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# ON THE JONES POLYNOMIALS OF CHECKERBOARD COLORABLE VIRTUAL LINKS

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## 1. Introduction

In 1996, L.H. Kauffman introduced the notion of a virtual knot, which was motivated by the study of knots in a thickened surface and abstract Gauss codes, cf. [8, 9]. M. Goussarov, M. Polyak, and O. Viro [1] proved that the natural map from the category of classical knots to the category of virtual knots is injective; namely, if two classical knot diagrams are equivalent as virtual knots, then they are equivalent as classical knots. Thus, virtual knot theory is a generalization of knot theory. In [1], virtual knots are used to study of finite type invariants.

Kauffman defined the Jones polynomial of a virtual knot, which is also called the normalized bracket polynomial or the  $f$ -polynomial (cf. [9]). In this paper, according to [9], we call it the  $f$ -polynomial instead of the Jones polynomial, since the definition is different from Jones' in [2, 3]. Finite type invariants derived from the  $f$ -polynomials are studied in [9]. For example, the following results appear in [9]: (1) If  $f_K(A)$  denotes the  $f$ -polynomial of a virtual link  $K$ , the coefficient  $v_n(K)$  of  $x^n$  in the power series expansion of  $f_K(e^x)$  is a Vassiliev invariant of order  $n$ . (2) When the notion  $v_n$  for a “singular” virtual link  $G$  is generalized in the obvious way, the Vassiliev invariant  $v_n(G)$  depends only on the chord diagram associated with  $G$  (cf. Corollary 14 of [9]).

The  $f$ -polynomial of a virtual link is quite different from the  $f$ -polynomial of a classical link. For a Laurent polynomial  $f$  in the variable  $A$ , we denote by  $\text{EXP}(f)$  the set of integers appearing as exponents of  $f$ . For example, if  $f = 3A^{-2} + 6A - 7A^5$ , then  $\text{EXP}(f) = \{-2, 1, 5\}$ . For the  $f$ -polynomial  $f$  of a classical link with  $n$  components, it is well known that  $\text{EXP}(f) \subset 4\mathbf{Z}$  if  $n$  is odd and  $\text{EXP}(f) \subset 4\mathbf{Z} + 2$  if  $n$  is even ([2], [7]). However, this is not true for a virtual knot/link in general. In this paper we introduce the notion of *checkerboard coloring* of a virtual link diagram as a generalization of checkerboard coloring of a classical link diagram.

**Theorem 1.** *Let  $f$  be the  $f$ -polynomial of a virtual link  $L$  with  $n$  components. Suppose that  $L$  has a virtual link diagram which admits a checkerboard coloring. Then  $\text{EXP}(f) \subset 4\mathbf{Z}$  if  $n$  is odd, and  $\text{EXP}(f) \subset 4\mathbf{Z} + 2$  if  $n$  is even.*

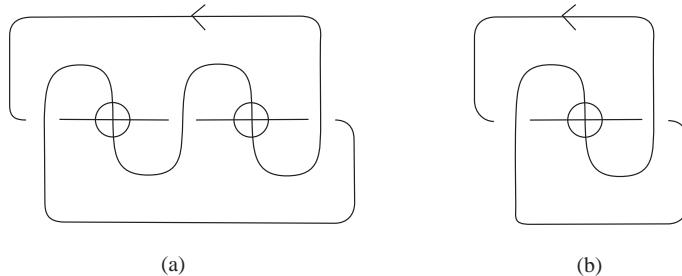


Fig. 1.

For example the virtual knot diagram illustrated in Fig. 1 (a) admits a checkerboard coloring, and the  $f$ -polynomial is  $A^4 + A^{12} - A^{16}$ . So  $\text{EXP}(f) \subset 4\mathbf{Z}$ . On the other hand, the virtual knot diagram illustrated in Fig. 1 (b) does not admit a checkerboard coloring, and the  $f$ -polynomial is  $-A^{10} + A^6 + A^4$ . Theorem 1 implies that this diagram is not equivalent to any diagram that admits a checkerboard coloring.

If a virtual link diagram is alternating (the definition is given later), then the diagram admits a checkerboard coloring. Thus we have the following.

**Corollary 2.** *Let  $f$  be the  $f$ -polynomial of a virtual link  $L$  with  $n$  components. Suppose that  $L$  has an alternating virtual link diagram. Then  $\text{EXP}(f) \subset 4\mathbf{Z}$  if  $n$  is odd, and  $\text{EXP}(f) \subset 4\mathbf{Z} + 2$  if  $n$  is even.*

By this corollary, we see that the virtual knot represented by Fig. 1 (b) is not equivalent to any alternating diagram.

## 2. Virtual link diagram and abstract link diagram

A *virtual link diagram* is a closed oriented 1-manifold generically immersed in  $\mathbf{R}^2$  such that each double point is labeled to be (1) a *real* crossing which is indicated as usual in classical knot theory or (2) a *virtual* crossing which is indicated by a small circle around the double point. The moves of virtual link diagrams illustrated in Fig. 2 are called *generalized Reidemeister moves*. Two virtual link diagrams are said to be *equivalent* if they are related by a finite sequence of generalized Reidemeister moves. We call the equivalence class of a virtual link diagram a *virtual link*.

A pair  $P = (\Sigma, D)$  of a compact oriented surface  $\Sigma$  and a link diagram  $D$  on  $\Sigma$  is called an *abstract link diagram* (ALD) if  $|D|$  is a deformation retract of  $\Sigma$ , where  $|D|$  is a graph obtained from  $D$  by replacing each crossing point with a vertex. If  $D$  is oriented,  $P$  is said to be *oriented*. Unless otherwise stated, we assume that an ALD is oriented. For an ALD,  $P = (\Sigma, D)$ , if there is an orientation preserving embedding  $f: \Sigma \rightarrow F$  into a closed oriented surface  $F$ ,  $f(D)$  is a link diagram on  $F$ . We call it a *link diagram realization* of  $P$  on  $F$ . In Fig. 3, we show two abstract link diagrams

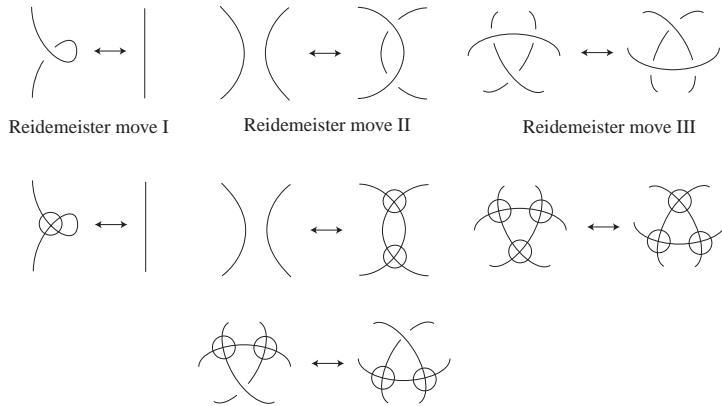


Fig. 2.

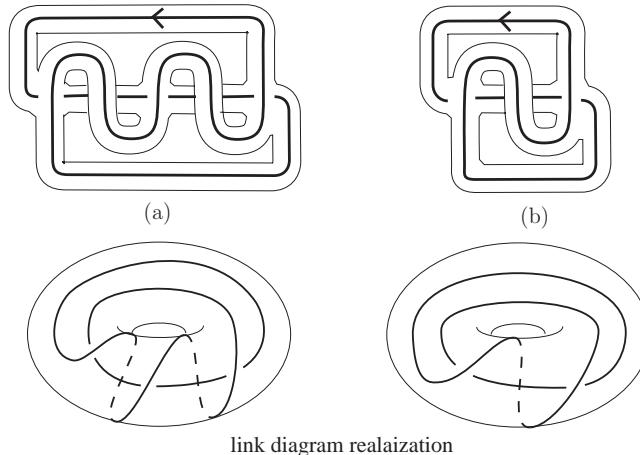


Fig. 3.

and their link diagram realizations. Two ALDs,  $P = (\Sigma, D)$  and  $P' = (\Sigma', D')$ , are related by an *abstract Reidemeister move* (of type I, II or III) if there exist link diagram realizations  $f: \Sigma \rightarrow F$  and  $f': \Sigma' \rightarrow F$  into the same closed oriented surface  $F$  such that the link diagrams  $f(D)$  and  $f'(D')$  on  $F$  are related by a Reidemeister move (of type I, II or III) on  $F$ . Two ALDs are said to be *equivalent* if they are related by a finite sequence of abstract Reidemeister moves. We call the equivalence class of an ALD *an abstract link*.

In [6] a map

$$\varphi: \{\text{virtual link diagrams}\} \longrightarrow \{\text{ALDs}\}$$

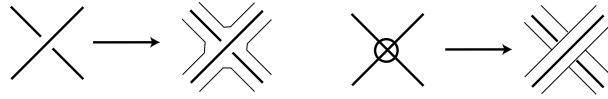


Fig. 4.

was defined. The idea of this map is illustrated in Fig. 4. Refer to [6] for the definition. We call  $\varphi(D)$  an *ALD associated with* a virtual link diagram  $D$ . The ALDs in Fig. 3 (a) and (b) are ALDs associated with the virtual link diagrams in Fig. 1 (a) and (b) respectively.

**Theorem 3** ([6]). *The map  $\varphi$  induces a bijection*

$$\Phi: \{\text{virtual links}\} \longrightarrow \{\text{abstract links}\}.$$

Let  $P = (\Sigma, D)$  be an ALD. A *checkerboard coloring* of  $P$  is a coloring of all the components of  $\Sigma - |D|$  by two colors, say black and white, such that any two components of  $\Sigma - |D|$  that share an edge have different colors.

We say that a virtual link diagram *admits a checkerboard coloring* or is *checkerboard colorable* if the associated ALD admits a checkerboard coloring.

### 3. The $f$ -polynomials of abstract link diagrams

There is a unique map

$$\langle \quad \rangle: \{\text{unoriented ALDs}\} \longrightarrow \Lambda = \mathbf{Z}[A, A^{-1}]$$

satisfying the following rules.

- (i)  $\langle T \rangle = 1$  where  $T$  is a one-component trivial ALD,
- (ii)  $\langle T \amalg P \rangle = (-A^2 - A^{-2})\langle P \rangle$  if  $P$  is not empty, where  $T \amalg P$  is the disjoint union of  $T$  and  $P$ , and

$$\text{(iii)} \quad \langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \rangle = A \langle \begin{array}{c} \text{circle} \\ \text{circle} \end{array} \rangle + A^{-1} \langle \begin{array}{c} \text{circle} \\ \text{circle} \end{array} \rangle.$$

The map  $\langle \quad \rangle$  is invariant under abstract Reidemeister moves II and III. We call it the *Kauffman bracket polynomial* of ALD, cf. [4].

Let  $P = (\Sigma, D)$  be an unoriented ALD. Replacing the neighborhood of a crossing point as in Fig. 5, we have another unoriented ALD. We call it an unoriented ALD obtained from  $D$  by doing an *A-splice* or a *B-splice* at the crossing point. An unoriented trivial ALD obtained from  $P$  by doing an A-splice or a B-splice at each crossing point is called a *state* of  $P$ . From the definition of  $\langle \quad \rangle$ , we see

$$\langle P \rangle = \sum_S A^{\sharp(S)} (-A^2 - A^{-2})^{\sharp(S)-1},$$

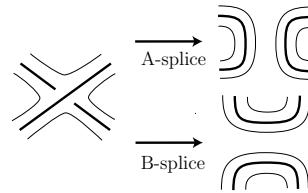


Fig. 5.

where  $S$  runs over all of states of  $P$ ,  $\natural(S)$  is the number of A-splices minus that of B-splices used for obtaining  $S$  and  $\sharp(S)$  is the number of components of  $S$ .

For an ALD,  $P = (\Sigma, D)$ , the writhe  $\omega(P)$  is defined by the number of positive crossings minus the number of negative crossings of  $D$ . Then we define the *normalized bracket polynomial* or the *f-polynomial* of  $P$  by

$$f_P(A) = (-A^3)^{-\omega(P)} \langle P \rangle.$$

This value is preserved under abstract Reidemeister moves of type I. Thus this is an invariant of an abstract link. This invariant was defined in [4], where it was called the Jones polynomial of  $P$ . It should be noted that the bijection  $\Phi$  preserves the *f*-polynomial.

#### 4. Proof of Theorem 1

Let  $p$  be a crossing point of an ALD,  $P = (\Sigma, D)$ . Let  $P_0 = (\Sigma_0, D_0)$  and  $P_\infty = (\Sigma_\infty, D_\infty)$  be ALDs obtained from  $P$  by splicing at  $p$  orientation coherently and orientation incoherently, respectively. Note that  $D_\infty$  does not inherit an orientation from  $D$ . The crossing point  $p$  is either (i) a self-intersection of an immersed loop of  $D$  or (ii) an intersection of two immersed loops. Let  $\alpha$  and  $\alpha'$  be the immersed open arcs obtained from the loop (in case (i)) or from the two loops (in case (ii)) by removing (the small neighborhood of)  $p$ . Choose one of them, say  $\alpha$ , and we give an orientation to  $D_\infty$  which is induced from that of  $D$  except  $\alpha$  (and hence the orientation is reversed on  $\alpha$ ). Let  $C$  be the set of crossing points of  $D$ , except  $p$ , such that the sign of the crossing point is preserved when we consider the new diagram  $D_\infty$ ; in other words, at each crossing point belonging to  $C$ , both of the two intersecting arcs are contained in  $D - \alpha$  or both of them are in  $\alpha$ . Let  $C'$  be the set of crossing points of  $D$ , except  $p$ , such that the sign of the crossing point changes in  $D$  and  $D_\infty$ ; in other words, at each crossing point belonging to  $C'$ , one of the two intersecting arcs is contained in  $D - \alpha$  and the other is in  $\alpha$ . Let  $k$  (or  $l$ , resp.) be the number of positive crossings of  $C$  (resp.  $C'$ ) minus the number of negative crossings of  $C$  (resp.  $C'$ ).

**Lemma 4.** *In the above situation, let  $f$ ,  $f_0$  and  $f_\infty$  be the  $f$ -polynomials of  $P$ ,  $P_0$  and  $P_\infty$ , respectively. Then we have*

$$f = \begin{cases} -A^{-2}f_0 - (-A^3)^{-2l}A^{-4}f_\infty, & \text{if } p \text{ is a positive crossing,} \\ -A^{+2}f_0 - (-A^3)^{-2l}A^{+4}f_\infty, & \text{if } p \text{ is a negative crossing.} \end{cases}$$

Proof. If  $p$  is a positive crossing, then  $\omega(D) = k + l + 1$ ,  $\omega(D_0) = k + l$  and  $\omega(D_\infty) = k - l$ . Since  $\langle P \rangle = A\langle P_0 \rangle + A^{-1}\langle P_\infty \rangle$ , we have the result. The case where  $p$  is a negative crossing is proved by a similar argument.  $\square$

REMARK. In Remark of Section 5 of [9, page 677], an equation which is similar to Lemma 4 is given. However, it seems to be forgotten there to take account of the term  $(-A^3)^{-2l}$ . In consequence, the recursion formula of Theorem 13 of [9] is as follows:

$$v_n(G_*) = \sum_{k=0}^{n-1} \frac{2^{n-k}}{(n-k)!} \{(1 - (-1)^{n-k})v_k(G_0) + \{(2 - 3l)^{n-k} - (-2 - 3l)^{n-k}\}v_k(G_\infty)\}.$$

By this formula, Corollary 14 of [9] is still true.

**Corollary 5** (cf. Theorem 13 of [9]). *Let  $f$  be the  $f$ -polynomial of an ALD with  $n$  components. Then  $f(1) = (-2)^{n-1}$ . In particular,  $f$ -polynomials of ALDs are not zero.*

Proof. It follows from Lemma 4 by induction on the number of (real) crossing points.  $\square$

Since  $\Phi$  preserves the  $f$ -polynomials, Theorem 1 is equivalent to the following theorem.

**Theorem 6.** *Let  $f$  be the  $f$ -polynomial of an ALD,  $P = (\Sigma, D)$ , with  $n$  components. Suppose that  $P$  admits a checkerboard coloring. Then  $\text{EXP}(f) \subset 4\mathbf{Z}$  if  $n$  is odd, and  $\text{EXP}(f) \subset 4\mathbf{Z} + 2$  if  $n$  is even.*

Proof. For a state  $S$  of  $P$ , we define  $I(S)$  by

$$I(S) = A^{\sharp(S)}(-A^2 - A^{-2})^{\sharp(S)-1}$$

so that the bracket polynomial of  $P$  is the sum of  $I(S)$  over all states of  $P$ . Let  $\text{ind}(S)$  be the value in  $\mathbf{Z}_4 = \{0, 1, 2, 3\}$  such that  $\text{EXP}(I(S)) \subset 4\mathbf{Z} + \text{ind}(S)$ .

Every state of  $P$  has a unique checkerboard coloring induced from the checkerboard coloring of  $P$ , see Fig. 6. (Fig. 7 shows an example of an ALD with a checkerboard coloring and a state with the induced checkerboard coloring.) Using this fact,

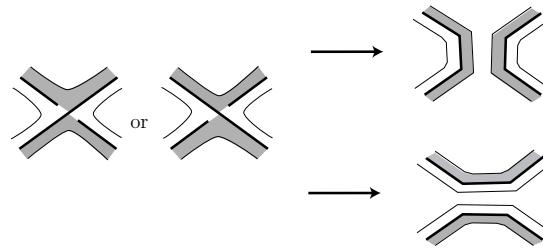


Fig. 6.

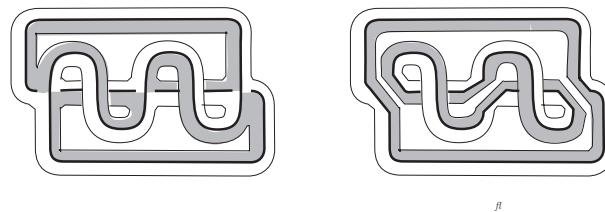


Fig. 7.



Fig. 8.

we prove that  $\text{ind}(S) = \text{ind}(S')$  for any states  $S$  and  $S'$  of  $P$ . It is sufficient to prove this equality in the special case that  $S$  and  $S'$  differ in a single 2-disk  $E$  as in Fig. 8, where  $E$  is a neighborhood of a crossing point of  $D$ . There are three possibilities for the connection of  $S$  outside  $E$  as in Fig. 9. However, the case (C) does not occur because such a state does not have a checkerboard coloring induced from the checkerboard coloring of  $P$ . In both cases (A) and (B), we have  $I(S') = A^{\sharp(S) \pm 2}(-A^2 - A^{-2})^{\sharp(S)-1 \pm 1}$  and hence  $\text{ind}(S) = \text{ind}(S')$ .

Now we have that  $\text{EXP}(f) \subset 4\mathbf{Z} + i$  where  $i = \text{ind}(S)$  for any state  $S$  of  $P$ . We denote this number  $i$  by  $\text{ind}(f)$ . The remaining task is to prove that this index  $i$  is 0 if  $n$  is odd, and 2 if  $n$  is even. It is proved by induction on the number of (real) crossing points of  $P$ . If  $P$  has no real crossing points, then it is obvious by the definition of the  $f$ -polynomial. Suppose that  $P$  has a crossing point. For this crossing point, let  $P_0$  and  $P_\infty$  be ALDs as in Lemma 4. Note that  $P_0$  and  $P_\infty$  admit checkerboard colorings. Hence  $\text{EXP}(f_0) \subset 4\mathbf{Z} + \text{ind}(f_0)$  and  $\text{EXP}(f_\infty) \subset 4\mathbf{Z} + \text{ind}(f_\infty)$ . Since  $f \neq 0$  and  $f_0 \neq 0$  by Corollary 5, it follows from the equation in Lemma 4 that  $\text{ind}(f) \equiv \text{ind}(f_0) + 2 \pmod{4}$ . The ALD  $P_0$  has fewer crossing points than  $P$  and admits a checkerboard

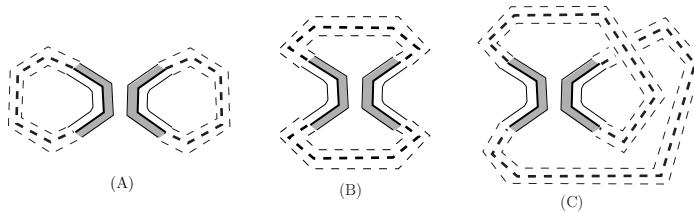


Fig. 9.

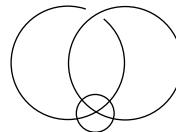


Fig. 10.

coloring. By the inductive hypothesis,  $\text{ind}(f_0)$  is 0 if  $n'$  is odd, and 2 if  $n'$  is even, where  $n'$  is the number of components of  $P_0$ . Since  $n' = n \pm 1$ , we have that  $\text{ind}(f)$  is 0 if  $n$  is odd, and 2 if  $n$  is even.  $\square$

## 5. Alternating virtual link diagrams and ALDs

An ALD or a virtual link diagram is said to be *alternating* if an over-crossings and under-crossings alternate as one travels along each component of the diagram. Note that the virtual link diagram in Fig. 10 is not alternating.

**Lemma 7.** *For an ALD,  $P = (\Sigma, D)$ , the following conditions are equivalent.*

- (i) *By applying crossing changes,  $P$  changes into an alternating ALD.*
- (ii)  *$P$  admits a checkerboard coloring.*

**Proof.** If  $P$  admits a checkerboard coloring, change each real crossing according to the coloring as in the leftmost figure of Fig. 6. Conversely, if  $P$  is an alternating ALD, then give a checkerboard coloring near each crossing point as in the figure used above, which is extended to a checkerboard coloring of  $P$ .  $\square$

**Proof of Corollary 2.** It follows from Theorem 1 and Lemma 7.  $\square$

**REMARK.** M.B. Thistlethwaite [11] and K. Murasugi [10] showed that the  $f$ -polynomial (Jones polynomial) of a non-split alternating link is alternating, namely, it is in a form of  $A^\alpha \sum c_i A^{4i}$  such that  $c_i c_j \geq 0$  for  $i \equiv j \pmod{2}$  and  $c_i c_j \leq 0$  for  $i \not\equiv j \pmod{2}$ . This result is not true for virtual knots. The  $f$ -polynomial of a

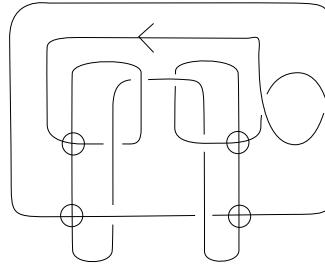


Fig. 11.

virtual knot in Fig. 11 is  $A^{12} + 3A^{16} - 4A^{20} + 3A^{24} - 4A^{28} + 4A^{32} - 3A^{36} + A^{40}$ .

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### References

- [1] M. Goussarov, M. Polyak, and O. Viro: *Finite-type invariants of classical and virtual knots*, Topology **39** (2000), 1045–1068.
- [2] V.F.R. Jones: *A polynomial invariant for knots via von Neumann algebra*, Bull. Amer. Math. Soc. **12** (1985), 103–111.
- [3] V.F.R. Jones: *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. **126** (1987), 335–388.
- [4] N. Kamada: *Alternating link diagrams on compact oriented surfaces*, preprint (1995).
- [5] N. Kamada: *The crossing number of alternating link diagrams of a surface*, Proceedings of Knots 96, World Scientific Publishing Co., 1997, 377–382.
- [6] N. Kamada and S. Kamada: *Abstract link diagrams and virtual knots*, J. Knot Theory Ramifications **9** (2000), 93–106.
- [7] L.H. Kauffman: *State models and the Jones polynomial*, Topology **26** (1987), 395–407.
- [8] L.H. Kauffman: Talks at: the MSRI Meeting in January 1997: AMS Meeting at the University of Maryland College Park in March 1997; Knots in Hellas Meeting in Delphi, Greece in July 1998; APCTP-NANKAI Symposium on Yang-Baxter Systems, Non-Linear Models and Applications at Seoul, Korea in October 1998.
- [9] L.H. Kauffman: *Virtual Knot Theory*, Europ. J. Combinatorics **20** (1999), 663–690.
- [10] K. Murasugi: *On invariants of graphs with applications to knot theory*, Trans. Amer. Math. Soc. **314** (1989), 1–49.
- [11] M.B. Thistlethwaite: *A spanning tree expansion of the Jones polynomial*, Topology **26** (1987), 297–309.

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