Kamada, N.
Osaka J. Math.
39 (2002), 325–333

ON THE JONES POLYNOMIALS OF CHECKERBOARD COLORABLE VIRTUAL LINKS

NAOKO KAMADA

(Received August 23, 2000)

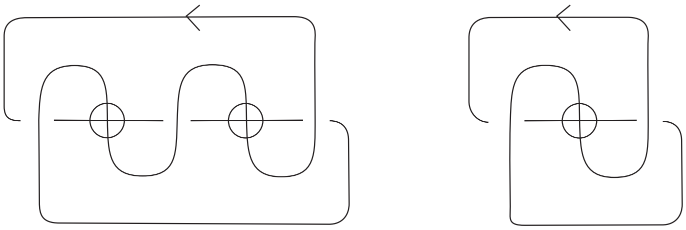
1. Introduction

In 1996, L.H. Kauffman introduced the notion of a virtual knot, which was motivated by the study of knots in a thickened surface and abstract Gauss codes, cf. [8, 9]. M. Goussarov, M. Polyak, and O. Viro [1] proved that the natural map from the category of classical knots to the category of virtual knots is injective; namely, if two classical knot diagrams are equivalent as virtual knots, then they are equivalent as classical knots. Thus, virtual knot theory is a generalization of knot theory. In [1], virtual knots are used to study of finite type invariants.

Kauffman defined the Jones polynomial of a virtual knot, which is also called the normalized bracket polynomial or the f-polynomial (cf. [9]). In this paper, according to [9], we call it the f-polynomial instead of the Jones polynomial, since the definition is different from Jones' in [2, 3]. Finite type invariants derived from the f-polynomials are studied in [9]. For example, the follwing results appear in [9]: (1) If $f_K(A)$ denotes the f-polynomial of a virtual link K, the coefficient $v_n(K)$ of x^n in the power series expansion of $f_K(e^x)$ is a Vassiliev invariant of order n. (2) When the notion v_n for a "singular" virtual link G is generalized in the obvious way, the Vassiliev invariant $v_n(G)$ depends only on the chord diagram associated with G (cf. Corollary 14 of [9]).

The f-polynomial of a virtual link is quite different from the f-polynomial of a classical link. For a Laurent polynomial f in the variable A, we denote by $\mathrm{EXP}(f)$ the set of integers appearing as exponents of f. For example, if $f = 3A^{-2} + 6A - 7A^5$, then $\mathrm{EXP}(f) = \{-2, 1, 5\}$. For the f-polynomial f of a classical link with n components, it is well known that $\mathrm{EXP}(f) \subset 4\mathbf{Z}$ if n is odd and $\mathrm{EXP}(f) \subset 4\mathbf{Z} + 2$ if n is even ([2], [7]). However, this is not true for a virtual knot/link in general. In this paper we introduce the notion of *checkerboard coloring* of a virtual link diagram as a generalization of checkerboard coloring of a classical link diagram.

Theorem 1. *Let f be the f-polynomial of a virtual link L with n components. Suppose that L has a virtual link diagram which admits a checkerboard coloring. Then* $\mathrm{EXP}(f) \subset 4\mathbf{Z}$ *if n is odd, and* $\mathrm{EXP}(f) \subset 4\mathbf{Z} + 2$ *if n is even.*



(a) (b)

Fig. 1.

For example the virtual knot diagram illustrated in Fig. 1 (a) admits a checkerboard coloring, and the f-polynomial is $A^4 + A^{12} - A^{16}$. So $\mathrm{EXP}(f) \subset 4\mathbf{Z}$. On the other hand, the virtual knot diagram illustrated in Fig. 1 (b) does not admit a checkerboard coloring, and the f-polynomial is $-A^{10} + A^6 + A^4$. Theorem 1 implies that this diagram is not equivalent to any diagram that admits a checkerboard coloring.

If a virtual link diagram is alternating (the definition is given later), then the diagram admits a checkerboard coloring. Thus we have the following.

Corollary 2. *Let f be the f-polynomial of a virtual link L with n components. Suppose that L has an alternating virtual link diagram. Then* $\mathrm{EXP}(f) \subset 4\mathbf{Z}$ *if n is odd, and* $\mathrm{EXP}(f) \subset 4\mathbf{Z} + 2$ *if n is even.*

By this corollary, we see that the virtual knot represented by Fig. 1 (b) is not equivalent to any alternating diagram.

2. Virtual link diagram and abstract link diagram

A *virtual link diagram* is a closed oriented 1-manifold generically immersed in $\mathbf{R}^2$ such that each double point is labeled to be (1) a *real* crossing which is indicated as usual in classical knot theory or (2) a *virtual* crossing which is indicated by a small circle around the double point. The moves of virtual link diagrams illustrated in Fig. 2 are called *generalized Reidemeister moves*. Two virtual link diagrams are said to be *equivalent* if they are related by a finite sequence of generalized Reidemeister moves. We call the equivalence class of a virtual link diagram *a virtual link*.

A pair $P = (\Sigma, D)$ of a compact oriented surface Σ and a link diagram D on Σ is called an *abstract link diagram* (ALD) if $|D|$ is a deformation retract of Σ, where $|D|$ is a graph obtained from D by replacing each crossing point with a vertex. If D is oriented, P is said to be *oriented*. Unless otherwise stated, we assume that an ALD is oriented. For an ALD, $P = (\Sigma, D)$, if there is an orientation preserving embedding $f : \Sigma \to F$ into a closed oriented surface F, $f(D)$ is a link diagram on F. We call it a *link diagram realization* of P on F. In Fig. 3, we show two abstract link diagrams

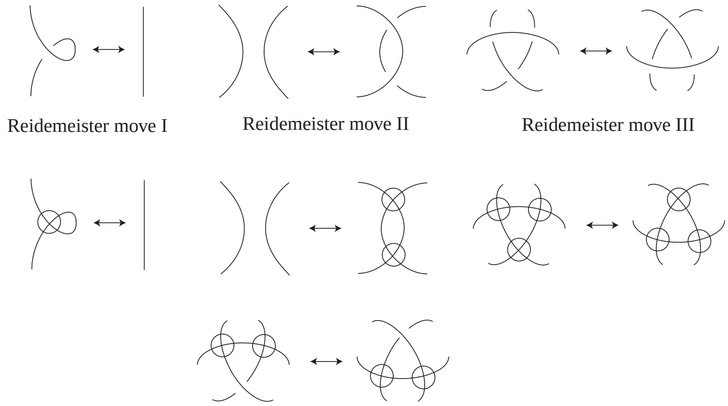


Fig. 2.

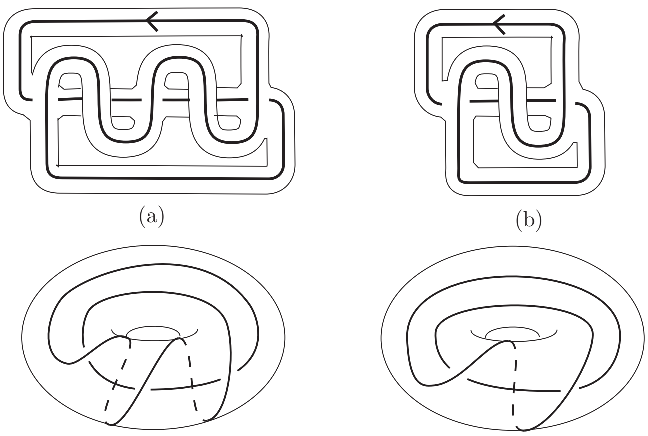


Fig. 3.

and their link diagram realizations. Two ALDs, $P = (\Sigma, D)$ and $P' = (\Sigma', D')$, are related by an *abstract Reidemeister move* (of type I, II or III) if there exist link diagram ralizations $f : \Sigma \to F$ and $f' : \Sigma' \to F$ into the same closed oriented surface F such that the link diagrams $f(D)$ and $f'(D')$ on F are related by a Reidemeister move (of type I, II or III) on F. Two ALDs are said to be *equivalent* if they are related by a finite sequence of abstract Reidemeister moves. We call the equivalence class of an ALD *an abstract link*.

In [6] a map

$$\varphi\colon \{\text{virtual link diagrams}\} \longrightarrow \{\text{ALDs}\}$$

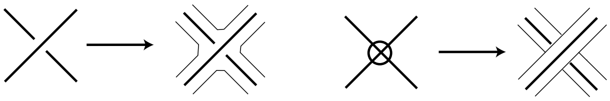


Fig. 4.

was defined. The idea of this map is illustrated in Fig. 4. Refer to [6] for the definition. We call $\varphi(D)$ an *ALD associated with* a virtual link diagram D. The ALDs in Fig. 3 (a) and (b) are ALDs associated with the virtual link diagrams in Fig. 1 (a) and (b) respectively.

Theorem 3 ([6]). *The map* φ *induces a bijection*

$$\Phi\colon \{\textit{virtual links}\} \longrightarrow \{\textit{abstract links}\}.$$

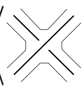
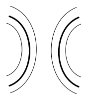
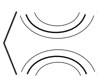
Let $P = (\Sigma, D)$ be an ALD. A *checkerboard coloring* of P is a coloring of all the components of $\Sigma - |D|$ by two colors, say black and white, such that any two components of $\Sigma - |D|$ that share an edge have different colors.

We say that a virtual link diagram *admits a checkerboard coloring* or is *checkerboard colorable* if the associated ALD admits a checkerboard coloring.

3. The f-polynomials of abstract link diagrams

There is a unique map

$$\langle\quad\rangle\colon \{\text{unoriented ALDs}\} \longrightarrow \Lambda = \mathbf{Z}[A, A^{-1}]$$

satisfying the following rules.
(i) $\langle T\rangle = 1$ where T is a one-component trivial ALD,
(ii) $\langle T \amalg P\rangle = (-A^2 - A^{-2})\langle P\rangle$ if P is not empty, where $T \amalg P$ is the disjoint union of T and P, and
(iii) $\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$.

The map $\langle\quad\rangle$ is invariant under abstract Reidemeister moves II and III. We call it the *Kauffman bracket polynomial* of ALD, cf. [4].

Let $P = (\Sigma, D)$ be an unoriented ALD. Replacing the neighborhood of a crossing point as in Fig. 5, we have another unoriented ALD. We call it an unoriented ALD obtained from D by doing an *A-splice* or a *B-splice* at the crossing point. An unoriented trivial ALD obtained from P by doing an A-splice or a B-splice at each crossing point is called a *state* of P. From the definition of $\langle\quad\rangle$, we see

$$\langle P\rangle = \sum_S A^{\natural(S)}(-A^2 - A^{-2})^{\sharp(S)-1},$$

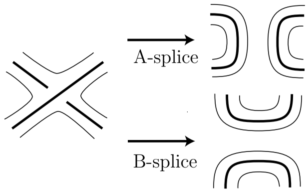


Fig. 5.

where S runs over all of states of P, $\natural(S)$ is the number of A-splices minus that of B-splices used for obtaining S and $\sharp(S)$ is the number of components of S.

For an ALD, $P = (\Sigma, D)$, the writhe $\omega(P)$ is defined by the number of positive crossings minus the number of negative crossings of D. Then we define the *normalized bracket polynomial* or the *f-polynomial* of P by

$$f_P(A) = (-A^3)^{-\omega(P)}\langle P\rangle.$$

This value is preserved under abstract Reidemeister moves of type I. Thus this is an invariant of an abstract link. This invariant was defined in [4], where it was called the Jones polynomial of P. It should be noted that the bijection Φ preserves the f-polynomial.

4. Proof of Theorem 1

Let p be a crossing point of an ALD, $P = (\Sigma, D)$. Let $P_0 = (\Sigma_0, D_0)$ and $P_\infty = (\Sigma_\infty, D_\infty)$ be ALDs obtained from P by splicing at p orientation coherently and orientation incoherently, respectively. Note that D_∞ does not inherit an orientation from D. The crossing point p is either (i) a self-intersection of an immersed loop of D or (ii) an intersection of two immersed loops. Let α and α' be the immersed open arcs obtained from the loop (in case (i)) or from the two loops (in case (ii)) by removing (the small neighborhood of) p. Choose one of them, say α, and we give an orientation to D_∞ which is induced from that of D except α (and hence the orientation is reversed on α). Let C be the set of crossing points of D, except p, such that the sign of the crossing point is preserved when we consider the new diagram D_∞; in other words, at each crossing point belonging to C, both of the two intersecting arcs are contained in $D - \alpha$ or both of them are in α. Let C' be the set of crossing points of D, except p, such that the sign of the crossing point changes in D and D_∞; in other words, at each crossing point belonging to C', one of the two intersecting arcs is contained in $D - \alpha$ and the other is in α. Let k (or l, resp.) be the number of positive crossings of C (resp. C') minus the number of negative crossings of C (resp. C').

Lemma 4. *In the above situation, let f, f_0 and f_∞ be the f-polynomials of P, P_0 and P_∞, respectively. Then we have*

$$f = \begin{cases} -A^{-2} f_0 - (-A^3)^{-2l} A^{-4} f_\infty, & \text{if } p \text{ is a positive crossing,} \\ -A^{+2} f_0 - (-A^3)^{-2l} A^{+4} f_\infty, & \text{if } p \text{ is a negative crossing.} \end{cases}$$

Proof. If p is a positive crossing, then $\omega(D) = k + l + 1$, $\omega(D_0) = k + l$ and $\omega(D_\infty) = k - l$. Since $\langle P \rangle = A\langle P_0 \rangle + A^{-1}\langle P_\infty \rangle$, we have the result. The case where p is a negative crossing is proved by a similar argument. □

REMARK. In Remark of Section 5 of [9, page 677], an equation which is similar to Lemma 4 is given. However, it seems to be forgotten there to take account of the term $(-A^3)^{-2l}$. In consequence, the recursion formula of Theorem 13 of [9] is as follows:

$$v_n(G_*) = \sum_{k=0}^{n-1} \frac{2^{n-k}}{(n-k)!} \{(1 - (-1)^{n-k}) v_k(G_0) + \{(2 - 3l)^{n-k} - (-2 - 3l)^{n-k}\} v_k(G_\infty)\}.$$

By this formula, Corollary 14 of [9] is still true.

Corollary 5 (cf. Theorem 13 of [9]). *Let f be the f-polynomial of an ALD with n components. Then $f(1) = (-2)^{n-1}$. In particular, f-polynomials of ALDs are not zero.*

Proof. It follows from Lemma 4 by induction on the number of (real) crossing points. □

Since Φ preserves the f-polynomials, Theorem 1 is equivalent to the following theorem.

Theorem 6. *Let f be the f-polynomial of an ALD, $P = (\Sigma, D)$, with n components. Suppose that P admits a checkerboard coloring. Then $\mathrm{EXP}(f) \subset 4\mathbf{Z}$ if n is odd, and $\mathrm{EXP}(f) \subset 4\mathbf{Z} + 2$ if n is even.*

Proof. For a state S of P, we define $I(S)$ by

$$I(S) = A^{\natural(S)} (-A^2 - A^{-2})^{\sharp(S)-1}$$

so that the bracket polynomial of P is the sum of $I(S)$ over all states of P. Let $\mathrm{ind}(S)$ be the value in $\mathbf{Z}_4 = \{0, 1, 2, 3\}$ such that $\mathrm{EXP}(I(S)) \subset 4\mathbf{Z} + \mathrm{ind}(S)$.

Every state of P has a unique checkerboard coloring induced from the checkerboard coloring of P, see Fig. 6. (Fig. 7 shows an example of an ALD with a checkerboard coloring and a state with the induced checkerboard coloring.) Using this fact,

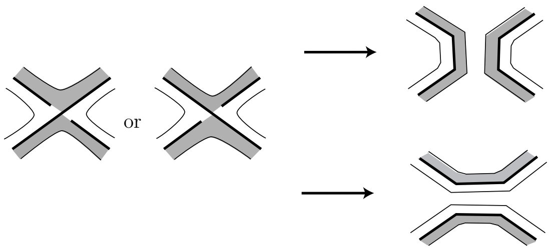


Fig. 6.

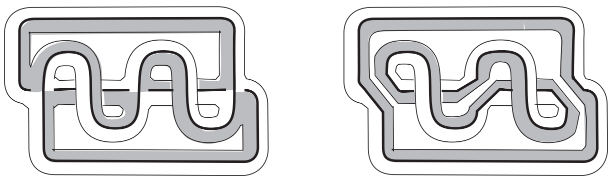


Fig. 7.

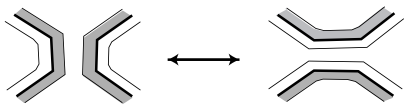


Fig. 8.

we prove that $\mathrm{ind}(S) = \mathrm{ind}(S')$ for any states S and S' of P. It is sufficient to prove this equality in the special case that S and S' differ in a single 2-disk E as in Fig. 8, where E is a neighborhood of a crossing point of D. There are three possibilities for the connection of S outside E as in Fig. 9. However, the case (C) does not occur because such a state does not have a checkerboard coloring induced from the checkerboard coloring of P. In both cases (A) and (B), we have $I(S') = A^{\natural(S)\pm 2}(-A^2 - A^{-2})^{\sharp(S)-1\pm 1}$ and hence $\mathrm{ind}(S) = \mathrm{ind}(S')$.

Now we have that $\mathrm{EXP}(f) \subset 4\mathbf{Z} + i$ where $i = \mathrm{ind}(S)$ for any state S of P. We denote this number i by $\mathrm{ind}(f)$. The remaining task is to prove that this index i is 0 if n is odd, and 2 if n is even. It is proved by induction on the number of (real) crossing points of P. If P has no real crossing points, then it is obvious by the definition of the f-polynomial. Suppose that P has a crossing point. For this crossing point, let P_0 and P_∞ be ALDs as in Lemma 4. Note that P_0 and P_∞ admit checkerboard colorings. Hence $\mathrm{EXP}(f_0) \subset 4\mathbf{Z} + \mathrm{ind}(f_0)$ and $\mathrm{EXP}(f_\infty) \subset 4\mathbf{Z} + \mathrm{ind}(f_\infty)$. Since $f \neq 0$ and $f_0 \neq 0$ by Corollary 5, it follows from the equation in Lemma 4 that $\mathrm{ind}(f) \equiv \mathrm{ind}(f_0) + 2 \pmod 4$. The ALD P_0 has fewer crossing points than P and admits a checkerboard

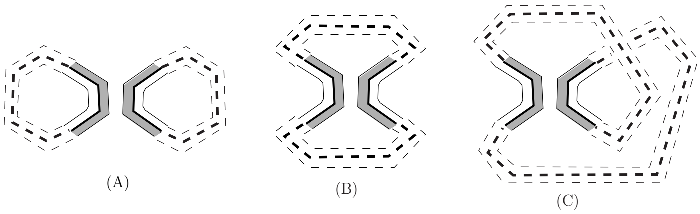


Fig. 9.

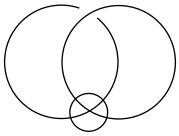


Fig. 10.

coloring. By the inductive hypothesis, ind(f_0) is 0 if n' is odd, and 2 if n' is even, where n' is the number of components of P_0. Since $n' = n \pm 1$, we have that ind(f) is 0 if n is odd, and 2 if n is even. □

5. Alternating virtual link diagrams and ALDs

An ALD or a virtual link diagram is said to be *alternating* if an over-crossings and under-crossings alternate as one travels along each component of the diagram. Note that the virtual link diagram in Fig. 10 is not alternating.

Lemma 7. *For an ALD, $P = (\Sigma, D)$, the following conditions are equivalent.*
(i) *By applying crossing changes, P changes into an alternating ALD.*
(ii) *P admits a checkerboard coloring.*

Proof. If P admits a checkerboard coloring, change each real crossing according to the coloring as in the leftmost figure of Fig. 6. Conversely, if P is an alternating ALD, then give a checkerboard coloring near each crossing point as in the figure used above, which is extended to a checkerboard coloring of P. □

Proof of Corollary 2. It follows from Theorem 1 and Lemma 7. □

Remark. M.B. Thistlethwaite [11] and K. Murasugi [10] showed that the f-polynomial (Jones polynomial) of a non-split alternating link is alternating, namely, it is in a form of $A^{\alpha} \sum c_i A^{4i}$ such that $c_i c_j \geq 0$ for $i \equiv j \pmod 2$ and $c_i c_j \leq 0$ for $i \not\equiv j \pmod 2$. This result is not true for virtual knots. The f-polynomial of a

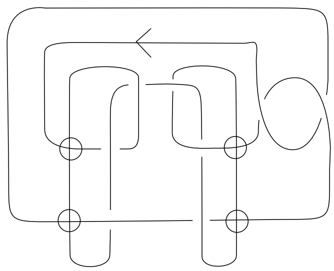


Fig. 11.

virtual knot in Fig. 11 is $A^{12} + 3A^{16} - 4A^{20} + 3A^{24} - 4A^{28} + 4A^{32} - 3A^{36} + A^{40}$.

References

[1] M. Goussarov, M. Polyak, and O. Viro: *Finite-type invariants of classical and virtual knots*, Topology **39** (2000), 1045–1068.

[2] V.F.R. Jones: *A polynomial invariant for knots via von Neumann algebra*, Bull. Amer. Math. Soc. **12** (1985), 103–111.

[3] V.F.R. Jones: *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. **126** (1987), 335–388.

[4] N. Kamada: *Alternating link diagrams on compact oriented surfaces*, preprint (1995).

[5] N. Kamada: *The crossing number of alternating link diagrams of a surface*, Proceedings of Knots 96, World Scientific Publishing Co., 1997, 377–382.

[6] N. Kamada and S. Kamada: *Abstract link diagrams and virtual knots*, J. Knot Theory Ramifications **9** (2000), 93–106.

[7] L.H. Kauffman: *State models and the Jones polynomial,* Topology **26** (1987), 395–407.

[8] L.H. Kauffman: Talks at: the MSRI Meeting in January 1997: AMS Meeting at the University of Maryland College Park in March 1997; Knots in Hellas Meeting in Delphi, Greece in July 1998; APCTP-NANKAI Symposium on Yang-Baxter Systems, Non-Linear Models and Applications at Seoul, Korea in October 1998.

[9] L.H. Kauffman: *Virtual Knot Theory*, Europ. J. Combinatorics **20** (1999), 663–690.

[10] K. Murasugi: *On invariants of graphs with applications to knot theory,* Trans. Amer. Math. Soc. **314** (1989), 1–49.

[11] M.B. Thistlethwaite: *A spanning tree expansion of the Jones polynomial,* Topology **26** (1987), 297–309.

Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka, 558-8585, Japan