

Title	A Heegaard-diagram of the 3-sphere
Author(s)	Ochiai, Mitsuyuki
Citation	Osaka Journal of Mathematics. 1985, 22(4), p. 871-873
Version Type	VoR
URL	<a href="https://doi.org/10.18910/10460">https://doi.org/10.18910/10460</a>
rights	
Note	

*Osaka University Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

Osaka University

## A HEEGAARD-DIAGRAM OF THE 3-SPHERE

MITSUYUKI OCHIAI

(Received October 23, 1984)

### 1. Introduction

As an algorithm for recognizing  $S^3$  in 3-manifolds, there exists the Whitehead algorithm. This algorithm is partly true by Homma-Ochiai-Takahashi [4], but in general not true by Viro [8], Ochiai [7], and Morikawa [6]. That is, Whitehead conjecture [9], [1], which asserts that all Heegaard diagram, other than the canonical one, have always waves, is true in the case of genus two, but not true in the case of genus greater than two. All already known counterexamples to the conjecture was constructed as Heegaard diagrams of 2-fold branched coverings branched along knot diagrams of the trivial knot. In this paper, we construct such a counterexample through the different method using presentations of the trivial group, and the resulting diagram has two interesting properties different from already known examples.

### 2. A new counterexample to Whitehead conjecture

For all definitions of Heegaard diagrams, complete systems of meridians, band moves, waves, and others we refer to [4].

At first, let's choose a trivial group

$$G = \{X, Y, Z; XY^2X^{-1} = Y^3, YX^2Y^{-1} = X^3, Z = 1\}$$

It will be noticed that the group  $G$  is trivial by Crowell-Fox [3] and by Birman-Hilden [2] there are no Heegaard diagrams which have relators of  $G$  as that of the fundamental groups induced by them. Hence we may change the relators of  $G$  and get a new presentation of the trivial group

$$H = \{X, Y, Z; XY^2X^{-1}Y^{-1}ZY^{-1}ZY^{-1}Z = 1, \\ YX^2Y^{-1}X^{-1}Z^{-1}X^{-1}Z^{-1}X^{-1}Z^{-1} = 1, \\ ZX YX^{-1}Y^{-1} = 1\}$$

It will be noticed that  $H$  is obtained from  $G$  by the trial and error method and that the relators of  $H$  is induced by some Heegaard diagrams of  $S^3$ . Next

we may change the diagram through a band move, and the resulting diagram  $\psi$  is obtained which is illustrated in Figure 1. Let  $\psi$  denote  $(F; v, w)$ . Then one complete system of meridians of  $F$ ,  $v$  is illustrated as in Figure 1. Moreover, the Heegaard surface  $F$  is obtained from Figure 1 by identifying meridians  $\partial A, \partial C, \partial E$  with  $\partial B, \partial D, \partial F$ , respectively, where  $A, B, C, D, E, F$  are meridian-disks in Figure 1. At the same time, another complete system of meridians,  $w$  is obtained as follows; identify points  $A_1, A_2, \dots, A_9$  with points  $B_1, B_2, \dots, B_9$ , points  $C_1, C_2, \dots, C_{14}$  with points  $D_1, D_2, \dots, D_{14}$ , and points  $E_1, E_2, \dots, E_{15}$  with  $F_1, F_2, \dots, F_{15}$ , respectively. The last Heegaard diagram has as the fundamental group the following group  $H'$ ;

$$H' = \{X, Y, Z; XZY^2Z^{-1}X^{-1}(Y^{-1}Z^{-1})^3 = 1, \\ ZY(XZ)^2Y^{-1}Z^{-1}X^{-3} = 1, \\ XZY^3Z^{-1}X^{-1}(Y^{-1}Z^{-1})^4 = 1\}$$

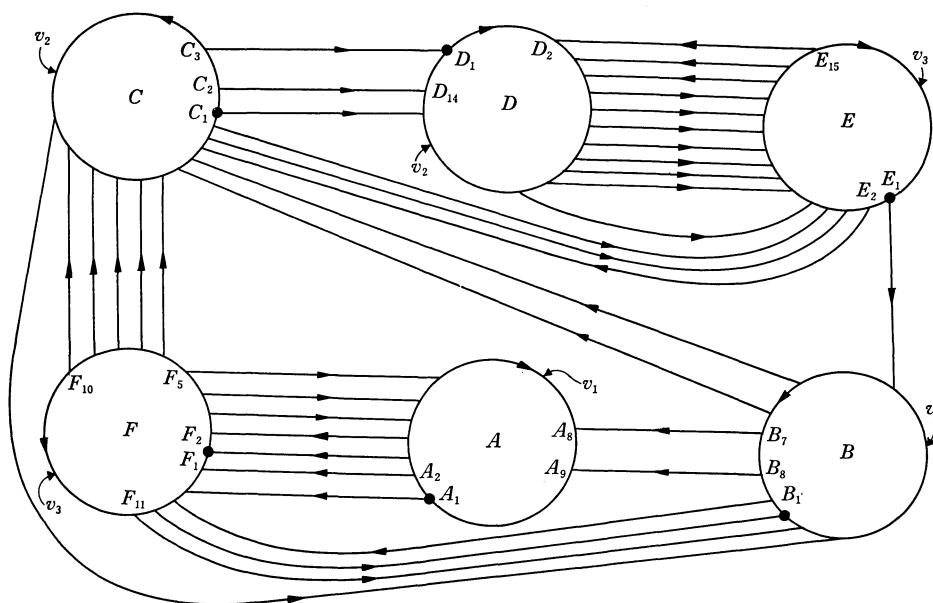


Figure 1.

By the way, the fundamental group of the dual diagram of  $\psi$  is the following;

$$H'' = \{A, B, C; ABC^2 = C^3BA, (AB)^2AC = CA(BA)^3, \\ CABC^2 = (AB)^3ACBA\}$$

It will be noticed that both of  $H'$  and  $H''$  are not simply trivial different from Kaneto's example [5]. The Heegaard diagram  $\psi$  has no waves and so is an counterexample to Whitehead conjecture. Moreover, different from already

known examples,  $\psi$  does not permit us to reduce directly Heegaard genus of it, but permit us to construct another Heegaard diagram which has arbitrary many intersections of one complete system of meridians and another one. Such a Heegaard diagram has as the fundamental group the following;

$$H(n, m) = \{X, Y, Z; XZY^{n-1}Z^{-1}X^{-1}(Y^{-1}Z^{-1})^n = 1, \\ ZY(XZ)^{m-1}Y^{-1}Z^{-1}X^{-m} = 1, \\ XZY^nZ^{-1}X^{-1}(Y^{-1}Z^{-1})^{n+1} = 1\}$$

It will be noticed that the construction method described above permit us to make homotopy 3-spheres with complicated presentations of the fundamental group such that they have Heegaard diagrams of genus three and might be homotopy 3-spheres other than the 3-sphere. To construct fake homotopy 3-spheres by the method, it must be impossible without making use of computer.

---

#### References

- [1] J. Birman: *Heegaard splittings, diagrams and sewings for closed, orientable 3-manifolds*, Lecture Notes for CBMS Conference at Blacksburg, Va., Oct., (1977), 8–12.
- [2] J. Birman and H. Hilden: *Heegaard splittings of branched coverings of  $S^3$* , Trans. Amer. Math. Soc. **213** (1975), 315–352.
- [3] R. Crowell and R. Fox: *Introduction to knot theory*, Ginn and Company, New York, 1963.
- [4] T. Homma, M. Ochiai, and M. Takahashi: *An algorithm for recognizing  $S^3$  in 3-manifolds with Heegaard splittings of genus two*, Osaka J. Math. **17** (1980), 625–648.
- [5] T. Kaneto: *On presentations of the fundamental group of the 3-sphere associated with Heegaard diagrams*, J. Math. Soc. Japan **33** (1981), 147–158.
- [6] O. Morikawa: *A counterexample to a conjecture of Whitehead*, Math. Sem. Notes Kobe Univ. **8** (1980), 295–298.
- [7] M. Ochiai: *A counterexample to a conjecture of Whitehead and Volodin-Kuznetsov-Fomenko*, J. Math. Soc. Japan **31** (1979), 687–691.
- [8] O. Ja. Viro: *The Volodin-Kuznetsov-Fomenko conjecture on Heegaard diagrams is false*, Uspeki Math. Nauk **32** (1977), 175–176.
- [9] J.H.C. Whitehead: *On certain sets of elements in a free group*, Proc. London Math. Soc. **41** (1936), 48–56.

Department of Mathematics  
Faculty of Science  
Osaka University  
Toyonaka, Osaka 560  
Japan

