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# ON A NEW DIMENSION ESTIMATE OF THE GLOBAL ATTRACTOR FOR CHEMOTAXIS-GROWTH SYSTEMS

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**Abstract**

In this paper we continue systematic study of the dimension estimate of the global attractor for the chemotaxis-growth system. Using nonnegativity of solutions we manage significantly to improve dimension estimates with respect to the chemotactic parameter.

## 1. Introduction

In this paper we consider the following initial value problem for a chemotaxis-growth system of equations:

$$\tag{1.1} \begin{cases} \dfrac{\partial u}{\partial t} = a \triangle u - \nabla \cdot \{u \nabla \chi(\rho)\} + f(u) & \text{in} \quad \Omega \times (0, \infty), \\ \dfrac{\partial \rho}{\partial t} = b \triangle \rho - c\rho + du & \text{in} \quad \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad \rho(x, 0) = \rho_0(x) & \text{in} \quad \Omega. \end{cases}$$

This problem arises in mathematical biology, where $u(x, t)$ and $\rho(x, t)$ denote the population density of biological individuals and the concentration of a chemical substance, respectively, at the position $x \in \Omega \subset \mathbb{R}^2$ and time $t \in [0, \infty)$. The mobility of individuals is characterized by two effects: one is random walking, and the other is the directed movement with the tendency to move toward higher concentration of the chemical substance. This phenomenon is called chemotaxis in biology (for details see Budrene and Berg [5] or Murray [15]). The constants $a > 0$ and $b > 0$ are the diffusion rates of $u$ and $\rho$, respectively; $c > 0$ and $d > 0$ are the degradation and production rates of $\rho$, respectively. The function $\chi(\rho)$ is a sensitivity function due to chemotaxis. The function $f(u)$ denotes a growth rate of $u$. In this paper we consider the case when $\Omega \subset \mathbb{R}^2$

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is a bounded convex domain. For simplicity, $\chi(\rho)$ is assumed to be a linear,

$$\chi(\rho) = \nu\rho \tag{1.2}$$

with a chemotactic coefficient $\nu > 0$, and $f(u)$ is assumed to be a cubic function

$$f(u) = f u^2(1-u) \tag{1.3}$$

with a growth coefficient $f > 0$, respectively.

In order to study aggregating patterns due to chemotaxis and growth, there are several contributions not only from experiments but also from mathematical analysis. Budrene and Berg [5] experimentally observed that *E. coli* bacteria form complex spatio-temporal colony patterns. In order to understand theoretically such a chemotactic pattern formation, several models have been proposed, e.g., in [3, 10, 13, 16, 20]. Mimura and Tsujikawa presented in [14] a model (1.1), which is rather simple in the sense that it is characterized by only four effects: diffusion, chemotaxis, production of a chemical substance, and growth. In the absence of the growth term $f(u)$, (1.1) reduce to the Keller-Segel equations [12] modeling the initiation of aggregating patterns of slime mold.

The formation of the colony patterns by chemotaxis is considered as to be a prototype of various phenomena of Self-Organization, cf. [11, 17]. According to description by Synergetics due to Haken [11], the chemical substance plays the role of a conductor which leads the individuals and is itself produced by them cooperatively. The fractal dimension of the attractor then corresponds to a reduction of the degrees of freedom in the process of pattern formation which is called the slaving principle.

The authors have already established in the previous paper [7] the upper and lower estimate

$$C_1 \nu d \leq \dim \mathfrak{A} \leq C_2(\nu d)^6, \tag{1.4}$$

of the fractal dimension $\dim \mathfrak{A}$ of the global attractor $\mathfrak{A}$ for (1.1) by applying the technique given in [2, 4, 19]. There we have not used an important property, nonnegativity, of solutions to (1.1), with the intention of comparing the results with those in the approximation case discussed in [8], where we have not proved nonnegativity of approximate solutions. However, by utilizing the nonnegativity of solutions, we can revise the upper estimate shown in [7] to the lower polynomial order of the coefficients in the equation (1.1).

The paper is organized as follows: In Section 2 we present the revised upper estimate of $\dim \mathfrak{A}$. Then, in Section 3, we state the main result of this paper.

REMARK. Numerical approximation to (1.1) by positivity-preserving scheme will be studied in the forthcoming paper [9].

## 2. Revised upper estimate

As is already well known, the system (1.1) possesses a global attractor, see [1, 2, 7, 18]. In this section we present an upper estimate for the dimension of the global attractor. To this end we follow [4, 19] and recall some basic facts.

Let $X$ be a Hilbert space with inner product $(\cdot\,,\,\cdot)_X$ and norm $\|\cdot\|_X$, let $\mathfrak{X}$ be a compact subset of $X$, and $S$ a continuous nonlinear operator acting on $\mathfrak{X}$. Then, $S$ is said to be uniformly quasidifferentiable [4, Definition 10.1.3] on $\mathfrak{X}$ in the norm of $X$ if, for each $U \in \mathfrak{X}$, there exists a linear operator $W = S'(U)$ in $X$, called the quasidifferential, such that

$$\|S(U_1) - S(U) - W(U_1 - U)\|_X \le \gamma(\|U_1 - U\|_X)\|U_1 - U\|_X \tag{2.1}$$

holds for any $U_1 \in \mathfrak{X}$, where the function $\gamma(\zeta)$ is independent of $U$ and $U_1$ and satisfies $\gamma(\zeta) \to 0$ as $\zeta \to 0$.

Next consider a continuous dynamical system $(S_t, \mathfrak{X}, X)$. According to [19], the global attractor of $(S_t, \mathfrak{X}, X)$ is given by $\mathfrak{A} = \bigcap_{0\le t<\infty} S_t\mathfrak{X}$.

Assume that, for each $t \ge 0$, $S_t$ is uniformly quasidifferentiable and that, for each $U_0 \in \mathfrak{X}$, the quasidifferential $W_t = S_t'(U_0)$ is generated by the evolution equation

$$\frac{dV}{dt} = -\mathcal{A}(U(t))V, \tag{2.2}$$

where $U(t) = S_tU_0$. It is supposed that the operators $\mathcal{A}(U)$ are densely defined, closed linear operators acting on $X$ and are defined for all $U \in \mathfrak{X}$, and that the domains $\mathcal{D}(A(U)) \equiv \mathcal{D}$ are constant. Then, by Babin and Vishik [4, Theorem 10.1.1], $\dim\mathfrak{A}$ is estimated from above by the smallest integer $N$

$$\dim\mathfrak{A} \le N \tag{2.3}$$

satisfying

$$q_N < 0. \tag{2.4}$$

Here, the number $q_N$ is defined by

$$q_N = \liminf_{T\to\infty} \sup_{U_0\in\mathfrak{A}} \frac{1}{T}\int_0^T \inf_{\{\phi_j\}} \sum_{j=0}^{N} (-\mathcal{A}(U(t))\phi_j, \phi_j)_X \, dt, \tag{2.5}$$

$U(t) = S_tU_0$, and $\{\phi_j\} = \{\phi_j \in \mathcal{D}\}_{j=1,2,\dots}$ are arbitrary orthonormal systems in $X$.

Now we will apply (2.3)–(2.5) for the system (1.1).

In a similar manner as in [18], and thanks to nonnegativity of solutions, we can establish the following a priori estimates for the solutions in the global attractor $\mathfrak{A}$ for (1.1):

$$\|u(t)\|_{L^1} = \int_\Omega u(x,t)\,dx \le 2|\Omega|; \tag{2.6}$$

$$\int_0^T \|u(t)\|_{L^2}^2\,dt \le |\Omega|\left(T + \frac{4}{f}\right); \tag{2.7}$$

$$\int_0^T \|u(t)\|_{L^3}^3\,dt \le |\Omega|\left(T + \frac{6}{f}\right); \tag{2.8}$$

$$\|\rho(t)\|_{L^2}^2 \le \frac{d^2(4c+2f)}{c^2 f}|\Omega|; \tag{2.9}$$

$$\|\nabla\rho(t)\|_{L^2}^2 \le \frac{d^2(4c+f)}{bcf}|\Omega|; \tag{2.10}$$

$$\int_0^T \|\triangle\rho(t)\|_{L^2}^2\,dt \le \frac{d^2}{b^2}|\Omega|\left(T + \frac{8c+f}{cf}\right). \tag{2.11}$$

Next we exploit that, for each fixed $t \ge 0$, the operator $S_t$ is uniformly quasi-differentiable on $\mathfrak{A}$ in $X$. For each $U_0 = \begin{bmatrix} u_0 \\ \rho_0 \end{bmatrix} \in \mathfrak{A}$, the quasidifferential $W_t = S_t'(U_0)\colon X \to X$ is generated by the linearization equation for (1.1):

$$\begin{cases} \dfrac{\partial v}{\partial t} = a\,\triangle v - \nu\nabla\cdot(v\nabla\rho + u\nabla\eta) + f(2u - 3u^2)v & \text{in } \quad \Omega\times(0,\infty), \\ \dfrac{\partial \eta}{\partial t} = b\,\triangle\eta - c\eta + dv & \text{in } \quad \Omega\times(0,\infty), \\ \dfrac{\partial v}{\partial n} = \dfrac{\partial \eta}{\partial n} = 0 & \text{on } \quad \partial\Omega\times(0,\infty), \end{cases} \tag{2.12}$$

where $\begin{bmatrix} u \\ \rho \end{bmatrix} = \begin{bmatrix} u(t) \\ \rho(t) \end{bmatrix} = S_t\begin{bmatrix} u_0 \\ \rho_0 \end{bmatrix}$ is a solution of (1.1) whose trajectory is contained in $\mathfrak{A}$. We omit the proof here.

Now we will apply (2.3)–(2.5) to obtain an upper bound for $\dim\mathfrak{A}$.

Let us define the family of operators

$$\mathcal{A}(U)V = \begin{bmatrix} -a\,\triangle v - f(2u - 3u^2)v + \nu\nabla\cdot(v\nabla\rho + u\nabla\eta) \\ -dv - b\,\triangle\eta + c\eta \end{bmatrix}, \tag{2.13}$$

where

$$U = \begin{bmatrix} u \\ \rho \end{bmatrix} \in \mathfrak{A}, \quad V = \begin{bmatrix} v \\ \eta \end{bmatrix} \in H_N^2(\Omega)\times H_N^3(\Omega).$$

Then we can see that $\mathcal{A}(U)$ is a densely defined, closed linear operator acting on $X = L^2(\Omega)\times H^1(\Omega)$ and is defined for every $U \in \mathfrak{X} \subset H^2(\Omega)\times H^3(\Omega)$, and that the domains $\mathcal{D}(A(U)) \equiv \mathcal{D} = H^2_N(\Omega) \times H^3_N(\Omega)$.

Let

$$\left\{\phi_j = \begin{bmatrix} y_j \\ \xi_j \end{bmatrix} \in \mathcal{D}\right\}_{j=1,2,\dots} \tag{2.14}$$

be an orthonormal system in $X$. Hereafter, the inner product of $X$ is given by

$$(\phi, \phi')_X = \langle y, y'\rangle_{L^2} + \zeta\langle \Lambda\xi, \Lambda\xi'\rangle_{L^2}, \quad \phi = \begin{bmatrix} y \\ \xi \end{bmatrix},\ \phi' = \begin{bmatrix} y' \\ \xi' \end{bmatrix} \in X, \tag{2.15}$$

where $\Lambda = (-\triangle + 1)^{1/2}$, and $\zeta > 0$ is an arbitrary fixed number, which will be specified below.

Now we calculate $q_N$. In a similar manner as in [7, Equations (2.13)–(2.18)] we have

$$\begin{aligned}
&(-\mathcal{A}(U(t))\phi_j, \phi_j)_X \\
&\quad = -a\|\nabla y_j\|^2_{L^2} + \nu\langle y_j\nabla\rho, \nabla y_j\rangle_{L^2} + \nu\langle u\nabla\xi_j, \nabla y_j\rangle_{L^2} + f\langle (2u - 3u^2)y_j, y_j\rangle_{L^2} \\
&\qquad - \zeta b\|\nabla\Lambda\xi_j\|^2_{L^2} - \zeta c\|\Lambda\xi_j\|^2_{L^2} + \zeta d\langle y_j, \Lambda^2\xi_j\rangle_{L^2} \\
&\quad \le -a\|\nabla y_j\|^2_{L^2} + \frac{\nu}{2}\|\triangle\rho\|_{L^2}\|y_j\|^2_{L^4} + \nu\|u\|_{L^3}\|\nabla\xi_j\|_{L^6}\|\nabla y_j\|_{L^2} + \frac{f}{3}\|y_j\|^2_{L^2} \\
&\qquad - \zeta b\|\nabla\Lambda\xi_j\|^2_{L^2} - \zeta c\|\Lambda\xi_j\|^2_{L^2} + \zeta d\|y_j\|_{L^2}\|\Lambda^2\xi_j\|_{L^2} \\
&\quad \le -\frac{a}{2}\|\Lambda y_j\|^2_{L^2} - \frac{3\zeta b}{4}\|\Lambda^2\xi_j\|^2_{L^2} + \frac{C_2^2\nu^2}{a}\|u\|^2_{L^3}\|\Lambda\xi_j\|^{2/3}_{L^2}\|\Lambda^2\xi_j\|^{4/3}_{L^2} \\
&\qquad + \left(a + \frac{f}{3} + \frac{C_1^2\nu^2}{4a}\|\triangle\rho\|^2_{L^2}\right)\|y_j\|^2_{L^2} + \zeta\left(b - c + \frac{\zeta d^2}{b}\right)\|\Lambda\xi_j\|^2_{L^2}.
\end{aligned} \tag{2.16}$$

Here, $C_1$ and $C_2$ are some positive constants determined from embedding theorems, and hence may depend on $\Omega$ but are independent of the coefficients in (1.1).

Summing up in $j$,

$$\begin{aligned}
&\sum_{j=1}^{N}(-\mathcal{A}(U(t))\phi_j, \phi_j)_X \\
&\le -\frac{a}{2}\sum_{j=1}^{N}\|\Lambda y_j\|^2_{L^2} - \zeta b\sum_{j=1}^{N}\|\Lambda^2\xi_j\|^2_{L^2} \\
&\quad + \frac{C_2^2\nu^2}{a}\|u\|^2_{L^3}\sum_{j=1}^{N}\|\Lambda\xi_j\|^{2/3}_{L^2}\|\Lambda^2\xi_j\|^{4/3}_{L^2} + \left\{a + b - c + \frac{f}{3} + \frac{\zeta d^2}{b} + \frac{C_1^2\nu^2}{4a}\|\triangle\rho\|^2_{L^2}\right\}N.
\end{aligned} \tag{2.17}$$

Here we used the fact that

$$\|y_j\|_{L^2}^2 + \zeta\|\Lambda\xi_j\|_{L^2}^2 = 1, \tag{2.18}$$

and hence

$$\sum_{j=1}^N \|y_j\|_{L^2}^2 \le N, \quad \sum_{j=1}^N \|\Lambda\xi_j\|_{L^2}^2 \le \zeta^{-1}N. \tag{2.19}$$

Integration with respect to $t \in (0, T)$ and dividing by $T$ yield that

$$\begin{aligned}
&\frac{1}{T}\int_0^T \sum_{j=1}^N (-\mathcal{A}(U(t))\phi_j, \phi_j)_{L^2\times H^1}\, dt \\
&\le -\frac{a}{2}\sum_{j=1}^N \|\Lambda y_j\|_{L^2}^2 - \frac{3\zeta b}{4}\sum_{j=1}^N \|\Lambda^2\xi_j\|_{L^2}^2 + \frac{C_2^2\nu^2}{a}K_u \sum_{j=1}^N \|\Lambda\xi_j\|_{L^2}^{2/3}\|\Lambda^2\xi_j\|_{L^2}^{4/3} \\
&\quad + \left\{a + b - c + \frac{f}{3} + \frac{\zeta d^2}{b} + \frac{C_1^2\nu^2}{4a}K_\rho\right\}N,
\end{aligned} \tag{2.20}$$

where

$$K_u = \frac{1}{T}\int_0^T \|u(t)\|_{L^3}^2\, dt \le |\Omega|^{2/3}\left(1 + \frac{6}{fT}\right)^{2/3}, \tag{2.21}$$

$$K_\rho = \frac{1}{T}\int_0^T \|\triangle\rho(t)\|_{L^2}^2\, dt \le \frac{d^2}{b^2}|\Omega|\left(1 + \frac{8c + f}{cfT}\right), \tag{2.22}$$

from (2.8) and (2.11). Applying Hölder's inequality and (2.19) to the third term on the right-hand side of (2.20) and choosing $\zeta = \nu^2/(ab)$, we have

$$\begin{aligned}
&\frac{1}{T}\int_0^T \sum_{j=1}^N (-\mathcal{A}(U(t))\phi_j, \phi_j)_{L^2\times H^1}\, dt \\
&\le -\frac{1/2}{1/a + 1/b}\sum_{j=1}^N (\|\Lambda y_j\|_{L^2}^2 + \zeta\|\Lambda^2\xi_j\|_{L^2}^2) \\
&\quad + \left\{a + b - c + \frac{f}{3} + \frac{\nu^2 d^2}{ab^2} + \frac{C_1^2\nu^2}{4a}K_\rho + \frac{64C_2^6 b}{27}K_u^3\right\}N.
\end{aligned} \tag{2.23}$$

By the similar discussion as in [6, 7], we obtain

$$
\begin{aligned}
\sum_{j=1}^{N}(\|\Lambda y_j\|_{L^2}^2+\zeta\|\Lambda^2\xi_j\|_{L^2}^2) &\geq \sum_{j=1}^{N/2}\lambda_j(\|y_j\|_{L^2}^2+\zeta\|\Lambda\xi_j\|_{L^2}^2)\\
&=\sum_{j=1}^{N/2}\lambda_j \geq C_0\left(\frac{N}{2}\right)^2.
\end{aligned}
\tag{2.24}
$$

Here, $\lambda_1,\ldots,\lambda_N$ are the first $N$ eigenvalues of $-\triangle_N+1=\Lambda^2$, and $C_0$ is some positive number.

Then we have

$$
\begin{aligned}
&\frac{1}{T}\int_0^T\sum_{j=1}^{N}(-\mathcal{A}(U(t))\phi_j,\phi_j)_X\,dt\\
&\leq -\frac{1/2}{1/a+1/b}\frac{C_0}{4}N^2\\
&\quad+\left\{a+b-c+\frac{f}{3}+\frac{\nu^2d^2}{ab^2}+\frac{C_1^2\nu^2}{4a}K_\rho+\frac{64C_2^6b}{27}K_u^3\right\}N.
\end{aligned}
\tag{2.25}
$$

Using (2.21) and (2.22) and taking limits, we obtain

$$
\begin{aligned}
q_N &\leq -\frac{C_0/8}{a^{-1}+b^{-1}}N^2\\
&\quad+\left\{a+b-c+\frac{f}{3}+\frac{\nu^2d^2}{ab^2}+\frac{C_1^2\nu^2d^2}{4ab^2}|\Omega|+\frac{64C_2^6b}{27}|\Omega|^2\right\}N.
\end{aligned}
\tag{2.26}
$$

The smallest integer $N$ satisfying $q_N<0$ will be at least

$$
\begin{aligned}
N &> \left\{a+b-c+\frac{f}{3}+\frac{\nu^2d^2}{ab^2}+\frac{C_1^2\nu^2d^2}{4ab^2}|\Omega|+\frac{64C_2^6b}{27}|\Omega|^2\right\}\Big/\frac{C_0/8}{a^{-1}+b^{-1}}\\
&\sim O((\nu d)^2).
\end{aligned}
\tag{2.27}
$$

Thus, the number on the right-hand side gives the estimate from above for $\dim\mathfrak{A}$.

## 3. Main result

We have already obtained in [7, Section 3] a lower bound for $\dim\mathfrak{A}$. Combining this bound with the result given in the preceding section, we can state main result of the paper.

**Theorem 3.1.** *The dimension of the global attractor $\mathfrak{A}$ satisfy the estimate*:

$$C_1\nu d\leq\dim\mathfrak{A}\leq C_2(\nu d)^2 \tag{3.1}$$

*with some positive constants $C_1$ and $C_2$.*

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