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ON THE NON-TRIVIALITY OF THE GREEK LETTER ELEMENTS IN THE ADAMS-NOVIKOV E_2-TERM

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1. Introduction

For a prime p, there is a spectral sequence, called the Adams-Novikov spectral sequence, converging to the stable homotopy ring of spheres localized at p, $\pi_{*(p)}^S$, whose E_2-term is isomorphic to $\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*)$, where BP denotes the Brown-Peterson spectrum at p([2]).

In [1], the elements $\alpha_t^{(n)}$ were defined in $\mathrm{Ext}^n_{BP_*(BP)}(BP_*, BP_*)$ for every positive integer n, t. (In [1], these elements were denoted by $\eta(v_n^t)$ instead of $\alpha_t^{(n)}$.) Here $\alpha^{(n)}$ stands for the n-th letter of Greek alphabet and we call them Greek letter elements.

For $n\leq 3$, it has already been prvoed that these elements are represented by non-trivial elements in $\pi_{*(P)}^S$ if $p\geq 2n$ ([3], [4], [1], [2]) but in the case of $n\geq 4$, we have had few information on them yet.

The purpose of this paper is to prove the non-triviality of $\alpha_t^{(n)}$ in $\mathrm{Ext}^n_{BP_*(BP)}(BP_*, BP_*)$ for $n\geq 4$ under suitalbe restrictions on p, t and we succeed for $p\geq n$ and $1\leq t\leq p-1$. Moreover we also prove p does not devide them.

In the next section, we recall the necessary information on BP and state our results proved in §3.

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2. Recollections on BP and Statement of results

Let BP denote the Brown-Peterson spectrum at a prime p ([2]). Then $BP_*=\boldsymbol{Z}_{(p)}[v_1, v_2, \cdots]$ and $BP_*(BP)=BP_*[t_1, t_2, \cdots]$ where the generators v_m and t_m are defined as follows.

$BP_*\otimes\boldsymbol{Q}=\boldsymbol{Q}[\lambda_1, \lambda_2, \cdots]$ for canonical generators λ_m, $|\lambda_m|=2(p^m-1)$. Then v_m are determined inductively by

$$p\lambda_m=\sum_{0\leq i\leq m}\lambda_i v_{m-i}^{p^i} \quad (\lambda_0=1, v_0=p) \quad ([2]\ \mathrm{A}\ 2.2.2)$$

and t_m by

$$\eta_R(\lambda_m) = \sum_{0\le i\le m} \lambda_i\, t_{m-i}^{p^i} \quad (t_0 = 1) \quad ([2]\ \text{A } 2.1.27)$$

where η_R: $BP_*\otimes \boldsymbol{Q}\to BP_*(BP)\otimes \boldsymbol{Q}$ is the right unit of the Hopf algebroid $(BP_*, BP_*(BP))$ tensored with $\boldsymbol{Q}$.

Under the above choice of generators we have

Theorem 2.1. ([2] A. 2.2.5).

$$\sum_{i,j\ge 0}^{F} t_i\, \eta_R(v_j)^{p^i} = \sum_{i,j\ge 0}^{F} v_i\, t_j^{p^i}\,,$$

Theorem 2.2 ([2] A 2.1.27).

$$\sum_{i>0}^{F} \Delta(t_i) = \sum_{\substack{i,j\ge 0\\ i+j>0}}^{F} t_i\otimes t_j^{p^i}\,,$$

where $\sum^F$ *denotes the formal group sum associated with* BP *and* Δ: $BP_*(BP)\to BP_*(BP)\underset{BP_*}{\otimes} BP_*(BP)$ *is the coproduct of* $BP_*(BP)$.

Let I_m be the ideal in BP_* generated by $p, v_1, \cdots, v_{m-1}$. Using 2.1 we see easily that I_m is an invariant ideal in BP_*. In fact we have the following theorem.

Theorem 2.3. ([2] 4.3.2). *Let* $I_m=(p, v_1, \cdots, v_{m-1})$

(a) I_m *is invariant.*

(b) *For* $m>0$.

$$\mathrm{Ext}^0(BP_*/I_m) = \boldsymbol{F}_p[v_m]\,,$$

and

$$\mathrm{Ext}^0(BP_*) = \boldsymbol{Z}_{(p)}\,.$$

(c) $0\to \Sigma^{2(p^m-1)} BP_*/I_m \xrightarrow{v_m} BP_*/I_m \to BP_*/I_{m+1}\to 0$
is a short exact sequence of comodules.

(d) *The only invairant prime ideals in* BP_* *are the* I_m *for* $0\le m\le\infty$.

(From here we abbreviate $\mathrm{Ext}(M)$ for $\mathrm{Ext}_{BP_*(BP)}(BP_*, M)$.)

This result allows us to define Greek letter elements.

We consider the short exact sequence given by (c) which leads to a long exact sequence of Ext and let

$$\delta_m\colon \mathrm{Ext}^s(BP_*/I_{m+1}) \to \mathrm{Ext}^{s+1}(BP_*/I_m)$$

denote the connecting homomorphism of the resulting long exact sequence.

DEFINITION. *For* $t, n>0$, *let*

$$\alpha_t^{(n)} = \delta_0\,\delta_1\cdots\delta_{n-1}(v_n^t)\in \mathrm{Ext}^n(BP_*)\,.$$

We now state our results. Let

$$\varphi_n\colon \mathrm{Ext}^s(BP_*) \to \mathrm{Ext}^s(BP_*/(I_{n-1}+I_n^{p-n+1}))$$

be the homomorphism induced by the natural projection $BP_*\to BP_*/(I_{n-1}+I_n^{p-n+1})$. (From now on we always assume $p\geq n\geq 3$.) Then we have

Theorem 2.4. $\varphi_n(\alpha_t^{(n)})\neq 0$ *if* $3\leq n\leq p$ *and* $1\leq t\leq p-1$.

As an immediate consequence of 2.4 we have

Corollary 2.5. $\alpha_t^{(n)}\neq 0$ *if* $3\leq n\leq p$ *and* $1\leq t\leq p-1$. *Moreover* p *does not devide them.*

REMARK. *For* $n\leq 3$ *we have much more general results than* 2.4. (*See* [1], [2].)

The rest of this section is devoted to describing the cobar construction which we need in the next section.

Let (A, Γ) be a Hopf algebroid such that Γ is flat over A. Then the category of (left) Γ-comodules becomes an abelian category with enough injectives, so we can define $\mathrm{Ext}_\Gamma^s(L, M)$ for (left) Γ-comodules L, M as the s-th right derived functor of $\mathrm{Hom}_\Gamma(L, M)$.

In the case of $L=A$, these Ext groups can be computed as the homology of the cobar complex $C_\Gamma(M)$ defined below.

DEFINITION. *Let* $\varepsilon\colon \Gamma\to A$ *be the counit and* $\bar{\Gamma}=\ker\varepsilon$. *The cobar complex* $C_\Gamma(M)$ *is defined by* $C_\Gamma^s(M)=\bar{\Gamma}^{\otimes s}\underset{A}{\otimes} M$ *with the differential* $d\colon C_\Gamma^s(M)\to C_\Gamma^{s+1}(M)$ *given by*

$$\begin{aligned} &d(\gamma_1\otimes\cdots\otimes\gamma_s\otimes m)\\ &\quad=\sum_{1\leq i\leq s}(-1)^i\sum\gamma_1\otimes\cdots\otimes\gamma_{i-1}\otimes\gamma_i'\otimes\gamma_i''\otimes\gamma_{i+1}\otimes\cdots\otimes\gamma_s\otimes m\\ &\quad\quad+(-1)^{s+1}\sum\gamma_1\otimes\cdots\otimes\gamma_s\otimes m'\otimes m'' \end{aligned}$$

for $\gamma_1, \cdots, \gamma_s\in\bar{\Gamma}$ *and* $m\in M$ *where* $\bar{\Gamma}^{\otimes s}$ *denotes the s-fold tensor product of* $\bar{\Gamma}$ *over* A, $\Delta(\gamma_i)=1\otimes\gamma_i+\sum\gamma_i'\otimes\gamma_i''+\gamma_i\otimes 1$ *and* $\psi(m)=1\otimes m+\sum m'\otimes m''$. ($\Delta$ *denotes the coproduct of* Γ *and* ψ *denotes the coaction map of* M). *The element* $\gamma_1\otimes\cdots\otimes\gamma_s\otimes m$ *will be denoted by* $\gamma_1|\cdots|\gamma_s|m$.

Then the following isomorphism holds.

Theorem 2.6 ([2] A 1.2.12).

$$\mathrm{Ext}_\Gamma^s(A, \Gamma)\cong H^s(C_\Gamma(M)).$$

Finally we define a certain quotient complex of $C_\Gamma(A)$ associated with a sequence of non negative integers $(a_1, \cdots, a_l)$ if $\Gamma=A[\gamma_1, \cdots, \gamma_m]$ (m may be infinity.) and γ_i is primitive modulo $(\gamma_1, \cdots, \gamma_{i-1})$ for all i.

Let $E=(e_1, e_2, \cdots)$ be a sequence of non negative integers such that $e_i=0$ for all but a finite number of i. We introduce an order between such sequences by saying that $E<F\,(=(f_1, f_2, \cdots))$ iff there is a positive integer i such that $e_j=f_j$ for $j<i$ and $e_i<f_i$. Let $E+F$ denote a sequence $(e_1+f_1, e_2+f_2, \cdots)$ and $\gamma^E=\gamma_1^{e_1}\cdots\gamma_m^{e_m}\in\Gamma$.

DEFINITION.

$$C_\Gamma((a_1, \cdots, a_l)) = \frac{C_\Gamma(A)}{\bigoplus_{s\geq 1} A\{\gamma^{E_1}|\cdots|\gamma^{E_s};\ E_1+\cdots+E_s>(a_1, \cdots, a_l)\}}$$

where $A\{\cdot\}$ denotes the submodule of $C_\Gamma(A)$ generated by the indicated generators.

(Clearly $C_\Gamma((a_1, \cdots, a_l))$ depends on the choice of γ_i but we do not indicate the generators in our notation because our choice is always evident in this paper.)

Now we show that $C_\Gamma((a_1, \cdots, a_l))$ is a quotient complex of $C_\Gamma(A)$. By our assumption

$$\Delta(\gamma_i) = 1\otimes\gamma_i+\sum \gamma_i'\otimes\gamma_i''+\gamma_i\otimes 1$$

where $\gamma_i'\in(\gamma_1, \cdots, \gamma_{i-1})$ or $\gamma_i''\in(\gamma_1, \cdots, \gamma_{i-1})$. Thus $\Delta(\gamma_i)\in A\{\gamma^F\otimes\gamma^G;\ F+G\geq(0, \cdots, 0, \overset{i}{\smile}{1})\}$ and more generally we have $\Delta(\gamma^E)\in A\{\gamma^F\otimes\gamma^G;\ F+G\geq E\}$ since Δ is an algebra homomorphism and $(\gamma^F\otimes\gamma^G)\,(\gamma^{F'}\otimes\gamma^{G'})=\gamma^{F+F'}\otimes\gamma^{G+G'}$. Therefore $\bigoplus_{s\geq 1} A\{\gamma^{E_1}|\cdots|\gamma^{E_s};\ E_1+\cdots+E_s>(a_1, \cdots, a_l)\}$ is a subcomplex of $C_\Gamma(A)$ as desired.

3. Proof of Theorem 2.4

Let $C(n, m)$ (resp. $D(n, m)$) denote

$$C_{BP_*(BP)/J_{n,m}}((p^{n-2}+1, p^{n-3}, \cdots, p, 1))$$
$$(\text{resp. } C_{BP_*(BP)/K_{n,m}}((p^{n-2}+1, p^{n-3}, \cdots, p, 1)))$$

where $J_{n,m}=I_m+I_{n-1}^{m+1}+I_n^{p-n+m+1}$ and $K_{n,m}=I_m+I_{n-1}^{m+2}+I_n^{p-n+m+2}$.

(Note that $BP_*(BP)=BP_*[t_1, t_2, \cdots]$ and $\Delta(t_i)$ has the form

$$(3.1)\qquad \Delta(t_i) = 1\otimes t_i+t_i\otimes 1 \quad \text{in} \quad BP_*(BP)\underset{BP_*}{\otimes}BP_*(BP)/(t_1, \cdots, t_{i-1})$$

for degree reasons.) It is obvious that the sequence

$$0 \to C(n, m) \overset{v_m}{\to} D(n, m) \to C(n, m+1) \to 0$$

is a short exact sequence of complexes and letting

$$\tilde{\delta}_m\colon H^s(C(n, m+1)) \to H^{s+1}(C(n, m))$$

denote the corresponding connecting homomorphism we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^s(BP_*/I_{m+1}) & \xrightarrow{\delta_m} & \mathrm{Ext}^{s+1}(BP_*/I_m) \\ \psi_{m+1}\downarrow & & \downarrow\psi_m \\ H^s(C(n, m+1)) & \xrightarrow[\tilde{\delta}_m]{} & H^{s+1}(C(n, m)) \end{array}$$

where ψ_m is the homomorphism induced by the natural projection $C_{BP_*(BP)}(BP_*/I_m) \to C(n, m)$.

Thus it is sufficient to show 3.2 below for the proof of 2.4 since ψ_0 factors through

$$\varphi_n\colon \mathrm{Ext}^s(BP_*) \to \mathrm{Ext}^s(BP_*/(I_{n-1}+I_n^{p-n+1}))\,.$$

Proposition 3.2. *$\tilde{\alpha}_t^{(n)} \neq 0$ in $H^n(C(n, 0))$ if $1 \le t \le p-1$ where $\tilde{\alpha}_t^{(n)}$ denotes the element $\psi_0(\alpha_t^{(n)}) = \tilde{\delta}_0 \cdots \tilde{\delta}_{n-1}\, \psi_n(v_n^t) \in H^n(C(n, 0))$.*

In order to prove 3.2 we begin with giving an explicit representative for $\tilde{\alpha}_t^{(n)}$ and this requires some formulas on η_R of BP.

Lemma 3.3 ([2] 4.3.21).

$$\eta_R(v_m) \equiv v_m + v_{m-1}\, t_1^{p^{m-1}} - v_{m-1}^p\, t_1 \quad \mathrm{mod}\ I_{m-1}\,.$$

Lemma 3.4.

$$\eta_R(v_m) \equiv \sum_{0 \le i \le m} v_i\, t_{m-i}^{p^i} \quad \mathrm{mod}\ I_m^p\,.$$

For the proof of 3.4 we first prove the following simple fact about the formal group law associated with BP.

Lemma 3.5.

$$X +_F Y = X + Y \quad \text{in} \quad BP_*[[X, Y]]/(X, Y)^p\,.$$

Proof. Note that $X +_F Y$ has the form

$$X + Y + \sum_{i,j \ge 1} a_{i,j}\, X^i\, Y^j \quad \text{in} \quad BP_*[[X, Y]]$$

where $a_{i,j} = a_{j,i} \in BP_{2(i+j-1)}$.

Considering the degree of $a_{i,j}$, it is clear that $a_{i,j} = 0$ if $i+j < p$ so we get the desired result. $\square$

Proof of 3.4. In the degree of $\eta_R(v_m)$, the left hand side of 2.1 is congruent to $\eta_R(v_m)$ modulo I_m^p by 2.3 (a) and 3.5.

The right hand side of 2.1 is congruent to $\sum_{0 \le i \le m} v_i\, t_{m-i}^{p^i}$ modulo I_m^p and the result follows. □

We now describe a representative for $\tilde{\alpha}_t^{(n)}$.

Lemma 3.6. *$\tilde{\alpha}_t^{(n)} \in H^n(C(n, 0))$ is represented by a cocycle*

$$-t\frac{(p-1)!}{(p-n)!}v_n^{t-1}v_{n-1}^{p-n}t_{n-1}|t_{n-2}^p|\cdots|t_1^{p^{n-2}}|t_1 \in C^n(n, 0)\,.$$

Proof. In $D(n, n-1)$,

$$\begin{aligned} d(v_n^t) &= \eta_R(v_n)^t - v_n^t \\ &= (v_n + v_{n-1}t_1^{p^{n-1}} - v_{n-1}^p t_1)^t - v_n^t \quad \text{(by 3.3)} \\ &= -tv_n^{t-1}v_{n-1}^p t_1\,. \end{aligned}$$

So we have

$$\tilde{\delta}_{n-1}(v_n^t) = -tv_n^{t-1}v_{n-1}^{p-1}t_1 \in H^1(C(n, n-1))\,.$$

(We often abuse the same notation for a cocycle and its representing element in the cohomology.)

In $D(n, n-2)$,

$$\begin{aligned} d(v_n^{t-1}v_{n-1}^{p-1}t_1) &= d(v_n^{t-1}v_{n-1}^{p-1})|t_1 \quad \text{(by 3.1)} \\ &= \{\eta_R(v_n)^{t-1}\eta_R(v_{n-1})^{p-1} - v_n^{t-1}v_{n-1}^{p-1}\}|t_1 \\ &= \{v_n^{t-1}(v_{n-1} + v_{n-2}t_1^{p^{n-2}})^{p-1} - v_n^t v_{n-1}^{p-1}\}|t_1 \quad \text{(by 3.4)} \\ &= (p-1)v_n^{t-1}v_{n-1}^{p-2}v_{n-2}t_1^{p^{n-2}}|t_1 \end{aligned}$$

and thus

$$\tilde{\delta}_{n-2}\tilde{\delta}_{n-1}(v_n^t) = -t(p-1)v_n^{t-1}v_{n-1}^{p-2}t_1^{p^{n-2}}|t_1 \in H^2(C(n, n-2))\,.$$

More generally, by induction on k, we can easily show

$$\begin{aligned} \text{(3.7)} \qquad \tilde{\delta}_{n-k}\cdots\tilde{\delta}_{n-1}(v_n^t) &= -\frac{t(p-1)!}{(p-k)!}v_n^{t-1}v_{n-1}^{p-k}t_{k-1}^{p^{n-k}}|\cdots|t_1^{p^{n-2}}|t_1 \\ &\in H^k(C(n, n-k)) \quad \text{for all} \quad k, 2 \le k \le n\,. \end{aligned}$$

Let $k=n$ in 3.7 then we obtain the lemma. □

Next we define a subcomplex of $C(n, 0)$ which will be denoted by $\overline{C(n, 0)}$.

Let $(P(v_{n-1}, v_n)/(v_{n-1}^{p-n+1}), P(v_{n-1}, v_n, t_1, \cdots, t_n)/(v_{n-1}^{p-n+1}))$ be the sub-Hopf algebroid of $(BP_*/J_{n,0}, BP_*(BP)/J_{n,0})$ where $P(\cdot)$ denotes the polynomial algebra which has the indicated generators over $\boldsymbol{F}_p$. We define

$$\overline{C(n, 0)} = C_{P(v_{n-1}, v_n, t_1, \cdots, t_n)/(v_{n-1}^{p-n+1})}((p^{n-2}+1, p^{n-3}, \cdots, p, 1))$$

and let

$$B(n, m) = C_{P(t_1,\cdots,t_m)}((p^{n-2}+1, p^{n-3}, \cdots, p, 1))$$

where $P(t_1, \cdots, t_m)$ is considered as a Hopf algebra over $\boldsymbol{F}_p$ whose coproduct is given by $\Delta(t_i)=\sum_{0\leq j\leq i} t_j\otimes t_{i-j}^{p^j}(1\leq i\leq m)$. Then the following isomorphism of differential graded algebras holds.

$$\overline{C(n, 0)} \cong P(v_{n-1}, v_n)/(v_{n-1}^{p-n+1})\underset{\boldsymbol{F}_p}{\otimes} B(n, n) . \tag{3.8}$$

This follows from 3.4 and the formulas on the coproduct of $BP_*(BP)$ given by the next lemma.

Lemma 3.9 ([2] 4.3.15). *For* $m\geq 1$

$$\Delta(t_m) = \sum_{0\leq i\leq m} t_i\otimes t_{m-i}^{p^i} \text{ in } BP_*(BP)\underset{BP_*}{\otimes} BP_*(BP)/I_m ,$$

and

$$\Delta(t_{m+1}) = \sum_{0\leq i\leq m+1} t_i\otimes t_{m+1-i}^{p^i}$$
$$\text{in } BP_*(BP)\underset{BP_*}{\otimes} BP_*(BP)/(I_m+BP_*\{t_1^{e_1}\otimes t_1^{e_2};\ e_1+e_2\geq p^m\}) .$$

Now note that $|\alpha_t^{(n)}|<|v_{n+1}|$ for $t\leq p-1$ and $C(n, 0)$ is equal to the subcomplex $\overline{C(n, 0)}$ defined above in the internal degree less than $|v_{n+1}|$ and therefore 3.2 is equivalent to

Proposition 3.10. $t_{n-1}|t_{n-2}^p|\cdots|t_1^{p^{n-2}}|t_1\neq 0$ *in* $H^n(B(n, n-1))$

by 3.6 and 3.8 since $B(n, n-1)=B(n, n)$ in the internal degree less than $|t_n|$ $(>|t_{n-1}|t_{n-2}^p|\cdots|t_1^{p^{n-2}}|t_1|$ for $p\geq n)$.

In order to show 3.10 we need the following lemma proved at the end of this section.

Lemma 3.11. *There is a spectral sequence converging to* $H^*(B(n, m))$ *with*

$$E_2^{a,b} = H^b(C_{P(t_m)}(\boldsymbol{F}_p))\otimes H^a(B(n, m-1))/R_{a,b}$$

and

$$d_r\colon E_r^{a,b} \to E_r^{a+r,b-r+1}$$

where $P(t_m)$ *is considered as a Hopf algebra over* $\boldsymbol{F}_p$ *with* t_m *primitive and*

$$R_{a,b} = \boldsymbol{F}_p\{x\otimes y\in H^b(C_{P(t_m)}(\boldsymbol{F}_p))\otimes H^a(B(n, m-1));\ \textit{Both } x \textit{ and } y \textit{ have representative cocycles } \tilde{x} \textit{ and } \tilde{y} \textit{ such that } \tilde{x}\tilde{y} = 0 \textit{ in } B^{a+b}(n, m)\} .$$

Moreover this spectral sequence has the third grading induced by the internal degree in the cohomology which is preserved by all differentials.

Proof of 3.10. First note that

$$t_1^{p^{n-2}} | t_1 \neq 0 \text{ in } H^2(B(n, 1))$$

by 3.12 below since $B(n, 1)$ is a direct summand of $C_{p(t_1)}(\boldsymbol{F}_p)$ as a complex. (Recall our assumption $n \geq 3$ which assures $p^{n-2} > 1$.)

Lemma 3.12.

$$H^*(C_{p(t_m)}(\boldsymbol{F}_p)) = E(h_{m,0}, h_{m,1}, \cdots) \otimes P(b_{m,0}, b_{m,1}, \cdots)$$

where $h_{m,i}$ (resp. $b_{m,i}$) is represented by $t_m^{p^i}$ (resp. $\frac{1}{p} \sum_{0<j<p} \binom{p}{j} t_m^{p^i j} | t_m^{p^i(p-j)}$) and $E(\cdot)$ denotes the exterior algebra which has the indicated generators over $\boldsymbol{F}_p$.

Proof. This result is obtained by a routine calculation. □

Now suppose

$$t_{m-1}^{p^{n-m}} | \cdots | t_1^{p^{n-2}} | t_1 \neq 0 \text{ in } H^m(B(n, m-1)) \tag{3.13}$$

holds for some m, $1 < m \leq n-1$. Then the element $t_m^{p^{n-m-1}} \otimes t_{m-1}^{p^{n-m}} | \cdots | t_1^{p^{n-2}} | t_1$ $(\in H^1(C_{p(t_m)}(\boldsymbol{F}_p)) \otimes H^m(B(n, m-1)))$ defines a non-trivial element in the E_2-term of the spectral sequence given by 3.11 which is clearly a permanent cycle and moreover there is no differential killing this element as observed below.

Let c_m denote the internal degree of the above element then

$$c_m = 2(p-1)\{mp^{n-2} + (m-1)p^{n-3} + \cdots + 2p^{n-m} + p^{n-m-1} + 1\} \tag{3.14}$$

and it is enough to prove $E_r^{m-r, r, c_m} = 0$ for all $r \geq 2$.

Using 3.11 and 3.12 we can identify the E_r with an appropriate subquotient of $\bigotimes_{\substack{1 \leq i \leq m \\ j \geq 0}} (E(h_{i,j}) \otimes P(b_{i,j}))$ and let $c_{i_1, j_1, \cdots, i_l, j_l, i'_1, j'_1, \cdots, i'_s, j'_s}$ denote the internal degree of $h_{i_1, j_1} \cdots h_{i_l, j_l} b_{i'_1, j'_1} \cdots b_{i'_s, j'_s}$ $(1 \leq i_1 \leq \cdots \leq i_l \leq m,\ 1 \leq i'_1 \leq \cdots \leq i'_s \leq m)$ then

$$\begin{aligned} &c_{i_1, j_1, \cdots, i_l, j_l, i'_1, j'_1, \cdots, i'_s, j'_s} \\ &\quad = 2(p-1)\{\sum_{1 \leq k \leq l} p^{j_k}(p^{i_k - 1} + \cdots + p + 1) + \sum_{1 \leq k \leq s} p^{j'_k + 1}(p^{i'_k - 1} + \cdots + p + 1)\}. \end{aligned} \tag{3.15}$$

Comparing 3.14 with 3.15 we see easily that $c_m = c_{i_1, j_1, \cdots, i_l, j_l, i'_1, j'_1, \cdots, i'_s, j'_s}$ does not hold for $l + 2s \leq m$ under our assumption $m < n \leq p\ (\geq 3)$ and consequently $E_r^{m-r, r, c_m} = 0$ for all $r \geq 2$. Therefore

$$t_m^{p^{n-m-1}} | \cdots | t_1^{p^{n-2}} | t_1 \neq 0 \text{ in } H^{m+1}(B(n, m))$$

and by induction on m we have shown 3.13 for all m, $1 < m \leq n$.

Letting $m = n$ in 3.13 we get 3.10 and thus complete the proof of 2.4 assuming 3.11. □

Proof of 3.11. We begin with recalling the construction of the Cartan-Eilenberg spectral sequence for the following cocentral Hopf algebra extension

$$P(t_1, \cdots, t_{m-1}) \xrightarrow{f} P(t_1, \cdots, t_m) \xrightarrow{g} P(t_m). \quad \text{(cf. [2] A 1.3.17)}$$

We first define a decreasing filtration on $C_{p(t_1,\cdots,t_m)}(\boldsymbol{F}_p)$ by

$$\begin{aligned}(3.16)\qquad \widetilde{F}^{a,b} &= \boldsymbol{F}_p\{t^{E_1}|\cdots|t^{E_{a+b}};\ \text{at least } a \text{ of the } t^{E_i} \text{ lie in } \ker g\}\\ &\subset C^{a+b}_{p(t_1,\cdots,t_m)}(\boldsymbol{F}_p)\end{aligned}$$

and let $\widetilde{E}_r$ denote the E_r-term of the spectral sequence associated with this filtration.

Next define a homomorphism

$$\widetilde{h}_{a,b}\colon C^b_{p(t_m)}(\boldsymbol{F}_p)\otimes C^a_{p(t_1,\cdots,t_{m-1})}(\boldsymbol{F}_p) \to \widetilde{E}^{a,b}_0$$

which is given by

$$\begin{aligned}\widetilde{h}_{a,b}(t_m^{e_1}|\cdots|t_m^{e_b}\otimes t^{E_1}|\cdots|t^{E_a}) &= t_m^{e_1}|\cdots|t_m^{e_b}|t^{E_1}|\cdots|t^{E_a}\\ &\in \widetilde{F}^{a,b}/\widetilde{F}^{a+1,b-1} = \widetilde{E}^{a,b}_0\end{aligned}$$

for $t_m^{e_1}|\cdots|t_m^{e_b}\in C^b_{p(t_m)}(\boldsymbol{F}_p)$ and $t^{E_1}|\cdots|t^{E_a}\in C^a_{p(t_1,\cdots,t_{m-1})}(\boldsymbol{F}_p)$. If we consider $C_{p(t_m)}(\boldsymbol{F}_p)\otimes C^a_{p(t_1,\cdots,t_{m-1})}(\boldsymbol{F}_p)$ as a complex with its differential $d\otimes 1$ then $\widetilde{h}_{a,*}$ becomes a chain map and induces

$$\widetilde{h}'_{a,b}\colon H^b(C_{p(t_m)}(\boldsymbol{F}_p))\otimes C^a_{p(t_1,\cdots,t_{m-1})}(\boldsymbol{F}_p) \to \widetilde{E}^{a,b}_1.$$

Moreover we can prove $\widetilde{h}'$ is an isomorphism and if we consider $H^b(C_{p(t_m)}(\boldsymbol{F}_p))\otimes C_{p(t_1,\cdots,t_{m-1})}(\boldsymbol{F}_p)$ as a complex with its differential $(-1)^b\, 1\otimes d$ then $\widetilde{h}'_{*,b}$ is also a chain map.

Hence we obtain an isomorphism

$$\widetilde{h}''_{a,b}\colon H^b(C_{p(t_m)}(\boldsymbol{F}_p))\otimes H^a(C_{p(t_1,\cdots,t_{m-1})}(\boldsymbol{F}_p)) \xrightarrow[\cong]{} \widetilde{E}^{a,b}_2$$

induced by $\widetilde{h}'$.

Therefore we have a spectral sequence converging to $H^*(C_{p(t_1,\cdots,t_m)}(\boldsymbol{F}_p))$ whose E_2-term is isomorphic to $H^*(C_{p(t_m)}(\boldsymbol{F}_p))\otimes H^*(C_{p(t_1,\cdots,t_{m-1})}(\boldsymbol{F}_p))$. This spectral sequence is called the Cartan-Eilenberg spectral sequence.

We now turn to our case. It is trivial that $\widetilde{F}^{a,b}$ given by 3.16 also defines a decreasing filtration on $B(n, m)$ naturally. Thus we obtain a spectral sequence E_r converging to $H^*(B(n, m))$ and a homomorphism

$$h''_{a,b}\colon H^b(C_{p(t_m)}(\boldsymbol{F}_p))\otimes H^a(B(n, m-1))/R_{a,b} \to E^{a,b}_2$$

induced by a chain map

$$h'_{*,b}\colon H^b(C_{p(t_m)}(\boldsymbol{F}_p))\otimes B(n, m-1)/R'_{*,b} \to E^{*,b}_1$$

where

$$R'_{a,b}=\boldsymbol{F}_p\{x\otimes y\in H^b(C_{p(t_m)}(\boldsymbol{F}_p))\otimes B^a(n,m-1);\ x \text{ has a representative cocycle } \tilde{x} \text{ such that } \tilde{x}y=0 \text{ in } B^{a+b}(n,m)\}$$

and h' (resp. h'') is the map induced by $\tilde{h}'$ (resp. $\tilde{h}''$) naturally. So we will show that h' is an isomorphism.

Let

$$R''=\boldsymbol{F}_p\{x\otimes y\in C_{p(t_m)}(\boldsymbol{F}_p)\otimes B(n,m-1);\ xy=0 \text{ in } B(n,m)\}\ .$$

It is easy to see that $C_{p(t_m)}(\boldsymbol{F}_p)\otimes B(n,m-1)/R''$ (resp. E_0) is a direct summand of $C_{p(t_m)}(\boldsymbol{F}_p)\otimes C_{p(t_1,\cdots,t_{m-1})}(\boldsymbol{F}_p)$ (resp. $\tilde{E}_0$) as a complex where $C_{p(t_m)}(\boldsymbol{F}_p)\otimes C_{p(t_1,\cdots,t_{m-1})}(\boldsymbol{F}_p)$ is endowed with the differential $d\otimes 1$ and $C_{p(t_m)}(\boldsymbol{F}_p)\otimes B(n,m-1)/R''$ with the induced one, and moreover through $\tilde{h}$, $C_{p(t_m)}(\boldsymbol{F}_p)\otimes B(n,m-1)/R''$ corresponds to E_0 and another summand of $C_{p(t_m)}(\boldsymbol{F}_p)\otimes C_{p(t_1,\cdots,t_{m-1})}(\boldsymbol{F}_p)$ corresponds to another one of $\tilde{E}_0$.

Hence the fact $\tilde{h}'$ is an isomorphism implies h' is also an isomorphism and we complete the proof of 3.11. □

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