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# WIENER FUNCTIONALS AND PROBABILITY LIMIT THEOREMS I: THE CENTRAL LIMIT THEOREMS

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## 1. Introduction

Our object is to study limit theorems in relation to functionals built on a dynamical system generated by the flow of Gaussian white noise or equivalently functionals subordinate to a real Gaussian stationary process

$$\xi(t) = \int_{-\infty}^{\infty} \exp i\lambda t \, d\beta(\lambda) , \tag{1.1}$$

with $E\xi(t)=0$, complex spectral measure $d\beta$, and spectral measure $d\sigma(\lambda)=E|d\beta(\lambda)|^2$, which is absolutely continuous with respect to Lebesgue measure, $d\sigma(\lambda)=f(\lambda)d\lambda$.

Define $\mathcal{L}_{k,p}(1\leq k<\infty,\ 0<p\leq\infty)$ to be the set of complex symmetric Borel functions $h$ on $\boldsymbol{R}^k$ satisfying (i) $\overline{h(\lambda)}=h(-\lambda)$ (ii) $h\in L^p(d^k\sigma)$, $d^k\sigma=d\sigma(\lambda_1)d\sigma(\lambda_2)\cdots d\sigma(\lambda_k)$, $\lambda=(\lambda_1,\cdots,\lambda_k)\in R^k$. A real-valued second order strictly stationary process $X(t)$, subordinate to $\xi$, with zero mean is represented by the Ito-Wiener expansion

$$X(t) = \sum_{k=1}^{\infty} X_k(t) , \quad X_k(t) = \int c_k(\lambda)e_k(\lambda,t)d^k\beta , \tag{1.2}$$

where $c_k\in\mathcal{L}_{k,2}$, $e_k(\lambda, t)=\exp i\bar{\lambda}t$, $\bar{\lambda}=\lambda_1+\cdots+\lambda_k$, $d^k\beta=d\beta(\lambda_1)\cdots d\beta(\lambda_k)$; the $k$-fold multiple Ito Integral ([2], [3]) in (1.2) is understood in the usual way ([6], [7], [8]). Throughout the paper the whole space as an integration domain is suppressed in an integral sign. $\boldsymbol{R}^k$ is such an example for the integral in (1.2).

Summarizing notational conventions: Constants will be denoted by $c$, $c_1$, $c_2$, $\cdots$ which are not always the same for each appearance. Given non-negative $f(x)$, $g(x)$, we use $f(x)\asymp g(x)$ to indicate that there exist constants $c_1$, $c_2>0$ such that $c_1f(x)\leq g(x)\leq c_2f(x)$ on a specified region.

To formulate the main theorem introduce an integral transform which maps $u\in L^1(d^k\sigma)$ $(k\geq 1)$ to $\varphi(u;\lambda)\in L^1(\boldsymbol{R})$, the space of Lebesgue integrable functions on $\boldsymbol{R}$:

$$\varphi(u;\lambda)=k!\int u(\lambda-\bar{\lambda}',\lambda_1,\cdots,\lambda_{k-1})f(\lambda-\bar{\lambda}')f(\lambda_1)\cdots f(\lambda_{k-1})d\lambda_1\cdots d\lambda_{k-1},$$
$$\bar{\lambda}'=\lambda_1+\cdots+\lambda_{k-1}\quad(k\geq 2),\quad \lambda'=(\lambda_1,\cdots,\lambda_{k-1}),$$
$$\varphi(u;\lambda)=u(\lambda)f(\lambda)\quad(k=1).$$

Our main theorem is

**Theorem 1.** *Suppose that the following conditions* (i)–(iv) *are satisfied.*

( i ) *$f(\lambda)$ is bounded.*

(ii) $V(T)=V(\int_0^T X(t)dt)\asymp T$, *as* $T\to\infty$,

*where $V$ denotes variance.*

(iii) $\lim\limits_{n\to\infty}\overline{\lim\limits_{T\to\infty}}\frac{1}{T}\Delta V_n(T)=0$

*where*

$$\Delta V_n(T)=V\left(\int_0^T R_n(t)dt\right),\quad R_n(t)=X(t)-S_n(t),$$
$$V_n(T)=V\left(\int_0^T S_n(t)dt\right),\quad S_n(t)=\sum_{k=1}^{n-1}X_k(t).$$

(iv) *For every $\varepsilon>0$, $k\geq 1$*

$$\lim_{h\downarrow 0}\Phi([\delta|c_k|^2];h)/h=0,\quad h=1/T,$$

*where*

$$\delta[|c_k|^2]=|c_k|^2-|c_k|^2\wedge(\varepsilon T^{1/3})$$

*and $\Phi(|c_k|^2;h)$ is the functional of $|c_k|^2$ defined by*

$$\Phi(|c_k|^2;h)=\int_0^h\varphi(|c_k|^2;\lambda)d\lambda.$$

*Then*

$$\text{dist}\,\bar{X}(T)\to N(0,1)\quad(\textit{weakly}),\quad \textit{as}\ T\to\infty,\tag{1.3}$$

*with*
$$\bar{X}(T)=\frac{1}{\sqrt{V(T)}}\int_0^T X(t)dy,$$

*where* dist *denotes probability distribution, and $N(0,1)$ the normal law with zero mean and variance* 1.

This theorem anounced in [8], [9] refines the central limit theorem in [6]. It will be easy to see that the conditions of Theorem 1 ensure the convergence of finite-dimensional distributions of the process $X_T(t)$, $0\leq t<\infty$, to the corresponding ones of standard Brownian motion, as $T\to\infty$, where

$$X_T(t)=\frac{1}{\sqrt{V(T)}}\int_0^{Tt}X(s)ds,\qquad 0\leq t<\infty.$$

The functional central limit theorem, which will be studied in a forthcoming paper, would require further regularity conditions imposed on $c_k$, $f(\lambda)$.

## 2. Preliminaries

Given real random variables $\xi_i$ $(1 \leq i \leq m)$, each with the $m$th moment, the cumulant $S(\xi_1, \cdots, \xi_m)$ as a multi-linear symmetric functional of them is defined by

$$S(\xi_1, \cdots, \xi_m) = i^{-m}\left(\frac{\partial^m}{\partial\alpha_1 \cdots \partial\alpha_m}\right)\log E\{\exp i[\alpha_1\xi_1 + \cdots + \alpha_m\xi_m]\}\big|_{\alpha_1 = \cdots = \alpha_m = 0} .$$

When $\xi_1, \cdots, \xi_m$ consist of $m(1)$ $\eta_1$'s, $\cdots$, $m(k)$ $\eta_k$'s, with $m(1) + \cdots + m(k) = m$, we use $S_{m(1)\cdots m(k)}(\eta_1, \cdots, \eta_k)$ instead of $S(\xi_1, \cdots, \xi_m)$.

Consider a family of functions $\mathcal{F} = \{f_i,\ 1 \leq i \leq p\}$, $f_i \in L^2(d^{l(i)}\sigma)$, with $\sum_{i=1}^{p} l(i) = 2m$ $(m = 1, 2, \cdots)$, and corresponding integrals with kernels $f_i$

$$I_i = I(f_i) = \int f_i d^{l(i)}\beta , \qquad 1 \leq i \leq p .$$

The arguments involved in $f_i$, $1 \leq i \leq p$, together form a set of $2m$ letters. Make couples $(a_1, b_1)$, $(a_2, b_2)$, $\cdots$ of letters taken at once from the $2m$ letters, demanding that the letters in each couple are from different kernels e.g. $a_1$ is from $f_k$ but $b_1$ from $f_l$, $1 \leq k \neq l \leq p$, we want to make as many couples as possible. When we could get just $m$ couples, $\Gamma = \{(a_1, b_1), (a_2, b_2), \cdots, (a_m, b_m)\}$ (a complete set of couples) the coupling is complete. Whether there exists a complete coupling depends on the composition of the numbers $l(1), \cdots, l(p)$. For example, if $l(1)$ is too large compared with the others, a complete coupling does not exist. If $l(1), \cdots, l(p)$ are well balanced, there may be several complete couplings which yield different complete sets of couples.

Given a complete set of couples $\Gamma$, make substitutions $a_1 \to \lambda_1$, $b_1 \to -\lambda_1$, $a_m \to \lambda_m$, $b_m \to -\lambda_m$ in the arguments of the product $f_1 f_2 \cdots f_p$, to obtain a function of $\lambda_1, \cdots, \lambda_m$, $\mathcal{F}(\Gamma; \lambda_1, \cdots, \lambda_m)$. $\mathcal{F}(\Gamma; \lambda_1, \cdots, \lambda_m)$ is called a $p$-fold kernel composed of $f_1, \cdots, f_p$ and will also be denoted by $\mathcal{K}(\Gamma; f_1, \cdots, f_p)$; $f_i$ $(1 \leq i \leq p)$ is said to be concerned with $\mathcal{K}(\Gamma; f_1, \cdots, f_p)$, $f_k$, $f_l$ $(k \neq l)$ are said to be connected by $\Gamma$ if there exists a couple $(a_i, b_i)$ with letters $a_i$, $b_i$ from either $f_k$ or $f_l$.

To proceed further, represent $f_j$, $1 \leq j \leq p$, by distinct $p$ points on the plane, again denoted by $f_j$. They are called vertices. If $f_k$, $f_l$ $(k \neq l)$ are connected, draw a segment connecting them, which will be called an edge. To avoid configurational complexity, the points representing vertices are so chosen that on an edge there are no other vertices than its end vertices.

The figure composed of all the edges and vertices is called the graph corresponding to a complete coupling $\Gamma$, and is again denoted by $\Gamma$. A subset of the edges and vertices form a subgraph of $\Gamma$. A figure composed of successively connected vertices and edges connecting them is called a polygonal line.

In general, $f_k$, $f_l$ $(k \neq l)$ are said to be connected if there is a polygonal line starting at $f_k$ and ending in $f_l$. If vertices in $\Gamma$ are connected each other, $\Gamma$ is called a connected graph, and $\mathcal{F}(\Gamma; \lambda_1, \cdots, \lambda_m)$ a connected kernel. In general, a disconnected graph $\Gamma$ is a disjoint union of connected subgraphs $\Gamma_0, \cdots, \Gamma_s$, and the corresponding kernel $\mathcal{K}(\Gamma_1; f_1, \cdots, f_p)$ is a product of connected kernels corresponding to $\Gamma_0, \cdots, \Gamma_s$. If $f_k$, $f_l$ $(k \neq l)$ are connected, the total number of couples $(a_j, b_j)$ connecting $f_k$, $f_l$ is called the multiplicity of the edge connecting $f_k, f_l$.

Let us denote by $\mathfrak{G}(\mathcal{F})$ and $\mathfrak{C}(\mathcal{F})$ respectively the set of graphs and connected ones based on $\mathcal{F} = \{f_i \colon 1 \leq i \leq p\}$. Obviously $\mathfrak{C}(\mathcal{F}) \subset \mathfrak{G}(\mathcal{F})$. When $l(1) + \cdots + l(p)$ is odd, $\mathfrak{G}(\mathcal{F})$, $\mathfrak{C}(\mathcal{F})$ are empty. *We have the following formulas* ([4], [8])

$$E(I(f_1) \cdots I(f_p)) = \sum_{\Gamma \in \mathfrak{G}(\mathcal{F})} \int \mathcal{F}(\Gamma; \lambda) d^m \sigma\,, \tag{2.1}$$

$$S(I(f_1) \cdots I(f_p)) = \sum_{\Gamma \in \mathfrak{C}(\mathcal{F})} \int \mathcal{F}(\Gamma; \lambda) d^m \sigma\,, \tag{2.2}$$

*where* $m = (l(1) + \cdots + l(p))/2$ *if* $l(1) + \cdots + l(p)$ *is even, and* $\mathcal{F}(\Gamma; \lambda)$ *are abbreviations of* $\mathcal{K}(\Gamma; f_1, \cdots, f_l)$, *while if it is odd the right-hand members stand for zero.*

Proof. The product $I(f_1) \cdots I(f_p)$ can be represented as a sum of homegeneous polynomials (p. 388, [9]). $E(I(f_1) \cdots I(f_p))$ is then given by the nonrandom term of the sum. This proves (2.1).

The formula (2.2) follows in turn from (2.1). We will prove it by induction. When $p=1$, the right-hand side of (2.2) is equal to zero since $\mathfrak{C}(\mathcal{F})$ is empty, whereas by the equality $S(I(f_1)) = E(I(f_1))$, the left-hand side is also equal to zero.

Let $p \geq 2$, assume (2.2) up to the $(p-1)$th step, and notice that there hold the relations between moments and cumlants ([5]):

$$\begin{aligned} E(I(f_1) \cdots I(f_p)) - S(I(f_1), \cdots, I(f_p)) = {} & S(I(f_1)) S(I(f_2), \cdots, I(f_p)) + \cdots \\ & + S(I(f_1), I(f_2)) S(I(f_3), \cdots, I(f_p)) + \cdots \end{aligned} \tag{2.3}$$

Suppose that $l(1) + \cdots + l(p)$ is odd, then on the right-hand side of this equality every term vanishes by the induction hypothesis, while on the left-hand side the first term vanishes, so does the second one. This means that (2.2) is true.

Suppose next that $l(1) + \cdots + l(p)$ is even, and using the induction hypothesis rewrite every term on the right-hand member of (2, 3) in terms of integrals with connected kernels; then they give rise to the sum

$$\sum_{\Gamma \in \mathfrak{D}} \int \mathcal{F}(\Gamma; \lambda) d^m \sigma\,, \quad m = (l(1) + \cdots + l(p))/2\,,$$

where $\mathfrak{D}$ is a set of disconnected graphs $\Gamma$. Here in view of the composition of terms on the right-hand side of (2.3), one recognizes that $\mathfrak{D}$ must be the totality of disconnected graphs formed by $\mathcal{F}$. Therefore by (2.1), (2.3) we have

$$S(I(f_1), \cdots, I(f_p)) = \sum_{\Gamma \in \mathfrak{G}(\mathcal{F})} \int \mathcal{F}(\Gamma; \lambda) d^m\sigma - \sum_{\Gamma \in \mathfrak{D}} \int \mathcal{F}(\Gamma; \lambda) d^m\sigma$$
$$= \sum_{\Gamma \in \mathfrak{C}(\mathcal{F})} \int \mathcal{F}(\Gamma; \lambda) d^m\sigma ,$$

which proves (2.2).

Taking into account of (1.2)

$$E\{X_k(t+\tau)X_k(t)\} = k! \int |c_k(\lambda_1, \cdots, \lambda_k)|^2 \exp i[\lambda_1 + \cdots + \lambda_k]\tau d^k\sigma$$
$$= \int_{-\infty}^{\infty} e^{i\lambda\tau} f_{2k}(\lambda) d\lambda , \quad f_{2k}(\lambda) = \varphi(|c_k|^2; \lambda) .$$

So that the spectral densities of $X(t)$ and $R_n(t)$ are respectively

$$\sum_{k=1}^{\infty} f_{2k}(\lambda) \quad \text{and} \quad \sum_{k \geqq n} f_{2k}(\lambda) .$$

Define

$$\theta_k(\lambda) = c_k(\lambda)/|c_k(\lambda)| \quad \text{if} \quad c_k(\lambda) \neq 0 , \quad = 1 \quad \text{if} \quad c_k(\lambda) = 0 , \tag{2.4}$$

and write

$$c_k(\lambda) = c_k^\varepsilon(\lambda) + \Delta c_k(\lambda) \qquad (k \geq 1) ,$$

where

$$c_k^\varepsilon(\lambda) = (|c_k(\lambda)| \wedge \eta)\theta_k(\lambda) , \quad \eta = \sqrt{\varepsilon}\, T^{1/6} .$$

Obviously $\theta_k(\lambda) \in \mathcal{L}_{k,\infty}$.

For later use we set up several fundamental propositions (I–V) mostly pertaining to the expression (1.2).

I. *Suppose that*

$$\Phi(|c_k|^2; h) = O(h) , \qquad as \quad h \to +0 , \tag{2.5}$$

*then* $\Phi(|\Delta c_k|^2; h) = o(h)$ *if and only if* $\Phi(\delta[|c_k|^2]; h) = o(h)$, $h = 1/T$.

Proof. Suppose

$$\Phi(|\Delta c_k|^2; h) = o(h) .$$

Since

$$|c_k| - |c_k| \wedge \eta = |\Delta c_k| , \quad \delta[|c_k|^2] \leq |\Delta c_k|(2|c_k|) , \tag{2.6}$$

the Schwartz inequality yields

$$\varphi(\delta[|c_k|^2]; \lambda) \leq 2\varphi^{1/2}(|\Delta c_k|^2; \lambda)\varphi^{1/2}(|c_k|^2; \lambda),$$

whence by (2.5)

$$\Phi(\delta[|c_k|^2]; h) \leq 2\{\Phi(|c_k|^2; h)\}^{1/2}\{\Phi(|\Delta c_k|^2; h)\}^{1/2} = o(h)$$

Suppose

$$\Phi(\delta[|c_k|^2]; h) = o(h).$$

Since by (2.6)

$$|\Delta c_k|^2 \leq (|c_k| - |c_k| \wedge \eta)(|c_k| + |c_k| \wedge \eta) = \delta[|c_k|^2],$$

one obtains

$$\Phi(|\Delta c_k|^2; h) \leq \Phi(\delta[|c_k|^2]; h) = o(h).$$

II. *Let $X(t)$ be a real second order stationary process, with zero mean and spectral density $\varphi(\lambda)$. Put*

$$v(T) = V(\int_0^T X(t)dt), \quad \Phi(h) = \int_0^h \varphi(\lambda)d\lambda.$$

*Then*

$$2\left(\frac{2}{\pi}\right)^2 \varliminf_{h \downarrow 0} \Phi(h)/h \leq \varliminf_{T\to\infty} v(T)/T, \tag{2.7}$$

$$\varlimsup_{T\to\infty} v(T)/T \leq 18 \varlimsup_{h\downarrow 0} \Phi(h)/h \leq 3\pi^2 \varlimsup_{T\to\infty} v(T)/T. \tag{2.8}$$

Proof. Write

$$\frac{v(T)}{2T} = \int_0^\infty K(\lambda)d\lambda = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_0^{1/T} K(\lambda)d\lambda, \quad I_2 = \int_{1/T}^{\delta} K(\lambda)d\lambda, \quad I_3 = \int_\delta^\infty K(\lambda)d\lambda, \quad \delta > 0,$$

$$K(\lambda) = \left(\frac{\sin \lambda T/2}{\lambda/2}\right)^2 \frac{\varphi(\lambda)}{T}.$$

Then obviously

$$I_3 = O(1/T) \tag{2.9}$$

and since

$$2T\pi \leq \left|\frac{\sin \lambda T/2}{\lambda/2}\right| \leq T \qquad \text{if} \quad 0 \leq \lambda \leq 1/T, \tag{2.10}$$

one gets

$$\left(\frac{2}{\pi}\right)^2 \Phi(h)/h \leq I_1 \leq \Phi(h)/h, \quad h = 1/T. \tag{2.11}$$

On the other hand, since

$$K(\lambda) \leq 4\varphi/\lambda^2 T,$$

by partial integration

$$(2.12) \quad \begin{aligned} I_2 &\leq 4T^{-1}\int_{1/T}^{\delta} \varphi(\lambda)/\lambda^2 d\lambda \leq 4\Phi(\delta)/T\delta^2 + 8T^{-1} \sup_{0<h\leq\delta} (\Phi(h)/h) \int_{1/T}^{\infty} \lambda^{-2} d\lambda \\ &= O(1/T) + 8 \sup_{0<h\leq\delta} (\Phi(h))/h . \end{aligned}$$

On making $T \to \infty$, and then $\delta \downarrow 0$

$$(2.13) \qquad \overline{\lim_{T\to\infty}}\, I_2 \leq 8 \overline{\lim_{h\downarrow 0}}\, \Phi(h)/h .$$

(2.11) implies (2.7) and the second inequality of (2.8), while (2.9), (2.11), (2.13) do the first one of (2.8).

III. *Write*

$$X_k(t) = X_k^{\varepsilon}(t) + \Delta X_k(t), \qquad 1 \leq k < \infty,$$

$$X_k^{\varepsilon}(t) = \int c_k^{\varepsilon}(\lambda) e_k(\lambda, t) d^k\beta, \quad \Delta X_k(t) = \int \Delta c_k(\lambda) e_k(\lambda, t) d^k\beta .$$

*Then under the assumptions of the theorem*

$$\lim_{T\to\infty} V\Big(\frac{1}{\sqrt{T}} \int_0^T \Delta X_k(t) dt\Big) = 0$$

Proof. The proof goes after that of the preceding proposition. Write

$$\frac{1}{2T} V\Big(\int_0^T \Delta X_k(t) dt\Big) = I_1 + I_2,$$

where

$$I_1 = \int_0^{rh} K(\lambda) d\lambda, \quad I_2 = \int_{rh}^{\infty} K(\lambda) d\lambda,$$

$$K(\lambda) = \Big(\frac{\sin \lambda T/2}{\lambda/2}\Big)^2 \varphi(|\Delta c_k|^2; \lambda)/T, \quad h = 1/T,$$

and $r$ is a positive parameter.

Put

$$\Phi(\varepsilon, x) = \int_0^x \varphi(|\Delta c_k|^2; \lambda) d\lambda .$$

Then as in the proof of the preceding proposition

$$I_1 \leq r \frac{1}{rh} \Phi(\varepsilon \sqrt[3]{r}, rh) \to 0, \qquad \text{as} \quad h \to +0 .$$

On the other hand, since $\varphi(|\Delta c_k|^2; \lambda) \leq \varphi(|c_k|^2; \lambda)$ and $\Phi(|c_k|^2; x) \leq cx$ on $[0, \infty)$.

$$I_2 \leq 4h\int_{rh}^{\infty}\varphi(|c_k|^2;\lambda)\lambda^{-2}d\lambda = 4h\Phi(|c_k|^2;x)x^{-2}]_{rh}^{\infty}$$
$$+8h\int_{rh}^{\infty}\Phi(|c_k|^2;x)x^{-3}dx \leq 8ch\int_{rh}^{\infty}x^{-2}dx \leq 8c/r\,.$$

On making $T\to\infty$, and then $r\to\infty$, we obtain the desired conclusion.

Let $|\cdot|$ be the Enclidean norm on $\boldsymbol{R}^d$ and write $x^m$ for the multipower $x_1^{m_1}\cdots x_d^{m_d}$ of $x=(x_1,\cdots,x_d)\in\boldsymbol{R}^d$, with $m=(m_1,\cdots,m_d)$ a multiindex of integer entries $m_1,\cdots,m_d\geq 0$.

In relation to the method of moments we mention without proof an elememtary proposition.

IV. *Let* $\{\xi_n, n\geq 1\}$ *be a sequence of* $\boldsymbol{R}^d$*-valued random variables whose distributions* $\mathcal{F}=\{F_n, n\geq 1\}$ *satisfy the conditions*

$$\sup_{n\geq 1}\int|x|^p dF_n<\infty$$

*for any* $p>0$.

*Then*

(i) $\mathcal{F}$ *is relatively compact.*

(ii) *For every* $q>0$, $\{|\xi_n|^q, n\geq 1\}$ *is uniformly integrable.*

(iii) *Let* $G$ *be a limit point of the sequence* $\mathcal{F}$, *and* $F_{n(l)}$ *be a subsequence of* $\mathcal{F}$ *such that*

$$F_{n(k)}\to G\,,\qquad as\quad n(k)\to\infty\,.$$

*Then* $G$ *has the moment of an arbitrary multi-power* $m$ *and*

$$\lim_{k\to\infty}\int x^m dF_{n(k)}=\int x^m dG\,.$$

REMARK. The reference topology in the above statement is the usual weak one. By (i) $\mathcal{F}$ certainly has at least a limit point.

DEFINITION. Define $\mathcal{D}_0$ to be the set of sequences on $[0,\infty)$ tending to $\infty$.

V. *Under the assumptions of the theorem there exists a natural number* $n_1$, *such that*

(i) $c_1<V_n(T)/T<c_2$ *for any* $n\geq n_1$, $T\geq 1$ *with* $c_1$, $c_2>0$, *independent of* $n$, $T$, *where* $V_n(T)=V(\int_0^T S_n(t)dt)$,

(ii) *for any* $D'\in\mathcal{D}_0$, *there exists a subsequence* $D$ *of* $D'$ *with* $D\in\mathcal{D}_0$ *such that there exist the limits*

$$\text{(a)}\quad \rho_n\equiv\lim_{\substack{T\to\infty\\ T\in D}}V_n(T)/V(T)\qquad(n\geq n_1)\,,$$

(b) $\lim_{\substack{T\to\infty \\ T\in D}} v_k(T)/V_n(T) \qquad (n\geq n_1,\ k\geq 1)$,

*where*

$$v_k(T)=V(\int_0^T x_k(t)dt)\,,\quad V_n(T)=V(\int_0^T S_n(t)dt)\,,$$

(c) $\lim_{n\to\infty}\rho_n=1.$

Proof. $V_n(T)$, $V(T)$ are continuous on $[0, \infty)$ and positive for $T>0$, and $\Delta V_n(T)\downarrow 0$, locally uniformly on $[0, \infty)$, as $n\to\infty$. Therefore by the condition (iii) of Theorem 1, given $\varepsilon>0$, there exists $n_0=n_0(\varepsilon)$ such that

$$\sup_{1\leq T<\infty} \Delta V_n(T)/T<\varepsilon$$

whenever $n\geq n_0$.

Let

$$l=\inf_{T\geqq 1} V(T)/T\,,\quad m=\sup_{T\geqq 1} V(T)/T\,,$$

then obviously $0<l\leq m<\infty$, and

$$\sup_{T\geq 1} V_n(T)/T\leq m \tag{2.14}$$

for any $n$. On the other hand

$$V_n(T)/T=V(T)/T-\Delta V_n(T)/T>l-l/2=l/2 \tag{2.15}$$

for $n\geq n_0(l/2)\equiv n_1$ and $T\geq 1$. (2.14), (2.15) imply (i). (a), (b) in (ii) follow from (i) by applying diagonal procedures.

Observe that

$$1\geq V_n(T)/V(T)=1-\frac{\Delta V_n(T)/T}{V(T)/T}\geq 1-\frac{\varepsilon}{l}$$

for all $T\geq 1$, $n\geq n_0(\varepsilon)$. This proves (c).

DEFINITION. Define $\mathscr{D}_1$ to be the set of sequences $D\in\mathscr{D}_0$ for which the limits in (a), (b) exist.

## 3. Proof of Theorem 1

Take $n\geq n_1$ and using the notations in Section 2 write

$$\begin{aligned}
\eta_k(T)&=\frac{1}{\sqrt{V_n(T)}}\int_0^T X_k(t)dt\,,\\
\eta_k^\varepsilon(T)&=\frac{1}{\sqrt{V_n(T)}}\int_0^T X_k^\varepsilon(t)dt\,,\quad \Delta\eta_k(T)=\frac{1}{\sqrt{V_n(T)}}\int_0^T \Delta X_k(t)dt\,,\\
Y(T)&=(\eta_1(T),\cdots,\eta_n(T))\,,\\
Y^\varepsilon(T)&=(\eta_1^\varepsilon(T),\cdots,\eta_n^\varepsilon(T))\,,\quad \Delta Y(T)=(\Delta\eta_1(T),\cdots,\Delta\eta_n(T))\,.
\end{aligned} \tag{3.1}$$

Then

$$Y(T) = Y^{\varepsilon}(T) + \Delta Y(T),$$

(3.2)
$$\frac{1}{\sqrt{V_n(T)}} \int_0^T S_n(t)dt = \sum_{j=1}^{n} \eta_j(T),$$

(3.3)
$$E|Y(T)|^2 = 1,$$

and by III, Section 2

(3.4)
$$\lim_{T\to\infty} V(\Delta Y(T)) = 0 \qquad \text{for every} \quad \varepsilon > 0,$$

where $V(\Delta Y(T)) = \sum_{k=1}^{n} V(\Delta \eta_k(T))$.

**Lemma 1.**

(3.5)
$$\lim_{T\to\infty} \sup_{0<\varepsilon\leq 1} |S_{k(1)\cdots k(n)}(\eta_1^{\varepsilon}(T), \cdots, \eta_n^{\varepsilon}(T))| = 0, \qquad \text{if} \quad k(1)+\cdots+k(n) \geq 4,$$

(3.6)
$$\sup_{T\geq 1} |S_{k(1)\cdots k(n)}(\eta_1^{\varepsilon}(T), \cdots, \eta_n^{\varepsilon}(T))| \leq c\varepsilon^{3/2}, \qquad \text{if} \quad k(1)+\cdots+k(n) = 3,$$

*where c is a positive constant independent of* $T$, $\varepsilon$.

Proof. Step 1. Proof of (3.6). For the proof of (3.6) we are sufficed to show that

(3.6)′
$$\sup_{T\geq 1} S(\eta_k^{\varepsilon}(T), \eta_l^{\varepsilon}(T), \eta_m^{\varepsilon}(T)) \leq c\varepsilon^{3/2}, \qquad 1 \leq k, l, m \leq n.$$

Denote by $\boldsymbol{x} = (x_1, \cdots, x_k) \in R^k$, $\boldsymbol{y} = (y_1, \cdots, y_l) \in R^l$, $\boldsymbol{z} = (z_1, \cdots, z_m) \in R^m$ respectively the arguments of $c_k^{\varepsilon}$, $c_l^{\varepsilon}$, $c_m^{\varepsilon}$. Then

$$S(T) \equiv S(\eta_k^{\varepsilon}, \eta_l^{\varepsilon}, \varepsilon_m^{\varepsilon})$$

(3.7)
$$= \left(\frac{1}{\sqrt{V_n(T)}}\right)^3 S\left(\int c_k^{\varepsilon}(\boldsymbol{x})\mathcal{D}_T(\bar{\boldsymbol{x}})d^k\beta, \int c_l^{\varepsilon}(\boldsymbol{y})\mathcal{D}_T(\bar{\boldsymbol{y}})d^l\beta, \int c_m^{\varepsilon}(\boldsymbol{z})\mathcal{D}_T(\bar{\boldsymbol{z}})d^m\beta\right),$$

where

$$\bar{\boldsymbol{x}} = \sum_{j=1}^{k} x_j, \quad \text{etc.,} \quad \mathcal{D}_T(x) = \frac{e^{iTx}-1}{ix}.$$

By the formula (2.2), the right-hand member of the last equation is represented as a sum of integrals involving connected kernels corresponding to connected graphs. Those connected kernels are given birth through couplings among the components of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$. To be precise, write $k=b+c$, $l=c+a$, $m=a+b$ and suppose that $a$ components of $\boldsymbol{y}$ are connected with $a$ ones of $\boldsymbol{z}$, $b$ components of $\boldsymbol{z}$ with $b$ ones of $\boldsymbol{x}$ and similarly for $c$. Next make the substitutions denoted by $Q = Q(\lambda^{(a)}, \mu^{(b)}, \nu^{(c)})$, with $\lambda^{(a)} = (\lambda_1, \cdots, \lambda_a) \in R^a$ etc.: substitute $\mu_1, \cdots, \mu_b, \nu_1, \cdots, \nu_c$ into the components of $\boldsymbol{x}$, $-\nu_1 \cdots, -\nu_c, \lambda_1, \cdots, \lambda_a$ into those of $\boldsymbol{y}$, and $-\lambda_1, \cdots, -\lambda_a, -\mu_1, \cdots, -\mu_b$ into those of $\boldsymbol{z}$. $Q$ are restricted to the set $\mathfrak{C}$ of those substitutions which give rise to connected kernels. Let $N=N(Q)$ be the number of edges of the connected graph corresponding to $Q$. Then $N$ is equal to the number of positive $a, b, c$. The right-hand side of (3.7) is rewritten as

$$S(T)=\left(\frac{1}{\sqrt{V_n(T)}}\right)^3 \sum_{\substack{b+c=k,\ c+a=l \\ a+b=m \\ Q\in\mathfrak{C}}} \int Q(\lambda^{(a)}, \mu^{(b)}, \nu^{(c)})\{c_k^{\varepsilon}(\boldsymbol{x}) \tag{3.8}$$

$$\times c_l^{\varepsilon}(\boldsymbol{y})c_m^{\varepsilon}(\boldsymbol{z})\mathcal{D}_T(\bar{\boldsymbol{x}})\mathcal{D}_T(\bar{\boldsymbol{y}})\mathcal{D}_T(\bar{\boldsymbol{z}})\}\ d^a\sigma d^b\sigma d^c\sigma\,,$$

where $d^a\sigma=d\sigma(\lambda_1)\cdots d\sigma(\lambda_a)$ etc. Since $|c_k^{\varepsilon}|\leq\sqrt{\varepsilon}\,T^{1/6}$ etc., the absolute value of a typical term on the right-hand member of (3.8) is less than

$$I=\varepsilon^{3/2}T^{1/2}\left(\frac{1}{\sqrt{V_n(T)}}\right)^3 |D_T(\overline{\mu^{(b)}}+\overline{\nu^{(c)}})D_T(-\overline{\nu^{(c)}}+\overline{\lambda^{(a)}})D_T(-\overline{\lambda^{(a)}}-\overline{\mu^{(b)}})|$$

$$\times d^a\sigma d^b\sigma d^c\sigma\,,$$

$$D_T(x)=\frac{\sin Tx/2}{x/2}\,,$$

where integers $a, b, c$ are so chosen that $0\leq a, b, c$, $b+c=k$, $c+a=l$, $a+b=m$, and the arising kernels $|D_T(\overline{\mu^{(b)}}+\overline{\nu^{(c)}})D_T(-\overline{\nu^{(c)}}+\overline{\lambda^{(a)}})D_T(-\overline{\lambda^{(a)}}-\overline{\mu^{(b)}})|$ be connected. Writing

$$u=\overline{\lambda^{(a)}}\,,\quad v=\overline{\mu^{(b)}}\,,\quad w=\overline{\nu^{(c)}}\,,$$

one obtains

$$I=\varepsilon^{3/2}T^{1/2}\left(\frac{1}{\sqrt{V_n(T)}}\right)^3\int |D_T(l_1)D_T(l_2)D_T(l_3)|\,f_a(u)f_b(v)f_c(w)dudvdw\,,$$

where

$$l_1=v+w\,,\quad l_2=-w+u\,,\quad l_3=-u-v$$

$$f_a(u)=f^{a*}(u)\qquad \text{(the $a$-fold convolution of $f$) etc.}$$

The linear functions $l_1, l_2, l_3$ are linearly dependent, actually

$$l_1+l_2+l_3\equiv 0\,.$$

However any two of them are linearly independent. This is a consequence of the connectedness of the graph.

There arise two cases:

A. One of $a, b, c$ vanishes,

B. $a, b, c>0$.

Case A ($N=2$). With no loss of generality, assume $c=0$, $a\cdot b>0$. Then in view of the obvious inequalities

(3.9) $$||f^{m*}||_1 \leq ||f||_1^m\,,\quad ||f^{m*}||_\infty \leq ||f||_\infty ||f||_1^{m-1} \qquad (m\geq 1)$$

one gets

$$I \leq K_1 \varepsilon^{3/2} T^{3/2} \left(\frac{1}{\sqrt{V_n(T)}}\right)^3, \quad K_1 = ||f||_1^{a+b-2} ||f||_\infty^2 ||\Psi^{(2)}||_1 < \infty\,,$$

where $||\cdot||_1$ denotes $L^1$-norm. Here we made use of the fact ([1]) that the function

$$\Psi^{(n)}(x_1, \cdots, x_n) = \prod_{k=1}^n \left|\frac{\sin x_k/2}{x_k/2}\right| \left|\frac{\sin(x_1+\cdots+x_n)/2}{(x_1+\cdots+x_n)/2}\right|$$

belongs to $L^1(\boldsymbol{R}^n)$ $(n\geq 1)$. Since $V_n(T) \asymp T$, the last inequality implies

(3.10) $$\sup_{T\geq 1} |S(T)| \leq c\varepsilon^{3/2}\,.$$

Case B ($N=3$). Choose a linear function $\tilde{l}_3$ of $u$, $v$, $w$ such that $l_1$, $l_2$, $\tilde{l}_3$ are linearly independent, and consider a transformation of variables $u$, $v$, $w$ to $x_1$, $x_2$, $x_3$:

$$x_1 = l_1\,, \quad x_2 = l_2\,, \quad x_3 = \tilde{l}_3\,.$$

We may assume that $|\partial(x_1, x_2, x_3)/\partial(u, v, w)| = 1$.

The inverse to this transformation takes the form

$$u = u^*(x_1, x_2) + a_1 x_3\,, \quad v = v^*(x_1, x_2) + a_2 x_3$$
$$w = w^*(x_1, x_2) + a_3 x_3$$

where $u^*$, $v^*$, $w^*$, are linear in $x_1$, $x_2$. At least one of $a_1$, $a_2$, $a_3$ is unequal to zero. Let $a_3 \neq 0$. Then

$$\begin{aligned} I &= \varepsilon^{3/2} T^{1/2} \left(\frac{1}{\sqrt{V_n(T)}}\right)^3 \int |D_T(x_1) D_T(x_2) D_T(x_1+x_2)|\, ||f_a||_\infty ||f_b||_\infty dx_1 dx_2 \\ &\qquad \times \int f_c(a_3 x_3 + w^*(x_1, x_2)) dx_3 \\ &\leq K_2 \varepsilon^{3/2} T^{3/2} \left(\frac{1}{\sqrt{V_n(T)}}\right)^3, \end{aligned}$$

with

$$K_2 = ||f||_1^{a+b+c-2} ||f||_\infty^2 ||\Psi^{(2)}||_1 \frac{1}{|a_3|}\,.$$

So that

(3.11) $$\sup_{T\geq 1} |S(T)| \leq c\varepsilon^{3/2}\,,$$

that is the same conclusion as (3.10),

(3.10), (3.11) together complete the proof of (3.6).

Step 2. Proof of (3.5). Let $S(T) = S(\eta_{k(1)}^{\varepsilon}, \cdots, \eta_{k(p)}^{\varepsilon})$ $(p\geq 4)$. Then

$$
\begin{aligned}
S(T) &= \left(\frac{1}{\sqrt{V_n(T)}}\right)^p S\left(\int c^{\varepsilon}_{k(1)}(\boldsymbol{w})D_T(\bar{\boldsymbol{w}})d^{k(1)}\beta, \cdots, \int c_{k(p)}(\boldsymbol{z})D_T(\bar{\boldsymbol{z}})d^{k(p)}\beta\right) \\
&= \left(\frac{1}{\sqrt{V_n(T)}}\right)^p \sum_{Q\in\mathfrak{C}} \int Q\{c^{\varepsilon}_{k(1)}(\boldsymbol{w})\cdots c^{\varepsilon}_{k(p)}(\boldsymbol{z})D_T(\bar{\boldsymbol{w}})\cdots D_T(\bar{\boldsymbol{z}})\}d^s\sigma ,
\end{aligned} \tag{3.12}
$$

where $s=(k(1)+\cdots+k(p))/2$, $Q$ denotes a complete coupling of the arguments of $c_{k(1)}, \cdots, c_{k(p)}$, say $\boldsymbol{w}\in\boldsymbol{R}^{k(1)}, \cdots, \boldsymbol{z}\in\boldsymbol{R}^{k(p)}$, and the summation means that $Q$ runs over $\mathfrak{C}$, the set of connected couplings.

From now on, to avoid notational complexity we deal with the case $p=4$. In this case a typical term on the right-hand side of (3.12) is of the form

$$
\begin{gathered}
\left(\frac{1}{\sqrt{V_n(T)}}\right)^4 \int c^{\varepsilon}_j(\boldsymbol{w})c^{\varepsilon}_k(\boldsymbol{x})c^{\varepsilon}_l(\boldsymbol{y})c^{\varepsilon}_m(\boldsymbol{z})D_T(l_1)D_T(l_2)D_T(l_3)D_T(l_4)d^s\sigma , \\
\begin{aligned}
& s = (j+k+l+m)/2 , \qquad 1\leq j,\, k,\, l,\, m\leq n , \\
& \boldsymbol{w} = (\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g}) , \quad \boldsymbol{x} = (-\boldsymbol{e}, \boldsymbol{h}, \boldsymbol{i}) , \quad \boldsymbol{y} = (-\boldsymbol{f}, -\boldsymbol{h}, \boldsymbol{j}) , \\
& \boldsymbol{z} = (-\boldsymbol{g}, -\boldsymbol{j}, -\boldsymbol{i}) , \\
& l_1 = \bar{\boldsymbol{e}}+\bar{\boldsymbol{f}}+\bar{\boldsymbol{g}}, \cdots, l_4 = -\bar{\boldsymbol{g}}-\bar{\boldsymbol{j}}-\bar{\boldsymbol{i}} ,
\end{aligned}
\end{gathered} \tag{3.13}
$$

where the six vectors $\boldsymbol{e}, \cdots, \boldsymbol{j}$ are so chosen that their dimensionalities $d(\boldsymbol{e})$, $\cdots, d(\boldsymbol{j})$ satisfy that $0\leq d(\boldsymbol{e}), \cdots, d(\boldsymbol{j})$, $d(\boldsymbol{e})+d(\boldsymbol{f})+d(\boldsymbol{g})=j, \cdots, d(\boldsymbol{g})+d(\boldsymbol{j})+d(\boldsymbol{i})=m$ and moreover that the arising coupling be connected.

Since $|c_j(\boldsymbol{w})|\leq\sqrt{\varepsilon}\,T^{1/6}$ etc., the absolute value of (3.13) does not exceed

$$
\begin{aligned}
I = (\sqrt{\varepsilon}\,T^{1/6})^4\left(\frac{1}{\sqrt{V_n(T)}}\right)^4 \int & |D_T(l_1)D_T(l_2)D_T(l_3)D_T(l_4)| \\
& \times g_1(e)g_2(f)\cdots g_6(j)dedf\cdots dj ,
\end{aligned} \tag{3.14}
$$

where

$$
\begin{aligned}
& l_1 = e+f+g , \quad l_2 = -e+h+i , \quad l_3 = -f-h+j \\
& l_4 = -g-j-i , \quad e = \bar{\boldsymbol{e}}, \cdots, j=\bar{\boldsymbol{j}} , \\
& g_1 = f^{d(e)*}, \cdots, g_6 = f^{d(j)*} \qquad \text{(convolutions of } f).
\end{aligned}
$$

Depending on the composition of $d(\boldsymbol{e}), \cdots, d(\boldsymbol{j})$, there arise several cases. However, there is no essential change of technicality for different cases. So that we will restrict ourselves to the case that $d(\boldsymbol{e}), \cdots, d(\boldsymbol{j})>0$. $l_p (1\leq p\leq 4)$ are linearly dependent functions of 6 variables $e, f, \cdots, j$, actually

$$
\sum_{p=1}^{4} l_p \equiv 0 ,
$$

but any three of them are linearly independent. Choose three linear functions $\tilde{l}_4$, $\tilde{l}_5$, $\tilde{l}_6$ such that these together with $l_1$, $l_2$, $l_3$ form a linearly independent set.

To compute the integral in (3.14), make a linear transformation from

$e, \cdots, j$ to $x_1, \cdots, x_6$:

$$x_i = l_i \, (1 \leq i \leq 3) \,, \quad x_j = \tilde{l}_j \, (4 \leq j \leq 6) \,. \tag{3.15}$$

The inverse transformation is

$$e = e_0 + e_1 \,, \quad f = f_0 + f_1, \cdots, j = j_0 + j_1 \,, \tag{3.16}$$

where $e_0, \cdots, j_0$ are linear functions of $x_1, x_2, x_3$, whereas $e_1, \cdots, j_1$, are those of $x_4, x_5, x_6$. Write

$$\begin{pmatrix} e_1 \\ f_1 \\ \vdots \\ j_1 \end{pmatrix} = ||a_{ij}|| \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix}, \qquad 1 \leq i \leq 6 \,, \quad 4 \leq j \leq 6 \,, \tag{3.17}$$

and with no loss of generality assume that the square matrix $A = ||a_{i,j+3}||$, $1 \leq i$, $j \leq 3$ is non-singular. Using these transformations and (3.9)

$$\begin{aligned} &\int \prod_{p=1}^{4} |D_T(l_p)| g_p dedf \cdots dj \\ &\quad = |\det D \det A|^{-1} \int \prod_{j=1}^{3} |D_T(x_j) D_T(x_1+x_2+x_3)| \, dx_1 dx_2 dx_3 \\ &\qquad \times ||g_4||_\infty ||g_5||_\infty ||g_6||_\infty \int g_1(e_0+u) g_2(f_0+v) g_3(g_0+w) dudvdw \\ &\quad \leq |\det D \det A|^{-1} \prod_{p=4}^{6} ||g_p||_\infty \prod_{q=1}^{3} ||g_q||_1 ||\Psi^{(3)}||_1 T \end{aligned} \tag{3.18}$$

where
$$D = \partial(l_1, \cdots, \tilde{l}_6)/\partial(e, \cdots, j) \,.$$

Insert this into (3.14), then (3.12), (3.13) imply that

$$\sup_{0<\varepsilon\leq 1} |S(T)| \leq \text{const} \, (T^{1/6})^4 \left(\frac{1}{\sqrt{T}}\right)^4 T = \text{const} \, T^{-1/3} \,.$$

In general, by the same device as above one gets

$$\sup_{0<\varepsilon\leq 1} |S(T)| \leq \text{const} \, T^{1-p/3} \qquad (p \geq 4) \,.$$

This completes Step 2.

Proof of Theorem 1. Since $V(\bar{X}(T)) = 1$, $\{\text{dist}\, \bar{X}(T), T \geq 1\}$ is relatively compact. Therefore if we denote by $\boldsymbol{M}$ the set of limit points of $\{\text{dist}\, \bar{X}(T), T \geq 1\}$ as $T \to \infty$, $\boldsymbol{M}$ is non-empty. Let $L \in \boldsymbol{M}$, then there is a $D_0 \in \mathcal{D}_0$ such that

$$\lim_{\substack{T \to \infty \\ T \in D_0}} \text{dist}\, \bar{X}(T) = L \,.$$

By $V$, Section 2 we can find a subsequence $D_1$ of $D_0$, $D_1 \in \mathcal{D}_1$ such that

$$\lim_{\substack{T\to\infty \\ T\in D_1}} \operatorname{dist} \bar{X}(T) = L\,.$$

Define $\boldsymbol{N}$ to be the set of limit points of $\{\operatorname{dist} Y(T),\ T\in D_1\}$, as $T\to\infty$ on $D_1$. Since by (3.3) $\{\operatorname{dist} Y(T),\ T\in D_1\}$ is relatively compact, $\boldsymbol{N}$ is non-empty. Let $P\in\boldsymbol{N}$, then by $V$, Section 2 there exists a subsequence $D_2$ of $D_1$ such that $D_2\in\mathcal{D}_1$ and

$$\lim_{\substack{T\to\infty \\ T\in D_2}} \operatorname{dist} Y(T) = P\,. \tag{3.19}$$

(3.4) implies that

$$\lim_{\substack{T\to\infty \\ T\in D_2}} \operatorname{dist} Y^{\varepsilon}(T) = P \qquad \text{for every} \quad 0<\varepsilon\leq 1. \tag{3.20}$$

Suppose $Y=(\eta_1, \cdots, \eta_n)$ is a random variable with probability distribution $P$. Taking into account of relations between moments and cumulants, (3.5), (3.6) mean that each sequence $\{Y^{\varepsilon}(T),\ T\in D_2\}$, $0<\varepsilon\leq 1$ satisfies the conditions of IV, Section 2, while by (3.19), (3.20) its limit $P$ is independent of $\varepsilon$. Therefore combination of (3.5), (3.6) and IV gives us

$$\begin{aligned} &S_{k(1)\cdots k(n)}(\eta_1, \cdots, \eta_n) = 0 && \text{if} \quad k(1)+\cdots+k(n)\geq 4\,, \\ &|S_{k(1)\cdots k(n)}(\eta_1, \cdots, \eta_n)| \leq c\varepsilon^{3/2} && \text{if} \quad k(1)+\cdots+k(n) = 3\,. \end{aligned} \tag{3.21}$$

$0<\varepsilon\leq 1$ being arbitrary the last inequality means that

$$S_{k(1)\cdots k(n)}(\eta_1, \cdots, \eta_n) = 0 \qquad \text{if} \quad k(1)+\cdots+k(n) = 3\,. \tag{3.22}$$

Moreover, if $1\leq i\neq j\leq n$, (3.4) implies that

$$E(\eta_i^{\varepsilon}(T)\eta_j^{\varepsilon}(T)) = 0\,, \quad \lim_{\substack{T\to\infty \\ T\in D_2}} E(\eta_j^{\varepsilon}(T))^2 = b_j^2$$

where $b_j^2 = \lim_{\substack{T\to\infty \\ T\in D_2}} v_j(T)/V_n(T)$, whose existence is assured in V, Section 2. So that by the same reasoning as above

$$E(\eta_i\eta_j) = 0 \ \ (1\leq i\neq j\leq n)\,, \quad E(\eta_j^2) = b_j^2\,. \tag{3.23}$$

(3.21), (3.22), (3.23) imply that $P=N(0, B)$, the normal law with zero mean and covariance matrix $B=||\delta_{ij}b_j^2||$.

Write

$$\bar{X}(T) = \bar{S}_n(T)+\Delta\bar{S}_n(T)\,,$$

$$\bar{S}_n(T) = \frac{1}{\sqrt{V(T)}}\int_0^T S_n(t)dt = \{V_n(T)/V(T)\}^{1/2}\frac{1}{\sqrt{V_n(T)}}\int_0^T S_n(t)dt\,,$$

$$\Delta\bar{S}_n(T) = \frac{1}{\sqrt{V(T)}} \int_0^T R_n(t)dt\,,$$

and we are going to derive the conclusion of the theorem.

Observe that

$$\begin{aligned}&|E \exp iz\bar{X}(T) - \exp(-z^2/2)|\\ &\qquad \le |E \exp iz\bar{S}_n(T) - \exp(-z^2/2)| + |E \exp iz\bar{S}_n(T)[\exp iz\Delta\bar{S}_n(T) - 1]|\,,\end{aligned}$$

and the second term on the right does not exceed

$$|z|E|\Delta\bar{S}_n(T)| \le |z| \left(\frac{\Delta V_n(T)}{V(T)}\right)^{1/2}.$$

Then

$$\begin{aligned}&\lim_{T\in D_2, T\to\infty} |E \exp iz\bar{X}(T) - \exp(-z^2/2)|\\ &\qquad \le \lim_{\substack{T\to\infty\\ T\in D_0}} |E \exp iz\bar{S}_n(T) - \exp(-z^2/2)| + |z| \lim_{\substack{T\to\infty\\ T\in D_2}} \sqrt{\frac{\Delta V_n(T)}{V(T)}} \qquad (3.24)\\ &\qquad = |\exp(-\rho_n z^2/2) - \exp(-z^2/2)| + |z|(1-\rho_n)^{1/2}\,,\end{aligned}$$

where we have used the fact that $\bar{S}_n(T) \to \sqrt{\rho_n}\ (\eta_1 + \cdots + \eta_n)$ in distribution by (3.19) as $T\to\infty$ on $D_2$, and $\sqrt{\rho_n}\ (\eta_1 + \cdots + \eta_n) \in N(0, \rho_n)$ since $\sum_{j=1}^n b_j^2 = 1$. The left-hand side of (3.24) is the discrepancy between the characteristic function of $L$ and $\exp(-z^2/2)$, while on making $n\to\infty$, the right-hand side tends to zero. So that $L = N(0, 1)$, or $\boldsymbol{M}$ consists of a single element $N(0, 1)$. Since $\{\text{dist}\,\bar{X}(T), T \ge 1\}$ is compact, this means that the conclusion (1.3) is true. This completes the proof of Theorem 1.

## 4. Random fields

We consider a direct extension of Theorem 1 to random fields. Let $X(x)$, $x \in \boldsymbol{R}^d$ be a strictly stationary random field subordinate to a strictly stationary real Gaussian random field

$$\xi(x) = \int \exp i\lambda\cdot x d\beta\,, \quad x, \lambda \in \boldsymbol{R}^d \qquad (4.1)$$

with $E\xi(x) = 0$, complex spectral random measure $d\beta$ and spectral measure $d\sigma(\lambda) = E|d\beta(\lambda)|^2$, which is absolutely continuous with respect to Lebesque measure on $R^d$, $d\sigma(\lambda) = f(\lambda)d\lambda$. Then similarly to (1.2), $X(x)$ is represented by the Ito-Wiener expansion

$$X(x) = \sum_{k=1}^{\infty} X_k(x) \qquad (4.2)$$

$$X_k(x) = \int c_k(\boldsymbol{\lambda}) e_k(\boldsymbol{\lambda}, x) d^k\beta ,$$

where $\boldsymbol{\lambda} = (\lambda^1, \cdots, \lambda^k)$, $d^k\beta = d\beta(\lambda^1) \cdots d\beta(\lambda^k)$, $\lambda^j \in R^d$ $(1 \le j \le k)$, $e_k(\boldsymbol{\lambda}, x) = \exp i\bar{\boldsymbol{\lambda}} \cdot x$, $\bar{\boldsymbol{\lambda}} \cdot x$ is the inner product of $\bar{\boldsymbol{\lambda}} = \lambda^1 + \cdots + \lambda^k$ with $x$, and each $c_k(\boldsymbol{\lambda})$ $(1 \le k < \infty)$ is symmetric in $\lambda^1, \cdots, \lambda^k$, subject to the conditions that $\overline{c_k(\boldsymbol{\lambda})} = c_k(-\boldsymbol{\lambda})$,

$$||c_k||_2^2 = \int |c_k(\boldsymbol{\lambda})|^2 d^k\sigma < \infty , \quad d^k\sigma = d\sigma(\lambda^1) \cdots d\sigma(\lambda^k) .$$

Let $a, b \in \boldsymbol{R}^d$, $a = (a_1, \cdots, a_d)$ etc., and 0, 1 respectively be the zero $d$-vector and the $d$-vector $x = (x_1, \cdots, x_d)$ with $x_i = 1$ $(1 \le i \le d)$. Write $a \le b$, $a \to \infty$, $a \to 0$ $(+0)$ respectively to abbreviate the relations, $a_i \le b_i$, $a_i \to \infty$, $a_i \to 0\,(+0)$ $(1 \le i \le d)$. For $\delta > 0$, $a \in \boldsymbol{R}^d$ we use the abbreviation $|a| \le \delta$ $(|a| \ge \delta)$ if $|a_i| \le \delta$ $(|a_i| \ge \delta)$ $(1 \le i \le d)$. For example $0 \le |h| \le \delta$ $(h = (h_1, h_2))$ means $0 \le |h_1|, |h_2| \le \delta$.

As in Section 1 define

$$\Phi(|c_k|^2; h) = \int_{0 \le \lambda \le h} \varphi(|c_k|^2; \lambda) d\lambda , \qquad 0 \le h \in \boldsymbol{R}^d ,$$

$$\varphi(|c_k|^2; \lambda) = k! \int |c_k(\lambda - \bar{\lambda}', \lambda^1, \cdots, \lambda^{k-1})|^2 f(\lambda - \bar{\lambda}') \times f(\lambda^1) \cdots f(\lambda^{k-1}) d\lambda^1 \cdots d\lambda^{k-1} , \qquad \lambda \in R^d ,$$

$$\lambda' = (\lambda^1, \cdots, \lambda^{k-1}) , \quad \bar{\lambda}' = \lambda^1 + \cdots + \lambda^{k-1} \in R^d .$$

As a direct extension of Theorem 1 we have

**Theorem 2.** *Suppose that $X(x)$ in* (4.2) *satisfies the following conditions* (i)–(iv).

( i ) *$f(\lambda)$ is bounded.*

(ii) $V(T) = V(\int_{0 \le x \le T} X(x) dx) \asymp |Q(T)|$, *as* $T \to \infty$,

*where $T \in \boldsymbol{R}^d$, $T > 0$, and $|Q(T)|$ is the volume of $Q(T) = \{x \in R^d : 0 \le x \le T\}$.*

(iii) $$\lim_{n \to \infty} \varlimsup_{|Q(T)| \to \infty} \frac{1}{|Q(T)|} \Delta V_n(T) = 0 ,$$

*where*

$$\Delta V_n(T) = V(\int_{0 \le x \le T} R_n(x) dx) , \quad R_n(x) = X(x) - S_n(x) , \quad S_n(x) = \sum_{k=1}^{n-1} X_k(x) .$$

(iv) *For every $\varepsilon > 0$, $k \ge 1$*

$$\lim_{h \to +0} \Phi(\delta[|c_k|^2]; h) / |Q(h)| = 0 ,$$

*where*

$$\delta[|c_k|]^2=|c_k|^2-|c_k|^2\wedge(\varepsilon|Q(T)|^{1/3}),\quad h=(h_1,\cdots,h_d),\quad h_i=1/T_i \quad (1\leq i\leq d).$$

*Then,*

$$\operatorname{dist}\bar{X}(T)\to N(0,1)\qquad (\textit{weakly}),\quad \textit{as } T\to\infty,$$

*where*

$$\bar{X}(T)=\frac{1}{\sqrt{V(T)}}\int_{0\leq x\leq T}X(x)dx.$$

In the same spirit as in Section 3 we clarify first relations (propositions VI–VIII) between the growth of $V(T)$ and behavior of spectral density of $X(x)$. The proof of Theorem 2 is essentially similar to that of Theorem 1. Stress is made on the features specific to the dimensionality $d>1$, but to avoid notational complexity, we restrict ourselves to the case $d=2$, and put $\boldsymbol{E}=\boldsymbol{R}^2$. So we are dealing with a random field $X(x)$ defined by (4.2) with $x\in E$, $\boldsymbol{\lambda}$ an $\boldsymbol{E}^k$-vector, i.e. a vector with $k$ components from $\boldsymbol{E}$.

VI. *Let $\{X(x), x\in\boldsymbol{E}\}$ be a square-integrable strictly stationary real random field with mean zero and spectral density $\varphi(\lambda)=\varphi(\lambda_1,\lambda_2)$, $\lambda=(\lambda_1,\lambda_2)\in\boldsymbol{E}$. Then*

$$\varliminf_{h\to\infty}\frac{\Phi(h)}{|Q(h)|}\leq c_1\varliminf_{T\to\infty}\frac{v(T)}{|Q(T)|},\tag{4.3}$$

$$\varlimsup_{h\to\infty}\frac{\Phi(h)}{|Q(h)|}\leq c_2\varlimsup_{T\to\infty}\frac{v(T)}{|Q(T)|},\tag{4.4}$$

*with numerical constants $c_1$, $c_2>0$, where*

$$v(T)=V(\int_{0\leq x\leq T}X(x)dx),\quad \Phi(h)=\int_{-h\leq\lambda\leq h}\varphi(\lambda)d\lambda,\qquad h=(h_1,h_2)>0.$$

Proof. Since

$$v(T)=\int\boldsymbol{D}_T^2(\lambda)\varphi(\lambda)d\lambda,\quad \boldsymbol{D}_T^2(\lambda)=D_{T_1}^2(\lambda_1)D_{T_2}^2(\lambda_2),\qquad T=(T_1,T_2),$$

by (2.10), we obtain

$$\left(\frac{2}{\pi}\right)^4\frac{\Phi(h)}{|Q(h)|}\leq\frac{v(T)}{|Q(T)|},\quad h=(h_1,h_2),\quad h_i=1/T_i\qquad (i=1,2),$$

which implies (4.3), (4.4).

VII. *Let $\{X(x), X\in\boldsymbol{E}\}$ be as in* VI. *Then*

(4.5) $$\overline{\lim_{T\to\infty}}\frac{v(T)}{|Q(T)|}<\infty$$

*if and only if*

(4.6) $$\overline{\lim_{h\downarrow 0}}\frac{\Phi(h)}{|Q(h)|}<\infty\,,\quad \overline{\lim_{h\downarrow 0}}\frac{1}{h}\Psi(h)<\infty\,,$$

*where*

$$\Psi(h)=\int_{|\lambda_1|\leq h}d\lambda_1\int_{-\infty}^{\infty}\frac{\varphi(\lambda)}{1+\lambda_2^2}d\lambda_2\,,\quad \lambda=(\lambda_1,\lambda_2)\,.$$

*In this case*

(4.7) $$\overline{\lim_{T\to\infty}}\frac{v(T)}{|Q(T)|}\leq c_1\overline{\lim_{h\to+0}}\frac{\Phi(h)}{|Q(h)|}\,,$$

*with a numerical constant* $c_1>0$.

Proof. Suppose that (4.5) is true. Then there is a constant $c_2>0$ such that

$$\int \boldsymbol{D}_T^2(\lambda)\varphi(\lambda)d\lambda\leq c_2T_1T_2 \qquad (T\geq 1)\,.$$

Multiply $e^{-T_2}$ and integrate over $1\leq T_2<\infty$ on the both sides of the last inequality and use the fact that

$$\int_1^\infty D_{T_2}^2(x)e^{-T_2}dT_2=2e^{-1}\left\{1-\frac{\cos x-x\sin x}{1+x^2}\right\}\frac{1}{x^2}\asymp\frac{1}{1+x^2}$$

on $0\leq x<\infty$. Then with a constant $c_3>0$

$$\int_{-\infty}^{\infty}D_{T_1}^2(\lambda_1)d\lambda_1\int_{-\infty}^{\infty}\frac{\varphi(\lambda)}{1+\lambda_2^2}d\lambda_2<c_3T_1\,,\quad T_1\geq 1\,.$$

From this, appealing to (2.8), we obtain the second inequality in (4.6), whereas the first one is obvious by the proof of VI.

Suppose that (4.6) holds true, and write

$$\frac{1}{|Q(T)|}v(T)=\frac{1}{|Q(T)|}\Big(\int_{0\leq|\lambda|\leq\delta}+\int_{\substack{|\lambda_1|\leq\delta\\|\lambda_2|>\delta}}+\int_{\substack{|\lambda_1|>\delta\\|\lambda_2|\leq\delta}}+\int_{|\lambda|>\delta}\Big)\boldsymbol{D}_T^2(\lambda)\varphi d\lambda\,.$$

Denote by $I_1$, $I_2$, $I_3$ and $I_4$ respectively the first, second, third, and forth term of the last expression, and we will show that $I_2$, $I_3$, $I_4$ tend to zero, as $T\to\infty$.

First,

(4.8) $$I_k\leq 16\frac{h_1h_2}{\delta^4}\int\varphi d\lambda\to 0\,,\qquad \text{as}\quad T\to\infty\,.$$

Second, if we write

$$I_2=J_1+J_2\,,$$
$$J_1=\frac{1}{Q(T)}\int_{\substack{0\le|\lambda_1|\le h_1\\|\lambda_2|>\delta}}\boldsymbol{D}_T^2(\lambda)\varphi d\lambda\,,\quad J_2=\frac{1}{|Q(T)|}\int_{\substack{h_1\le|\lambda_1|\le\delta\\|\lambda_2|>\delta}}\boldsymbol{D}_T^2(\lambda)\varphi d\lambda\,,$$

then

$$J_1\le c_4\frac{h_2}{h_1}\int_{0\le|\lambda_1|\le h_1}d\lambda_1\int_{-\infty}^{\infty}\frac{\varphi}{1+\lambda_2^2}d\lambda_2\,, \tag{4.9}$$

and by partial integration with respect to $\lambda_1$

$$\begin{aligned}J_2&\le c_5h_1h_2\int_{h_1\le|\lambda_1|\le\delta}\lambda_1^{-2}d\lambda_1\int_{-\infty}^{\infty}\varphi/(1+\lambda_2^2)d\lambda_2\\&\le c_5h_2\{h_1\Psi(\delta)/\delta^2+2h_1\int_{h_1}^{\delta}\frac{\Psi(x)}{x^3}dx\}\le c_6h_2\{h_1\Psi(\delta)/\delta^2+2\sup_{0<x\le\delta}\Psi(x)/x\}\,,\end{aligned}\tag{4.10}$$

whence by (4.6), (4.9), (4.10) and by symmetry

$$\lim_{T\to\infty}I_2=\lim_{T\to\infty}I_3=0\,. \tag{4.11}$$

Third, turning to $I_1$, write

$$I_1=\frac{1}{|Q(T)|}\Bigg(\int_{\substack{0\le|\lambda_1|\le h_1\\0\le|\lambda_2|\le h_2}}+\int_{\substack{0\le|\lambda_1|\le h_1\\h_2\le|\lambda_2|\le\delta}}+\int_{\substack{0\le|\lambda_2|\le h_2\\h_1\le|\lambda_1|\le\delta}}+\int_{\substack{h_1\le|\lambda_1|\le\delta\\h_2\le|\lambda_2|\le\delta}}\Bigg)\boldsymbol{D}_T^2(\lambda)\varphi d\lambda\,,$$

and denote by $K_1$, $K_2$, $K_3$, and $K_4$ respectively the first, second, third, and forth term on the right-hand side. Obviously

$$\overline{\lim_{T\to\infty}}K_1\le\overline{\lim_{h\to+0}}\frac{\Phi(h)}{|Q(h)|}\,. \tag{4.12}$$

By integration by parts with respect to $\lambda_2$

$$K_2\le4\frac{h_2}{h_1}\int_{0\le|\lambda_1|\le h_1}d\lambda_1\Big\{\delta^{-2}\int_{|\lambda_2|\le\delta}\varphi(\lambda)d\lambda_2+2\int_{h_2}^{\delta}y^{-3}dy\int_{|\lambda_2|\le y}\varphi(\lambda)d\lambda_2\Big\}\,.$$

This implies

$$\begin{aligned}K_2&\le4\frac{h_2}{\delta}\frac{\Phi((h_1,\delta))}{h_1\delta}+8\frac{h_2}{h_1}\int_{h_2\le y\le\delta}\frac{\Phi((h_1,y))}{y^3}dy\le4\frac{h_2}{\delta}\frac{\Phi((h_1,\delta))}{h_1\delta}\\&\quad+8\sup_{0<|h|\le\delta}\frac{\Phi(h)}{|Q(h)|}h_2\int_{h_2}^{\infty}\frac{dy}{y^2}\le4\Big(\frac{h_2}{\delta}+2\Big)\sup_{0<|h|\le\delta}\frac{\Phi(h)}{|Q(h)|}\,,\\&\Phi((x,y))=\int_{\substack{|\lambda_1|\le x\\|\lambda_2|\le y}}\varphi(\lambda)d\lambda\,,\qquad x,y>0\,,\end{aligned}\tag{4.13}$$

whence by symmetry

$$\overline{\lim_{T\to\infty}} K_2 = \overline{\lim_{T\to\infty}} K_3 \leq 8 \sup_{0<|h|\leq\delta} \frac{\Phi(h)}{|Q(h)|}$$

for every $\delta>0$. This implies that

$$\overline{\lim_{\delta\downarrow 0}}\, \overline{\lim_{T\to\infty}} K_j \leq 2 \overline{\lim_{h\to+0}} \frac{\Phi(h)}{|Q(h)|} \qquad (j=2, 3)\,. \tag{4.14}$$

Finally

$$K_4 \leq \frac{4^2}{|Q(T)|} \int_{h_1\leq\lambda_1\leq\delta} \lambda_1^{-2} d\lambda_1 \int_{h_2\leq|\lambda_2|\leq\delta} \varphi\lambda_2^{-2} d\lambda_2\,.$$

Apply integration by parts to the interior and exterior integral of the last expression and we have

$$\begin{aligned} K_4 \leq \frac{4^2}{|Q(T)|} \Big\{ & \frac{1}{\delta^4} \int_{|\lambda|\leq\delta} \varphi d\lambda + 2\int_{h_1}^{\delta} \frac{\Phi((x, \delta))}{x^3\delta^2} dx \\ & + 2\int_{h_2}^{\delta} \frac{\Phi((\delta, y))}{\delta^2 y^3} dy + 4\int_{h_1}^{\delta}\int_{h_2}^{\delta} \frac{\Phi((x, y))}{x^3 y^3} dx dy \Big\}\,. \end{aligned} \tag{4.15}$$

Therefore

$$K_4 \leq 4^2 \frac{|Q(h)|}{\delta^4} \Phi((\delta, \delta)) + \frac{32}{\delta}(h_1+h_2) \sup_{0<|h|\leq\delta} \frac{\Phi(h)}{|Q(h)|} + 64 \sup_{0<|h|\leq\delta} \frac{\Phi(h)}{|Q(h)|}\,, \tag{4.16}$$

whence

$$\overline{\lim_{\delta\downarrow 0}}\, \overline{\lim_{T\to\infty}} K_4 \leq 64 \overline{\lim_{h\to+0}} \frac{\Phi(h)}{|Q(h)|}\,. \tag{4.17}$$

Combination of (4.8), (4.11), (4.12), (4.14) and (4.17) proves that (4.6) implies (4.5), (4.7). This completes the proof of VII.

Define $\theta_k(\boldsymbol{\lambda})$ after the definition of $\theta_k(\lambda)$, put

$$\begin{aligned} c_k^{\varepsilon}(\boldsymbol{\lambda}) &= (|c_k(\boldsymbol{\lambda})| \wedge \eta)\theta_k(\boldsymbol{\lambda})\,, \quad \eta = \sqrt{\varepsilon}\,|Q(T)|^{1/6}\,, \\ c_h(\boldsymbol{\lambda}) &= c_k^{\varepsilon}(\boldsymbol{\lambda}) + \Delta c_k(\boldsymbol{\lambda}) \qquad (k\geq 1)\,, \end{aligned}$$

and define

$$\begin{aligned} \Phi(|\Delta c_k|^2; (x, y)) &= \int_{\substack{0\leq|\lambda_1|\leq x \\ 0\leq|\lambda_2|\leq y}} \varphi(|\Delta c_k|^2; \lambda) d\lambda\,, \qquad (0\leq x, y)\,, \\ \Phi(\varepsilon, h) &= \Phi(|\Delta c_k|^2; (h_1, h_2))\,, \qquad h=(h_1, h_2)>0\,. \end{aligned} \tag{4.18}$$

Then, in the same way as in *I*

(4.19) $\Phi(\varepsilon, h)=o(|Q(h)|)$ if and only if $\Phi(\delta[|c_k|^2]; h)=o(|Q(h)|)$, as $h\downarrow 0$, $h=(h_1, h_2)\in E$, $h_i=1/T_i$ $(1\leq i\leq 2)$ provided that $\Phi(|c_k|^2; h)=O(|Q(h)|)$, as $h\to+0$.

It is easy to see that

$$(4.20)\quad \text{(i)}\quad \Phi(0, h) = \Phi(|c_k|^2; h) \quad \text{(ii)}\quad \Phi(|\Delta c_k|^2; h) \leq \Phi(0, h), \qquad 0 \leq h \in \boldsymbol{E}.$$

Write

$$X_k(x) = X_k^{\varepsilon}(x) + \Delta X_k(x), \qquad 1 \leq k < \infty,$$
$$X_k^{\varepsilon}(x) = \int c_k^{\varepsilon}(\boldsymbol{\lambda}) e_k(\boldsymbol{\lambda}, x) d^k\beta, \quad \Delta X_k(x) = \int \Delta c_k(\boldsymbol{\lambda}) e_k(\boldsymbol{\lambda}, x) d^k\beta,$$

Then we have

VIII. *Under the assumptions of Theorem* 2

$$\lim_{T\to\infty} V\left(\frac{1}{\sqrt{|Q(T)|}} \int_{0\leq x\leq T} \Delta X_k(x) dx\right) = 0.$$

Proof. The proof is a modification of that of III. Let us write in the abbreviated vector notations introduced in the beginning of Section 4

$$(4.21)\quad \begin{aligned} &V\left(\frac{1}{\sqrt{|Q(T)|}} \int_{0\leq x\leq T} \Delta X_k(x) dx\right) \\ &= \frac{1}{|Q(T)|}\left( \int_{0\leq|\lambda|\leq\delta} + \int_{\substack{|\lambda_1|\leq\delta \\ |\lambda_2|>\delta}} + \int_{\substack{|\lambda_1|>\delta \\ |\lambda_2|\leq\delta}} + \int_{|\lambda|>\delta}\right) \boldsymbol{D}_T^2(\lambda)\varphi(|\Delta c_k|^2; \lambda) d\lambda, \end{aligned}$$

and notice that $\varphi(|\Delta c_k|^2; \lambda) \leq \varphi(|c_k|^2; \lambda)$, and by VII, (4.6) holds true with $\varphi(\lambda)$, $\Phi(h)$ respectively replaced by $\varphi(|c_k|^2; \lambda)$, $\Phi(|c_k|^2; h)$. Then, if we denote by $I_1^*$, $I_2^*$, $I_3^*$, and $I_4^*$ respectively the first, second, third, and forth term on the right-hand member of (4.21), the passage from (4.8) to (4.11) implies that

$$(4.22)\qquad \lim_{T\to\infty} (I_2^* + I_3^* + I_4^*) = 0.$$

To show that

$$(4.23)\qquad \lim_{T\to\infty} I_1^* = 0$$

represent $I_1^*$ as the sum of integrals $K_j^*$ $(1 \leq j \leq 4)$ which are of the same type as $K_j$ in (4.12)–(4.17) except that this time the integrand is $\boldsymbol{D}_T^2(\lambda)\varphi(|\Delta c_k|^2; \lambda)$ instead of $\boldsymbol{D}_T^2(\lambda)\varphi(\lambda)$.

First, by (4.12), (4.19)

$$(4.24)\qquad \overline{\lim_{T\to\infty}}\, K_1^* \leq \lim_{h\to+0} \frac{\Phi(\varepsilon, h)}{|Q(h)|} = 0.$$

Second, consulting (4.13) one obtains

$$K_2^* \leq 4\left(\frac{h_2}{\delta}+2\right) \sup_{0<|h|\leq\delta} \frac{\Phi(\varepsilon, h)}{|Q(h)|},$$

whence by (4.19)

$$\lim_{\delta\downarrow 0} \overline{\lim_{T\to\infty}} K_2^* = \lim_{\delta\downarrow 0} \overline{\lim_{T\to\infty}} K_3^* = 0. \tag{4.25}$$

Estimate $K_4^*$ after (4.15), (4.16) to have

$$K_4^* \leq 4(4L_1+8L_2+16L_3)$$

where

$$L_1 = \frac{|Q(h)|}{\delta^4}\Phi(0, (\delta, \delta)) \to 0,$$

$$L_2 = \frac{1}{\delta}(h_1+h_2) \sup_{0<|h|\leq\delta} \frac{\Phi(0, h)}{|Q(h)|} \to 0, \qquad \text{as} \quad T\to\infty, \tag{4.26}$$

and

$$L_3 = \frac{1}{|Q(T)|} \int_{h_1}^{\delta}\int_{h_2}^{\delta} \Phi(|\Delta c_k|^2; (x, y))/x^3y^3dxdy.$$

We are sufficed to show

$$\lim_{T\to\infty} L_3 = 0. \tag{4.27}$$

For this purpose rewrite $L_3$ in the form

$$L_3 = \frac{1}{|Q(T)|}\left(\int_{h_1}^{rh_1}\int_{h_2}^{rh_2} + \int_{h_1}^{rh_1}\int_{rh_2}^{\delta} + \int_{rh_1}^{\delta}\int_{h_2}^{rh_2} + \int_{rh_1}^{\delta}\int_{rh_2}^{\delta}\right) \times\Phi(|\Delta c_k|^2; (x, y))/x^3y^3dxdy, \tag{4.28}$$

with parameter $r>1$. Denote by $M_i$, $1\leq i\leq 4$, the $i$th term on the right-hand member of (4.28). By the monotonicity of $\varphi$, $\varphi(|c_k|^2; \lambda)\leq\varphi(|d_k|^2; \lambda)$ provided $|c_k|^2\leq|d_k|^2$, and the equality $|\Delta c_k|=|c_k|-|c_k|\sqrt{\varepsilon}(1/h_1h_2)^{1/6}$ we know that for $0\leq x\leq rh_1$, $0\leq y\leq rh_2$,

$$\Phi(|\Delta c_k|^2; (x, y)) = \int_{\substack{|\lambda_1|\leq x\\|\lambda_2|\leq y}} \varphi(|\Delta c_k|^2; \lambda)d\lambda \leq \Phi(\varepsilon, h'), \qquad h' = rh,$$

whence for $h_1\leq x\leq rh_1$, $h_2\leq y\leq rh_2$

$$\frac{\Phi(|\Delta c_k|^2; (x, y))}{xy} \leq r^2\frac{\Phi(\varepsilon, h')}{|Q(h')|}$$

This implies

$$M_1 \leq r^2 \frac{\Phi(\varepsilon, h')}{|Q(h')|} |Q(h)| \int_{h_1}^{\infty} \int_{h_2}^{\infty} (xy)^{-2} dxdy = r^2 \frac{\Phi(\varepsilon, h')}{|Q(h')|}. \tag{4.29}$$

On the other hand, since the common integrand of $M_2$, $M_3$, $M_4$ satisfies, on the working regions of the variables, the inequality

$$\frac{\Phi(|\Delta c_k|^2; (x, y))}{x^3 y^3} \leq \sup_{0<|h|\leq\delta} \frac{\Phi(|c_k|^2; h)}{|Q(h)|} \frac{1}{(xy)^2},$$

one gets

$$\begin{aligned} M_2 + M_3 &\leq \sup_{0<|h|\leq\delta} \frac{\Phi(|c_k|^2; h)}{|Q(h)|} |Q(h)| \left( \int_{h_1}^{rh_1} \int_{rh_2}^{\infty} + \int_{rh_1}^{\infty} \int_{h_2}^{rh_2} \right) \frac{dxdy}{(xy)^2} \\ &\leq \frac{2}{r} \sup_{0<|h|\leq\delta} \frac{\Phi(|c_k|^2; h)}{|Q(h)|} \end{aligned} \tag{4.30}$$

and similarly

$$M_4 \leq \frac{1}{r^2} \sup_{0<|h|\leq\delta} \frac{\Phi(|c_k|^2; h)}{|Q(h)|}. \tag{4.31}$$

Keeping (4.19), (4.20) in mind, successively make $T \to \infty$, $r \to \infty$ in (4.29)–(4.31). Then one concludes that

$$\lim_{T\to\infty} L_3 = 0, \tag{4.32}$$

as was requested. This completes the proof.

Proof of Theorem 2. In the notations of Theorem 2 put

$$V_n(T) = V(\int_{0\leq x\leq T} S_n(x)dx),$$

$$v_k(T) = V(\int_{0\leq x\leq T} X_k(x)dx), \qquad 1 \leq T \in \boldsymbol{E}, \quad n, k \geq 1.$$

Under the assumptions of Theorem 2, $V_n(T)$, $v_k(T)$ behave similarly to the corresponding quantities in Section 3. Thus, there exists an $n_1$ such that $V_n(T) \asymp |Q(T)|$ for $T \geq 1$, whenever $n \geq n_1$. Define

$$\eta_k = \frac{1}{\sqrt{V_n(T)}} \int_{0\leq x\leq T} X_k(x)dx,$$

$$\eta_k^\varepsilon(T) = \frac{1}{\sqrt{V_n(T)}} \int_{0\leq x\leq T} X_k^\varepsilon(x)dx, \ \Delta\eta_k(T) = \frac{1}{\sqrt{V_n(T)}} \int_{0\leq x\leq T} \Delta X_k(x)dx,$$

$$n \geq n_1, \quad k \geq 1.$$

Then, in view of VIII, for the proof of the theorem, we are sufficed to show that for all $n \geq n_1$

$$\lim_{T\to\infty}\sup_{0<\varepsilon\le 1}|S_{k(1)\cdots k(n)}(\eta_1^\varepsilon(T),\cdots,\eta_n^\varepsilon(T))|=0\,, \quad \text{if} \quad k(1)+\cdots+k(n)\ge 4\,, \tag{4.33}$$

$$\sup_{T\ge 1}|S_{k(1)\cdots k(n)}(\eta_1^\varepsilon(T),\cdots,\eta_n^\varepsilon(T))|\le \text{const}\,\varepsilon^{3/2}\,, \quad \text{if} \quad k(1)+\cdots+k(n)=3\,. \tag{4.34}$$

Before going to derive these observe that

$$\eta_k^\varepsilon(T)=\frac{1}{\sqrt{V_n(T)}}\int_{0\le x\le T}dx\int c_k^\varepsilon(\boldsymbol{\lambda})e_k(\boldsymbol{\lambda},x)d^k\beta$$
$$=\frac{1}{\sqrt{V_n(T)}}\int c_k^\varepsilon(\boldsymbol{\lambda})\boldsymbol{D}_T(\bar{\boldsymbol{\lambda}})d^k\beta\,,$$

where

$$\boldsymbol{D}_T(\bar{\boldsymbol{\lambda}})=D_{T_1}((\bar{\boldsymbol{\lambda}})_1)D_{T_2}((\bar{\boldsymbol{\lambda}})_2)\,,$$

and $(\bar{\boldsymbol{\lambda}})_i$ $(i=1,2)$ is the $i$th component of $\bar{\boldsymbol{\lambda}}\in\boldsymbol{R}^2$.

The same computation principles as before ((2.1), (2.2)) in terms of kernels apply as well to the moments and cumulants of multiple integrals with respect to $d\beta$. Consider, for example, $S_{k(1)\cdots k(n)}(\eta_1^\varepsilon(T),\cdots,\eta_n^\varepsilon(T))$ with $k(1)+\cdots+k(n)=4$. In the same way as in (3.12), it is a sum of integrals with connected kernels of the form (c.f. (3.13))

$$\left(\frac{1}{\sqrt{V_n(T)}}\right)^4\int c_j^\varepsilon(\boldsymbol{w})c_k^\varepsilon(\boldsymbol{x})c_l^\varepsilon(\boldsymbol{y})c_m^\varepsilon(\boldsymbol{z})\prod_{p=1}^4\boldsymbol{D}_T(l_p)d^s\sigma\,, \tag{4.35}$$
$$s=(j+k+l+m)/2\,,\quad 1\le j,k,l,m\le n\,,$$

$$\boldsymbol{w}=(\boldsymbol{e},\boldsymbol{f},\boldsymbol{g}),\quad \boldsymbol{x}=(-\boldsymbol{e},\boldsymbol{h},\boldsymbol{i})\,,\quad \boldsymbol{y}=(-\boldsymbol{f},-\boldsymbol{h},\boldsymbol{j})\,,\quad \boldsymbol{z}=(-\boldsymbol{g},-\boldsymbol{j},-\boldsymbol{i})$$
$$l_1=\bar{\boldsymbol{e}}+\bar{\boldsymbol{f}}+\bar{\boldsymbol{g}},\ \cdots,\ l_4=-\bar{\boldsymbol{g}}-\bar{\boldsymbol{j}}-\bar{\boldsymbol{i}}\,,$$

where $\boldsymbol{w}$ is an $\boldsymbol{E}^j$-vector consisting of an $\boldsymbol{E}^{d(\boldsymbol{e})}$-vector $\boldsymbol{e}$, $\boldsymbol{E}^{d(\boldsymbol{f})}$-vector $\boldsymbol{f}$, $E^{d(\boldsymbol{g})}$-vector $\boldsymbol{g}$ as components with respective dimensions $d(\boldsymbol{e})$, $d(\boldsymbol{f})$, $d(\boldsymbol{g})$, $\bar{\boldsymbol{e}}$ etc. are the sums of the component vectors of $\boldsymbol{e}$ etc., similarly for $\boldsymbol{x},\cdots,\boldsymbol{z}$, so that $l_p$ $(1\le p\le 4)$ are $\boldsymbol{E}$-vectors. As in (3.13), $0\le d(\boldsymbol{e}),\cdots,d(\boldsymbol{j})$, $d(\boldsymbol{e})+\cdots+d(\boldsymbol{g})=j,\cdots,d(\boldsymbol{g})+\cdots+d(\boldsymbol{i})=m$, and moreover $d(\boldsymbol{e}),\cdots,d(\boldsymbol{j})$ must be so chosen that the arising kernel be connected.

Since $|c_j^\varepsilon(\boldsymbol{w})|\le\sqrt{\varepsilon}\,|Q(T)|^{1/6}$ etc., the absolute value of (4.35) does not exceed

$$I=(\sqrt{\varepsilon}\,|Q(T)|^{1/6})^4\left(\frac{1}{\sqrt{V_n(T)}}\right)^4|\prod_{p=1}^4\boldsymbol{D}_T(l_p)|\,g_1(e)\cdots g_6(j)de\cdots dj\,, \tag{4.36}$$

where

$$l_1=e+f+g,\ \cdots,\ l_4=-g-j-i\,,$$

$$e=\bar{e},\ \cdots,j=\bar{j},\ g_1=f^{d(e)*},\cdots,g_6=f^{d(j)*},$$

and $de$ etc. are Lebesgue measure on $\boldsymbol{E}$. Similarly as in (3.14) we have thus 6 base $\boldsymbol{E}$-vectors $e,\cdots,j$, $\boldsymbol{E}$-vectors $l_1,\cdots,l_4$ as their linear combinations which satisfy $l_1+\cdots+l_4=0$, and bounded $L^1$-functions on $\boldsymbol{E}$, $g_p$ $(1\leq p\leq 6)$. There are several varieties of permissible compositions for $d(\boldsymbol{e}),\cdots,d(\boldsymbol{j})$. However, by the same reason as in (3.13) we confine ourselves to the case $d(\boldsymbol{e}),\cdots,d(\boldsymbol{j})>0$ and proceed along the same line as before. Choose linear transformations $\tilde{l}_4$, $\tilde{l}_5$, $\tilde{l}_6$ of the base $\boldsymbol{E}$-vectors, in order that these together with $l_1$, $l_2$, $l_3$ form a linearly independent set. By means of these 6 linear functions, make a linear transformation from $\boldsymbol{E}^6$ onto itself, and consider its inverse, with the same representation as in (3.15), (3.16). Obviously this transformation resolves into a linear transformation $\tilde{D}$ from $\boldsymbol{R}^{12}$ onto itself. With no loss of generality we assume that $e_1,\cdots,j_1$ satisfy the same relation as (3.17) with non-singular $A$, which determines a linear map from $\boldsymbol{E}^3$ onto itself, or equivalently a linear map $\tilde{A}$ from $R^6$ onto itself. Then, corresponding to (3.18), this time we have

$$\begin{aligned}
&\int_{E^6}\Big|\prod_{p=1}^{4}\boldsymbol{D}_T(l_p)\Big|g_p de\cdots dj\\
&\leq c_1\prod_{j=4}^{6}\|g_j\|_\infty\int_{E^3}\prod_{j=1}^{3}|\boldsymbol{D}_T(x_j)|\,|\boldsymbol{D}_T(x_1+x_2+x_3)|\,dx_1dx_2dx_3\\
&\qquad\times\int_{E^3}g_1(e_0+u)g_2(f_0+v)g_3(g_0+w)dudvdw\\
&=c_2\int_{R^3}\prod_{j=1}^{3}|D_{T_1}(x_{j1})|\,|D_{T_1}(x_{11}+x_{21}+x_{31})|\,dx_{11}dx_{21}dx_{31}\\
&\qquad\times\int_{R^3}\prod_{j=1}^{3}|D_{T_2}(x_{j2})|\,|D_{T_2}(x_{12}+x_{22}+x_{32})|\,dx_{12}dx_{22}dx_{32}\\
&=c_3\|\Psi^{(3)}\|_1^2\,|Q(T)|,
\end{aligned}\tag{4.37}$$

where we have used the notational convention $x_j=(x_{j1},x_{j2})\in E$ $(1\leq j\leq 3)$, and $c_j$ $(1\leq j\leq 3)$ depend only on $\det\tilde{D}$, $\det\tilde{A}$, $\|g_p\|_1$, $(1\leq p\leq 3)$, $\|g_q\|_\infty$ $(4\leq q\leq 6)$.

Collecting (4.35), (4.36) and (4.37) we conclude that if $k(1)+\cdots+k(n)=4$

$$\begin{aligned}
|S_{k(1)\cdots k(n)}(\eta_1^\varepsilon(T),\cdots,\eta_n^\varepsilon(T))|&\leq c_4(\sqrt{\varepsilon})^4(|Q(T)|^{1/6})^4\Big(\frac{1}{\sqrt{|Q(T)|}}\Big)^4|Q(T)|\\
&=c_4(\sqrt{\varepsilon})^4|Q(T)|^{-1/3}.
\end{aligned}$$

In general, by the same device, we draw the conclusion that if $k(1)+\cdots+k(n)=p\geq 3$

$$|S_{k(1)\cdots k(n)}(\eta_1^\varepsilon(T),\cdots,\eta_n^\varepsilon(T))|\leq c_4(\sqrt{\varepsilon})^p|Q(T)|^{1-p/3},$$

where $c_4$ is independent of $\varepsilon$ and $T$. This implies (4.33), (4.34). Therefore, by the same arguments as in the proof of Theorem 1, immediately follows that of Theorem 2.

## 5. Asymptotic independence and supplementary remarks

In relation to the limit theorems in Section 3, 4 we draw our attention to a structural interrelation between $X_k(t)$, $1 \leq k < \infty$, of (1.2) and (4.2). For this purpose we make

DEFINITION 5.1. Let $X(t) = \{x_1(t), \cdots, x_k(t)\}$, $t \in [0, \infty)$, be an $R^k$-valued stochastic process such that dist $X(t)$, $0 \leq t < \infty$ form a relatively compact set under the weak topology.

The components of $X(t)$ are asymptotically independent as $t \to \infty$ if and only if

$$\lim_{t \to \infty} \{E(\prod_{j=1}^{k} f_j(x_j(t))) - \prod_{j=1}^{k} Ef_j(x_j(t))\} = 0 \tag{5.1}$$

for any choice of $f_j \in C_b(\boldsymbol{R})$, where $C_b(\boldsymbol{R})$ is the set of real continuous bounded functions on $R$.

Notice that the relative compactness of dist. $X(t)$, $0 \leq t < \infty$, implies that (5.1) is true if only it is so for an arbitrary choice of $f_j \in \mathcal{S}(\boldsymbol{R})$, the Schwartz space of all real-valued rapidly decreasing infinitely differentiable functions on $\boldsymbol{R}$.

IX. *Let $X(t) = (x_1(t), \cdots, x_k(t))$, $0 \leq t < \infty$, be an $\boldsymbol{R}^k$-valued stochastic process, and suppose that each $X(t)$ has the moment of an arbitrary order and satisfies the conditions that there exists a non-negative sequence $\lambda_{2n}$, $1 \leq n < \infty$, such that*

(i)

$$\begin{aligned} &\max_{1 \leq j \leq k} \overline{\lim_{t \to \infty}} E(x_j^{2m}(t)) = \frac{(2m)!}{2^m m!} \lambda_{2m} \\ &\lim_{m \to \infty} a^m \frac{\lambda_{2m}}{m!} = 0 \qquad \text{for any} \quad a > 0\,, \end{aligned} \tag{5.2}$$

(ii)

$$\lim_{t \to \infty} \{E(x_1(t)^{m_1} \cdots x_k(t)^{m_k}) - E(x_1(t)^{m_1}) \cdots E(x_k(t)^{m_k})\} = 0 \tag{5.3}$$

*for an arbitrary set of non-negative integers $m_1, \cdots, m_k$.*

*Then the components of $X(t)$ are asymptotically independent, as $t \to \infty$.*

Sketch of the proof. For notational simplicity we deal with the case $k=3$. First notice that by (i), we can find a $t_0 \geq 0$ such that dist $X(t)$, $t_0 \leq t < \infty$, is relatively compact.

If $f \in \mathcal{S}(\boldsymbol{R})$, then for $A > 0$

$$f_A(x) = \frac{1}{2\pi} \int_{-A}^{A} \hat{f}(u) \left(1 - \frac{|u|}{A}\right) e^{ixu} du\,,$$

where

$$\hat{f}(u)=\int f(x)e^{-ixu}dx\,,\quad f_A=f*\frac{1}{2\pi A}D_A^2(x)\,,$$

with the Dirichlet kernel $D_A$ in Section 3.

Define

$$S_l(t,n)=\sum_{j=0}^{2n-1}\frac{(iA_l(t))^j}{j!}\,,\quad A_l(t)=u_lx_l(t)\,,\qquad 1\leq l\leq 3\,,$$

$$\Delta_l(t,n)=\exp\{iA_l(t)\}-S_l(t,n)\,.$$

Then, for $m=1,2,\cdots$, $|u_l|\leq A$ $(1\leq l\leq 3)$, by (5.2)

$$\overline{\lim_{t\to\infty}}\,E|\Delta_l(t,n)|^m\leq\mu_A(m,n)\,,$$

where

$$\mu_A(m,n)=\left\{\left(\frac{mA^2}{2}\right)^n\frac{1}{n!}\right\}^m\lambda_{2mn}\,.$$

By (5.2), after elementary computations, $\mu_A(m,n)\to 0$, as $n\to\infty$, for any $m\geq 1$.

By elementary estimations we obtain for any $n$

$$\overline{\lim_{t\to\infty}}\sup_{|u_j|\leq A}|E\{\exp i[u_1x_1(t)+\cdots+u_3x_3(t)]\}$$
$$-E\{\exp iu_1x_1(t)\}\cdots E\{\exp iu_3x_3(t)\}|\leq c(\mu_A^{1/3}(3,n)+\mu_A(1,n)),$$

where $c$ is a constant depending on $A$, but independent of $n$. Insert this into the obvious inequality

$$|E(g_1(x_1(t))\cdots g_3(x_3(t)))-E(g_1(x_1(t)))\cdots E(g_3(x_3(t)))|$$
$$\leq\prod_{l=1}^{3}\|\hat{f}_l\|_\infty\int_{-A}^{A}\cdots\int_{-A}^{A}|E\exp i[u_1x_1(t)+\cdots+u_3x_3(t)]$$
$$-E\exp iu_1x_1(t)\cdots E\exp iu_3x_3(t)|\,du\,,$$

where $f_j\in\mathcal{S}(\boldsymbol{R})$, $g_j=(f_j)_A$, $1\leq j\leq 3$, to have

$$\lim_{t\to\infty}|E(g_1(x_1(t)\cdots g_3(x_3(t)))-E(g_1(x_1(t)))\cdots E(g_3(x_3(t)))|=0\,.$$

Since $g_j\to f_j$ locally uniformly, as $A\to\infty$, and $\{\mathrm{dist}\,X(t),t\geq t_0\}$ is weakly relatively compact, (5.1) holds true for any $f_j\in\mathcal{S}(\boldsymbol{R})$. This completes the proof of Proposition IX.

Let $F_l$ be a limit point of $\mathrm{dist}\,x_l(t)$, as $t\to\infty$ $(1\leq l\leq k)$, and a sequence $t_n\uparrow\infty$, $n\to\infty$, be such that

$$\mathrm{dist}\,x_l(t_n)\to F_l\quad(\text{weakly})\,,\qquad n\to\infty\,.$$

Then there exists

$$\nu_{e,m} = \lim_{n\to\infty} Ex_l^m(t_n) = \int x^m dF_l(x)\,, \qquad m\geq 1\,.$$

By (5.2), we can find a constant $c_1>0$ such that $a^n \frac{\lambda_{2n}}{n!}<c_1$, $n\geq 1$, for any $a>0$. This means that $\nu_{2n}^{-1/2n}>c_2/n$ with some $c_2>0$, independent of $n$. Since

$$\sum_{n\geq 1}^{\infty} \nu_{2n}^{-1/2n} \geq c_2 \sum_{n\geq 1} n^{-1} = \infty\,,$$

the Hamburger moment problem

$$\nu_{lm} = \int x^m dF_l(x)\,, \qquad m\geq 0$$

is determined.

**Theorem 3.** *Take an arbitrary $k\geq 2$ and define $\bar{X}(T)=(\bar{X}_1(T), \cdots, \bar{X}_k(T))$, where*

$$\text{(a)} \quad \bar{X}_j(T) = \frac{1}{\sqrt{V(T)}}\int_0^T X_j(t)dt\,, \qquad 1\leq T<\infty\,,$$

*in the notations of Theorem* 1, *or*

$$\text{(b)} \quad \bar{X}_j(T) = \frac{1}{\sqrt{V(T)}}\int_{0\leq x\leq T} X_j(x)dx\,, \qquad 1\leq T\in \boldsymbol{R}^d\,,$$

*in the notations of Theorem* 2, $1\leq j\leq k$.

*Then, under the conditions of Theorem* 1, *Theorem* 2, *the components of $\bar{X}(T)$ are asymptotically independent as $T\to\infty$.*

Proof. Assume the conditions of Theorem 1 and deal with the case (a), the other being done in a similar fashion.

Define

$$\bar{X}_j^\varepsilon(T) = \frac{1}{\sqrt{V(T)}}\int_0^T X_j^\varepsilon(t)dt\,, \qquad 1\leq j\leq k\,,$$

$$\bar{X}^\varepsilon(T) = (\bar{X}_1^\varepsilon(T), \cdots, \bar{X}_k^\varepsilon T))\,,$$

and write

$$\bar{X}(T) = \bar{X}^\varepsilon(T) + \Delta\bar{X}(T)\,.$$

For $T\to\infty$, $\bar{X}(T)$, $\bar{X}^\varepsilon(T)$, $\Delta\bar{X}(T)$ behave in the same manner as $Y(T)$, $Y^\varepsilon(T)$, $\Delta Y(T)$ in Section 3. Thus, {dist $\bar{X}(T)$, $T\geq 1$} is relatively compact; for $1\leq j\leq k$

$$|S_m(\bar{X}_j^\varepsilon(T))| \leq c(\sqrt{\varepsilon})^m T^{1-m/3} \qquad \text{if} \quad m\geq 3\,, \tag{5.4}$$

where $S_m(\cdot)$ is the $m$th cumulant (c.f. Section 2); dist $\bar{X}(T)$ is equi-convergent with $\bar{X}^\varepsilon(T)$, as $T\to\infty$, for any $\varepsilon>0$. (5.4) implies that

(5.5) $$\sup_{\substack{0<\varepsilon\le 1\\ 1\le T}} E|\bar{X}^{\varepsilon}(T)|^{m}<\infty\,, \qquad \text{for any} \quad m\ge 1\,.$$

We will show that

(5.6) $$\lim_{T\to\infty} S(\{\bar{X}^{\varepsilon}_{j_1}(T)\}^{m_1},\ \cdots,\ \{\bar{X}^{\varepsilon}_{j_l}(T)\}^{m_l})=0\,,$$
$$2\le l\le k\,,\quad m_1,\cdots,m_l\ge 1\,,\quad 1\le j_1<\cdots<j_l\le k\,.$$

For this purpose, to obtain an expansion of $\{\bar{X}^{\varepsilon}_{q}(T)\}^{m}$, $1\le q\le k$, into Ito's multiple integrals, introduce $P=||p_{ij}||$, an $m\times m$ symmetric matrix with integer entries $p_{ij}\ge 0$ such that $p_{ii}=0$, $o\le p_i\le q$ $(1\le i\le m)$, where $p_i=\sum_{j=1}^{m} p_{ij}\le q$, and take $m(m-1)/2$ independent variables $x_{ij}\in \boldsymbol{R}^{p_{ij}}$, $1\le i<j\le m, p_{ij}=d(x_{ij})$.

Let $a_1(\lambda),\cdots,a_m(\lambda)$, $\lambda\in\boldsymbol{R}^q$ be such that $a_1(\lambda)=\cdots=a_m(\lambda)=\gamma_q(\lambda)$, $\gamma_q(\lambda)=c^{\varepsilon}_q(\lambda)\mathcal{D}_T(\bar{\lambda})/\sqrt{V(T)}$, $\lambda\in\boldsymbol{R}^q$. Choose $p_1$ arguments of $a_1$ and replace them by $\boldsymbol{x}_1=(x_{12},x_{13},\cdots,x_{1m})\in\boldsymbol{R}^{p_1}$, choose $p_2$ ones of $a_2$ and replace them by $\boldsymbol{x}_2=(x_{21},x_{23},\cdots,x_{2m})\in\boldsymbol{R}^{p_2}$, and so on, where $x_{ji}$ is defined to be $-x_{ij}$ if $i<j$, to get successively $\gamma_q(\boldsymbol{x}_1,\lambda_1),\cdots,\gamma_q(\boldsymbol{x}_m,\lambda_m)$, $\lambda_1\in\boldsymbol{R}^{u_1},\cdots,\lambda_m\in\boldsymbol{R}^{u_m}$, with $u_i=q-p_i$ $(1\le i\le m)$. By the multiplication rule for Ito's multiple integrals (p. 53, [8], p. 388 [9])

(5.7.1) $$\{\bar{X}^{\varepsilon}_{q}(T)\}^{m}=\kappa_0+\sum_{P}\kappa(P)\int c_{qP}(\lambda_1,\cdots,\lambda_m)d^{s}\beta$$

(5.7.2) $$c_{qP}=\int\gamma_q(\boldsymbol{x}_1,\lambda_1)\cdots\gamma_q(\boldsymbol{x}_m,\lambda_m)d^{r}\sigma\,,$$
$$r=\sum_{i=1}^{m}p_i/2\,,\quad s=u_1+\cdots+u_m\,,$$

where $\kappa_0$ is non-random and the summation in (5.7.1) is taken over all the matrices $P$ satisfying $s=u_1+\cdots+u_m\ge 1$ in addition to the above-mentioned restrictions, while the integration in (5.7.2) is over $\boldsymbol{R}^r$ with respect to $d^r\sigma=\prod_{1\le i<j\le m}d^{p_{ij}}\sigma(x_{ij})$. Those $x_{ij}$ for which $p_{ij}=0$ are to be dropped out of the above descriptions. Similarly only those $\lambda_i$ $(1\le i\le m)$ for which $u_i>0$ actually appear on the right-hand sides of (5.7.1), (5.7.2), the others being fictitious. The concrete expression of $\kappa(P)$, which is immaterial for the present use, can be shown to be given by (p. 53, [8])

$$\kappa(P)=\left\{\left\{\prod_{i=1}^{m}\binom{m}{p_i}\frac{p_i!}{\prod_{j=1}^{m}p_{ij}!}\right\}\prod_{1\le i<j\le m}p_{ij}!\,.\right.$$

Generally speaking, the graph corresponding to $P$ contains several connected subgraphs each of which gives rise to a connected-kernel integral, thus

$\int c_{qP}(\lambda_1, \cdots, \lambda_m)d^s\beta$ contains these integrals as non-random factors. If we denote by $K(\gamma_q, P)$ the product of these integrals and take it outside the integral sign, we have the representation

$$(5.7.1)' \qquad \{\bar{X}_q^\varepsilon(T)\}^m = \kappa_0 + \sum_P \kappa(P)K(\gamma_q, P)\int \tilde{c}_{qP}d^s\beta\,,$$

where $\tilde{c}_{qP}$ is obtained by dropping those $\gamma_q$ which have gone with $K(\gamma_q, P)$ outside the integral sign.

For the aid of understanding here is exhibited a simple example. Let $q=2$, $m=6$, and

$$P = \begin{Vmatrix} P_1 & 0 \\ 0 & P_2 \end{Vmatrix} \qquad P_1 = \begin{Vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{Vmatrix} \qquad P_2 = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix},$$

then

$$K(\gamma_2, P) = \int_{R^6} \gamma_2(x, z)\gamma_2(-x, y)\gamma_2(-y, -z)d^3\sigma\,,$$

$$\tilde{c}_{2P}(\lambda) = \int_{R^2} \gamma_2(x, \lambda_1)\gamma_2(-x, y)\gamma_2(-y, \lambda_2)d^2\sigma,$$

$$\lambda = (\lambda_1, \lambda_2) \in \boldsymbol{R}^2\,.$$

A cumulant $S(\xi_1, \cdots, \xi_l)$ is multi-linear and identically zero if at least one of $\xi_1, \cdots, \xi_l$ is non-random. Therefore, in view of (5.7.1)′, the cumulant in (5.6) is expanded into a sum of constant multiples of expressions of the form

$$(5.8) \qquad K_1(\gamma_{j1})\cdots K_l(\gamma_{jl})\int F(\tilde{c}_1, \cdots, \tilde{c}_l; \nu)d^s\sigma(\nu)\,,$$

where $F(\tilde{c}_1, \cdots, \tilde{c}_l; \nu)$ is a connected kernel composed of functions of the types

$$(5.9) \qquad \tilde{c}_1(\lambda) = \int \gamma_{j1}(\boldsymbol{x}_1, \lambda_1)\cdots\gamma_{j1}(\boldsymbol{x}_m, \lambda_m)d^{r_1}\sigma,\ \lambda = (\lambda_1, \cdots, \lambda_m)\,,$$

$$\tilde{c}_2(\lambda) = \int \gamma_{j2}(\boldsymbol{y}_1, \mu_1)\cdots\gamma_{j2}(\boldsymbol{y}_m, \mu_m)d^{r_2}\sigma,\ \mu = (\mu_1, \cdots, \mu_m)\,,$$

$$\cdots\cdots,$$

$K_1(\gamma_{j1})$ a product of integrals with connected kernels composed of $\gamma_{j1}$, and similarly for the other $K$'s. Then $\int Fd^s\sigma$ in (5.8) results in an integral with a kernel $G$ composed of $\gamma_{j1}, \cdots, \gamma_{jl}$, through a complete coupling of arguments involved in $\gamma$'s. Resolve $G$ into a product of connected kernels $G_1, \cdots, G_g$ composed of $\gamma_{j1}, \cdots, \gamma_{jl}$:

$$G = G_1 \cdots G_g . \tag{5.10}$$

Let $d(G_p)$, $1 \leq p \leq g$, be the degree of $G_p$, i.e. the number of $\gamma$'s concerned with $G_p$ (c.f. 2). On the right-hand sides of (5.9) one can find a $\gamma_{jk}$ $(1 \leq k \leq l)$ which is concerned with $G_p$ and in its arguments contains a non-fictitious component of some of $\lambda$, $\mu$, $\cdots$, otherwise $G_p$ would be absorbed in the kernel of $K_1(\gamma_{j1}) \cdots K_l(\gamma_{jl})$. With no loss of generality, assume that $\lambda_1$ is such a non-fictitious component; $\gamma_{j1}(\boldsymbol{x}, \lambda_1)$ is concerned with $G_p$. Think of a coupling procedure making a connected kernel $G_p$, then one knows that there is a $\gamma_{jt}$, $1 \leq t \leq l$, such that $t \neq 1$ and $\gamma_{jt}$ is concerned with $G_p$. Then, in order that $G_p$ be connected, $G_p$ must be concerned with the other $\gamma$. This means that $d(G_p) \geq 3$.

By (5.8), (5.9), the cumulant in (5.6) is expanded into a sum of constant multiples of expressions of the form

$$K_1(\gamma_{j1}) \cdots K_l(\gamma_{jl}) \left( \int G_1 d^{s_1} \sigma \right) \cdots \left( \int G_g d^{s_g} \sigma \right) . \tag{5.11}$$

$K_1(\gamma_{j1}) \cdots K(\gamma_{jl})$ is represented as a product of integrals $\int G' d^u \sigma$ with connected kernels $G'$ each of which is composed of a single one of $\gamma_{j1}, \cdots, \gamma_{jl}$. It may happen that $d(G')=2$, and $G'$ is concerned with $\gamma_{jk}$. Then

$$\int G' d^u \sigma = E(\bar{X}^{\varepsilon}_{jk}(T))^2 \leq 1 .$$

If $d_0 \equiv d(G_p)$ $(1 \leq p \leq g)$, then since $d_0 \geq 3$, $\int G_p d^{s_p} \sigma$ is of the same type as a connected-kernel summand of $S_{k(1) \cdots k(n)}(\eta^{\varepsilon}_1(T), \cdots, \eta^{\varepsilon}_n(T))$, with $k(1)+\cdots+k(n)= d_0$, in Lemma 1. So that by Lemma 1 one obtains

$$L = \varlimsup_{T \to \infty} | S(\{\bar{X}^{\varepsilon}_{j1}(T)\}^{m_1}, \cdots, \{\bar{X}^{\varepsilon}_{jl}(T)\}^{m_l}) | \leq c\varepsilon ,$$

where $c>0$ is a constant independent of $\varepsilon$. The same is true with $\int G' d^u \sigma$ if $d(G') \geq 3$. By the relative compactness of $\{\operatorname{dist} X(T), T \geq 1\}$, equi-convergence of $\bar{X}$ and $\bar{X}^{\varepsilon}(T)$, (5.5), and IV, one can find a sequence $\{T_n\} \in \mathcal{D}_0$, such that there exist

$$F = \lim_{n \to \infty} \operatorname{dist} \bar{X}(T_n) = \lim_{n \to \infty} \bar{X}^{\varepsilon}(T_n) ,$$
$$L = \lim_{n \to \infty} | S(\{\bar{X}^{\varepsilon}_{j1}(T_n)\}^{m_1}, \cdots, \{\bar{X}^{\varepsilon}_{jl}(T_n)\}^{m_l}) | ,$$

and

$$\lim_{n \to \infty} S(\{\bar{X}^{\varepsilon}_{j1}(T_n)\}^{m_1}, \cdots, \{\bar{X}^{\varepsilon}_{jl}(T_n)\}^{m_l}) = S(\xi^{m_1}_{j1}, \cdots, \xi^{m_l}_{jl}) ,$$

where $(\xi_1, \cdots, \xi_l)$ is distributed according to $F$. Therefore $L$ is independent of $\varepsilon$. Since $\varepsilon$, $0 \leq \varepsilon \leq 1$, is arbitrary, this implies that $L=0$. This completes the proof of (5.6).

By the known functional relations between cumulants and moments (c.f. [5]) (5.6) implies that

$$(5.12)\quad \lim_{T\to\infty}\{E(\{\bar{X}_1^\varepsilon(T)\}^{m_1}\cdots\{\bar{X}_k^\varepsilon(T)\}^{m_k})-E\{\bar{X}_1^\varepsilon(T)\}^{m_1}\cdots E\{\bar{X}_k^\varepsilon(T)\}^{m_k}\}=0\,.$$

As we have seen in the proof of Theorem 1

$$\lim_{T\to\infty} S_m(\bar{X}_j^\varepsilon(T))=0\,,\qquad \text{for}\quad m\geq 3,\ 1\leq j\leq k\,.$$

Then again by the relations between cumulants and moments

$$\overline{\lim_{T\to\infty}}\,\{\bar{X}_j^\varepsilon(T)\}^{2n}\leq\frac{(2n)!}{2^n n!}\,\overline{\lim_{T\to\infty}}\,\{\bar{X}_j^\varepsilon(T)\}^2\leq\frac{(2n)!}{2^n n!}\,.$$

Appealing to IX, the components of $\bar{X}^\varepsilon(T)$ are asymptotically independent, so are those of $\bar{X}(T)$, as $T\to\infty$. This completes the proof of Theorem 3.

At the final stage of this section, we will describe sufficient conditions of practical use which guarantee the realization of some of assumptions in Theorem 1, Theorem 2. Although those conditions are confined to the frame work of Theorem 1, it is an easy task to modify them to be adapted to Theorem 2.

In the notations of 1 define

$$\Psi_k(h)=k!\int f(\lambda_1)\cdots f(\lambda_{k-1})d\lambda_1\cdots d\lambda_{k-1}$$
$$\times\sup_x\int_x^{x+h}|c_k(\lambda,\lambda_1,\cdots,\lambda_{k-1})|^2 f(\lambda)d\lambda\,,\qquad h>0\,,\quad k\geq 2\,.$$

Then obviously

$$\Phi(|c_k|^2;h)\leq\Psi_k(h)\,.$$

Applying the first inequality in (2.8) with $X(t)$ replaced by $R_n(t)$ one obtains:

In order that the assumption (iii), Theorem 1 be true it is sufficient that

$$\text{(A)}\qquad \lim_{n\to\infty}\overline{\lim_{h\downarrow 0}}\frac{1}{h}\sum_{k\geq n}\Phi(|c_k|^2;h)=0$$

or more strongly

$$\text{(B)}\qquad \lim_{n\to\infty}\overline{\lim_{h\downarrow 0}}\frac{1}{h}\sum_{k\geq n}\Psi_k(h)=0\,.$$

Let $a$, $b>0$, and suppose $x$ satisfies $-(a+b)<x<a+b$. Then one can find a $y$ such that $-b<y<b$, $-a<x-y<a$.

Define $C_{eb}(\boldsymbol{R})$ to be the set of real even bounded continuous functions on $\boldsymbol{R}$, and for non-negative $g\in C_{eb}(\boldsymbol{R})$ define $z(g)=\inf(x\colon x\geq 0,\ g(x)=0)$. If

$g$, $h\in\mathcal{C}_{eb}(\boldsymbol{R})\cap L^1(\boldsymbol{R})$, are non-negative, $z(g*h)\geq z(g)+z(h)$. This is a consequence of addition of independent random variables with constant multiples of $g$, $h$ as their density functions, or the above algebraic fact combined with the convolution $g*h$.

Put

$$l_k(a)=k!\operatorname*{ess\ inf}_{|\lambda_1+\cdots+\lambda_k|\leq a}|c_k(\lambda)|^2, \qquad a>0, \quad k\geq 1.$$

Then

$$\varphi(|c_k|^2;\lambda)\geq l_k(a)f_k(\lambda), \quad |\lambda|\leq a, \qquad f_k=f^{k*}. \tag{5.13}$$

$c_k$ $(k\geq 1)$ is said to satisfy $(L)$ if there exists $a_k>0$ such that $l_k(a_k)>0$. Since $f_k\in\mathcal{C}_{eb}(R)$ for $k\geq 2$, and $f_2(0)=||f||_2^2>0$, $\delta_1=z(f_2)>0$, $\delta_k=z(f_{2k})\geq k\delta_1$. Suppose that $c_{2k}\not\equiv 0$ and it satisfies $(L)$, then by (5.13)

$$\varphi(|c_{2k}|^2;\lambda)\geq\alpha \qquad \text{for} \quad |\lambda|\leq\beta,$$

where

$$\alpha=\inf_{|\lambda|\leq\delta_k/2}f_{2k}(\lambda)l_{2k}(a_{2k})>0, \quad \beta=a_{2k}\wedge\delta_k/2.$$

Then, since

$$\lim_{h\downarrow 0}\Phi(|c_k{}^2|;h)/h\geq\alpha$$

by (2.7)

$$\varliminf_{T\to\infty}v_k(T)/T>0,$$

whence

$$\varliminf_{T\to\infty}V(T)/T>0.$$

There exists a $\lambda_0>0$ such that $f_3(\lambda_0)>0$. Suppose that $c_{2k+3}\not\equiv 0$ $(k\geq 1)$, $(L)$ is satisfied for $c_{2k+3}$ and $k$ is so large that $k\delta_1-\lambda_0>0$. Then arguing as above $f_{2k+3}(x)$ is positive on $I=(-k\delta_1-\lambda_0,\ k\delta_1-\lambda_0)$ and, since one can find out $\varepsilon>0$ such that $I\supset[-\varepsilon,\varepsilon]$,

$$\alpha=\inf_{|\lambda|\leq\varepsilon}f_{2k+3}(\lambda)l_{2k+3}(a_{2k+3})>0.$$

Therefore having

$$\varphi(|c_{2k+3}|^2;\lambda)\geq\alpha \qquad \text{for} \quad |\lambda|\leq a_{2k+3}\wedge\varepsilon$$

one obtains as above

$$\varliminf_{T\to\infty}V_{2k+3}(T)/T>0,$$

whence also

$$\varliminf_{T\to\infty}V(T)/T>0.$$

$c_k$ $(k \geq 1)$ is said to satisfy $(U)$ if there exists a $b_k > 0$ such that

$$u_k = k! \operatorname*{ess\ sup}_{|\lambda_1+\cdots+\lambda_k| \leq b_k} |c_k(\lambda)|^2 < \infty .$$

If this is the case

$$\varphi(|c_k|^2; \lambda) \leq u_k \|f_k\|_\infty , \quad |\lambda| \leq b_k ,$$
$$\varlimsup_{h \downarrow 0} \Phi(|c_k|^2; h)/h < \infty ,$$

whence by (2.8)

$$\varlimsup_{T \to \infty} v_k(T)/T < \infty ,$$

and moreover $\Phi(\delta[|c_k|^2]; h) = 0$ for all sufficiently small $h$, because $\delta[|c_k|^2](\lambda) = 0$ for $|\lambda_j + \cdots + \lambda_k| \leq b$ and sufficiently small $h = 1/T$. If all $c_k$, $1 \leq k < \infty$, satisfy $(U)$ in such a way that there exists a $b_0 > 0$ such that $b_k > b_0$ for all $k \geq 1$, then

$$\frac{1}{h} \sum_{k=n}^{\infty} \Phi(|c_k|^2; h) \leq \sum_{k=n}^{\infty} u_k \|f_k\|_\infty .$$

Summarizing these we are led to the conclusions:

(C) If the expansion of $X(t)$ contains a summand of even degree whose kernel satisfies $(L)$, or it does infinitely many summands of odd degrees whose kernels satisfy $(L)$, then

$$\varliminf_{T \to \infty} V(T)/T > 0 .$$

(D) If every $c_k$, $1 \leq k \leq n$, satisfies $(U)$

$$\varlimsup_{T \to \infty} V_n(T)/T < \infty .$$

(E) If all $c_k$, $1 \leq k < \infty$, satisfy $(U)$ in such a way that there exists a $b_0 > 0$ such that $b_k > b_0$ for $k \geq 1$ and moreover

$$\sum_{n=1}^{\infty} u_k \|f_k\|_\infty < \infty ,$$

then the both conditions

$$\text{(A)} \quad \text{and} \quad \varlimsup_{T \to \infty} V(T)/T < \infty$$

are satisfied.

(F) If (C), (E) in the above are satisfied, the conditions (ii), (iii), and (iv) of Theorem 1 are satisfied.

## References

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