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ON FIXED POINT FREE INVOLUTIONS OF T^3

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1. Introduction. In 1962, Y. Tao [4] proved the following theorem in this *Osaka Mathematical Journal*:

Theorem. *If h is a fixed point free involution of $S^1 \times S^2$, and if M is the 3-manifoldobtained by identifyingx and hx in S^1 X S^2, then M is either homeomorphic to* (1) $S^1 \times S^2$, *or* (2) *the* 3-*dimensional Klein Bottle, or* (3) $S^1 \times P^2$, *or* (4) $P^3 \# P^3$.

In order to prove the theorem, Tao used a result of Livesay [2] and simple cut and paste techniques. The question naturally arises as to whether or not Tao's method can be applied to classify the orbit spaces of fixed point free involutions on any manifold of the form $S^1 \times F$, where F is a compact surface. We answer this question affirmatively in the case when F is the 2-dimensional torus T^2. In particular, we shall show that if h is a fixed point free involution on the 3-dimensional torus $T^3 = S^1 \times T^2$, then pasting the points equivalent under h, we must obtain either T^3, or $S^1 X K^2$, or K^3, or the torus bundle over S^1 obtained from $[0,1]$ X T^2 by identifying the boundaries with a homeomorphism h of period two such that $h(m) = m^{-1}$ and $h(l) = ml$, where $(m, /)$ is a meridian-longitude system for T^2.

2. **Preliminaries.** The interior of a topological mainfold M will be denoted by *int M* and the boundary by ∂M. The n-dimensional sphere, torus and Klein bottle will be denoted by S^n, T^n, and K^n, respectively.

Since we may assume [3] that T^3 has a fixed triangulation and that h acts piecewise linearly on this triangulation, the objects in this paper (maps, neighborhoods, simple closed curves, etc.) should always be considered from the polyhedral point of view.

We shall think of T^3, K^3 and $S^1 X K^2$ as obtained from $[0,1]$ X T^2 by identifying 0 X T^2 with 1 X T^2. Thus, if (m, l) denotes a meridian-longitude pair for T^2 and $m_i = i \times m$, $l_i = i \times l (i=0,1)$, then identifying $0 \times T^2$ with 1 x T^2 so that m_0, l_0 gets glued onto m_1, l_1, respectively, results in a manifold homeomorphic to T^3. For K^3 we must identify m_0, l_0 with m_1^{-1} and l_1^{-1}, respectively, and for S^1 X K^2, m_0, l_0 identifies with m_1^{-1} and l_1, respectively.

3. Classification of T^3/h

Theorem 1. *If h is a fixed point free involution on T^3, then there is a torus T in T^3 which does not separate T^3 and T has the property that either $T=hT$ or $T \cap hT=\phi$.*

Proof. Let $T=0\times T^2$. If $hT\neq T$ and $T\cap hT\neq\phi$, then, by using small isotopic deformations of T whenever necessary, we may suppose that $T\cap hT$ consists of a finite number of disjoint simple closed curves. If / is a component of $T\cap hT$,then J satisfies one of the following three properties:

(i) J is homotopically trivial on both T and hT.

(ii) J is homotopically trivial on one of T or hT, but not both.

(iii) / is homotopically non-trivial on both T and hT.

If J is a simple closed curve in $T\cap hT$ such that J bounds a disc D on T or hT with the property that int $D\cap(T\cap hT)=\phi$, then D is called an *innermost* disc with respect of $T\cap hT$. Our next step is to eliminate all simple closed curves in $T\cap hT$ which bound innermost discs *on* hT and satisfy (i).

Suppose / is a component of $T\cap hT$ satisfying (i) and bounding an innermost disc $D\subset hT$. We denote by E the disc on T bounded by / and let /' be a simple closed curve in $T-E$, sufficiently close to /, such that the annulus A on T bounded by / U J' has the property that $A\cap hT=J$. We now choose a disc D' so close to D that D' satisfies $D'\cap T=\partial D'=J'$ and $D'\cap hT=D'\cap hD'=\phi$. This choice of D' is possible since D is innermost and h is fixed point free. Since we only replaced the disc A U E by the disc D', the torus $T'=[T-(E\cup A)]$ U U does not separate T^3. It follows that $T'\cap hT'$ contains fewer components of type (i) which bound innermost discs on hT. We repeat this process until we obtain a non-separating torus T'' with the property that no component of $T''\cap hT''$ which satisfies (i) bounds an innermost disc on hT''.

For the sake of convenience, we shall again denote our adjusted torus T'' by T. Suppose / is a component of $T\cap hT$ satisfying (ii) and bounding a disc D on hT. If D is not innermost, then there is a compoent J' of $T\cap hT$ with $J'\subset$ int D so that the disc $D'\subset$ int D bounded by J' is innermost. By our previous reduction argument, J' cannot satisfy (i), and hence, J' must be non-trivial on T. But this is impossible since $D'\cap T=J'$and T is incompressible in T^3. Similarly, D cannot be innermost on hT.

If J is trivial on T and non-trivial on hT, then hJ is trivial *on* hT and non-trivial on T which, by the above argument, is impossible. We may conclude that $T\cap hT$ contains no curves satisfying (ii). Furthermore, since all curves satisfying (i) and bounding innermost discs have been removed, either $T\cap hT=\phi$ or the components of $T\cap hT$ must all satisfy (iii).

Since Γ does not separate T^3, it is possible that $T\cap hT$ contains exactly one component. We shall consider this case first.

A. $T\cap hT$ consists of exactly one simple closed curve J. Since $hJ=J$, there is a sufficiently small regular neighborhood N of J such that $hN=N$, ∂N separates T^3 and $(T\cup hT)\cap\partial N$ consists of four disjoint simple closed curves $c_1, \cdots, c_4$. The set $\cup c_i$ divides dN into four annuli $a_1, \cdots, a_4$ and we assume that all subscripts have been arranged in order to satisfy $a_1\cap a_2=c_1\subset T$, $a_2\cap a_3=c_2\subset hT$, $a_3\cap a_4=c_3\subset T$ and $a_4\cap a_1=c_4\subset hT$ (figure 1; the diagram represents a "meridian" cut). The curves c_1 and c_3 divide T into two annuli A and B, and we let B denote the annulus containing J. Since $h^2=1$, $h(a_2\cup a_4)=a_2\cup a_4$. It follows that $T'=A\cup hA\cup a_2\cup a_4$ remains invariant under h. Thus, if T' does not separate T^3, then T' satisfies the conclusion of Theorem 1.

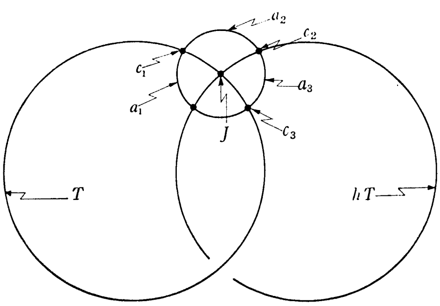

Figure 1.

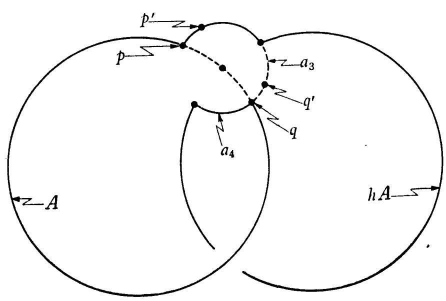

Figure 2.

Suppose T' separates T^3. Let U and V denote the components of T^3-T'. Let J' be a simple closed curve on T, transverse to c_1 and c_3, and let p and q be the points $c_1\cap J'$ and $c_3\cap J'$, respectively. Let p' and q' be two points in int a_2 and int a_3, close to p and q, respectively (figure 2). We may choose simple arcs a and β on a_2 and a_3, respectively, with $\partial\alpha=p\cup p'$, $\partial\beta=q\cup q'$, $\alpha\cap J'=p$, $\beta\cap J'=q$ and $(\alpha\cup\beta)\cap hA=\phi$. Since $a_3\cap T'=\partial a_3$, either int $a_3\subset U$ or int $a_3\subset V$, and we suppose that int $a_3\subset U$. Hence q' is a point in U and int $N\subset U$. If γ denotes the arc $\alpha\cup\beta\cup(J'-\text{int}(B\cap J'))$, then $\gamma\cap hA=\phi$ and we may push γ slightly off $a_2\cup a_3\cup A$, away from N and missing hA. Thus, we can obtain an arc missing T', with one end point in U and the other in V. It follows that T' cannot separate T^3.

B. $T\cap hT$ contains more than one component. If n is the number of components of $T\cap hT$, then $T\cap hT$ divides hT into n annuli, $A_1, \cdots, A_n$, such that $T\cap\text{int}\,A_i=\phi$. Each annulus A_i can satisfy one and only one of the following conditions:

(1) $\partial A_i\cap\partial hA_i=\partial A_i$

(2) $\partial A_i\cap\partial hA_i=\phi$

(3) $\partial A_i\cap\partial hA_i$ contains exactly one component of $T\cap hT$.

We consider each of these possibilities separately.

1. Suppose $A_i\subset hT$ is an annulus satisfying (1). Then ∂A_i divides T into

two annuli A and B. Since $T \cap$ int $A_i = \phi$, we may suppose, without loss of generality, that $hA_i = A$. Let $T_1 = A_i \cup A$and $T_2 = A_i \cup B$. Then at least one of the tori T_1 or T_2 does not separate T^3. For suppose both T_1 and T_2 separate T^3. Let U_1, V_1 and U_2, V_2 denote the components of $T^3 - T_1$ and $T^3 - T_2$, respectively. Since $B \cap T_1 = \partial B$, either int $B \subset U_1$, or int $Bd\ V_1$, and we suppose, without loss of generality, that int $B \subset U_1$. Similarly, we may suppose int Ad U_2. Let p be a point in int B and p_1, p_2 two points sufficiently close to p such that $p_1 \in U_1 \cap U_2$ and $p_2 \in U_1 \Pi V_2$. Let a be a simple arc in T^3 going from p_1 to p_2, and such that $\alpha \cap T = \phi$. Since p_1 lies in U_2 and p_2 in V_2, a must intersect T_2 and it can intersect T_1 only in int A_i. We may assume that $\alpha \cap A_i$consists of a finite number of piercing points. Since p_1 and p_2 both lie in U_1, and since each time we pierce through A_i we pass from U_1 into V_1 or vice versa, the number of piercing points in $a \Pi$ int A_i must be even. Let $q_1, \cdots, q_{2k}$ denote the components of $a \cap$ int A_i, and let the subscripts be ordered such that q_i precedes q_{i+1} when traversing a from p_1 to p_2. Let α_i denote the subarc of a from q_{2i-1} to q_{2i}. We now replace each α_i by a simple arc $\alpha'_i \subset$ int A_i from q_{2i-1} to q_{2i} such that $\alpha'_i \cap \alpha'_j = \phi$, if $i \neq j$. Let the new arc thus obtained be again denoted by a. Then pushing each α_i slightly off int A_i and into U_1, we may delete the intersection $\alpha \cap A_i$,keeping $\alpha \cap T = \phi$. Thus, we may obtain an arc joining p_1 in U_2 with p_2 in V_2 and missing T_2. This contradicts the assumption that T_2 separates T^3.

If T_1 does not separate T^3, then since $hT_1 = h(A_i \cup A) = A \cup A_i = TT_1$ is invariant under h and, therefore, satisfies the conclusion of Theorem 1.

If T_1 separates T^3, then T_2 does not separate T^3. Let J and K denote the components of ∂A_i. We may choose disjoint simple closed curves J' and K' on int B and sufficiently close J and K, respectively, such that the annuli $R_1 \subset B$ and $R_2 \subset B$ with $\partial R_1 = J \cup J'$, $\partial R_2 = K \cup K'$have the property that $hT \cap$ int R_i

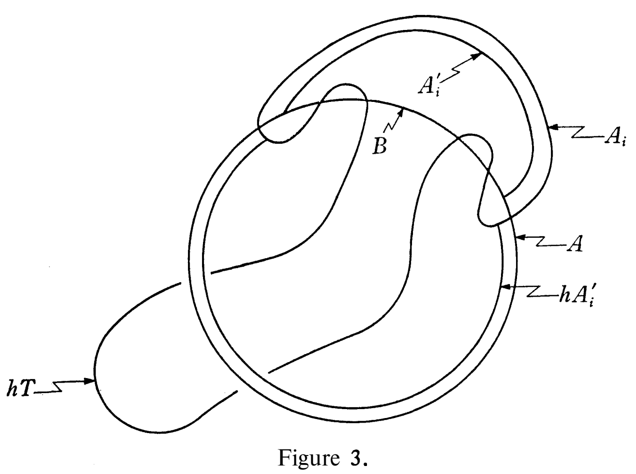

Figure 3.

$=\phi$. Let A_i' be an annulus sufficiently close to A_i and B (figure 3) so that A_i' satisfies the following conditions:

(i) $A_i' \cap T = A_i' \cap B = \partial A_i' = J' \cup K$ and $A_i' \cap hT = \phi$.

(ii) $T' = (B - \cup R_i) \cup A_i'$ does not separate T^3.

(iii) $A_i' \cap hA_i' = \phi$.

Since $T_2 = B \cup A_i$ does not separate T^3, condition (ii) can be easily satisfied for A_i' sufficiently close to A and J', K' close to J and K, respectively. Condition (iii) is obtainable since $T \cap \operatorname{int} A_i = \phi$, $hA_i' \cap T = \phi$ and $hA_i' \cap hT = h\partial A_i'$.

By our construction, the number of components of $T' \cap hT'$ is strictly less than the number of components of $T \cap hT$. Since the number of components in $T \Pi hT$ is finite, we can find - by proceeding with the above argument - a torus T'' which does not separate T^3 and satisfying exactly one of the following: (a) $hT'' = T''$, (b) $hT'' \cap T'' = \phi$, (c) $T'' \cap hT''$ contains exactly one simple closed curve, (d) $T'' \cap hT''$ has r components, $1 < r < n$, dividing hT'' into r annuli, A_1'', $\cdots, A''$, such that no A'' satisfies condition (1).

Both (a) and (b) satisfy the conclusion of Theorem 1. Case A applies if (c) holds. Thus, in order to complete our proof, we have only to consider (d). For convenience, we again denote T'' by T and A_i' by A_i.

2. Suppose no annulus $A_i \subset hT$ satisfies (1) and suppose that for some i, A_i satisfies (2). Then ∂A_i divides T into two annuli A and B. Since $T \cap \operatorname{int} A_i = \phi$ and $\partial A \cap \partial hA_i = \phi$, we may suppose, without loss of generality, that $hA_i \subset \operatorname{int} A$. Let $T_1 = A_i \cup A$, $T_2 = A_i \cup B$ and J, K the boundary components of A_i. As before, at least one of the tori, T_1 or T_2, does not separate T^3.

If T_1 does not separate T^3, let J' and K' be two simple closed curves on $\operatorname{int} A$, close to J and K, respectively, and A_i' an annulus sufficiently close to A_i such that the following conditions are satisfied:

(i) $A_i' \cap T = A_i' \cap A = \partial A_i' = J' \cup K'$ and $A_i' \cap hT = \phi$.

(ii) If R_1 and R_2 denote the annuli on A bounded by /,]' and K, K', respectively, then $hT \cap \operatorname{int} R_i = \phi$.

(iii) $T' = (A - \cup R_i) \cup A_i'$ does not separate T^3.

(iv) $A_i' \cap hA_i' = \phi$

As in argument 1. above, all these conditions can be satisfied. By construction, the number of components of $T' \cap hT'$ is strictly less than the number of components of $T \cap hT$.

If T_2 does not separate T^3, then J and K are not components of $T_2 \cap hT_2$ and, hence, the number of components of $T_2 \cap hT_2$ is strictly less than the number of components of $T \Pi hT$.

Since repeated application of the above algorithm reduces the number of annuli satisfying (2), we may now assume that there is a non-separating torus T in T^3 such that either $hT = T$ or $hT \cap T$ consists of a finite number of simple closed curves dividing hT into a finite number of annuli, with each annuli satisfying only condition (3).

3. Suppose each annulus $A_i \subset hT$ satisfies (3). Let / be a component of $T \cap hT$ which remains invariant under h, and let A_i and A_j denote the two annuli on hT with the property $A_i \cap A_j = J$. Let J_i and J_j denote the other boundary components of A_i and A_j, respectively. The curves J and J_i divide T into two annuli A and B, and we suppose, without loss of generality, that $hA_i \subset A$ and, therefore, $hA_j \subset B$. The set $J \cup J_j$ divides T into two annuli, B_1 and B_2, and we suppose that $hA_i \subset B_1$ (figure 4). For the same reasons as given in B.1., at least one of the tori $T_2 = A_j \cup B_1$ or $T_2 = A_j \cup B_2$ does not separate T^3.

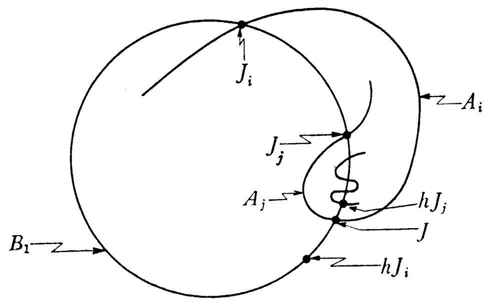

Figure 4.

If T_1 does not separate T^3, let J'_j and J' be two simple closed curves on int B_1 close *to Jj* and J, respectively, and A'_j an annulus with boundary $J'_j \cup J'$, and sufficiently close to A_j, such that the following hold:

(i) $A'_j \cap T = A'_j \cap B_1 = J'_j \cup J'$ and $A'_j \cap hT = \phi$.

(ii) If R_1 and R_2 denote the annuli on B_1 with boundary components J, J' and J_j, J'_j, respectively, then $hT \cap \text{int } R_i = \phi$.

(iii) $T' = (B_1 - \cup R_i) \cup A'_j$ does not separate T^3.

(iv) $A'_j \cap hA'_j = \phi$.

Again, all these conditions can be satisfied, and the number of components of $T' \cap hT'$ is strictly less than the number of components of $T \cap hT$.

If T_1 separates T^3, then T_2 cannot separate T^3. Hence, at least one of $B \cup A_i$ or $(B - \text{int } B_2) \cup A_i \cup A_j$ does not separate T^3.

If $B \cup A_i$ does not separate, let J' and J'_i be two simple closed curves on int B, close to J and J_i, respectively, and A'_i an annulus sufficiently close to A_i such that the following conditions are satisfied:

(i) $A'_i \cap T = A'_i \cap B = \partial A'_i = J' \cup J'_i$ and $A'_i \cap hT = \phi$.

(ii) If R_1 and R_2 denote the two annuli on B bounded by J, J' and J_i, J'_i, respectively, then $hT \cap \text{int } R_i = \phi$.

(iii) $T' = (B - \cup Ri) \cup A'_i$ does not separate T^3.

(iv) $A'_i \cap hA'_i = \phi$.

As in the previous cases, all these conditions are easily satisfied, and the number of components of $T\cap hT'$ is strictly less than the number of components of $T\cap hT$.

If $(B-\operatorname{int} B_2)\cup A_i\cup A_j$ does not separate, we simply set $T'=(B-\operatorname{int}B_2)\cup A_i\cup A_j$ and note that since $hJ_j\cap T'=\phi$, $T'\cap hT'$ contains fewer components than $T\cap hT$.

We have shown that there is always a torus T' in T^3 which does not separate T^3 and such that either $hT'=T'$ or $T'\cap hT'$ has fewer components than $T\cap hT$. Since the number of components of $T\cap hT$ is finite, a finite number of repetition of the above argument will yield a torus T'' which does not separate T^3 and such that either $T''\cap hT''=\phi$ or $T''=hT''$.

This proves Theorem 1.

Let $M_1=T^3$, $M_2=K^3$, $M_3=S^1\times K^2$ and M_4 the torus bundle over S^1 obtained from $[0,1]\times T^2$ by identifying $0\times T^2$ with $1\times T^2$ by a homeomorphism h of T^2 such that $h^2=1$, $h(m)=m^{-1}$ and $h(l)=ml$, where $(m, /)$ is a meridian-longitude system for T^2.

Theorem 2. *If h is a fixed point free involution on T^3, then T^3/h is homeomorphic to M_i forsome $i=1$, 2, 3 or 4.*

Proof. By Theorem 1, there is a torus $T\subset T^3$ which does not separate T^3 and satisfies either $T\cap hT=\phi$ or $T=hT$. We divide our proof according to these two possibilities.

A. We suppose that $hT=T$. Since T does not separate T^3 and the inclusion of T into $[0,1]$ X T induces an isomorphism on the fundamental groups, we have by Theorem 3.4 of [1] that the space obtained from T^3 by cutting T^3 open along T must be homeomorphic to $[0,1]$ X T. Thus, we may suppose that h is a fixed point free involution of the toroidal shell $[0,1]$ X T^2, leaving each boundary invariant.

Let m_i and l_i denote the meridian and longitude, respectively, of $i\times T^2$, $i=0.1$. Since h is of period two and $h(m_i)\cup h(l_i)$ is a pair of transverse simple closed curves on $i\times T^2$ intersecting in exactly one point, it follows that either $h(m_i)=m_i$ and $h(l_i)=l_i^{\pm 1}$, or $h(m_i)=m_i^{-1}$ and $h(l_i)=l_i^{\pm 1}$ or $h(l_i)=m_il_i$, or $h(m_i)=l_i$ and $h(l_i)=m_i$, or $h(m_i)=l_i^{-1}$ and $h(l_i)=m_i^{-1}$, or $h(m_i)=m_il_i$ and $h(l_i)=l_i^{-1}$, up to isotopy on $i\times T^2$. Using general positioning, we may assume that $m_i\cap h(m_i)$ consists of at most a finite number of crossing points. Since h is of even period and fixed point free, the number of crossing points in $m_i\cap h(m_i)$ cannot be odd. Hence, $h(m_i)=m_i^{\pm 1}$ and, similarily, $h(l_i)=l_i^{\pm 1}$, up to isotopy on ix T^2.

Let $V_i=D^2\times S^1$, where $i=0$, 1 and $D^2=\{(x,y)\in R^2\,|\,x^2+y^2\leq 1\}$. Then the space obtained by attaching $V_0\cup V_1$ to $[0, 1]$ X T^2 by gluing ∂V_i to ix T^2 such that the meridian and longitude of V_i gets identified with ra,- and l_i, respectively, is homeomorphic to the lens-space $L(0, 1)=S^1\times S^2$. Extending h radially to the core of V_i induces a piecewise linear extension h' of h to all of S^1 X S^2. By

[5], h' is equivalent to a standard rotation of $S^1 X S^2$. Therefore, h is equivalent to $e\times k$: $[0, 1] X T^2 \to [0, 1] X T^2$, where e is the identity on [0, 1] and k is one of the two standard fixed point free involutions of T^2. Since T^2/k is either homeomorphic to T^2 or K^2, matching again the boundaries of $[0, 1] X T^2$, we observe that T^3/h is either homeomorphic to $S^1 X T^2$ or $S^1 X K^2$.

B. We suppose that $T \cap hT = \phi$. Since T does not separate T^3, we again have by [1] that the set $T \cup hT$ divides T^3 into two components V and W, each homeomorphic to $[0, 1] X T^2$. We must consider two cases, namely $hV = W$ and $hV = V$.

If $hV = W$, then T^3/h may be viewed as being obtained from $[0, 1] X T^2$ by identifying the boundaries with the homeomorphism h. If m_i and l_i are as in case A, then either $h(m_0)=m_1$ and $h(l_0)=l_1^{+1}$, or $h(m_0)=m_1^{-1}$ and $h(l_0)=l_1^{\pm 1}$ or $h(l_0)=m_1 l_1$, or $h(m_0)=m_1 l_1$ and $h(l_0)=l_1^{-1}$, or $h(m_0)=l_1$ and $h(l_0)=m_1$, or $h(m_0)=l_1^{-1}$ and $h(l_0)=m_1^{-1}$, up to isotopy on T^2. Thus, for all but the last two cases it is clear that T^3/h is homeomorphic to M_i for some $i=1, 2, 3$, or 4. If $h(m_0)=l_1$ and $h(l_0)=m_1$, then T^3/h must be homeomorphic to T^3. In order to see this, let J be a (1, 1)-curve, $m_0 l_0$, on $O x T^2$ and let R be an annulus properly embedded in V such that $R \cap \partial V = \partial R = J \cup hJ$. Then the torus $T' = R \cup hR$ does not separate T^3 and case A applies. We use the exact same argument if $h(m_0)=l_1^{-1}$ and $h(l_0) = m_1^{-1}$ in order to obtain the non-separating torus T'. In this case $T^3/h = S^1 \times K^2$ up to homeomorphism since h is orientation reversing.

If $hV = V$ with $h(m_0)=m_1 l_1$ and $h(l_0)=l_1^{-1}$, then $h(m_0)$ is a non-trivial multiple in $V=[0, 1] x T^2$ of l_1 and m_1. Therefore, $h(h(m_0))$ must be homologous in V to a non-trivial multiple of $h(l_1)$ and $h(m_1)$. But $h(h(m_0))=m_0$ while $h(l_1)=l_0^{-1}$. Hence, *if* $hV=V$, we cannot have $h(m_0)=m_1 l_1$ and $h(l_0)=l_1^{-1}$. Similarily, $h(m_0) = m_1^{-1}$ and $h(l_0)=m_1 l_1$ is impossible.

If $h(m_0)=l_1$ and $h(l_0)=m_1$, let S^3 be obtained from $V=[0, 1]\times T^2$ by filling in ∂V with two solid tori V_0 and V_1 such that the meridian and longitude of V_0 identify with m_0 and l_0, respectively, while the meridian and longitude of V_1 identify with l_1 and m_1, respectively. Then h extends naturally to a fixed point free involution of S^3 which interchanges the core of V_0 with a core of V_1. However, by chapter 3 of [3], this situation is impossible. Similarily, we cannot have $h(m_0)=l_1^{-1}$ and $h(l_0)=m_1^{-1}$.

If $h(m_0)=m_1$ and $h(l_0)=l_1$, then as in case A let $S^1\times S^2$ be obtained from V by filling in ∂V with the two solid tori V_0 and V_1. We now extend h to all of $S^1\times S^2$ by extending h radially to the core of V_i. More precisely, if $(x, y, z) \in V_0$, where $(x, y) \in D^2$ and $z \in S^1$, let R denote the radius of $D^2 \times \{z\}$ from $(0, 0, z)$ to ∂V_0 passing through (x, y, z). Then, if (x', y', z) denotes the point $R \cap \partial V_0$, map (x, y, z) to the point (x, y, z'') in V_1, where (x, y, z'') lies on the radius from $h(x', y, z)=(x'', y'', z'')$ to $(0, 0, z'')$. If we again let h denote this extension, then h is a fixed point free involution on $S^1 X S^2$. It follows from [4] that h is conjugate to one of the four standard fixed point free rotations of $S^1\times S^2$. Hence, there is a torus T' in $[0, 1]\times T$ which is isotopic to $0\times T$ and invariant under h and case A applies. We argue analogously if $h(m_0)=m_1^{-1}$ and $h(l_0)=l_1^{\pm 1}$. This proves Theorem 2.

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