ON THE NON-TRIVIALITY OF THE GREEK LETTER ELEMENTS IN THE ADAMS-NOVIKOV $E_2$-TERM

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1. Introduction

For a prime $p$, there is a spectral sequence, called the Adams-Novikov spectral sequence, converging to the stable homotopy ring of spheres localized at $p$, $\pi^{s}_*(p)$, whose $E_2$-term is isomorphic to $\operatorname{Ext}_{BP_*BP}(BP_*, BP_*)$, where $BP$ denotes the Brown-Peterson spectrum at $p$ ([2]).

In [1], the elements $\alpha^{(n)}_t$ were defined in $\operatorname{Ext}_{BP_*BP}(BP_*, BP_*)$ for every positive integer $n$, $t$. (In [1], these elements were denoted by $\eta(t^i)$ instead of $\alpha^{(n)}_t$.) Here $\alpha^{(n)}_t$ stands for the $n$-th letter of Greek alphabet and we call them Greek letter elements.

For $n \leq 3$, it has already been proved that these elements are represented by non-trivial elements in $\pi^s_*(p)$ if $p > 2n$ ([3], [4], [1], [2]) but in the case of $n \geq 4$, we have had few information on them yet.

The purpose of this paper is to prove the non-triviality of $\alpha^{(n)}_t$ in $\operatorname{Ext}_{BP_*BP}(BP_*, BP_*)$ for $n \geq 4$ under suitable restrictions on $p$, $t$ and we succeed for $p \geq n$ and $1 \leq t \leq p - 1$. Moreover we also prove $p$ does not devide them.

In the next section, we recall the necessary information on $BP$ and state our results proved in §3.

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2. Recollections on $BP$ and Statement of results

Let $BP$ denote the Brown-Peterson spectrum at a prime $p$ ([2]). Then $BP_* = Z_{(p)}[v_1, v_2, \ldots]$ and $BP_*(BP) = BP_*[t_1, t_2, \ldots]$ where the generators $v_m$ and $t_m$ are defined as follows.

$BP_* \otimes Q = Q[\lambda_1, \lambda_2, \ldots]$ for canonical generators $\lambda_m$, $|\lambda_m| = 2(p^m - 1)$. Then $v_m$ are determined inductively by

$$p\lambda_m = \sum_{0 \leq i \leq m} \lambda_i v_m^{p^i} \quad (\lambda_0 = 1, v_0 = p) \quad ([2] A 2.2.2)$$

and $t_m$ by
$$\eta_R(\lambda_m) = \sum_{i \leq m} \lambda_i t_{m-i}^i \quad (t_0 = 1) \quad ([2] \text{ A 2.1.27})$$

where $\eta_R: BP* \otimes Q \to BP_*(BP) \otimes Q$ is the right unit of the Hopf algebroid $(BP_*, BP_*(BP))$ tensored with $Q$.

Under the above choice of generators we have

**Theorem 2.1.** ([2] A. 2.2.5).

$$\sum_{i \in \mathbb{Z}} t_i \eta_R(v_j)^{\phi_i} = \sum_{i \in \mathbb{Z}} v_i t_j^{\phi_i} ,$$

**Theorem 2.2** ([2] A 2.1.27).

$$\sum_{i \in \mathbb{Z}} \Delta(t_i) = \sum_{i \in \mathbb{Z}} t_i \otimes t_j^{\phi_i} ,$$

where $\sum^F$ denotes the formal group sum associated with $BP$ and $\Delta: BP_*(BP) \to BP_*(BP) \otimes BP_*(BP)$ is the coproduct of $BP_*(BP)$.

Let $I_m$ be the ideal in $BP_*$ generated by $p, v_1, \ldots, v_{m-1}$. Using 2.1 we see easily that $I_m$ is an invariant ideal in $BP_*$. In fact we have the following theorem.

**Theorem 2.3.** ([2] 4.3.2). Let $I_m = (p, v_1, \ldots, v_{m-1})$

(a) $I_m$ is invariant.

(b) For $m > 0$.

$$\text{Ext}^q(BP_*/I_m) = F_p[v_m] ,$$

and

$$\text{Ext}^q(BP_*) = Z(q) .$$

(c) $0 \to \sum_{i \geq 0} t_i^{m-1} BP_*/I_m \otimes BP_*/I_m \to BP_*/I_m \to BP_*/I_{m+1} \to 0$

is a short exact sequence of comodules.

(d) The only invariant prime ideals in $BP_*$ are the $I_m$ for $0 \leq m \leq \infty$.

(From here we abbreviate $\text{Ext}(M)$ for $\text{Ext}_{BP_*(BP)}(BP_*, M)$.)

This result allows us to define Greek letter elements.

We consider the short exact sequence given by (c) which leads to a long exact sequence of Ext and let

$$\delta_m: \text{Ext}^q(BP_*/I_{m+1}) \to \text{Ext}^{q+1}(BP_*/I_m)$$

denote the connecting homomorphism of the resulting long exact sequence.

**Definition.** For $t, n > 0$, let

$$\alpha_{(n)} = \delta_0 \delta_1 \cdots \delta_{n-1}(v_\lambda) \in \text{Ext}^n(BP_*) ,$$
We now state our results. Let
\[ \varphi_n : \text{Ext}^t(BP_*) \to \text{Ext}^t(BP_*/(I_{n-1} + I_{t-n+1}^t)) \]
be the homomorphism induced by the natural projection \( BP_*/(I_{n-1} + I_{t-n+1}^t) \). (From now on we always assume \( p \geq n \geq 3 \).) Then we have

**Theorem 2.4.** \( \varphi_n(\alpha_i^{(t)}) \neq 0 \) if \( 3 \leq n \leq p \) and \( 1 \leq t \leq p-1 \).

As an immediate consequence of 2.4 we have

**Corollary 2.5.** \( \alpha_i^{(t)} \neq 0 \) if \( 3 \leq n \leq p \) and \( 1 \leq t \leq p-1 \). Moreover \( p \) does not divide them.

**Remark.** For \( n \leq 3 \) we have much more general results than 2.4. (See [1], [2].)

The rest of this section is devoted to describing the cobar construction which we need in the next section.

Let \( (A, \Gamma) \) be a Hopf algebroid such that \( \Gamma \) is flat over \( A \). Then the category of (left) \( \Gamma \)-comodules becomes an abelian category with enough injectives, so we can define \( \text{Ext}^t_\Gamma(L, M) \) for (left) \( \Gamma \)-comodules \( L, M \) as the \( s \)-th right derived functor of \( \text{Hom}_\Gamma(L, M) \).

In the case of \( L = A \), these Ext groups can be computed as the homology of the cobar complex \( C_\Gamma(M) \) defined below.

**Definition.** Let \( \varepsilon : \Gamma \to A \) be the counit and \( \Gamma = \ker \varepsilon \). The cobar complex \( \text{C}_\Gamma(M) \) is defined by \( \text{C}_\Gamma(M) = \Gamma^\otimes_\Lambda \otimes M \) with the differential \( d : \text{C}^s_\Gamma(M) \to \text{C}^{s+1}_\Gamma(M) \) given by
\[
d(\gamma_1 \otimes \cdots \otimes \gamma_s \otimes m) \\
= \sum_{1 \leq i \leq s} (-1)^i \gamma_1 \otimes \cdots \otimes \gamma_i \otimes \gamma_i' \otimes \gamma_i'' \otimes \cdots \otimes \gamma_s \otimes m \\
+ (-1)^{s+1} \sum \gamma_1 \otimes \cdots \otimes \gamma_s \otimes m' \otimes m''
\]
for \( \gamma_1, \cdots, \gamma_s \in \Gamma \) and \( m \in M \) where \( \Gamma^\otimes_\Lambda \) denotes the \( s \)-fold tensor product of \( \Gamma \) over \( A \), \( \Delta(\gamma_i) = 1 \otimes \gamma_i \otimes \sum \gamma_i' \otimes \gamma_i'' \) + \( \gamma_i \otimes 1 \) and \( \varphi(m) = 1 \otimes m + \sum m' \otimes m'' \). (\( \Delta \) denotes the coproduct of \( \Gamma \) and \( \varphi \) denotes the coaction map of \( M \)). The element \( \gamma_1 \otimes \cdots \otimes \gamma_s \otimes m \) will be denoted by \( \gamma_1 | \cdots | \gamma_s | m \).

Then the following isomorphism holds.

**Theorem 2.6** ([2] A 1.2.12).
\[ \text{Ext}^t_\Gamma(A, \Gamma) \simeq H^t(\text{C}_\Gamma(M)) \]

Finally we define a certain quotient complex of \( \text{C}_\Gamma(A) \) associated with a sequence of non negative integers \( (a_1, \cdots, a_t) \) if \( \Gamma = A[\gamma_1, \cdots, \gamma_m] \) (\( m \) may be...
infinity.) and \( \gamma_i \) is primitive modulo \( (\gamma_1, \ldots, \gamma_{i-1}) \) for all \( i \).

Let \( E=(e_1, e_2, \ldots) \) be a sequence of non-negative integers such that \( e_i=0 \) for all but a finite number of \( i \). We introduce an order between such sequences by saying that \( E < F \) \((= (f_1, f_2, \ldots))\) iff there is a positive integer \( i \) such that \( e_j=f_j \) for \( j<i \) and \( e_i<f_i \). Let \( E+F \) denote a sequence \( (e_1+f_1, e_2+f_2, \ldots) \) and \( \gamma^F=\gamma^{f_1} \cdots \gamma^{f_m} \in \Gamma \).

**Definition.**

\[
C_\Gamma((a_1, \ldots, a_i)) = \frac{C_\Gamma(A)}{\bigoplus_{i \geq 1} A\{\gamma^{E_1|\ldots|\gamma^{E_s}; E_1+\cdots+E_s>(a_1, \ldots, a_i)}\}}
\]

where \( A\{\cdot\} \) denotes the submodule of \( C_\Gamma(A) \) generated by the indicated generators.

(Clearly \( C_\Gamma((a_1, \ldots, a_i)) \) depends on the choice of \( \gamma_i \) but we do not indicate the generators in our notation because our choice is always evident in this paper.)

Now we show that \( C_\Gamma((a_1, \ldots, a_i)) \) is a quotient complex of \( C_\Gamma(A) \). By our assumption

\[
\Delta(\gamma_i) = 1 \otimes \gamma_i + \sum \gamma'_i \otimes \gamma'_i + \gamma_i \otimes 1
\]

where \( \gamma'_i(\gamma_1, \ldots, \gamma_{i-1}) \) or \( \gamma'_i(\gamma_1, \ldots, \gamma_{i-1}) \). Thus \( \Delta(\gamma_i) \in A\{\gamma^F \otimes \gamma^G; F+G \geq (0, \ldots, 0, 1)\} \) and more generally we have \( \Delta(\gamma^F) \in A\{\gamma^F \otimes \gamma^G; F+G \geq E\} \) since \( \Delta \) is an algebra homomorphism and \( (\gamma^F \otimes \gamma^G)(\gamma^F' \otimes \gamma^G') = \gamma^{F+F'} \otimes \gamma^{G+G'} \). Therefore \( \bigoplus_{i \geq 1} A\{\gamma^{E_1|\ldots|\gamma^{E_s}; E_1+\cdots+E_s>(a_1, \ldots, a_i)}\} \) is a subcomplex of \( C_\Gamma(A) \) as desired.

### 3. Proof of Theorem 2.4

Let \( C(n, m) \) (resp. \( D(n, m) \)) denote

\[
C_{BP}(BP)/J_{n,m}((p^{n-2}+1, p^{n-3}, \ldots, p, 1))
\]

(resp. \( C_{BP}(BP)/K_{n,m}((p^{n-2}+1, p^{n-3}, \ldots, p, 1)) \))

where \( J_{n,m}=I_{m+1}+I_{m+m+1}+I_{m+n+m+1} \) and \( K_{n,m}=I_{m}+I_{m+1}+I_{m+n+m+2} \).

(Note that \( BP_\ast(BP)=BP_\ast[t_2, t_2, \ldots] \) and \( \Delta(t_i) \) has the form

\[
\Delta(t_i) = 1 \otimes t_i + t_i \otimes 1 \quad \text{in} \quad BP_\ast(BP) \otimes BP_\ast(BP) \)

for degree reasons.) It is obvious that the sequence

\[
0 \rightarrow C(n, m) \rightarrow D(n, m) \rightarrow C(n, m+1) \rightarrow 0
\]

is a short exact sequence of complexes and letting

\[
\delta_m: H^r(C(n, m+1)) \rightarrow H^{r+1}(C(n, m))
\]
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Denote the corresponding connecting homomorphism we have a commutative diagram

\[
\begin{array}{c}
\Ext^t(BP_*/I_{m+1}) \xrightarrow{\delta_m} \Ext^{t+1}(BP_*/I_m) \\
\downarrow \psi_{m+1} \quad \downarrow \psi_m \\
H^t(C(n, m+1)) \xrightarrow{\delta_m} H^{t+1}(C(n, m))
\end{array}
\]

where \(\psi_m\) is the homomorphism induced by the natural projection \(C_{BP_*(BP_*)} / I_m \rightarrow C(n, m)\).

Thus it is sufficient to show 3.2 below for the proof of 2.4 since \(\psi_0\) factors through

\[
\phi_n: \Ext^t(BP_*) \rightarrow \Ext^t(BP_*/(I_{n-1} + I_{n-n+1})).
\]

**Proposition 3.2.** \(\alpha^{(n)}_t = 0\) in \(H^t(C(n, 0))\) if \(1 \leq t \leq p - 1\) where \(\alpha^{(n)}_t\) denotes the element \(\psi(\alpha^{(n)}_t) = \delta_0 \cdots \delta_{n-1} \psi_0(\alpha^{(n)}_t) \in H^t(C(n, 0))\).

In order to prove 3.2 we begin with giving an explicit representative for \(\alpha^{(n)}_t\) and this requires some formulas on \(\eta_r\) of \(BP_\delta\).

**Lemma 3.3** ([2] 4.3.21).

\[
\eta_r(v_m) = v_m + v_{m-1} t_1^{r-1} - v_{m-1} t_1 \mod I_{m-1}.
\]

**Lemma 3.4.**

\[
\eta_r(v_m) = \sum_{0 \leq i \leq m} v_i \ t_m^i \mod I_m.
\]

For the proof of 3.4 we first prove the following simple fact about the formal group law associated with \(BP_\delta\).

**Lemma 3.5.**

\[
X +_F Y = X + Y \quad \text{in} \quad BP_\delta[[X, Y]]/(X, Y)^p.
\]

**Proof.** Note that \(X +_F Y\) has the form

\[
X + Y + \sum_{i \geq 1} a_{i, j} X^i Y^j \quad \text{in} \quad BP_\delta[[X, Y]]
\]

where \(a_{i, j} = a_{j, i} \in BP_{2(i+j-1)}\).

Considering the degree of \(a_{i, j}\), it is clear that \(a_{i, j} = 0\) if \(i + j < p\) so we get the desired result. \(\square\)

**Proof of 3.4.** In the degree of \(\eta_r(v_m)\), the left hand side of 2.1 is congruent to \(\eta_r(v_m)\) modulo \(I_m^p\) by 2.3 (a) and 3.5.

The right hand side of 2.1 is congruent to \(\sum_{0 \leq i \leq m} v_i \ t_m^i \mod I_m^p\) and the
result follows. □

We now describe a representative for $\check{\alpha}^{(n)}$.

**Lemma 3.6.** $\check{\alpha}^{(n)} \in H^n(C(n, 0))$ is represented by a cocycle

$$-t \frac{(p-1)!}{(p-n)!} v_{n-1}^{p-n-1} (t_{n-2} \cdots t_1) t_1 \in C^n(n, 0).$$

**Proof.** In $D(n, n-1)$,

$$d(v^t_n) = \eta_n(v^t_n) - v^t_n$$

$$= (v_n + v_{n-1} t_{n-1}^{p-1} - v_{n-1}^p t_1) - v^t_n \quad \text{(by 3.3)}$$

$$= -tv_n^{p-1} v_{n-1}^p t_1.$$

So we have

$$\tilde{\delta}_{n-1}(v^t_n) = -tv_n^{p-1} v_{n-1}^p t_1 \in H^1(C(n, n-1)).$$

(We often abuse the same notation for a cocycle and its representing element in the cohomology.)

In $D(n, n-2)$,

$$d(v_n^{p-1} v_{n-1}^p t_1) = d(v_n^{p-1} v_{n-1}^p t_1) | t_1 \quad \text{(by 3.1)}$$

$$= \{\eta_n(v_n^{p-1} v_{n-1}^p t_1) - v_n^{p-1} v_{n-1}^p t_1 \} | t_1$$

$$= \{v_n^{p-1}(v_{n-1} + v_{n-2} t_{n-2}^{p-1} - v_{n-2}^p t_1) - v_n^{p-1} v_{n-1}^p t_1 \} \quad \text{(by 3.4)}$$

$$= (p-1) v_n^{p-1} v_{n-1}^{p-2} v_{n-2} t_{n-2}^{p-2} t_1$$

and thus

$$\tilde{\delta}_{n-2} \tilde{\delta}_{n-1}(v^t_n) = -t(p-1) v_n^{p-1} v_{n-1}^{p-2} t_{n-2}^{p-2} t_1 t_1 \in H^2(C(n, n-2)).$$

More generally, by induction on $k$, we can easily show

$$\tilde{\delta}_{n-k} \cdots \tilde{\delta}_{n-1}(v^t_n) = -t \frac{(p-1)!}{(p-k)!} v_n^{p-1} v_{n-1}^{p-k} t_{n-2}^{p-k} t_{n-3}^{p-k-2} \cdots t_1^{p-k-2} | t_1 \in H^n(C(n, n-k)) \quad \text{for all } k, 2 \leq k \leq n.$$

Let $k=n$ in 3.7 then we obtain the lemma. □

Next we define a subcomplex of $C(n, 0)$ which will be denoted by $\overline{C(n, 0)}$. Let $C_n (v_{n-1}, v_n, t_1, \cdots, t_n) / (v_n^{p-1} + 1)$ be the sub-Hopf algebroid of $(BP_{*}/J_{*,0}, BP_{*}(BP)/J_{*,0})$ where $P(\cdot)$ denotes the polynomial algebra which has the indicated generators over $F_p$. We define

$$\overline{C(n, 0)} = C_n (v_{n-1}, v_n, t_1, \cdots, t_n) / (v_n^{p-1} + 1)$$

and let
\[ B(n, m) = C_p(t_1, \ldots, t_m) \left( (\mathfrak{p}^{n-2}+1, \mathfrak{p}^{n-3}, \ldots, \mathfrak{p}, 1) \right) \]

where \( P(t_1, \ldots, t_m) \) is considered as a Hopf algebra over \( F_p \) whose coproduct is given by \( \Delta(t_i) = \sum_{j \leq i} t_j \otimes t_{i-j} \) for \( 1 \leq i \leq m \). Then the following isomorphism of differential graded algebras holds.

\[ (3.8) \quad C(n, 0) \cong P(v_{n-1}, v_n)/(v_n^{p+1} - 1) \otimes B(n, n). \]

This follows from 3.4 and the formulas on the coproduct of \( BP_* (BP) \) given by the next lemma.

**Lemma 3.9** ([2] 4.3.15). For \( m \geq 1 \)

\[ \Delta(t_m) = \sum_{0 \leq i \leq m} t_i \otimes t_{m-i} \text{ in } BP_* (BP) \otimes BP_* (BP)/I, \]

and

\[ \Delta(t_{m+1}) = \sum_{0 \leq i \leq m+1} t_i \otimes t_{m+1-i} \text{ in } BP_* (BP) \otimes BP_* (BP)/(I_m + BP_* \{ t_i \otimes t_{i-1}; e_1 + e_2 \geq p^n \}). \]

Now note that \( |\alpha_t| < |v_{n+1}| \) for \( t \leq p - 1 \) and \( C(n, 0) \) is equal to the subcomplex \( C(n, 0) \) defined above in the internal degree less than \( |v_{n+1}| \) and therefore 3.2 is equivalent to

**Proposition 3.10.** \( t_{n-1} | t_{n-2} | \cdots | t_{1} | t_1 \neq 0 \text{ in } H^*(B(n, n-1)) \)

by 3.6 and 3.8 since \( B(n, n-1) = B(n, n) \) in the internal degree less than \( |t_n| \) \( (> |t_{n-1}| |t_{n-2}| \cdots |t_{1}^{p-2}| t_1 \) for \( p \geq n \).

In order to show 3.10 we need the following lemma proved at the end of this section.

**Lemma 3.11.** There is a spectral sequence converging to \( H^*(B(n, m)) \) with

\[ E_2^{a,b} = H^a(C_p(t_m) \langle F_p \rangle) \otimes H^b(B(n, m-1))/R_{a,b} \]

and

\[ d_r : E_r^{a,b} \to E_r^{a+r,b-r+1} \]

where \( P(t_m) \) is considered as a Hopf algebra over \( F_p \) with \( t_m \) primitive and

\[ R_{a,b} = F_p \{ x \otimes y \in H^a(C_p(t_m) \langle F_p \rangle) \otimes H^b(B(n, m-1)); \text{ Both } x \text{ and } y \text{ have } \}

representative cocycles \( \tilde{x} \) and \( \tilde{y} \) such that \( \tilde{x} \tilde{y} = 0 \text{ in } B^{a+b}(n, m) \} \).

Moreover this spectral sequence has the third grading induced by the internal degree in the cohomology which is preserved by all differentials.
Proof of 3.10. First note that
\( t_i^{s-2} | t_1 = 0 \) in \( H^2(B(n, 1)) \)
by 3.12 below since \( B(n, 1) \) is a direct summand of \( C_p(t_0) (F_p) \) as a complex.
(Recall our assumption \( n \geq 3 \) which assures \( p^{s-2} > 1 \).)

**Lemma 3.12.**

\[ H^*(C_p(t_0) (F_p)) = E(h_{m,0}, h_{m,1}, \ldots) \otimes P(b_{m,0}, b_{m,1}, \ldots) \]

where \( h_{m,i} \) (resp. \( b_{m,i} \)) is represented by \( t_{m,i}^i \) (resp. \( \frac{1}{p} \sum_{i < j < p} (p) t_{m,i}^i | t_{m,j}^{p(j-i)} \)) and \( E(\cdot) \)
denotes the exterior algebra which has the indicated generators over \( F_p \).

**Proof.** This result is obtained by a routine calculation. \( \square \)**

Now suppose

\[ (3.13) \quad t_{m-1}^{s-2} \cdots t_1^{s-2} | t_1 = 0 \text{ in } H^m(B(n, m-1)) \]
holds for some \( m, 1 < m \leq n-1 \). Then the element \( t_m^{s-m-1} \otimes t_{m-1}^{s-m} \cdots | t_1^{s-2} | t_1 \)
\((\in H^1(C_p(t_m) (F_p)) \otimes H^m(B(n, m-1)))\) defines a non-trivial element in the \( E_2 \)-term
of the spectral sequence given by 3.11 which is clearly a permanent cycle and moreover there is no differential killing this element as observed below.

Let \( c_m \) denote the internal degree of the above element then

\[ (3.14) \quad c_m = 2(p-1) \{ m p^{s-2} + (m-1) p^{s-3} + \cdots + 2 p^{s-m-1} + p^{s-m} + p^{s-m-1}+1 \} \]

and it is enough to prove \( E_r^{s-r,c} = 0 \) for all \( r \geq 2 \).

Using 3.11 and 3.12 we can identify the \( E_r \) with an appropriate subquotient
of \( \bigotimes_{i \leq t \leq m} (E(h_{i,j}) \otimes P(b_{i,j})) \) and let \( c_{i_1,i_2,\ldots,i_t,\bar{i}_1,\ldots,\bar{i}_s} \) denote the internal degree of
\( h_{i_1,i_2} \cdots h_{i_t,i_1} b_{i_t,i_1} \cdots b_{i_t,i_1} (1 \leq i_1 \leq \cdots \leq i_t \leq m, 1 \leq i_1 \leq \cdots \leq i_t \leq m) \) then

\[ (3.15) \quad c_{i_1,i_2,\ldots,i_t,\bar{i}_1,\ldots,\bar{i}_s} = 2(p-1) \{ \sum_{i \leq t \leq i_s} p^i(p^{i-1} + \cdots + p + 1) + \sum_{i \leq t \leq s} p^{i+1}(p^{i-1} + \cdots + p + 1) \}. \]

Comparing 3.14 with 3.15 we see easily that \( c_m = c_{i_1,i_2,\ldots,i_t,\bar{i}_1,\ldots,\bar{i}_s} \) does not hold
for \( l+2s \leq m \) under our assumption \( m < n \leq p \) (\( \geq 3 \)) and consequently
\( E_r^{s-r,c} = 0 \) for all \( r \geq 2 \). Therefore

\[ t_m^{s-m-1} \cdots | t_1^{s-2} | t_1 = 0 \text{ in } H^{m+1}(B(n, m)) \]
and by induction on \( m \) we have shown 3.13 for all \( m, 1 < m \leq n \).

Letting \( m = n \) in 3.13 we get 3.10 and thus complete the proof of 2.4 assuming 3.11. \( \square \)
Proof of 3.11. We begin with recalling the construction of the Cartan-
Eilenberg spectral sequence for the following cocentral Hopf algebra extension

$$P(t_1, \ldots, t_{m-1}) \xrightarrow{f} P(t_1, \ldots, t_m) \xrightarrow{g} P(t_m).$$


We first define a decreasing filtration on $C_{p(t_1, \ldots, t_m)}(F_p)$ by

$$F_{a,b} = F_p\{t^{E_1} \cdots t^{E_{a+b}}; \text{ at least } a \text{ of the } t^{E_i} \text{ lie in kerg}\} \subset C_{p(t_1, \ldots, t_m)}(F_p)$$

and let $\tilde{E}_r$ denote the $E_r$-term of the spectral sequence associated with this filtration.

Next define a homomorphism

$$\bar{h}_{a,b} : C_{p(t_m)}(F_p) \otimes C_{p(t_1, \ldots, t_{m-1})}(F_p) \to \tilde{E}_r^{a,b}$$

which is given by

$$\bar{h}_{a,b}(t_{m}^{E_1} \cdots t_{m}^{E_a} \otimes t_{m}^{E_1} \cdots t_{m}^{E_b}) = t_{m}^{E_1} \cdots t_{m}^{E_a} = \tilde{E}_r^{a,b}$$

for $t_{m}^{E_1} \cdots t_{m}^{E_a} \in C_{p(t_m)}(F_p)$ and $t_{m}^{E_1} \cdots t_{m}^{E_b} \in C_{p(t_1, \ldots, t_{m-1})}(F_p)$. If we consider

$$C_{p(t_m)}(F_p) \otimes C_{p(t_1, \ldots, t_{m-1})}(F_p)$$

as a complex with its differential $d \otimes 1$ then $\bar{h}_{a,b}$ becomes a chain map and induces

$$\bar{h}_r : H^r(C_{p(t_m)}(F_p)) \otimes H^r(C_{p(t_1, \ldots, t_{m-1})}(F_p)) \to \tilde{E}_r^{a,b}.$$ 

Moreover we can prove $\bar{h}$ is an isomorphism and if we consider $H^r(C_{p(t_m)}(F_p)) \otimes C_{p(t_1, \ldots, t_{m-1})}(F_p)$ as a complex with its differential $(-1)^b d \otimes d$ then $\bar{h}'_{a,b}$ is also a chain map.

Hence we obtain an isomorphism

$$\bar{h}_r' : H^r(C_{p(t_m)}(F_p)) \otimes H^r(C_{p(t_1, \ldots, t_{m-1})}(F_p)) \cong \tilde{E}_r^{a,b}.$$ 

induced by $\bar{h}'$.

Therefore we have a spectral sequence converging to $H^*(C_{p(t_1, \ldots, t_m)}(F_p))$ whose $E_2$-term is isomorphic to $H^*(C_{p(t_m)}(F_p)) \otimes H^*(C_{p(t_1, \ldots, t_{m-1})}(F_p))$. This spectral sequence is called the Cartan-Eilenberg spectral sequence.

We now turn to our case. It is trivial that $\tilde{E}_{a,b}$ given by 3.16 also defines a decreasing filtration on $B(n, m)$ naturally. Thus we obtain a spectral sequence $E_r$ converging to $H^*(B(n, m))$ and a homomorphism

$$h_{a,b}' : H^r(C_{p(t_m)}(F_p)) \otimes H^r(B(n, m-1))/R_{a,b} \to E_1^{a,b}$$

induced by a chain map

$$h_{a,b}' : H^r(C_{p(t_m)}(F_p)) \otimes B(n, m-1)/R_{a,b} \to E_1^{a,b}.$$
where
\[ R_{a,b}' = F_p \{ x \otimes y \in H^i(C_{p(t,m)}(F_p)) \otimes B^a(n, m-1); \ x \text{ has a representative cocycle } \bar{x} \text{ such that } \bar{x}y = 0 \text{ in } B^{a+b}(n, m) \} \]

and \( h' \) (resp. \( h'' \)) is the map induced by \( \tilde{h}' \) (resp. \( \tilde{h}'' \)) naturally. So we will show that \( h' \) is an isomorphism.

Let
\[ R'' = F_p \{ x \otimes y \in C_{p(t,m)}(F_p) \otimes B(n, m-1); \ xy = 0 \text{ in } B(n, m) \}. \]

It is easy to see that \( C_{p(t,m)}(F_p) \otimes B(n, m-1)/R'' \) (resp. \( E_0 \)) is a direct summand of \( C_{p(t,m)}(F_p) \otimes C_{p(t_1, \ldots, t_{m-1})}(F_p) \) (resp. \( E_0 \)) as a complex where \( C_{p(t,m)}(F_p) \otimes C_{p(t_1, \ldots, t_{m-1})}(F_p) \) is endowed with the differential \( d \otimes 1 \) and \( C_{p(t,m)}(F_p) \otimes B(n, m-1)/R'' \) with the induced one, and moreover through \( \tilde{h} \), \( C_{p(t,m)}(F_p) \otimes B(n, m-1)/R'' \) corresponds to \( E_0 \) and another summand of \( C_{p(t,m)}(F_p) \otimes C_{p(t_1, \ldots, t_{m-1})}(F_p) \) corresponds to another one of \( E_0 \).

Hence the fact \( \tilde{h}' \) is an isomorphism implies \( h' \) is also an isomorphism and we complete the proof of 3.11. □

References


