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ON THE JONES POLYNOMIALS OF CHECKERBOARD COLORABLE VIRTUAL LINKS

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1. Introduction

In 1996, L.H. Kauffman introduced the notion of a virtual knot, which was motivated by the study of knots in a thickened surface and abstract Gauss codes, cf. [8, 9]. M. Goussarov, M. Polyak, and O. Viro [1] proved that the natural map from the category of classical knots to the category of virtual knots is injective; namely, if two classical knot diagrams are equivalent as virtual knots, then they are equivalent as classical knots. Thus, virtual knot theory is a generalization of knot theory. In [1], virtual knots are used to study of finite type invariants.

Kauffman defined the Jones polynomial of a virtual knot, which is also called the normalized bracket polynomial or the f -polynomial (cf. [9]). In this paper, according to [9], we call it the f -polynomial instead of the Jones polynomial, since the definition is different from Jones' in [2, 3]. Finite type invariants derived from the f -polynomials are studied in [9]. For example, the following results appear in [9]: (1) If $f_K(A)$ denotes the f -polynomial of a virtual link K , the coefficient $v_n(K)$ of x^n in the power series expansion of $f_K(e^x)$ is a Vassiliev invariant of order n . (2) When the notion v_n for a “singular” virtual link G is generalized in the obvious way, the Vassiliev invariant $v_n(G)$ depends only on the chord diagram associated with G (cf. Corollary 14 of [9]).

The f -polynomial of a virtual link is quite different from the f -polynomial of a classical link. For a Laurent polynomial f in the variable A , we denote by $\text{EXP}(f)$ the set of integers appearing as exponents of f . For example, if $f = 3A^{-2} + 6A - 7A^5$, then $\text{EXP}(f) = \{-2, 1, 5\}$. For the f -polynomial f of a classical link with n components, it is well known that $\text{EXP}(f) \subset 4\mathbf{Z}$ if n is odd and $\text{EXP}(f) \subset 4\mathbf{Z} + 2$ if n is even ([2], [7]). However, this is not true for a virtual knot/link in general. In this paper we introduce the notion of *checkerboard coloring* of a virtual link diagram as a generalization of checkerboard coloring of a classical link diagram.

Theorem 1. *Let f be the f -polynomial of a virtual link L with n components. Suppose that L has a virtual link diagram which admits a checkerboard coloring. Then $\text{EXP}(f) \subset 4\mathbf{Z}$ if n is odd, and $\text{EXP}(f) \subset 4\mathbf{Z} + 2$ if n is even.*

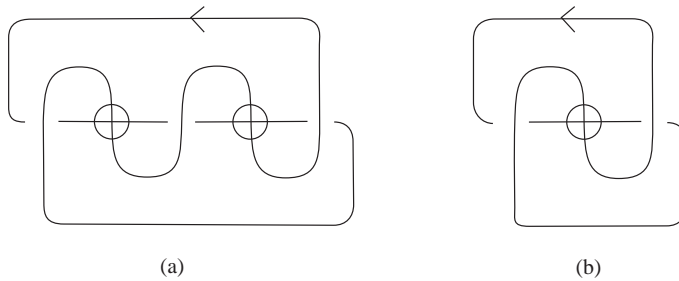


Fig. 1.

For example the virtual knot diagram illustrated in Fig. 1 (a) admits a checkerboard coloring, and the f -polynomial is $A^4 + A^{12} - A^{16}$. So $\text{EXP}(f) \subset 4\mathbb{Z}$. On the other hand, the virtual knot diagram illustrated in Fig. 1 (b) does not admit a checkerboard coloring, and the f -polynomial is $-A^{10} + A^6 + A^4$. Theorem 1 implies that this diagram is not equivalent to any diagram that admits a checkerboard coloring.

If a virtual link diagram is alternating (the definition is given later), then the diagram admits a checkerboard coloring. Thus we have the following.

Corollary 2. *Let f be the f -polynomial of a virtual link L with n components. Suppose that L has an alternating virtual link diagram. Then $\text{EXP}(f) \subset 4\mathbb{Z}$ if n is odd, and $\text{EXP}(f) \subset 4\mathbb{Z} + 2$ if n is even.*

By this corollary, we see that the virtual knot represented by Fig. 1 (b) is not equivalent to any alternating diagram.

2. Virtual link diagram and abstract link diagram

A *virtual link diagram* is a closed oriented 1-manifold generically immersed in \mathbb{R}^2 such that each double point is labeled to be (1) a *real* crossing which is indicated as usual in classical knot theory or (2) a *virtual* crossing which is indicated by a small circle around the double point. The moves of virtual link diagrams illustrated in Fig. 2 are called *generalized Reidemeister moves*. Two virtual link diagrams are said to be *equivalent* if they are related by a finite sequence of generalized Reidemeister moves. We call the equivalence class of a virtual link diagram a *virtual link*.

A pair $P = (\Sigma, D)$ of a compact oriented surface Σ and a link diagram D on Σ is called an *abstract link diagram* (ALD) if $|D|$ is a deformation retract of Σ , where $|D|$ is a graph obtained from D by replacing each crossing point with a vertex. If D is oriented, P is said to be *oriented*. Unless otherwise stated, we assume that an ALD is oriented. For an ALD, $P = (\Sigma, D)$, if there is an orientation preserving embedding $f: \Sigma \rightarrow F$ into a closed oriented surface F , $f(D)$ is a link diagram on F . We call it a *link diagram realization* of P on F . In Fig. 3, we show two abstract link diagrams

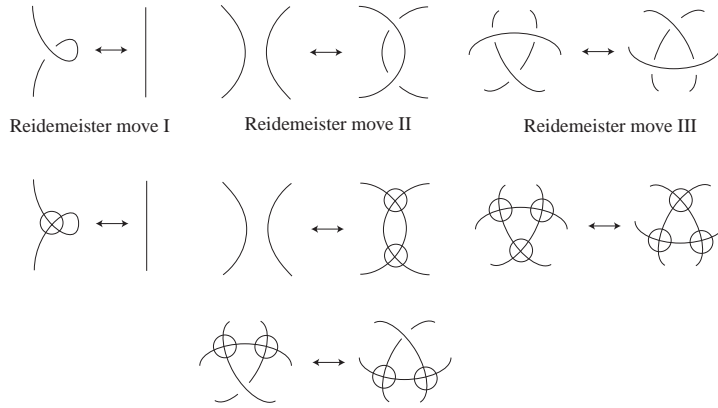


Fig. 2.

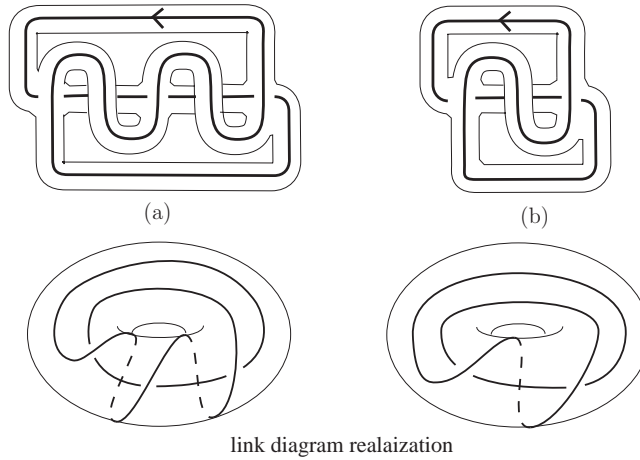


Fig. 3.

and their link diagram realizations. Two ALDs, $P = (\Sigma, D)$ and $P' = (\Sigma', D')$, are related by an *abstract Reidemeister move* (of type I, II or III) if there exist link diagram realizations $f: \Sigma \rightarrow F$ and $f': \Sigma' \rightarrow F$ into the same closed oriented surface F such that the link diagrams $f(D)$ and $f'(D')$ on F are related by a Reidemeister move (of type I, II or III) on F . Two ALDs are said to be *equivalent* if they are related by a finite sequence of abstract Reidemeister moves. We call the equivalence class of an ALD an *abstract link*.

In [6] a map

$$\varphi: \{\text{virtual link diagrams}\} \longrightarrow \{\text{ALDs}\}$$

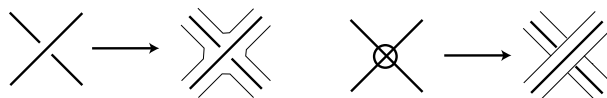


Fig. 4.

was defined. The idea of this map is illustrated in Fig. 4. Refer to [6] for the definition. We call $\varphi(D)$ an *ALD associated with a virtual link diagram D* . The ALDs in Fig. 3 (a) and (b) are ALDs associated with the virtual link diagrams in Fig. 1 (a) and (b) respectively.

Theorem 3 ([6]). *The map φ induces a bijection*

$$\Phi: \{\text{virtual links}\} \longrightarrow \{\text{abstract links}\}.$$

Let $P = (\Sigma, D)$ be an ALD. A *checkerboard coloring* of P is a coloring of all the components of $\Sigma - |D|$ by two colors, say black and white, such that any two components of $\Sigma - |D|$ that share an edge have different colors.

We say that a virtual link diagram *admits a checkerboard coloring* or is *checkerboard colorable* if the associated ALD admits a checkerboard coloring.

3. The f -polynomials of abstract link diagrams

There is a unique map

$$\langle \rangle: \{\text{unoriented ALDs}\} \longrightarrow \Lambda = \mathbf{Z}[A, A^{-1}]$$

satisfying the following rules.

- (i) $\langle T \rangle = 1$ where T is a one-component trivial ALD,
- (ii) $\langle T \amalg P \rangle = (-A^2 - A^{-2})\langle P \rangle$ if P is not empty, where $T \amalg P$ is the disjoint union of T and P , and

$$(iii) \left\langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right\rangle = A \left\langle \begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \diagdown \diagdown \\ \diagup \diagup \end{array} \right\rangle.$$

The map $\langle \rangle$ is invariant under abstract Reidemeister moves II and III. We call it the *Kauffman bracket polynomial* of ALD, cf. [4].

Let $P = (\Sigma, D)$ be an unoriented ALD. Replacing the neighborhood of a crossing point as in Fig. 5, we have another unoriented ALD. We call it an unoriented ALD obtained from D by doing an *A-splice* or a *B-splice* at the crossing point. An unoriented trivial ALD obtained from P by doing an A-splice or a B-splice at each crossing point is called a *state* of P . From the definition of $\langle \rangle$, we see

$$\langle P \rangle = \sum_S A^{\sharp(S)} (-A^2 - A^{-2})^{\sharp(S)-1},$$

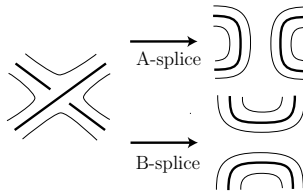


Fig. 5.

where S runs over all of states of P , $\sharp(S)$ is the number of A-splices minus that of B-splices used for obtaining S and $\#(S)$ is the number of components of S .

For an ALD, $P = (\Sigma, D)$, the writhe $\omega(P)$ is defined by the number of positive crossings minus the number of negative crossings of D . Then we define the *normalized bracket polynomial* or the *f-polynomial* of P by

$$f_P(A) = (-A^3)^{-\omega(P)} \langle P \rangle.$$

This value is preserved under abstract Reidemeister moves of type I. Thus this is an invariant of an abstract link. This invariant was defined in [4], where it was called the Jones polynomial of P . It should be noted that the bijection Φ preserves the f -polynomial.

4. Proof of Theorem 1

Let p be a crossing point of an ALD, $P = (\Sigma, D)$. Let $P_0 = (\Sigma_0, D_0)$ and $P_\infty = (\Sigma_\infty, D_\infty)$ be ALDs obtained from P by splicing at p orientation coherently and orientation incoherently, respectively. Note that D_∞ does not inherit an orientation from D . The crossing point p is either (i) a self-intersection of an immersed loop of D or (ii) an intersection of two immersed loops. Let α and α' be the immersed open arcs obtained from the loop (in case (i)) or from the two loops (in case (ii)) by removing (the small neighborhood of) p . Choose one of them, say α , and we give an orientation to D_∞ which is induced from that of D except α (and hence the orientation is reversed on α). Let C be the set of crossing points of D , except p , such that the sign of the crossing point is preserved when we consider the new diagram D_∞ ; in other words, at each crossing point belonging to C , both of the two intersecting arcs are contained in $D - \alpha$ or both of them are in α . Let C' be the set of crossing points of D , except p , such that the sign of the crossing point changes in D and D_∞ ; in other words, at each crossing point belonging to C' , one of the two intersecting arcs is contained in $D - \alpha$ and the other is in α . Let k (or l , resp.) be the number of positive crossings of C (resp. C') minus the number of negative crossings of C (resp. C').

Lemma 4. *In the above situation, let f , f_0 and f_∞ be the f -polynomials of P , P_0 and P_∞ , respectively. Then we have*

$$f = \begin{cases} -A^{-2}f_0 - (-A^3)^{-2l}A^{-4}f_\infty, & \text{if } p \text{ is a positive crossing,} \\ -A^{+2}f_0 - (-A^3)^{-2l}A^{+4}f_\infty, & \text{if } p \text{ is a negative crossing.} \end{cases}$$

Proof. If p is a positive crossing, then $\omega(D) = k + l + 1$, $\omega(D_0) = k + l$ and $\omega(D_\infty) = k - l$. Since $\langle P \rangle = A\langle P_0 \rangle + A^{-1}\langle P_\infty \rangle$, we have the result. The case where p is a negative crossing is proved by a similar argument. \square

REMARK. In Remark of Section 5 of [9, page 677], an equation which is similar to Lemma 4 is given. However, it seems to be forgotten there to take account of the term $(-A^3)^{-2l}$. In consequence, the recursion formula of Theorem 13 of [9] is as follows:

$$v_n(G_*) = \sum_{k=0}^{n-1} \frac{2^{n-k}}{(n-k)!} \{ (1 - (-1)^{n-k})v_k(G_0) + \{ (2 - 3l)^{n-k} - (-2 - 3l)^{n-k} \} v_k(G_\infty) \}.$$

By this formula, Corollary 14 of [9] is still true.

Corollary 5 (cf. Theorem 13 of [9]). *Let f be the f -polynomial of an ALD with n components. Then $f(1) = (-2)^{n-1}$. In particular, f -polynomials of ALDs are not zero.*

Proof. It follows from Lemma 4 by induction on the number of (real) crossing points. \square

Since Φ preserves the f -polynomials, Theorem 1 is equivalent to the following theorem.

Theorem 6. *Let f be the f -polynomial of an ALD, $P = (\Sigma, D)$, with n components. Suppose that P admits a checkerboard coloring. Then $\text{EXP}(f) \subset 4\mathbf{Z}$ if n is odd, and $\text{EXP}(f) \subset 4\mathbf{Z} + 2$ if n is even.*

Proof. For a state S of P , we define $I(S)$ by

$$I(S) = A^{\sharp(S)}(-A^2 - A^{-2})^{\#(S)-1}$$

so that the bracket polynomial of P is the sum of $I(S)$ over all states of P . Let $\text{ind}(S)$ be the value in $\mathbf{Z}_4 = \{0, 1, 2, 3\}$ such that $\text{EXP}(I(S)) \subset 4\mathbf{Z} + \text{ind}(S)$.

Every state of P has a unique checkerboard coloring induced from the checkerboard coloring of P , see Fig. 6. (Fig. 7 shows an example of an ALD with a checkerboard coloring and a state with the induced checkerboard coloring.) Using this fact,

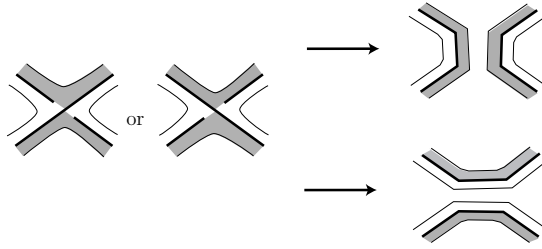


Fig. 6.

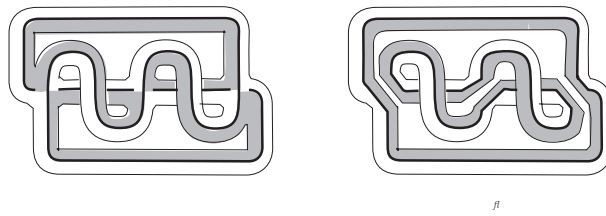


Fig. 7.

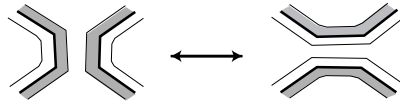


Fig. 8.

we prove that $\text{ind}(S) = \text{ind}(S')$ for any states S and S' of P . It is sufficient to prove this equality in the special case that S and S' differ in a single 2-disk E as in Fig. 8, where E is a neighborhood of a crossing point of D . There are three possibilities for the connection of S outside E as in Fig. 9. However, the case (C) does not occur because such a state does not have a checkerboard coloring induced from the checkerboard coloring of P . In both cases (A) and (B), we have $I(S') = A^{\sharp(S) \pm 2}(-A^2 - A^{-2})^{\sharp(S) - 1 \pm 1}$ and hence $\text{ind}(S) = \text{ind}(S')$.

Now we have that $\text{EXP}(f) \subset 4\mathbb{Z} + i$ where $i = \text{ind}(S)$ for any state S of P . We denote this number i by $\text{ind}(f)$. The remaining task is to prove that this index i is 0 if n is odd, and 2 if n is even. It is proved by induction on the number of (real) crossing points of P . If P has no real crossing points, then it is obvious by the definition of the f -polynomial. Suppose that P has a crossing point. For this crossing point, let P_0 and P_∞ be ALDs as in Lemma 4. Note that P_0 and P_∞ admit checkerboard colorings. Hence $\text{EXP}(f_0) \subset 4\mathbb{Z} + \text{ind}(f_0)$ and $\text{EXP}(f_\infty) \subset 4\mathbb{Z} + \text{ind}(f_\infty)$. Since $f \neq 0$ and $f_0 \neq 0$ by Corollary 5, it follows from the equation in Lemma 4 that $\text{ind}(f) \equiv \text{ind}(f_0) + 2 \pmod{4}$. The ALD P_0 has fewer crossing points than P and admits a checkerboard

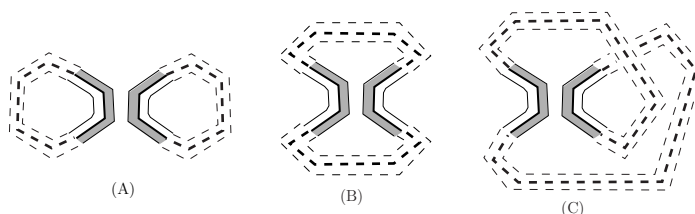


Fig. 9.

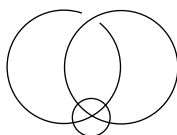


Fig. 10.

coloring. By the inductive hypothesis, $\text{ind}(f_0)$ is 0 if n' is odd, and 2 if n' is even, where n' is the number of components of P_0 . Since $n' = n \pm 1$, we have that $\text{ind}(f)$ is 0 if n is odd, and 2 if n is even. \square

5. Alternating virtual link diagrams and ALDs

An ALD or a virtual link diagram is said to be *alternating* if an over-crossings and under-crossings alternate as one travels along each component of the diagram. Note that the virtual link diagram in Fig. 10 is not alternating.

Lemma 7. *For an ALD, $P = (\Sigma, D)$, the following conditions are equivalent.*

- (i) *By applying crossing changes, P changes into an alternating ALD.*
- (ii) *P admits a checkerboard coloring.*

Proof. If P admits a checkerboard coloring, change each real crossing according to the coloring as in the leftmost figure of Fig. 6. Conversely, if P is an alternating ALD, then give a checkerboard coloring near each crossing point as in the figure used above, which is extended to a checkerboard coloring of P . \square

Proof of Corollary 2. It follows from Theorem 1 and Lemma 7. \square

REMARK. M.B. Thistlethwaite [11] and K. Murasugi [10] showed that the f -polynomial (Jones polynomial) of a non-split alternating link is alternating, namely, it is in a form of $A^\alpha \sum c_i A^{4i}$ such that $c_i c_j \geq 0$ for $i \equiv j \pmod{2}$ and $c_i c_j \leq 0$ for $i \not\equiv j \pmod{2}$. This result is not true for virtual knots. The f -polynomial of a

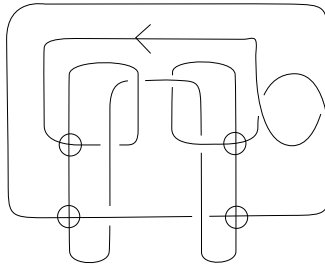


Fig. 11.

virtual knot in Fig. 11 is $A^{12} + 3A^{16} - 4A^{20} + 3A^{24} - 4A^{28} + 4A^{32} - 3A^{36} + A^{40}$.

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