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ON THE EQUIVALENCE BETWEEN TRACE AND CAPACITARY INEQUALITIES FOR THE ABSTRACT CONTRACTIVE SPACE OF BESSEL POTENTIALS

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Abstract

Let $F_{r,p} = V_{r,p}(L^p(X, m))$ be the abstract space of Bessel potentials and μ a positive smooth Radon measure on X. For $2 \le p \le q < \infty$, we give necessary and sufficient criteria for the boundedness of $V_{r,p}$ from $L^p(X, m)$ into $L^p(X, \mu)$, provided $F_{r,p}$ is contractive. Among others, we shall prove that the boundedness is equivalent to a capacitary type inequality. Further we give necessary and sufficient conditions for $F_{r,p}$ to be compactly embedded in $L^q(\mu)$. Our method relies essentially on establishing a *capacitary strong type inequality*.

1. Introduction

In this note we continue our investigations which we began in [2] about trace inequalities for operators associated to Dirichlet forms.

Let X be a locally compact separable metric space and m a positive Radon measure on X whose support is X.

Suppose that for each $t > 0$ we are given a symmetric strongly continuous contractive Markovian semigroup T_t acting on $L^2(X, m)$. Then by interpolation, T_t defines also a strongly continuous contractive Markovian semigroup acting on $L^p(X, m)$ for every $1 \le p < \infty$. We shall denote by $T_{t,p}$ the latter operator and by H_p its generator. Consider the gamma transform of $T_{t,p}$ [6]

$$(1)\qquad \begin{aligned} &V_{r,p} := L^p(X, m) \longrightarrow L^p(X, m),\\ &V_{r,p} = \frac{1}{\Gamma(r/2)} \int_0^\infty t^{(r/2)-1} \mathrm{e}^{-t} T_{t,p}\, dt, \quad \text{for every } r > 0. \end{aligned}$$

It is known [6] that such operator induces a set function called the (r, p)-capacity. Assume that every $f \in V_{r,p}(L^p(X, m))$ has been modified so as to become quasi-continuous, and let μ be a positive Radon measure on X charging no sets of zero (r, p)-capacity. Consider the operator

$$(2)\qquad V^{\mu}_{r,p} := L^p(X, m) \longrightarrow L^q(X, \mu), \ f \mapsto V_{r,p} f, \quad 1 < p \le q < \infty\cdot$$

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Let us emphasize that the latter operator is well defined because two quasi-continuous functions which coincide m-a.e., coincide quasi-everywhere, whence they coincide μ-a.e. since μ does not charge sets of zero capacity.

Our aim in this note is to give necessary and sufficient conditions for the operator $V^{\mu}_{r,p}$ to be bounded or even compact, provided $F_{r,p}$ is contractive. In particular we shall prove that the boundedness of $V^{\mu}_{r,p}$ is equivalent to some capacitary inequality. The importance of the result lies among others in the fact that one get the equivalence between the fact that $F_{r,p}$ is continuously 'embedded' into some Lebesgue space and capacitary-type inequality.

Moreover, once the criteria for the (p,q)-boundedness of $V^{\mu}_{r,p}$ are given, one can use the continuous embedding of $F_{r,p}$ into $L^q(\mu)$ as a starting point for defining traces of elements from $F_{r,p}$ on compact subsets. However, this is not the subject of this note. For an exposition of this approach in case $X = \mathbb{R}^d$ we refer the reader to [20, chap. 9].

We mention that such results are already known for the spaces of classical Bessel potentials, for operators V_r given by radially decreasing convolution kernel on $\mathbb{R}^d$ (see [1, 15]). For Dirichlet spaces as well as contractive Besov spaces, some of them are already known (see [2, 5, 21, 8, 9]).

The paper is organized as follows: in the next section we shall give the main tools and some preliminary results to handle the problem. In the last section, we prepare the capacitary strong type inequality and give the main result.

2. Preliminaries

Let X be a locally compact separable metric space and m a positive Radon measure on X whose support is X. For a positive Radon measure μ on X and $1 \le p < \infty$, we shall denote by $L^p(\mu)$ the real space of Borel measurable (equivalence classes of functions) f on X such that $\|f\|_{L^p(\mu)} := \left(\int_X |f|^p \, d\mu\right)^{1/p} < \infty$. For $p = \infty$, $L^\infty(\mu)$ is the space of measurable μ-almost everywhere bounded functions on X. The corresponding norm will be denoted by $\|\cdot\|_{L^\infty(\mu)}$. If $\mu = m$ the norm $\|\cdot\|_{L^p(\mu)}$ will be simply denoted by $\|\cdot\|_{L^p}$. Further the notation a.e. means m-almost everywhere.

The space of continuous functions with compact support on X (respectively vanishing at infinity) will be denoted by $C_c(X)$ (respectively $C_0(X)$). $C_0(X)$ is the space of continuous functions f on X such that for every $\epsilon > 0$ there is a compact subset K such that $|f| < \epsilon$ on K^c, the complement of K. Both spaces are equipped with the uniform norm, $\|\cdot\|_\infty$.

For every $t > 0$, let T_t be a symmetric strongly continuous contractive Markovian semigroup acting on L^2, i.e.:

1) $\lim_{t\to 0} \|T_t f - f\|_{L^2} = 0$, for every $f \in L^2$.
2) $\|T_t f\|_{L^2} \le \|f\|_{L^2}$ for every $f \in L^2$.
3) For every $f \in L^2$ such that $0 \le f \le 1$-a.e. we have $0 \le T_t f \le 1$-a.e.

It is known that for every $t > 0$ and every $1 \le p < \infty$, T_t decides a strongly continuous contractive Markovian semigroup on L^p which we denote by $T_{t,p}$. Whence the operator $V_{r,p}$, defined by formula (1), is also Markovian and contractive on every space L^p for $1 \leq p < \infty$.

From now on we shall definitively fix $r > 0$ and $2 \leq p < \infty$. By [4, Theorem 4.1], we have

$$V_{r,p} = (I - H_p)^{-r/2}. \tag{3}$$

Denote by $F_{r,p} := V_{r,p}(L^p)$. It is easily seen from (3) that $F_{r,p} = \mathrm{Dom}((I - H_p)^{r/2})$. For each $f \in F_{r,p}$ we define the norm

$$\|f\|_{r,p} := \|V_{r,p}^{-1} f\|_{L^p}. \tag{4}$$

Now since $V_{r,p}$ is injective on L^p, the space $\left(F_{r,p}, \|\cdot\|_{r,p}\right)$ is a reflexive Banach space. We shall denote by $F_{r,p}^*$ the dual space of $F_{r,p}$ and by $V_{r,p}^*$ the dual operator of $V_{r,p}$.

In the sequel we shall make the following two assumptions: First, we assume that the space $F_{r,p}$ is *regular*, this is

$$F_{r,p} \cap C_0(X) \text{ is dense both in } F_{r,p} \text{ and in } C_0(X) \tag{5}$$

with respect the relative norms.

Second, we assume that $F_{r,p}$ is *contractive*, i.e. for every normal contraction ψ and every $f \in F_{r,p}$, $\psi \circ f \in F_{r,p}$ and $\|\psi \circ f\|_{r,p} \leq \|f\|_{r,p}$. A normal contraction, ψ is a mapping $\mathbb{R} \longrightarrow \mathbb{R}$ such that $\psi(0) = 0$ and $|\psi(s) - \psi(t)| \leq |s - t|$ for every $s, t \in \mathbb{R}$.

Let us stress that the latter condition is satisfied for $p = 2$ and $r = 1$ because in this case $F_{1,2}$ is a Dirichlet space. It holds also true for every $0 < r < 1$ and every $p \geq 2$ under the conditions given in [11, Corollary 8]. Namely: The second order square field operator associated to the generator of T_t is positive.

Let us also emphasize that our assumptions are different from Kazumi-Shigekawa assumptions [14]. Indeed we do not suppose that $1 \in F_{r,p}$, neither the tightness of the capacity nor the existence of a subalgebra $\mathcal{D} \subset F_{r,p} \cap C(X)$ whose elements are bounded and that separates points of X.

Here are two examples of spaces satisfying our assumptions.

EXAMPLE 2.1. 1) The heat semigroup on manifolds: Let M_n be an n-dimensional connected complete Riemannian manifold with bounded geometry and positive injectivity radius (cf. [19, pp.282–284] for the definitions). Let Δ be the Laplace-Beltrami operator on M_n. Then it is known (cf. [18, 19]) that the corresponding heat semigroup $T_t, t > 0$ defines a strongly continuous contractive Markovian semigroup on L^p for $1 < p < \infty$. Moreover by the definition of Bessel spaces on manifolds [18, 19] and formula (3) we get

$$F_{r,p} = L^{r,p}(M_n),$$

the classical space of Bessel potentials on manifolds and in particular if r is an integer then $F_{k,p}$ is the Sobolev space on M_n. By [18, Theorem 4.3] $F_{r,p}$ is regular. Further by the chain rule the space $F_{1,p}$ is contractive for $1 < p < \infty$. Hence $L^{1,p}(M_n)$ satisfies our assumptions for every $2 \leq p < \infty$.

2) Lévy processes [12]: Let $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$, be a continuous negative definite function with representation

$$\psi(\xi) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(x \cdot \xi))\, d\nu(x),$$

where ν is Lévy measure integrating the function $1 \wedge |x|^2$. Consider the semigroup defined by

$$T_t u(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\xi} e^{-t\psi(\xi)} \hat{u}(\xi)\, d\xi,$$

for $u \in S(\mathbb{R}^d)$, the Schwartz space. Then (see [13]), for every $t > 0$, T_t decides a contractive strongly continuous Markovian semigroup on $L^p(\mathbb{R}^d)$ with generator H_p, for every $1 < p < \infty$. Define

$$H_p^{\psi,r} := (1 - H_p)^{-r/2}, \quad \text{for } r > 0.$$

Then by [12, Proposition 3.3.14], $S(\mathbb{R}^d)$ is dense in $H_p^{\psi,r}$ for every $r > 0$ and every $p \geq 2$ and by [12, Lemma 3.3.44], $H_p^{\psi,r}$ is contractive for $0 < r < 1$ and $2 \leq p < \infty$.

The (r, p)-capacity, which we denote by $\mathrm{Cap}_{r,p}$, is defined, for an open subset Ω, by (cf. [6])

$$\mathrm{Cap}_{r,p}(\Omega) = \inf\{\|f\|_{r,p}^p,\ \ f \in F_{r,p},\ \ f \geq 1 - \text{a.e. on } \Omega\}, \tag{6}$$

and for arbitrary subsets by

$$\mathrm{Cap}_{r,p}(\Omega) = \inf\{\mathrm{Cap}_{r,p}(\omega),\ \ \Omega \subset \omega,\ \ \omega \text{ open}\}. \tag{7}$$

We shall say that a property holds quasi-everywhere (q.e. for short) whenever it holds everywhere except on a set of (r, p)-capacity zero. A function f defined q.e. on X is said to be quasi-continuous if, for every $\epsilon > 0$ there is an open set Ω with $\mathrm{Cap}_{r,p}(\Omega) < \epsilon$ such that the restriction of f on $X \setminus \Omega$ is continuous.

We recall that, (see [6]) every $f \in F_{r,p}$ has a quasi-continuous modification $\tilde{f}$ and that if $\tilde{f} \geq 0$ a.e. then $\tilde{f} \geq 0$ q.e. Hence in the definition (6) we can change the assertion a.e. by q.e.

In the sequel we shall implicitly assume that elements from the space $F_{r,p}$ have been modified so as to become quasi-continuous.

By [6, Lemma 2], for every Borel subset B of finite (r,p)-capacity there is a unique positive element $e_B \in F_{r,p}$ such that

$$\text{Cap}_{r,p}(B) = \|e_B\|_{r,p}^p = \inf\{\|f\|_{r,p}^p,\ \ f \in F_{r,p},\ \ f \geq 1-\text{q.e. on } B\}. \tag{8}$$

The element e_B is called the (r,p)-equilibrium potential of B.

In what follows, we shall mean by capacity (respectively potential) the (r,p)-capacity (respectively the (r,p)-potential).

Along the note we shall assume that all measures under consideration are Radon measures and do not charge sets of zero capacity. We shall call these measures smooth Radon measures and denote them by S_0.

Proposition 2.1. *Let B be a Borel subset of X having finite capacity. Then $e_B = 1-$ q.e. on B.*

Proof. Since $F_{r,p}$ is contractive, then $\tilde{e}_B := (0 \vee e_B) \wedge 1 \in F_{r,p}$, $\|\tilde{e}_B\|_{r,p} \leq \|e_B\|_{r,p}$ and $\tilde{e}_B = 1$-q.e. on B. Hence $\tilde{e}_B$ is also a minimizer of $\|\cdot\|_{r,p}^p$ on the space $\mathcal{L}_B := \{f \in F_{r,p} \colon f \geq 1-\text{q.e. on } B\}$. Hence $\tilde{e}_B = e_B$-q.e. by the uniqueness of e_B, and the proof is finished. □

Let us consider the function $\mathcal{E}_{r,p}$ (see [14, p. 427]) defined on $F_{r,p} \times F_{r,p}$ by

$$\mathcal{E}_{r,p}(f,g) = \int_X V_{r,p}^{-1} f |V_{r,p}^{-1} g|^{p-1} \operatorname{sgn}\left(V_{r,p}^{-1} g\right)\, dm, \tag{9}$$

where 'sgn' is the usual sign-function.

Let p' be the conjugate of p. Consider the operator introduced in [14, 16] and defined by:

$$U_{r,p} := F_{r,p}^* \longrightarrow F_{r,p},\ \ \varphi \longmapsto V_{r,p}\left(\left|V_{r,p}^*\varphi\right|^{p'-1} \operatorname{sgn}\left(V_{r,p}^*\varphi\right)\right), \tag{10}$$

Then $U_{r,p}$ is well defined, bijective and continuous. Its inverse operator is given by

$$U_{r,p}^{-1} u = \left(V_{r,p}^*\right)^{-1}\left(\left|V_{r,p}^{-1} u\right|^{p-1} \operatorname{sgn}\left(V_{r,p}^{-1} u\right)\right). \tag{11}$$

Observing that $V_{r,p}$ is an isometry between the spaces L^p and $F_{r,p}$, we get

$$\mathcal{E}_{r,p}(f,g) = \left\langle V_{r,p}^{-1} f, V_{r,p}^* U_{r,p}^{-1} g\right\rangle_{L^p, L^{p'}} = \left\langle f, U_{r,p}^{-1} g\right\rangle_{F_{r,p}, F_{r,p}^*}. \tag{12}$$

Here $\langle\cdot\,,\,\cdot\rangle_{F_{r,p},F_{r,p}^*}$ is the duality product between $F_{r,p}$ and its dual space.

Let $\varphi \in F_{r,p}^*$. We say that φ is positive if $\langle f, \varphi\rangle_{F_{r,p},F_{r,p}^*} \geq 0$ for every positive $f \in F_{r,p}$. Let us denote by $F_{r,p}^{*,+}$ this set.

Proposition 2.2. *Let K be a compact subset of X. Then there is a unique positive Radon measure $\mu_K \in S$ such that* $\mathrm{supp}(\mu_K) \subset K$, $\mathrm{Cap}_{r,p}(K) = \mu_K(K)$ *and*

$$\mathcal{E}_{r,p}(f, e_K) = \int_X f\, d\mu_K, \quad \textit{for every } f \in F_{r,p} \cap C_0(X). \tag{13}$$

Here e_K is the equilibrium potential of K. We shall call μ_K the equilibrium measure of K.

Proof. Let K be compact. Arguing as [14, p.428], we realize that

$$\mathcal{E}_{r,p}(f, e_K) \geq 0 \tag{14}$$

for every $f \in F_{r,p}$ such that $f \geq 0$ q.e. on K. Now following Fukushima [7, p.75], for every $f \in F_{r,p} \cap C_0(X)$ set

$$I(f) := \mathcal{E}_{r,p}(f, e_K). \tag{15}$$

Let S be compact and $f_S \in F_{r,p} \cap C_0(X)$ be positive such that $f_S \geq 1$ on S. Then for every $f \in F_{r,p} \cap C_0(X)$ such that $\mathrm{supp}(f) \subset S$ we have $0 \leq |f| \leq \|f\|_\infty f_S$. Hence using the linearity of $\mathcal{E}_{r,p}$ with respect to the first argument together with (14) we get

$$|I(f)| \leq \|f\|_\infty I(f_S).$$

Now we shall extend I first from $F_{r,p} \cap C_c(X)$ to $C_c(X)$. Let $f \in C_c(X)$. Then there is a sequence $(f_k) \subset F_{r,p} \cap C_0(X)$ such that $\lim_{k\to\infty} \|f_k - f\|_\infty = 0$. Set

$$\tilde{f}_k := f_k - (((-k^{-1}) \vee f_k) \wedge k^{-1}).$$

Then by the contractivity property $(\tilde{f}_k) \subset F_{r,p} \cap C_0(X)$ and $\mathrm{supp}(\tilde{f}_k) \subset \mathrm{supp}(f)$. Moreover $\|f - \tilde{f}_k\|_\infty \leq 2k^{-1}$. Hence approximating f by $(\tilde{f}_k)$ in the uniform norm and making use of the inequality (14) we conclude that I has a unique extension as a positive functional, $\tilde{I}$, on $C_c(X)$. Thereby there is a unique positive Radon measure μ_K such that

$$\tilde{I}(f) = \int_X f\, d\mu_K \quad \text{for every } f \in C_c(X). \tag{16}$$

In particular, we have

$$\mathcal{E}_{r,p}(f, e_K) = \int_X f\, d\mu_K, \quad \text{for every } f \in F_{r,p} \cap C_c(X). \tag{17}$$

To show that $\mathrm{supp}(\mu_K) \subset K$ it suffices to prove that $\int_X f\, d\mu_K \geq 0$ for every $f \in C_c(X)$ such that $f \geq 0$ on K. Let $f \in F_{r,p} \cap C_c(X)$ be positive on K then by the latter identity and (14) we conclude that $\int_X f\, d\mu_K \geq 0$.

Now for general f, consider the $(\tilde{f}_k)$ constructed above. Let $\varphi \in F_{r,p} \cap C_c(X)$ be such that $\varphi = 1$ on K. set

$$g_k := \tilde{f}_k + 2k^{-1}\varphi.$$

Then $(g_k) \subset F_{r,p} \cap C_c(X)$ and $g_k \geq 0$ on K, so that $\int_X g_k \, d\mu_K \geq 0$. Observe that the supports of the g_k's are included in a fixed compact subset and that $\lim_{k\to\infty} \|g_k - f\|_\infty = 0$, yielding

$$0 \leq \lim_{k\to\infty} \int_X g_k \, d\mu_K = \int_X f \, d\mu_K, \tag{18}$$

and the result follows.

Now since μ_K has compact support, making use of equality (16), $\tilde{I}$ extends to $C_0(X)$ as a positive linear functional, $\tilde{\tilde{I}}$. Hence I extends to $C_0(X)$, in particular we achieve

$$\mathcal{E}_{r,p}(f, e_K) = \int_X f \, d\mu_K, \quad \text{for every } f \in F_{r,p} \cap C_0(X). \tag{19}$$

Finally with the help of the latter identity and the regularity property, we conclude that $\mu_K(B) = 0$ if $\mathrm{Cap}_{r,p}(B) = 0$ for every subset B, so $\mu_K \in S_0$. This property together with the property [6, (c)-p.45] imply that (19) extends to the whole space $F_{r,p}$. Finally, making use of Proposition 2.1, we get

$$\mu_K(K) = \int_X e_K \, d\mu_K = \mathcal{E}_{r,p}(e_K, e_K) = \mathrm{Cap}_{r,p}(K),$$

and the proof is finished. □

REMARK 2.1. Let $\mu \in S_0$ having compact support. Suppose that there is a constant C such that

$$\int_X |f| \, d\mu \leq C\|f\|_{r,p} \quad \text{for every } f \in F_{r,p}.$$

Then μ defines an element in the space $F_{r,p}^{*,+}$ as follows

$$L_\mu := F_{r,p} \longrightarrow \mathbb{R}, \quad L_\mu(f) = \int_X f \, d\mu.$$

We shall write

$$\langle f, \mu \rangle_{F_{r,p}, F_{r,p}^*} := L_\mu(f) = \int_X f \, d\mu, \tag{20}$$

and identify μ with L_μ. The element $U_{r,p}\mu$ will be called the potential of μ.

Arguing exactly as in the proof of Proposition 2.2 we achieve the following identity

$$\mathcal{E}_{r,p}(f, U_{r,p}\mu) = \int_X f\,d\mu, \quad \text{for every } f \in F_{r,p}. \tag{21}$$

If $\mu = \mu_K$ the equilibrium measure of a compact subset, the element $U_{r,p}\mu_K$ will be called the equilibrium potential of μ_K.

3. Trace inequalities and compactness criteria

Among other tools needed to study the boundedness of $V_{r,p}^{\mu}$, we shall need a *capacitary strong type inequality* [1, 8, 9, 10, 21]. Before giving the result we shall fix some notations. For every $f \in F_{r,p}$ and $t \geq 0$, we set

$$E_t := \{x : |f(x)| \geq t\}. \tag{22}$$

Theorem 3.1. *Let* $2 \leq p < \infty$ *and* $f \in F_{r,p} \cap C_0(X)$. *Then*

$$\int_0^\infty \mathrm{Cap}_{r,p}(E_t)\,dt^p \leq \frac{p^p}{(p-1)^{p-1}}\|f\|_{r,p}^p. \tag{23}$$

REMARK 3.1. We mention that a capacitary strong type inequality was first proved by Maz'ya [15, p.209] for the space $W^{1,p}(\mathbb{R}^d)$, $1 < p < \infty$ with the so called *condenser capacity* and with the sharp constant $\kappa_p := p^p/(p-1)^{p-1}$. The latter coincides with our constant for $p = 2$ and it was already observed by Fukushima-Uemura [8] that for $F_{1,2}$ the constant 4 in (23) is optimal.

Unfortunately, we do not know whether our constant is sharp or not in the general case.

Proof. We shall follow closely the proof of Adams-Hedberg [1], with a slight modification. Let $\lambda > 1$. Set

$$E_k := \{x : |f(x)| \geq \lambda^k\} \quad \text{for every } k \in \mathbb{Z}.$$

The inequality (23) follows from the inequality

$$\sum_{k=j}^{\infty} \lambda^{kp}\,\mathrm{Cap}_{r,p}(E_k) \leq \kappa_p \frac{\lambda}{\lambda-1}\|f\|_{r,p}^p, \quad \text{for every } j \in \mathbb{Z}, \text{ and every } \lambda > 1, \tag{24}$$

where $\kappa_p := (p/(p-1))^{p-1}$. Indeed, the latter inequality implies

$$\int_0^\infty \mathrm{Cap}_{r,p}\left(E_t\right) dt^p \leq (\lambda^p - 1)\sum_{k\in\mathbb{Z}} \lambda^{kp}\,\mathrm{Cap}_{r,p}(E_k)$$

$$\leq \kappa_p(\lambda^p - 1)\frac{\lambda}{\lambda - 1}\|f\|_{r,p}^p \quad \text{for all } \lambda > 1. \tag{25}$$

Now minimizing $(\lambda^p - 1)\lambda/(\lambda - 1)$ with respect to $\lambda > 1$ we get the desired inequality.

So our next aim is to prove the inequality (24).

Let $f \in F_{r,p} \cap C_0(X)$ be given. Then E_k is a compact subset and is empty for large k. Let μ_k be the equilibrium measure of E_k and e_k its equilibrium potential. For every $j \in \mathbb{Z}$, we denote by J_j the left side of (24). Then

$$\begin{aligned} J_j &= \sum_{k\geq j} \lambda^{kp} \int_{E_k} d\mu_k \leq \sum_{k\geq j} \lambda^{k(p-1)} \int_X |f|\, d\mu_k = \sum_{k\geq j} \lambda^{k(p-1)} \mathcal{E}_{r,p}(|f|, e_k) \\ &= \sum_{k\geq j} \lambda^{k(p-1)} \int_X V_{r,p}^{-1}|f||V_{r,p}^{-1}e_k|^{p-1} \operatorname{sgn}(V_{r,p}^{-1}e_k)\, dm \\ &\leq \|f\|_{r,p} \left\| \sum_{k\geq j} \lambda^{k(p-1)} |V_{r,p}^{-1}e_k|^{p-1} \right\|_{L^{p'}}, \end{aligned} \tag{26}$$

where the latter inequality is obtained from Hölder's inequality and the contraction property for the space $F_{r,p}$.

Since $V_{r,p}^{*}\mu_k \geq 0$ a.e. and thereby $V_{r,p}^{-1}e_k \geq 0$ a.e., we get

$$J_j \leq \|f\|_{r,p} \left\| \sum_{k\geq j} \lambda^{k(p-1)} \left(V_{r,p}^{-1}e_k\right)^{p-1} \right\|_{L^{p'}}. \tag{27}$$

For a.e. $x \in X$ and every $j \in \mathbb{Z}$, we shall set

$$A_j(x) := \left(V_{r,p}^{-1}e_j\right)^{p-1}(x), \quad \Lambda_j(x) := \sum_{k\geq j} \lambda^{k(p-1)} \left(V_{r,p}^{-1}e_k\right)^{p-1}(x) \text{ and } L_j := \|\Lambda_j\|_{L^{p'}}^{p'}.$$

Let us stress that both J_j and L_j are finite. Hence we conclude the proof by showing that

$$\|\Lambda_j\|_{L^{p'}}^{p'} \leq \frac{p}{p-1}\left(\frac{\lambda}{\lambda - 1}\right)^{p'-1} J_j, \tag{28}$$

for every $j \in \mathbb{Z}$ and every $\lambda > 1$.

For a.e. $x \in X$, let r_x be the function defined on the real line by $r_x(t) = \Lambda_j(x)$ for $j \leq t < j+1$. Then

$$\int_j^\infty (r_x(t))^{p'-1}\, dr_x = \sum_{k\geq j} (\Lambda_k(x))^{p'-1}(\Lambda_{k+1}(x) - \Lambda_k(x)) = -\frac{1}{p'}\left(\Lambda_j(x)\right)^{p'}, \tag{29}$$

yielding that

$$
\begin{aligned}
\big(\Lambda_j(x)\big)^{p'} &= p' \sum_{k\geq j} (\Lambda_k(x))^{p'-1}(\Lambda_k(x) - \Lambda_{k+1}(x)) \\
&= p' \sum_{k\geq j} (\Lambda_k(x))^{p'-1} \lambda^{k(p-1)} \big(V_{r,p}^{-1} e_k\big)^{p-1}. \qquad (30)
\end{aligned}
$$

By Hölder's inequality we get

$$
\begin{aligned}
\int_X \big(\Lambda_j(x)\big)^{p'}\, dm &= p' \int_X \sum_{k\geq j} \lambda^{kp(2-p')} A_k^{p'(2-p')} \Lambda_k^{p'-1} \lambda^{k(p'-1)} A_k^{(p'-1)^2}\, dm \\
&\leq p' \int_X \left(\sum_{k\geq j} \lambda^{kp} A_k^{p'} \right)^{2-p'} \left(\sum_{k\geq j} \Lambda_k \lambda^k A_k^{p'-1} \right)^{p'-1} dm \\
&\leq p' \left(\int_X \sum_{k\geq j} \lambda^{kp} A_k^{p'}\, dm \right)^{2-p'} \left(\int_X \sum_{k\geq j} \Lambda_k \lambda^k A_k^{p'-1}\, dm \right)^{p'-1} \\
&= p' L_{1,j}^{2-p'} L_{2,j}^{p'-1}, \qquad (31)
\end{aligned}
$$

where

$$
L_{1,j} := \int_X \sum_{k\geq j} \lambda^{kp} A_k^{p'}\, dm, \quad \text{and } L_{2,j} := \int_X \sum_{k\geq j} \Lambda_k \lambda^k A_k^{p'-1}\, dm.
$$

Let us observe that

$$
\int_X A_k^{p'}\, dm = \int_X (V_{r,p}^{-1} e_k)^p\, dm = \|e_k\|_{r,p}^p = \mathrm{Cap}_{r,p}(E_k),
$$

implying that $L_{1,j} = J_j$.

On the other hand we have

$$
\begin{aligned}
L_{2,j} &= \sum_{k\geq j} \lambda^k \sum_{l\geq k} \lambda^{l(p-1)} \int_X \big(V_{r,p}^{-1} e_l\big)^{p-1} V_{r,p}^{-1} e_k\, dm = \sum_{k\geq j} \lambda^k \sum_{l\geq k} \lambda^{l(p-1)} \mathcal{E}_{r,p}(e_k, e_l) \\
&= \sum_{k\geq j} \lambda^k \sum_{l\geq k} \lambda^{l(p-1)} \int_{E_l} e_k\, d\mu_l. \qquad (32)
\end{aligned}
$$

Now for every $l \geq k$, $E_l \subset E_k$. Hence by Proposition 2.1 we infer that e_k = 1-q.e. on E_l for every $l \geq k$. Thereby

$$
\int_{E_l} e_k\, d\mu_l = \mu_l(E_l) \quad \text{for every } l \geq k. \qquad (33)
$$

So we conclude that

$$L_{2,j} \leq \sum_{k\geq j} \lambda^k \sum_{l\geq k} \lambda^{l(p-1)} \mu_l(E_l) \leq \frac{\lambda}{\lambda - 1} J_j, \tag{34}$$

and the proof is completed. □

Next we shall use Theorem 3.1 to investigate the boundedness of the operator

$$V^{\mu}_{r,p} := L^p \longrightarrow L^q(\mu), \quad V^{\mu}_{r,p} f = V_{r,p} f. \tag{35}$$

Clearly this is equivalent to the boundedness of the operator

$$I^{\mu}_{r,p} := F_{r,p} \longrightarrow L^q(\mu), \tag{36}$$

whose to the m-equivalence class of $f \in F_{r,p}$ associates its μ-equivalence class. Let us emphasize that $I^{\mu}_{r,p}$ is not one-to-one (see [3, Remark 3.3]).

In the sequel we shall denote, for every compact subset K, by $\mu^K := 1_K \mu$, the restriction of μ to K.

Theorem 3.2. *Let $\mu \in S_0$ be a positive measure on X, $r > 0$ and $2 \leq p \leq q < \infty$. Then the following claims are equivalent*:
(i) *The operator $V^{\mu}_{r,p}$ is bounded from L^p into $L^q(\mu)$.*
(ii) *For every compact subset K of X, the measure $\mu^K \in F^{*,+}_{r,p}$ and there is a constant C_1 such that*

$$\left\| \left(V^{-1}_{r,p} U_{r,p} \mu^K \right)^{p-1} \right\|_{L^{p'}} \leq C_1 (\mu(K))^{1/q'} \tag{37}$$

(iii) *There is a constant C_2 such that*

$$\sup_{t>0} t(\mu(\{|f| \geq t\}))^{1/q} \leq C_2 \|f\|_{r,p}, \tag{38}$$

for every $f \in F_{r,p}$.
(iv) *There is a constant C_3 such that*

$$(\mu(K))^{1/q} \leq C_3 \left(\mathrm{Cap}_{r,p}(K) \right)^{1/p}, \tag{39}$$

for very compact subset K.

Moreover the constants C_j can be chosen so that

$$C_3 \leq C_2 \leq C_1 \leq \|V^{\mu}_{r,p}\|_{L^p, L^q(\mu)} \leq \frac{p^{p/q}}{(p-1)^{(p-1)/q}} \left(\frac{q}{p} \right)^{1/q} C_3 .$$

Inequality (38) may be seen as a weak-type Sobolev inequality [17, p.60] where the gradient energy form is replaced by the (r, p)-energy $\mathcal{E}_{r,p}$. Hence Theorem 3.2 says that the trace, the capacitary, and the weak-type Sobolev inequality are equivalent to each other.

We also infer from (i) $\Leftrightarrow$ (iii) that $V_{r,p}^{\mu}$ is of strong (p, q)-type if and only if it is of weak (p, q)-type.

Proof. (i) $\Rightarrow$ (ii): Let K be a compact subset. Then from the inequality

$$\int_X |f|\, d\mu^K \le \|V_{r,p}^{\mu}\|_{L^p,L^q(\mu)}(\mu(K))^{1/q'}\|f\|_{r,p} \quad \text{for every } f \in F_{r,p}, \tag{40}$$

we conclude that $\mu^K \in F_{r,p}^{*,+}$. Now set $e_K := U_{r,p}\mu^K$ and let $f \in L^p$. Then

$$\begin{aligned}\int_X f\big(V_{r,p}^{-1}e_K\big)^{p-1}\, dm &= \mathcal{E}_{r,p}\big(V_{r,p}f, e_K\big) = \int_X V_{r,p}f\, d\mu^K \\ &\le \|V_{r,p}f\|_{L^q(\mu)}(\mu(K))^{1/q'} \le \|V_{r,p}^{\mu}\|_{L^p,L^q(\mu)}\|f\|_{L^p}(\mu(K))^{1/q'},\end{aligned}$$

for every $f \in L^p$. Yielding $\big\|\big(V_{r,p}^{-1}e_K\big)^{p-1}\big\|_{L^{p'}} \le \|V_{r,p}^{\mu}\|_{L^p,L^q(\mu)}\big(\mu(K)\big)^{1/q'}$.

(ii) $\Rightarrow$ (iii): For every $t > 0$ and every $f \in F_{r,p} \cap C_0(X)$, we have

$$\begin{aligned} t\mu(E_t) &\le \int_X |f|\, d\mu_{E_t} = \mathcal{E}_{r,p}(|f|, e_{E_t}) \\ &= \int_X V_{r,p}^{-1}|f|\,\big(V_{r,p}^{-1}e_{E_t}\big)^{p-1}\, dm \le \|f\|_{r,p}\big\|\big(V_{r,p}^{-1}e_{E_t}\big)^{p-1}\big\|_{L^{p'}} \\ &\le C_1\|f\|_{r,p}(\mu(E_t))^{1/q'}, \end{aligned} \tag{41}$$

yielding $t(\mu(E_t))^{1/q} \le C_1\|f\|_{r,p}$ for every $t > 0$ and every $f \in F_{r,p} \cap C_0(X)$. For arbitrary $f \in F_{r,p}$ we get the result by approximation.

The proof of the implication (iii) $\Rightarrow$ (iv) is obvious, so we omit it.

(iv) $\Rightarrow$ (i): Let $f \in F_{r,p} \cap C_0(X)$. Then

$$\int_X |f|^q\, d\mu = \int_0^\infty \mu(E_t)\, dt^q \le C_3^q \int_0^\infty \big(\mathrm{Cap}_{r,p}(E_t)\big)^{q/p}\, dt^q. \tag{42}$$

Let us recall that $t\big(\mathrm{Cap}_{r,p}(E_t)\big)^{1/p} \le \|f\|_{r,p}$. Thus, since $q/p \ge 1$, we get

$$\int_0^\infty \big(\mathrm{Cap}_{r,p}(E_t)\big)^{q/p}\, dt^q \le \frac{q}{p}\|f\|_{r,p}^{q-p}\int_0^\infty \mathrm{Cap}_{r,p}(E_t)\, dt^p.$$

Finally we derive by Theorem 3.1

$$\int_X |f|^q\, d\mu \le \frac{q}{p}C_3^q\frac{p^p}{(p-1)^{p-1}}\|f\|_{r,p}^q\, . \tag{43}$$

For an arbitrary $f \in F_{r,p}$ the latter inequality is derived by standard approximation and the proof is finished. □

Corollary 3.1. *Let $2 \le p$ and $\mu \in S_0$ be positive. Assume that for every compact subset K, the measure $\mu^K \in F^{*,+}_{r,p}$ and that*

$$C := \sup_K \|U_{r,p}\mu^K\|_{L^\infty} < \infty. \tag{44}$$

Then $V^\mu_{r,p}$ is bounded from L^p into $L^p(\mu)$.

Let us observe that measures satisfying (44) extend the notion of potentially bounded measures in the linear case. Hence Corollary 3.1 generalizes [2, Corollary 3.1]

Proof. Let K be a compact subset of X. Set e_K the potential of μ^K. Then with the help of Remark 2.1 we get

$$\begin{aligned} \left\|\left(V^{-1}_{r,p}e_K\right)^{p-1}\right\|^{p'}_{L^{p'}} &= \int_X \left(V^{-1}_{r,p}e_k\right)^p\,dm = \mathcal{E}_{r,p}(e_K, e_K) \\ &= \int_X e_K\,d\mu^K \le C\mu(K), \end{aligned} \tag{45}$$

and the result follows using Theorem 3.2-ii. □

Now we will turn our attention to give necessary and sufficient conditions to the (p,q)-compactness of the operator $V^\mu_{r,p}$. We fix some notations: Let $\rho > 0$, $x_0 \in X$ and B_ρ the open ball of X centered at x_0 with radius ρ. We denote by $\mu_\rho := 1_{B_\rho}\mu$ and $V^\rho_{r,p} := V^{\mu_\rho}_{r,p}$.

Theorem 3.3. *Let $2 \le p \le q < \infty$. Then $V^\mu_{r,p}$ is compact from L^p into $L^q(\mu)$ if and only if the following two conditions are satisfied*:

i) *For every $\rho > 0$, $V^\rho_{r,p}$ is compact from L^p into $L^q(\mu_\rho)$.*

ii) $\lim_{\rho\to\infty} \sup\left\{\mu(K)\left(\mathrm{Cap}_{r,p}(K)\right)^{-q/p},\ K \subset X \setminus B_\rho,\ K \textit{ compact}\right\} = 0.$

Proof. Suppose that (i)–(ii) are satisfied. Fix $\epsilon > 0$ arbitrary small. Then by (ii), for every ρ, large enough and every compact subset $K \subset B^c_\rho := X \setminus B_\rho$, we have

$$(\mu(K))^{1/q} \le \epsilon \left(\mathrm{Cap}_{r,p} K\right)^{1/p}. \tag{46}$$

This implies, together with Theorem 3.2-(iv), that $V^{\mu^{B^c_\rho}}_{r,p}$ is bounded from L^p into $L^q(\mu^{B^c_\rho})$ and

$$\left\| V^{\mu^{B^c_\rho}}_{r,p} \right\|_{L^p, L^q\left(\mu^{B^c_\rho}\right)} \le \epsilon. \tag{47}$$

Now let $(f_n) \subset L^p$ such that $\|f_n\|_{L^p} \leq 1$ and (f_n) converges weakly to 0 in L^p. Then

$$\begin{aligned} \left\| V_{r,p}^{\mu} f_n \right\|_{L^q(\mu)}^q &\leq \left\| V_{r,p}^{\mu_\rho} f_n \right\|_{L^q(\mu_\rho)}^q + \int_{B_\rho^c} \left| V_{r,p}^{\mu^{B_\rho^c}} f_n \right|^q d\mu^{B_\rho^c} \\ &\leq \left\| V_{r,p}^{\mu_\rho} f_n \right\|_{L^q(\mu_\rho)}^q + \epsilon^q, \end{aligned} \tag{48}$$

where the latter inequality is obtained from (47). So passing to the limit and making use of assumption (i) we get

$$\lim_{n\to\infty} \left\| V_{r,p}^{\mu} f_n \right\|_{L^q(\mu)}^q \leq \epsilon^q,$$

for every ϵ arbitrary small, implying that $V_{r,p}^{\mu}$ is compact.

The converse: Let K be a compact subset, $f \in F_{r,p}$ such that $f \geq 1$-q.e. on K. Set $e_K := U_{r,p}\mu^K$. Then

$$\mu(K) \leq \int_X |f|\, d\mu^K = \mathcal{E}_{r,p}(|f|, e_K) \leq \|f\|_{r,p} \|e_K\|_{r,p}^{p-1}, \tag{49}$$

leading to

$$\mu(K) \leq \left(\mathrm{Cap}_{r,p}(K)\right)^{1/p} \|e_K\|_{r,p}^{p-1}. \tag{50}$$

Now for a compact subset $K_\rho \subset B_\rho^c$, set $\varphi_\rho := (\mu(K_\rho))^{-1/q'} 1_{K_\rho}$, where q' is the conjugate of q. Let $f \in L^q(\mu)$. Then

$$\left| \int_X f \varphi_\rho\, d\mu \right| \leq \left(\int_{B_\rho^c} |f|^q\, d\mu \right)^{1/q} \to 0 \quad \text{as } \rho \to \infty. \tag{51}$$

By assumption the adjoint operator of $V_{r,p}^{\mu}$, $\left(V_{r,p}^{\mu}\right)^* : L^{q'}(\mu) \to L^{p'}$ is compact. Thus $\lim_{\rho\to\infty} \left\| \left(V_{r,p}^{\mu}\right)^* \varphi_\rho \right\|_{L^{p'}} = 0$. Let us compute $\left(V_{r,p}^{\mu}\right)^* \varphi_\rho$. For convenience we shall omit, for the moment, the subscript ρ.

Let $f \in L^p$, then

$$\int_X \left(V_{r,p}^{\mu} f\right) \varphi\, d\mu = \int_X \left(V_{r,p} f\right) \varphi\, d\mu = (\mu(K))^{-1/q'} \int_X V_{r,p} f\, d\mu^K. \tag{52}$$

Since $V_{r,p}^{\mu}$ is bounded and μ^K has compact support we deduce from Remark 2.1 together with (52) that

$$\begin{aligned} \int_X V_{r,p}^{\mu} f \varphi\, d\mu &= (\mu(K))^{-1/q'} \mathcal{E}_{r,p}\left(V_{r,p} f, e_K\right) \\ &= (\mu(K))^{-1/q'} \int_X f \left(V_{r,p}^{-1} e_K\right)^{p-1} dm. \end{aligned} \tag{53}$$

Whence $(V^{\mu}_{r,p})^{*}\varphi = (\mu(K))^{-1/q'}\left(V^{-1}_{r,p}e_K\right)^{p-1}$ and

$$\|\left(V^{\mu}_{r,p}\right)^{*}\varphi\|_{L^{p'}} = (\mu(K))^{-1/q'}\left\|V^{-1}_{r,p}e_K\right\|^{p-1}_{p} = (\mu(K))^{-1/q'}\|e_K\|^{p-1}_{r,p}. \tag{54}$$

Finally putting all together we get

$$\begin{aligned}(\mu(K_\rho))^{1/q}\left(\mathrm{Cap}_{r,p}(K_\rho)\right)^{-1/p} &\le (\mu(K_\rho))^{1/q'}\|e_{K_\rho}\|^{p-1}_{r,p} = \|(V^{\mu}_{r,p})^{*}\varphi_\rho\|_{L^{p'}}\\ &\to 0 \quad \text{as } \rho\to\infty,\end{aligned} \tag{55}$$

and the proof is finished. □

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