z: x) \geqq 0$ for $0 < z < 1$ and (2) $\sum_{a \in A} \int s^a(z: x)\, dx = (1-z)^{-1}$. The proof of the assertion c) is trivial.

Corollary 4.9. *Assume that the dynamical system (F, μ) is mixing. Let*

$$\eta^{-1} = \textit{minimum in modulus of } \{z: z^{-1} \in \mathrm{Spec}(F) \textit{ and } z \neq 1\} .$$

Then η is the decay rate of correlation: for any $f \in BV$ and $g \in L^\infty$

$$\lim_{n \to \infty} (|\eta| + \varepsilon)^{-n} \{ \int f(x) g(F^n((x)\} d\mu - \int f d\mu \int g d\mu \} = 0$$

holds for any $\varepsilon > 0$.

The proof is trivial.

5. Zeta Function

5.1. Results and Ideas of Proof

In this section, we will prove Theorem C, then we see Theorem B as its corollary. For each fixed point $p \in I$ of F^n ($F^n(p) = p$), there exists a word $a_1 \cdots a_n$ such that $F^{i-1}(p) \in (a_i)$ ($a_i \in A$). We call such a periodic orbit p of type $a_1 \cdots a_n$. We put $F^{n\prime}(p) = \prod_{i=1}^n F'(a_i)$. The zeta function is given as follows:

$$\zeta(z) = \exp \{ \sum_{n=1}^{\infty} z^n n^{-1} \sum_{F^n(p)=p} |F^{n\prime}(p)|^{-1} \} .$$

Theorem C. *Suppose that the lower Lyapunov number ξ is positive. Then the zeta function $\zeta(z)$ is meromorphic in the domain $|z| < e^\xi$ and is expressed as*

$$\zeta(z) = (D(z))^{-1} \exp \{R(z)\} , \tag{14}$$

where $R(z)$ depends only on the orbits which pass fhrough the division points of the partition, and one can modify the partition so that the zero points of the Fredholm determinant coincide with the poles of the zeta function:

$$\{z: D(z)=0\,,\quad |z|<e^{\xi}\} = \{z: \zeta(z)=\infty\,,\quad |z|<e^{\xi}\}\,.$$

REMARK 5.1. a. Here the modification of the partitione means only a movement of some dividing points of the partition to the neighboring intervals. Note that $\zeta(z)$ may depend on the partition if the mapping F does not satisfy the endpoint condition: for example, as in figure (I), if $p\in\Delta I$, $F^{n\prime}(p)$ depends on the choice of the partition. Theorem C asserts that one can modify the partition for which $R(z)$ has no poles in $|z|<e^{\xi}$. The explicit form of $R(z)$ will be given in the way of the proof in 5–7.

b. The formula (14) also shows that one can remove the minimality condition from the assumption of Theorem A. Moreover, in the case of mappings which do not satisfy the endpoint condition, it turns out that the identity (14) remains valid for $\Phi_*(z)$ in place of $\Phi(z)$ with a certain modification $R_*(z)$ of $R(z)$.

Since $\exp[\mathrm{tr}\{\log(I-\Phi(z))\}]=D(z)$, the assertion (14) of Theorem C is equivalent to the identity

$$\zeta(z)=\exp\,\{\mathrm{tr}\sum_{k=1}^{\infty}k^{-1}\Phi^k(z)+R(z)\}\,,$$

where tr denotes the trace of a matrix. Hence its proof is reduced to study the correspoindence between the quantity $\mathrm{tr}\sum_{k=1}^{\infty}k^{-1}\Phi^k(z)$ and the periodic orbits. Let us see how the existence of periodic points is reflected in the quantities tr Φ and tr Φ^2 etc. We denote

$$c_k=\mathrm{coeff}\,(z^k\colon f(z))\quad\text{if}\quad f(z)=\sum_{n=0}^{\infty}c_nz^n\,.$$

EXAMPLE 1. As the simplest case, it follows from the difinition (6)

$$\begin{aligned}\sum_{\sigma}\mathrm{coeff}\,(z\colon \Phi(z)_{a^{\sigma},a^{\sigma}}) &= (F'(a))^{-1}\phi(a^+,a^+)-(F'(a))^{-1}\phi(a^-,a^-)\\ &= |F'(a)|^{-1}\delta[F(a^+)\underset{\mathrm{sgn}\,a}{<}a^+,\,F(a^-)\underset{\mathrm{sgn}\,a}{<}a^-]\\ &= |F'(a)|^{-1}\delta[F((a))\supset cl(a)]\\ &= \begin{cases}|F'(a)|^{-1} & \text{if there exists a fixed point of type } a,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Here the existence of the fixed points follows from the intermediate value theorem. In general cases, we can see the following version:

Lemma 5.1. *Suppose that a word w satisfies the inequalities $F(w^{\sigma})\underset{\mathrm{sgn}\,w}{>}w^{\sigma}$ for both $\sigma\in\{+,-\}$. Then we can find a peroidic orbit of type w.*

Proof. The intermediate value theorem is applicable for the mapping $F^{|w|}$ on the interval $[\{w^-\},\{w^+\}]$.

Example 2. As the second simplest case, if we assume $a^{\sigma}[2]\neq b$ for both $\sigma\in\{+,-\}$, then it is immediate to see

$$(15)\qquad \sum_{\tau}\sum_{\sigma}\operatorname{coeff}(z\colon \Phi(z)_{a^{\sigma},b^{\tau}})\times\operatorname{coeff}(z\colon \Phi(z))_{b^{\tau},a^{\sigma}})$$

$$=\begin{cases}|F'(ab)|^{-1} & \text{if there exists a periodic orbit of type } ab,\\ 0 & \text{otherwise.}\end{cases}$$

In fact, the left hand side of (15)

$$=|F'(ab)|^{-1}\sum_{\sigma}\sum_{\tau}\sigma\tau\{\delta[b^{\tau}\underset{\operatorname{sgn} a}{<}F(a^{\sigma})]-1/2\}\ \{\delta[a^{\sigma}\underset{\operatorname{sgn} b}{<}F(b^{\tau})]-1/2\}$$

$$=|F'(ab)|^{-1}\delta[F^2((ab))\supset cl(ab)]\,.$$

At this stage one might expect:

$$\sum_{\sigma_i}\prod_{i=1}^{n}\operatorname{coeff}(z\colon \Phi(z)_{a_i^{\sigma},a_{i+1}^{\sigma}})$$

$$=\begin{cases}|F'(w)|^{-1} & \text{if there exists a peroodic orbit of type } w,\\ 0 & \text{otherwise,}\end{cases}$$

where $w=a_1\cdots a_n$, $a_i^{\sigma}=a_i^{\sigma_i}\ (1\leqq i\leqq n)$ and $a_{n+1}^{\sigma}=a_1^{\sigma_1}$. But it is false;

Example 3. If $a^+[2]=b$, $\operatorname{sgn} a=\operatorname{sgn} b=+$, $b^+[2]>a^+$, $a^-[2]<b^-$ and $b^-[2]<a^-$, then the periodic orbit of type ab exists or not according as $a^+[3]>a^+$ or $a^+[3]<a^+$ Hence the equality in Example 2 fails to take place. Nevertheless, in either case, we obtain the correct expression if we take account of the coefficient of z^2 of $\Phi(z)_{a^+,a^+}$:

$$\sum_{\sigma}\sum_{\tau}\operatorname{coeff}(z\colon \Phi(z)_{a^{\sigma},b^{\tau}})\times\operatorname{coeff}(z\colon \Phi(z)_{b^{\tau},a^{\sigma}})+\operatorname{coeff}(z^2\colon \Phi(z)_{a^+,a^+})$$

$$=\begin{cases}|F'(ab)|^{-1} & \text{if there exists a periodic orbit of type } ab,\\ 0 & \text{otherwise.}\end{cases}$$

Similar treatment is necessary in the case where $a_i^{\sigma}[1,k]=a_i\cdots a_{i+k-1}$ for some $i(1\leqq i\leqq n)$, $\sigma\in\{+,-\}$ and $k\geqq 2$.

5.2. A Lemma for the computation of $\sum_{k=1}^{\infty}k^{-1}\Phi^k(z)$

Let us introduce an auxiliary matrix $\Psi(w)=\Psi(w\colon z)$ for a word w, which we will refer as the terms of $\sum_{k=1}^{\infty}\Phi^k(z)$ associated with the word w. Put

$$(16)\qquad \Psi(w)_{\tilde{a},\tilde{b}}=\sum\nolimits_{*}\prod_{j=1}^{l}\operatorname{coeff}(z^{i_{j+1}-i_j}\colon \Phi(z)_{\tilde{c}_j,\tilde{c}_{j+1}})$$

$$\times\delta[\tilde{c}_j[1,i_{j+1}-i_j]=w[i_j,i_{j+1}-1]]\,,$$

where the sum $\sum_{*}$ is taken over all decompositions of w into subwords, namely, it is taken over $1=i_1<i_2<\cdots<i_{l+1}=n+1$ ($|w|=n$) and $\sigma_2,\cdots,\sigma_l\in\{+,-\}$, and

$$\tilde{c}_j = \begin{cases} a & j=1\,, \\ w[i_j]^{\sigma_j} & 2\leqq j\leqq l\,, \\ \tilde{b} & j=l+1\,. \end{cases}$$

Remark that

$$\sum_w \Psi(w)_{\tilde{a},\tilde{b}} = \sum_{k=1}^{\infty} \Phi^k(z)_{\tilde{a},\tilde{b}}\,.$$

The following formula will be used repeatedly.

Lemma 5.2.

(17) $$\Psi(w)_{\tilde{a},\tilde{b}} = |F'(w)|^{-1}\chi(a,\, w\cdot\tilde{b})\delta[w[1]=\tilde{a}[1],\ \theta w\cdot\tilde{b}\in S^0]\,.$$

Proof. We will prove this lemma by induction on the length of word w.

1. Let $w=a$, then

$$\begin{aligned}\Psi(w)_{\tilde{a},\tilde{a}} &= \varepsilon(\tilde{a})\,(F'(a))^{-1}\phi(\tilde{a},\,\tilde{b}) \\ &= \varepsilon(\tilde{a})\,(F'(a))^{-1}[\{\delta[\tilde{b}\leqq a[2]^{-1}]-1/2\} \\ &= |F'(a)[|^{-1}\{\delta[a\tilde{b}<a\colon \varepsilon(\tilde{a})]-1/2\}\,.\end{aligned}$$

The last equality is proved by examining the four cases: (1) $\varepsilon(\tilde{a})=\operatorname{sgn}\tilde{a}=+$, (2) $\varepsilon(\tilde{a})=+$, $\operatorname{sgn}\tilde{a}=-$, (3) $\varepsilon(\tilde{a})=-$, $\operatorname{sgn}\tilde{a}=+$, and (4) $\varepsilon(\tilde{a})=\operatorname{sgn}\tilde{a}=-$.

2. Assume that the equation (17) holds for every words with length less than N and consider a word w with length N ($|w|=N$).

2.1. First we treat the case where $a[1,N]\neq w$. Let m be the minimal integer which satisfies $\tilde{a}[1,m]\neq w[1,m]$. Then there occurs two cases:

(i) $\tilde{a} \underset{\varepsilon(\tilde{a})}{>} w[1,\, m-1]w[m]^{\sigma}$ with both $\sigma\in\{+,-\}$,

(ii) $\tilde{a} \underset{\varepsilon(\tilde{a})}{<} w[1,\, m-1]w[m]^{\sigma}$ with both $\sigma\in\{+,-\}$,

and each case can be divided into three cases:

(a) $\theta w\cdot\tilde{b}\in S^0$,

(b) $\theta_u w\cdot\tilde{b}\in S^0$, but $\theta^{n-1}w\cdot\tilde{b}\notin S^0$ for some $n(2\leqq n\leqq m)$,

(c) $\theta^m w\cdot\notin S^0$.

Now we will calculate $\Psi(w)$ for each of the above six cases. Let us denote

$$\Psi(k,\,\sigma,\,\tilde{b}) = \Psi(\theta^{k-1}w)_{w[k]^{\sigma},\tilde{b}}$$

and

$$\Phi(\tilde{a},\,k,\,\sigma) = \operatorname{coeff}(z^{k-1}\colon \Phi(z)_{\tilde{a},w[k]\sigma})\,.$$

case i–a. By the inductive assumption, we obtain

$$\Psi(k,\,\sigma,\,\tilde{b}) = \sigma 2^{-1}|F'(\theta^{k-1}w)|^{-1} \quad (2\leqq k\leqq m)\,.$$

We also obtain

$$\Phi(\tilde{a}, k, \sigma) = \begin{cases} +2^{-}\varepsilon^{1}(\tilde{a})|F'(w[1, k-1])|^{-1} & \text{if } \ w[1, k-1]\cdot w]k]^{\sigma} <_{\varepsilon(\tilde{a})} \tilde{a}\,, \\ -2^{-1}\varepsilon(\tilde{a})|F'(w[1, k-1])|^{-1} & \text{otherwise.} \end{cases}$$

Then replacing the product terms of the right hand side of (16) by Φ for $j=1$ and Ψ for $2 \leqq j \leqq l$ respectively, we have

$$\begin{aligned} (18) \qquad \Psi(w)_{\tilde{a},\tilde{b}} &= \sum_{k=2}^{m} \sum_{\sigma} \Phi(\tilde{a}, k, \sigma)\Psi(k, \sigma, \tilde{b}) \\ &= \sum_{\sigma} \Phi(\tilde{a}, m, \sigma)\Psi(m, \sigma, \tilde{b}) \\ &= 2^{-1}\varepsilon(\tilde{a})|F'(w)|^{-1} \\ &= |F'(w)|^{-1}\{\delta[w\cdot\tilde{b}<\tilde{a}\colon \varepsilon(\tilde{a})]-1/2\} \\ &= |F'(w)|^{-1}\chi(\tilde{a}, w\cdot q)\delta[w[1]=\tilde{a}\,,\ \theta w\cdot\tilde{b}\in S^{0}]\,. \end{aligned}$$

Hence (17) follows.

case i—b. In this case

$$\Psi(k, \sigma, \tilde{b}) = \begin{cases} +2^{-1}\varepsilon(\tilde{a})|F'(\theta^{k-1}w)|^{-1} \\ \qquad \text{if } \ n+1 \leqq k \leqq m, \text{ or } k=n \text{ and } \theta^{n-1}w\cdot\tilde{b} <_{\sigma} w[n]^{\sigma}\,, \\ -2^{-1}\varepsilon(\tilde{a})|F'(\theta^{k-1}w)|^{-1} \\ \qquad \text{if } \ k=n \text{ and } \theta^{n-1}w\cdot\tilde{b} >_{\sigma} w[n]^{\sigma}\,, \\ 0 \qquad \text{if } \ 2 \leqq k < n\,. \end{cases}$$

Note that Ψ may take negative value and the cancellation takes place in the computation similar to (18) and we get:

$$\Psi(w)_{\tilde{a},\tilde{b}} = 0\,.$$

case i—c is trivial. We can prove the case (ii) in a similar way.

2.2. Now we assume that $\tilde{a}[1, N]=w$. Then

$$\Psi(w)_{\tilde{a},\tilde{b}} = \operatorname{coeff}(z^{N}\colon \Phi(z)_{\tilde{a},\tilde{b}}) + \sum_{k=1}^{N-2} \sum_{\sigma} \Phi(\tilde{a}, k, \sigma)\Psi(k, \sigma, \tilde{b})\,.$$

Thus, as in 2–1, we can prove (17). This proves the lemma.

5.3. Definition of $\psi[\tau]$

We now define several notations. For a word $w=a_1\cdots a_n$, we define

$$\pi^{k}w = a_{k+1}\cdots a_n a_1 \cdots a_k \qquad (0 \leqq k \leqq n-1)\,.$$

As in the definition of $\Psi(w)$, let us call the following quantity $Tr(w)$ the term of tr $\sum_{k=1}^{\infty} \Phi^{k}(z)$ associated with a word w:

$$Tr\,(w)=\sum_{n=0}^{|w|-1}\sum_{\sigma}\sum\nolimits_{*}\sum_{k=1}^{\infty}k^{-1}\prod_{j=1}^{k}\text{coeff}\,(z^{i_{j+1}-i_j}\colon\Phi^k(z)_{\tilde{c}_j,\tilde{c}_{j+1}})$$
$$\times\delta[\tilde{c}_j[1,\,i_{j+1}-i_j]=\pi^n w[i_j,\,i_{j+1}-1]]\,,$$

where the sum $\sum_*$ is taken over $1=i_1<i_2<\cdots<i_{k+1}=n+1$ and $\sigma_2,\cdot,\cdot,\cdot,\sigma_k\in\{+,-\}$, and

$$\tilde{c}_j=\begin{cases}(\pi^n w)\,[1]^{\sigma} & j=1 \text{ or } j=k+1\,,\\ (\pi^n w)\,[i_j]^{\sigma_j} & 2\leqq j\leqq k\,.\end{cases}$$

Note that $\sum_w Tr\,(w)=\sum_{k=1}^{\infty}\Phi^k(z)$. We will show that

$$Tr(w)=\begin{cases}|F'(w)|^{-1} & \text{if the there exists a periodic orbit of type } w,\\ 0 & \text{otherwise.}\end{cases}$$

If $w[|w|]^{\sigma}[2]\neq w[1]$ and $w[i]^{\sigma}[2]\neq w[i+1]$ for $\sigma\in\{+,-\}$ and any $i(1\leqq i\leqq|w|-1)$, then

$$Tr\,(w)=\sum_{\sigma}\text{coeff}\,(z^{|w|},\,\Phi^{|w|}(z)_{w[1]^{\sigma},w[1]^{\sigma}})=\sum_{\sigma}\Psi(w)_{w[1]^{\sigma},w[1]^{\sigma}}\,.$$

For other cases, as in Example 3, we need to consider higher codfficients. Let us use the following abbreviation for $\tau\in\{+,-\}$

$$w_n^{\tau}=w[n]^{\tau\,\text{sgn}\,\theta^n w}\qquad(1\leqq n\leqq|w|)\,,$$

Let

$$m_0=m_0[\tau,\,w]=2,$$

and for $i\geqq 1$

$$m_i=m_i[\tau,\,w]=\min\,\{N>m_{i-1}\colon w_n^{\tau}[1,\,|w|-n+2]=(\pi^{n-1}w)\,[1,\,|w|-n+2]\}\,,$$

if the minimum exists. Denote

$$k[\tau]=k[\tau,\,w]=\text{the number of } m_i\text{'s}\qquad(i\geqq 1)\,,$$

$$n_i=n_i[\tau,\,w]$$
$$=1+\max\,\{n<m_i\colon w_{m_i}^{\tau}[1,\,|w|-m_i+n+1]=(\pi^{m_i-1}w)\,[1,\,|w|-m_i+n+1]\}\,.$$

We put $m_1=1$ and $n_1=1$ if $k[\tau]=0$. Let

$$\psi_0[\tau\,\text{sgn}\,w]=\Psi(w)_{w[1]^{\tau},w[1]^{\tau}}\,,$$

and for $1\leqq i\leqq k[\tau]$ and $2\leqq j\leqq n_i$ put

$$\psi_{i,j}[\tau]=\begin{cases}\sum_{\sigma}\psi(j,\,\sigma,\,m_i,\tau)\phi(m_i,\,\tau,\,j,\,\sigma) & n_i<m_i\,,\\ \phi(m_i,\,\tau,\,m_i,\,\tau) & n_i=m_i\,,\end{cases}$$

where

$$\psi(j,,\,\sigma,\,m,\,\tau)=\text{the } (w_j^{\sigma},\,w_m^{\tau})\text{-component of the matrix}$$
$$\Psi(\pi^{j-1}w)\,|F'((\pi^{j-1}w)\,[1,\,m-j])\,|\,,$$

and

$$\phi(m, \tau, j, \sigma) = \text{coeff}\,(z^{|w|-m+j}:\ \Phi(z))_{w_m^\tau, w_j^\sigma} |F'((\pi^{m-1}w)\,[1,\ |w|-m+j])|\,.$$

Finally we put

$$\psi[\tau] = \psi_0[\tau \operatorname{sgn} w] + \sum_{i=1}^{k[\tau]} \sum_{j=2}^{n_i} \psi_{i,j}[\tau]\,.$$

Taking care of words w which satisfy $w[i,\ |w|+1] = w[1,\ |w|-i]$ for some $i(1 \leqq i \leqq |w|)$, we get

Lemma 5.3.

$$Tr\,(w) = \sum_\tau \psi[\tau]\,|F'(w)|^{-1}\,.$$

5.4. The calculation of $\psi(j, \sigma, m, \tau)$

We will calculate $\psi(j, \sigma, m, \tau)$.

ψ–1. The case $2 \leqq j \leqq n$ and $w[j+1, m-1]\cdot w_m^\tau \notin S^0$. It follows $\psi(j, \sigma, m, \tau) = 0$ for $\sigma \in \{+, -\}$ by Lemma 5.2.

ψ–2. The case $w[j+1,\ m-1]\cdot w_m^\pm \in S^0$. It also follows from Lemma 5.2 that

$$\begin{aligned}(19) \qquad \psi(j, \sigma, m, \tau) &= \{\delta[w[j,\ m-1]\cdot w_m^\tau \underset{\sigma(j)}{<} w_j^\sigma] - 1/2\} \\ &= \{\delta[w[1,\ m-1]\cdot w_m^\tau \underset{\sigma(j)}{<} w[1, j-1]\cdot w_j^\sigma] - 1/2\}\,,\end{aligned}$$

where $\sigma(j) = \sigma \operatorname{sgn} \theta^{j-1} w$. We divide the case into four subcases according to the values of $\psi(j, \sigma, m, \tau)$.

ψ–2–1. $\psi(j,\ \sigma,\ m,\ \tau) = +1/2$ for both $\sigma \in \{+, -\}$: It occurs when $w[j]^- < w[j, m-1]\cdot w_m^\tau < w[j]^+$ because of (19). In this case $w[j, m-1]\cdot w_m^\tau \in S^0$.

ψ–2–2. $\psi(j, +, m, \tau) = +1/2$ and $\psi(j, -, m, \tau) = -1/2$: It occures when $w[j, m-1]\cdot w_m^\tau \notin S^0$. Then by (19) $w[j, m-1]\cdot w_m^\tau \underset{\rho(j)}{\leqq} w[j]^{-\rho(j)} (\underset{\rho(j)}{<} w[j]^{\rho(j)})$, where $\rho(j) = \operatorname{sgn} \theta^{j-1} w$. Note that in this a case

$$w[j]^+ \geqq w_m^\tau[|w|-m+j+1,\ \infty) \geqq w[j]^-\,, \quad \text{if} \quad j < n\,,$$

and so

$$w[n,\ m-1] w_m^\tau \underset{\rho(n)}{\leqq} w_m^\tau[|w|-m+n+1,\ \infty)\,.$$

Since $w[n,\ m-1]\cdot w_m^\tau \in (w[n])$ and $w_m^\tau[|w|-m+n+1,\ \infty) \notin (w[n])$, we obtain

$$w_m^\tau[|w|-m+n+1,\ \infty) \underset{\rho(n)}{>} w[n]^{\rho(n)} (\underset{\rho(n)}{>} w[n]^{-\rho(n)})\,.$$

Hence,

$$w[1,\ m-1]\cdot w_m^\tau > w\cdot w[1,\ n-1]\cdot w[n]^{\rho(n)} (> w\cdot w[1,\ n-1]\cdot w[n]^{-\rho(n)})\,.$$

ψ–2–3. $\psi(j, +, m, \tau) = -1/2$ and $\psi(j, -, m, \tau) = +1/2$: It occure when

$$w[j,\ m-1]\cdot w_m^\tau \underset{\rho(j)}{\geqq} w[j]^{\rho(j)} (\underset{\rho(j)}{<} w[j]^{-\rho(j)})\,.$$

Notice also when $j<n$

$$w[1,\ m-1]\cdot w_m^\tau < w\cdot w[1,\ n-1]\cdot w[n]^{\rho(n)}(<w\cdot w[1,\ n-1]\cdot w[n]^{-\rho(n)})\,.$$

ψ–2–4. $\psi(j,\sigma,m,\tau)=-1/2$ for both $\sigma\in\{+,-\}$: It never occurs, since if then, one would have $w_j^+<w_j^-$.

5.5. The calculation of $\phi(m,\tau,j,\sigma)$

Next we will calculate $\phi(m,\tau,j,\sigma)$. By the definition, we get

$$\begin{aligned}\phi(m,\ \tau,\ j,\ \sigma) &= \{\delta[\theta^{m-1}w\cdot w[1,j-1]\cdot w_j^\sigma \underset{\tau(m)}{<} w_m^\tau]-1/2\}\\ &= \{\delta[w\cdot w[1,j-1]\cdot w_j^\sigma \underset{\tau(1)}{<} w[1,\ m-1]\cdot w_m^\tau]-1/2\}\ ,\end{aligned}$$

where $\tau(m)=\tau\,\mathrm{sgn}\,\theta^{m-1}w$. Let us classify the word w into four cases according to the values of $\phi(m,\tau,j,\sigma)$.

ϕ–1. $\phi(m,\tau,j,\sigma)=+1/2$ for both $\sigma\in\{+,-\}$: It occurs when $w_j^+, w_j^- <_{\tau'} w_m^\tau[|w|-m+j+1,\infty)$, where $\tau'=\tau\,\mathrm{sgn}\,w[1,j-1]$. Hence $j=n$ and

$$w[1,\ m-1]\cdot w^\tau \underset{\tau(1)}{>} w\cdot w[1\ ,j-1]\cdot w[j]^{\tau'}(\underset{\tau(1)}{>} w\cdot w[1,j-1]\cdot w[j]^{-\tau'})$$

ϕ–2. $\phi(m,\tau,j,\sigma)=-1/2$ for k both $\sigma\in\{+,-\}$: It occurs when $j=n$ and

$$w[1,m-1]\cdot w_m^\perp \underset{\tau(1)}{\leqq} w\cdot w[1,j-1]\cdot w[j]^{-\tau'}(\underset{\tau(1)}{\leqq} w\cdot w[1,j-1]\cdot w[j]^{\tau'})\,.$$

ϕ–3. $\phi(m,\tau,j,+)=-1//2$ and $\phi(m,\tau,j,-)=-1/2$: It occurs when $j<n$ and

$$w\cdot w[1,j-1]\cdot w[j]^- \underset{\tau(1)}{\geqq} w[1,m-1]\cdot w_m^\tau \underset{\tau(1)}{>} w\cdot w[1,j-1]\cdot w[j]^+\,.$$

Hence we get $\tau(1)=\tau\ \mathrm{sgn}\,w=-$.

ϕ–4. $\phi(m,\tau,j,+)=-1/2$ and $\phi(m,\tau,j,-)=+1/2$: As in ϕ–3, it follows $\tau\,\mathrm{sgn}\,w=+$.

5.6. The calculation of $\psi[\tau]$

Now we can calculate $\psi[\tau]$.

Lemma 5.4.

$$\begin{aligned}\psi[\tau] = \{&\delta[w\cdot w[1,\ n_1-1]\cdot w[n_1]^{\tau(n_1)} \underset{\tau(1)}{<} w[1,\ m_1-1]\cdot w_{m_1}^\tau]-1/2\\ &\times\delta[w[2,\ m_1-1[\cdot w_{m_1}^\tau\in S^0]\}\end{aligned}$$

Proof. case 1: $k[\tau]=0$. Then it is obvoius from Lemma 5.2.

case 2. $w[2,m_1-1]\cdot w_{m_1}^\tau\in S^0$. If n_1-m_1, the proof is trivial. On the other hand, if $n_1<m_1$, the case ψ–2–1 takes place for ψ and therefore

$$\sum_j \psi_{i,j}[\tau] = \sum_j \sum_\sigma \psi(j, \sigma, m_1, \tau)\phi(m_1, \tau, j, \sigma)$$
$$= \sum_\sigma \psi(n_1, \sigma, m_1, \tau)\phi(m_1, \tau, n_1, \sigma)$$
$$= \{\delta[w\cdot w[1, n_1-1]\cdot w[n_1]^{\tau(n_1)} \underset{\tau(1)}{<} w[1, m_1-1]\cdot w^{\tau}_{m_1}]-1/2\} .$$

case 3. In the case $w[2, m_1-1]w^{\tau}_{m_1} \notin S^0$ and that $w[j'+1, m_1-1]\cdot w^{\tau}_{m_1} \in S^0$ and $w[j', m_1-1]\cdot w^{\tau}_{m_1} \notin S^0$ for some j'. Then

$$\sum_j \psi_{i,j}[\tau] = \sum_\sigma \psi(j', \sigma, m_1, \tau)\phi(m_1, \tau, j', \sigma) + \sum_\sigma \psi(n_1, \sigma, m_1, \tau)\phi(m_1, \tau, n_1, \sigma) = 0 .$$

In fact, if there occurs the case ψ–2–2 and ϕ–3 for j', then $\psi_{i,j'}[\tau]=+1/2$. On the other hand $\tau \operatorname{sgn} w=-$ and

$$w[1, m_1-1]\cdot w^{\tau}_{m_1} > w\cdot w[1, n_1-1]\cdot w[n_1]^{-\operatorname{sgn}\theta^{n_1-1}w}$$
$$> w\cdot w[1, n_1-1]\cdot w[n_1]^{-\operatorname{sgn}\theta^{n_1-1}w} .$$

Thus, if $j=n_1$, the case ψ–2–1 and ϕ–2 holds. Hence $\psi_{1,n_1}[\tau]=-1/2$. The other possible cases are treated in similar ways.

case 4. If $n_i \leqq n_1$ for $i \geqq 2$, then $\psi_{i,j}[\tau]=0$ for any j, because ψ–1 holds.

case 5. For $i \geqq 2$ such that $n_i > n_1$, by the same reason as in case 3, $\sum \psi_{i,j}[\tau]=0$.

This proves the lemma.

Lemma 5.5. *If* $int(w) \neq \phi$, $w\cdot w[1, n_1-1]\cdot w[n_1]^{\tau(n_1)} <_{\tau(1)} w[1, m_1-1]\cdot w^{\tau}_{m_1}$ *is equivalent to* $w\cdot w^{\tau \operatorname{sgn} w} <_{\tau \operatorname{sgn} w} w^{\tau \operatorname{sgn} w}$.

The proof is trivial because $w[1, m_1-1]\cdot w^{\tau}_{m_1} < w^{\tau \operatorname{sgn} w}$ and $w[1, m_1-1]\cdot w^{\tau}_{m_1} \in (w[1, n_1-1])$.

5.7. The proof of Theorem C

Now we will prove Theorem *C*. For this purpose, we need to study periodic orbits passing through the division points. These points are the only candidates where the correspondence established in the previous subsections may fail. First of all note that if $w[2, m_1[\tau]-1]\cdot w^{\tau}_{m_1[\tau]} \in S^0$, then $w[2, m_1[\tau]-1]\cdot w^{-\tau}_{m_1[\tau]} \in S^0$, too. We get the following by Lemma 5.4.

Case 1. $\operatorname{int}(w)=\phi$. In this case, the periodic orbit of type w exists only when the periodic orbit pass through a division point.

1.1. When $w[2, m_1[\tau]-1]w^{\tau}_{m_1[\tau]} \notin S^0$ for both $\tau \in \{+, -\}$, then both $\psi[\tau]=0$.

1.2. When $w[2, m_1[\tau]-1]\cdot w^{\tau}_{m_1[\tau]} \notin S^0$ for both $\tau \in \{+, -\}$, then both $\psi[+]+\psi[-]=0$.

Figure of $F^{|w|}|_{(w)}$:

1. sgn $w=+$:

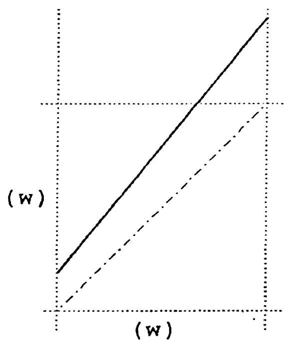


Fig. II

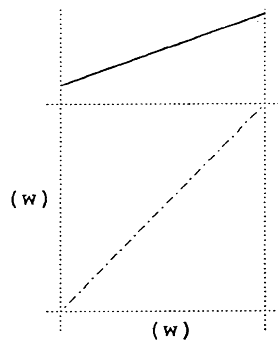


Fig. III

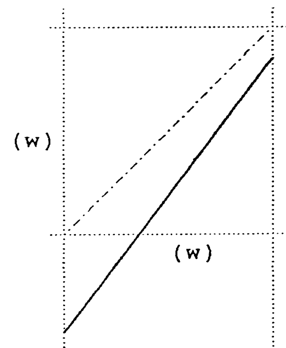


Fig. IV

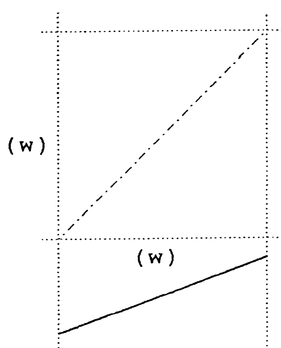


Fig. V

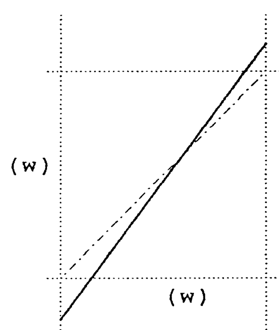


Fig. VI

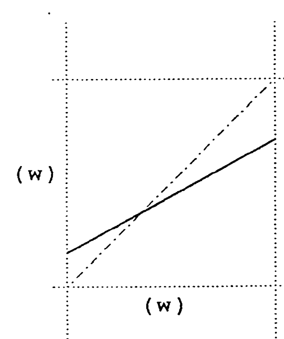


Fig. VII

2. sgn $w=-$:

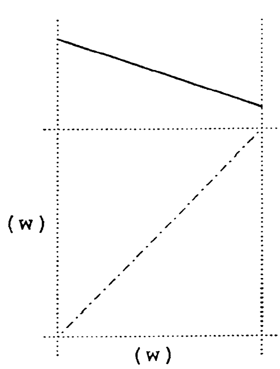


Fig. VIII

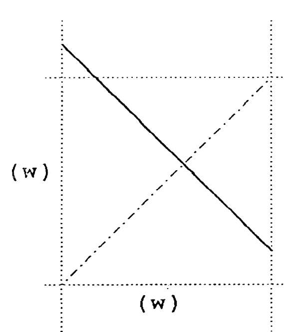


Fig. IX

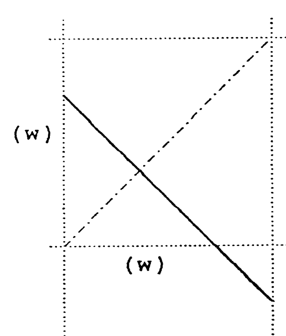


Fig. X

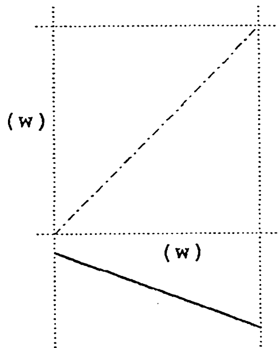


Fig. XI

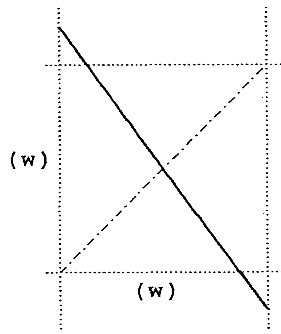


Fig. XII

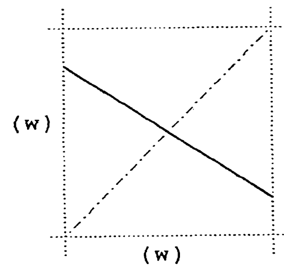


Fig. XIII

3. the case when the periodic orbit passing the boundary point exists:

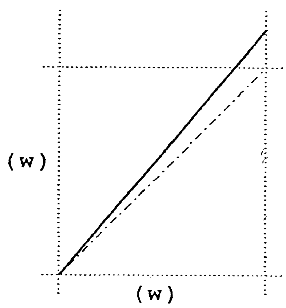


Fig. XIV

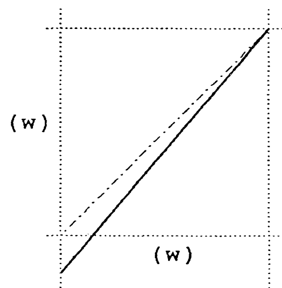


Fig. XV

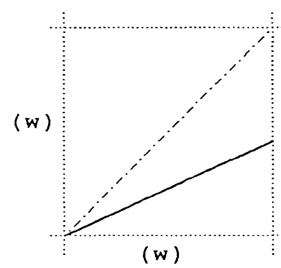


Fig. XVI

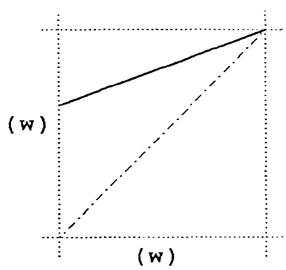


Fig. XVII

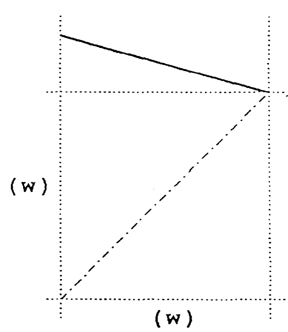


Fig. XVIII

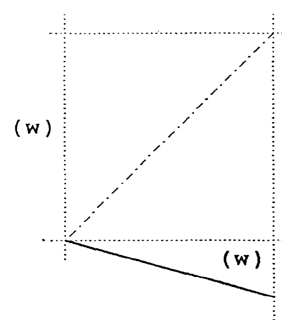


Fig. XIX

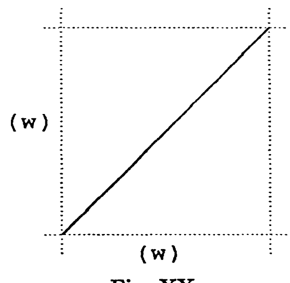


Fig. XX

case 2. $\text{int}(w)\neq\phi$. We will enumerate by Lemma 5.5 all the possible types of the graph of $F^{|w|}$ on (w) as figures (II)$\sim$(XX) among which the cases (VII), (XIII), (XVI), (XVII), (XX) never occur, since we assume the lower Lyapunov number $\xi>0$.

In the case of (II), (III), (IV), (V), (IX), (XI), we get $\psi[+]+\psi[-]=0$, and the periodic orbit of type w does not exist.

In the case of (VI), (VIII), (X), (XII), (XIII), we get $\psi[+]+\psi[-]=1$, and the periodic orbit of type w exists.

In the case of (VII), we get $\psi[+]+\psi[-]=-1$, and the periodic orbit of type w exists. For the rest of figures (XIV), (XV), (XVI), (XVII), (XVIII), (XIX), (XX), the periodic orbit of type w passes through the division points if it exists and

$$\psi[+]+\psi[-]=\begin{cases} 0 & \text{for the cases (XIV), (XV), (XVIII), (XIX)}, \\ -1 & \text{for the cases (XVI), (XVII), (XX)}. \end{cases}$$

Thus it turns out that the periodic points contributing to the quantitity $\sum_{k=1}^{\infty} k^{-1}\Phi^k(z)$ does not exhaust all the periodic orbits which pass through the boundary points and we should add extra terms to it, which will form the functions $R(z)$ and $R_*(z)$ in the statemnt of Theorem C. Now we will give the precise definition. They are defined as the sums $R(z)=\sum_{\tilde{w}\in\tilde{W}}c(\tilde{w})$ and $R_*(z)=\sum_{\tilde{w}\in\tilde{D}}c_*(\tilde{w})$.

The summands $c(\tilde{w})$ and $c_*(\tilde{w})$ are defined in the following way according to the nature of words $\tilde{w}$.

Case (a) $\text{int}(w)\neq\phi$ and $w\cdot w^{\sigma}=w^{\sigma}$ (this case occurs when $F(w)$ has the graph of type (XIV) and (XV)):

(a–1) If $w^{\sigma}\in S^0$, then $c(w^{\sigma})=|w|^{-1}z^{|w|}|F'(w)|^{-1}$ and $c_*(w^{\sigma})=0$.

(a–2) If $w^{\sigma}\notin S^0$, then $c(w^{\sigma})=0$ and $c_*(w^{\sigma})=-|w|^{-1}z^{|w|}|F'(w)|^{-1}$

Case (b) $\text{int}(w)\neq\phi$ and $w\cdot w^{\sigma\wedge}=w^{\sigma}$ and $w\cdot w^{\sigma\wedge}\in S^0$. Then $c(w^{\sigma})=c_*(w^{\sigma})=|w|^{-1}z^{|w|}|F'(w)|^{-1}$. This case occurs when $F(w)$ has the graph of type (XVIII) and (XIX). Notice here $\text{int}(w\cdot w)=\phi$.

Case (c) Otherwise we put $c(\tilde{w})=c_*(\tilde{w})=0$.

Thus we have exhausted all the periodic orbits which pass through the division points. In summary, we get

$$\begin{aligned}\zeta(z) &= \exp[-\text{tr}(\log(I-\Phi(z))+R(z)] \\ &= \exp[-\text{tr}(\log(I-\Phi_*(z))+R_*(z)]\,. \end{aligned} \tag{20}$$

REMARK 5.2. If F satisfies the endopint condition, $c(\tilde{w})=0$ for all words $\tilde{w}$. This shows $R(z)=0$, hence we get the proof of Theorem B.

We get by Theorem A,

$$\mathrm{Spec}\,(F) \cap \{z^{-1}\colon |z|<e^{\xi}\} = \{z^{-1}\colon \det(I-\Phi_*(z)) = 0,\ |z|<e^{\xi}\}\,.$$

In particular,

$$\mathrm{Spec}\,(F) \cap \{z^{-1}\colon |z|<e^{\xi_0}\} = \{z^{-1}\colon \zeta(z) = \infty,\ |z|<e^{\xi_0}\}$$

where

$$\xi_0 = \inf_{x\in I} \liminf_{n\to\infty} \cap\, n^{-1} \log |F^{n\prime}(x)|\ .$$

Let us see how $R_*(z)$ and $R_*(z)$ contributes to $\zeta(z)$. If $c(w^\sigma)$ or $c_*(w^\sigma)$ does not equal to zero, then by (20) $\zeta(z)$ has singularity at $z=|F'(w)|^{1/|w|}$. If, in addition, $w\cdot w^\sigma = w^\sigma$, then $|F'(w)|^{1/|w|} \geqq e^{\xi}$. This is the case (a) of the definition of $c(w^\sigma)$ and $c_*(w^\sigma)$. On the other hand, if $w\cdot w^{\sigma\wedge}=w^\sigma$ and $w\cdot w^{\sigma\wedge}\in S^0$, then it may happen

$$e^{\xi_0} \leqq |F'(w)|^{1/|w|} < e^{\xi}$$

Now we will show that one can reconstruct the partition so that

$$\{z\colon \zeta(z) = \infty\,,\ |z|<e^{\xi}\} = \{z\colon D(z) = 0\,,\ |z|<e^{\xi}\}\,.$$

For instance, let us assume there exists words w_1 and w_2 such that $w_1^+ = w_2^{-\wedge}$, the graph of $F(w_1)$ is of type (XVIII) and the graph of $F(w_2)$ is of type (XIX), Then it is obvious that $|F'(w_1\cdot w_2)| \geqq e^{|w_1\cdot w_2|\xi}$. Hence, $|F'(w_i)| \geqq e^{|w_i|\xi}$ at least for one of i's. Now we assume $|F'(w_1)| \geqq e^{|w_1|\xi}$. Then we can take a new partition so that $\{w_1^+\} \in S^0$. In general, we can take a partition of I for which, there exists no word w and $\sigma \in \{+, -\}$ such that $w\cdot w^\sigma = w^\sigma$ and $|F'(w)| < e^{|w|\xi}$. This is the desired partition. Thus Theorem C has been proved.

Remark 5.3. We can futher extend the definition to the case where $\xi \leqq 0$. In the case (a) of the definition of $R(z)$,if we change $|w|^{-1} z^{|w|} |F'(w)|^{-1}$ into

$$|w|^{-1} z^{|w|} |F'(w)|^{-1} \{\delta[F'(w)| > |1] + 2\delta[|F'(w)| \leqq 1]\}\ ,$$

then we can see that in the case (XX) there exists only one periodic orbit of type w.

6. Examples

Let us calculate the Fredholm determinant fo β-transformations, unimodal linear transformations and linear mod. one transformatious on the unit interval [0,1].

Example A(β-transformations, cf. [8]).

$F(x) = \lambda x$ mod. 1. (fig. XXI for $1<\lambda\leqq 2$ and fig. XXII for $\lambda>2$)

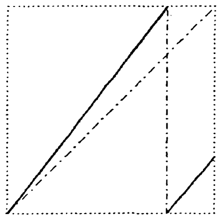


Fig. XXI

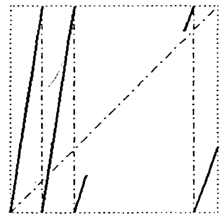


Fig. XXII

$$\Phi_*(\lambda z) = \begin{pmatrix} z/2, & , \; z/2, & z/2, & z/2 \\ z/2, & , \; z/2, & z/2, & z/2 \\ z/2, & , \; z/2, & z/2, & z/2 \\ z/2(1-z), & a(z), & a(z), & -z/2(1-z) \end{pmatrix} \begin{matrix} 0^- \\ 0^+ \\ 1^- \\ 1^+ \end{matrix}, \quad \begin{matrix} 0^- & 0^+ & 1^- & 1^+ \end{matrix}$$

where

$$a(z) = \sum_{n=1}^{\infty} \{\delta[1^+[n+1]-1/2\} z^n .$$

Then for $k<\lambda\leqq k+1$

$$D(\lambda z) = 1-\sum_{n=1}^{\infty} k^+[n] z^n ,$$

and the density function of the invariant measure

$$h(x) = C^{-1}\sum_{n=0}^{\infty} \lambda^{-n}\delta[x<k^+[n+1, \infty)] ,$$

where $C=\int h(x)dx$.

EXAMPLE B (Unumodal linear transformations, cf. [7]).

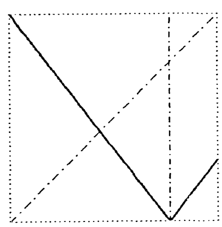


Fig. XXIII

$$F(x) = \begin{cases} \lambda x-1 & 0\leqq x<1/\lambda \quad \text{(fig. XXIII)} \\ \lambda x-1 & 1/\lambda\leqq x\leqq 1 , \end{cases}$$

where $1<\lambda\leqq 2$,

$$\Phi_*(\lambda z) = \begin{pmatrix} z/2, & z/2, & z/2, & z/2 \\ z/2, & z/2, & z/2, & z/2 \\ z/2, & z/2, & z/2, & z/2 \\ b(z), & c(z), & c(z), & -b(z) \end{pmatrix} \begin{matrix} 0^- \\ 0^+ \\ 1^- \\ 1^+ \end{matrix}. \quad \begin{matrix} 0^- & 0^+ & 1^- & 1^+ \end{matrix}$$

where

$$b(z)=\sum_{n=1}^{\infty}\{\delta[\operatorname{sgn} 1^+[1,n]=+]-1/2\}z^n .$$

$$c(z)=\sum_{n=1}^{\infty}\{\delta[\operatorname{sgn} 1^+[1,n]=+ \quad\text{and}\quad 1^+[n]=1]$$
$$+[\operatorname{sgn} 1^+[1,n]=- \quad\text{and}\quad 1^+[n]=0]-1/2\}z^n .$$

Then

$$D(\lambda z)=1-(3/2)z+b(z)-z(2b(z)+c(z))=1-\sum_{n=1}^{\infty}z^n \operatorname{sgn} 1^+[1,n] ,$$

and the density function of the invariant measure

$$h(x)=C^{-1}\sum_{n=0}^{\infty}\lambda^{-n}\delta[x<1^+[n+1,\infty)]\operatorname{sgn} 1^+[1,n] ,$$

where $C=\int h(x)\,dx$.

Example C (Linear mod. one transformations).

$F(x)=\lambda x+r$ mod. 1. (fig. 1 XXIV $1<\lambda\leqq 2$ and fig. XXV foa $\lambda>2$, $\lambda>2$, $0<r<1$).

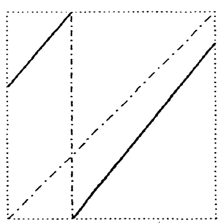


Fig. XXIV

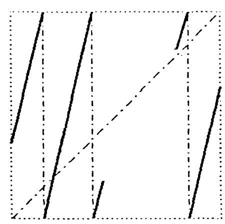


Fig. XXV

$$\Phi_*(\lambda z)=\begin{pmatrix} z/2(1-z), & d(z), & d(z), & -z/2(1-z) \\ z/2, & z/2, & z/2, & z/2 \\ z/2, & z/2, & z/2, & z/2 \\ z/2(1-z), & e(z), & e(z), & -z/2(1-z) \end{pmatrix}\begin{matrix} 0^- \\ 0^+ \\ 1^- \\ 1^+ , \end{matrix}$$
$$\qquad\qquad 0^- \qquad 0^+ \quad 1^- \qquad 1^+$$

where

$$d(z)=\sum_{n=1}^{\infty}=\{\delta[0^-[n+1]=0]-1/2\}z^n .$$

$$e(z)=\sum_{n=1}^{\infty}[\{\delta[1^+[n+1]=1]-1/2\}z^n .$$

Then for $k<\lambda\leqq k+1$

$$D(\lambda z)=(1-z)^{-1}\{1-\sum_{n=1}^{\infty}\{k^+[n]-0^-[n]\}z^n\} ,$$

and the desnity function of the invariant measure

$$h(z) = C^{-1}\sum_{n=1}^{\infty}\lambda^{-n}\{\delta[x<k^+[n+1,\ \infty)]-\delta[x<0^-[n+1,\ \infty]\}\ ,$$

where $C=\int h(x)\,dx$.

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