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<th>On the non-triviality of the Greek letter elements in the Adams-Novikov $E_2$-term</th>
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1. Introduction

For a prime \( p \), there is a spectral sequence, called the Adams-Novikov spectral sequence, converging to the stable homotopy ring of spheres localized at \( p \), \( \pi_*(BP) \), whose \( E_2 \)-term is isomorphic to \( \text{Ext}^{*}_{BP_*(BP)}(BP_*,BP_*) \), where \( BP \) denotes the Brown-Peterson spectrum at \( p \) ([2]).

In [1], the elements \( \alpha^{(t)} \) were defined in \( \text{Ext}^{*}_{BP_*(BP)}(BP_*,BP_*) \) for every positive integer \( n, t \). (In [1], these elements were denoted by \( \eta(v_t) \) instead of \( \alpha^{(t)} \).) Here \( \alpha^{(t)} \) stands for the \( n \)-th letter of Greek alphabet and we call them Greek letter elements.

For \( n \leq 3 \), it has already been proved that these elements are represented by non-trivial elements in \( \pi_*(BP) \) if \( p \geq 2n \) ([3], [4], [1], [2]) but in the case of \( n \geq 4 \), we have had few information on them yet.

The purpose of this paper is to prove the non-triviality of \( \alpha^{(t)} \) in \( \text{Ext}^{*}_{BP_*(BP)}(BP_*,BP_*) \) for \( n \geq 4 \) under suitable restrictions on \( p, t \) and we succeed for \( p \geq n \) and \( 1 \leq t \leq p-1 \). Moreover we also prove \( p \) does not devide them.

In the next section, we recall the necessary information on \( BP \) and state our results proved in §3.

I would like to thank Professor Akira Kono for useful conversations, kind encouragement and reading this manuscript.

2. Recollections on \( BP \) and Statement of results

Let \( BP \) denote the Brown-Peterson spectrum at a prime \( p \) ([2]). Then \( BP_*(BP)=BP_*(BP) \) where the generators \( v_m \) and \( t_m \) are defined as follows.

\[ BP_*(BP) \otimes Q=Q[\lambda_1, \lambda_2, \cdots] \text{for canonical generators } \lambda_m, |\lambda_m|=2(p^m-1). \]

Then \( v_m \) are determined inductively by

\[ p \lambda_m = \sum_{0 \leq i \leq m} \lambda_i v_{m-i} \quad (\lambda_0=1, v_0=p) \quad ([2] \ A 2.2.2) \]

and \( t_m \) by
\[ \eta_R(\lambda_m) = \sum_{i \leq m} \lambda_i t^i_m - i \quad (t_0 = 1) \quad ([2] \text{ A 2.1.27}) \]

where \( \eta_R : BP_* \otimes Q \to BP_* (BP) \otimes Q \) is the right unit of the Hopf algebroid \((BP_*, BP_* (BP))\) tensored with \(Q\).

Under the above choice of generators we have

**Theorem 2.1.** ([2] A 2.2.5).
\[ \sum_{i \geq 0} t^i \eta_R(v_j)^{t^i} = \sum_{i \geq 0} v_i t^i, \]

**Theorem 2.2** ([2] A 2.1.27).
\[ \sum_{i \geq 0} t^i \Delta(t_i) = \sum_{i \geq 0} t^i \otimes t^i_j, \]

where \( \sum^F \) denotes the formal group sum associated with \(BP\) and \( \Delta : BP_* (BP) \to BP_* (BP) \otimes BP_* (BP) \) is the coproduct of \( BP_* (BP) \).

Let \( I_m \) be the ideal in \( BP_* \) generated by \( p, v_1, \ldots, v_{m-1} \). Using 2.1 we see easily that \( I_m \) is an invariant ideal in \( BP_* \). In fact we have the following theorem.

**Theorem 2.3.** ([2] 4.3.2). Let \( I_m = (p, v_1, \ldots, v_{m-1}) \)

(a) \( I_m \) is invariant.

(b) For \( m > 0 \),
\[ \text{Ext}^0(BP_*/I_m) = F_p [v_m], \]

and
\[ \text{Ext}^0(BP_*) = Z_{(p)}. \]

(c) \( 0 \to \sum_{1 \leq i \leq m-1} BP_*/I_m \otimes v_i \to BP_*/I_m \to BP_*/I_{m+1} \to 0 \)

is a short exact sequence of comodules.

(d) The only invariant prime ideals in \( BP_* \) are the \( I_m \) for \( 0 \leq m \leq \infty \).

(From here we abbreviate \( \text{Ext}(M) \) for \( \text{Ext}_{BP_*(BP)}(BP_*, M) \).)

This result allows us to define Greek letter elements.

We consider the short exact sequence given by (c) which leads to a long exact sequence of \( \text{Ext} \) and let

\[ \delta_m : \text{Ext}^i(BP_*/I_{m+1}) \to \text{Ext}^{i+1}(BP_*/I_m) \]

denote the connecting homomorphism of the resulting long exact sequence.

**Definition.** For \( t, n > 0 \), let
\[ \alpha t^{(n)} = \delta_0 \delta_1 \cdots \delta_{n-1}(v_i) \in \text{Ext}^n(BP_*). \]
We now state our results. Let

\[ \varphi_n : \text{Ext}^t(BP_\ast) \to \text{Ext}^t(BP_\ast/(I_{n-1} + I_{n-1}^{n+1})) \]

be the homomorphism induced by the natural projection \( BP_\ast \to BP_\ast/(I_{n-1} + I_{n-1}^{n+1}) \). (From now on we always assume \( p \geq n \geq 3 \).) Then we have

**Theorem 2.4.** \( \varphi_n(\alpha^{(n)}_t) \neq 0 \) if \( 3 \leq n \leq p \) and \( 1 \leq t \leq p-1 \).

As an immediate consequence of 2.4 we have

**Corollary 2.5.** \( \alpha^{(n)}_t \neq 0 \) if \( 3 \leq n \leq p \) and \( 1 \leq t \leq p-1 \). Moreover \( p \) does not divide them.

**Remark.** For \( n \leq 3 \) we have much more general results than 2.4. (See [1], [2].)

The rest of this section is devoted to describing the cobar construction which we need in the next section.

Let \((A, \Gamma)\) be a Hopf algebroid such that \( \Gamma \) is flat over \( A \). Then the category of (left) \( \Gamma \)-comodules becomes an abelian category with enough injectives, so we can define \( \text{Ext}^t(L, M) \) for (left) \( \Gamma \)-comodules \( L, M \) as the \( s \)-th right derived functor of \( \text{Hom}_\Gamma(L, M) \).

In the case of \( L = A \), these Ext groups can be computed as the homology of the cobar complex \( C_\Gamma(M) \) defined below.

**Definition.** Let \( \varepsilon : \Gamma \to A \) be the counit and \( \Gamma = \ker \varepsilon \). The cobar complex \( C_\Gamma(M) \) is defined by \( C_\Gamma^t(M) = \Gamma^\otimes_s \otimes M \) with the differential \( d : C_\Gamma^t(M) \to C_\Gamma^{t+1}(M) \) given by

\[
d(\gamma_1 \otimes \cdots \otimes \gamma_s \otimes m) = \sum_{1 \leq i \leq s} (-1)^i \gamma_1 \otimes \cdots \otimes \gamma_{i-1} \otimes \gamma_i' \otimes \cdots \otimes \gamma_s \otimes m + (-1)^{s+1} \sum \gamma_1 \otimes \cdots \otimes \gamma_i \otimes m' \otimes m''
\]

for \( \gamma_1, \cdots, \gamma_s \in \Gamma \) and \( m \in M \) where \( \Gamma^\otimes_s \) denotes the \( s \)-fold tensor product of \( \Gamma \) over \( A \), \( \Delta(\gamma_i) = 1 \otimes \gamma_i + \sum \gamma_i' \otimes \gamma_i'' + \gamma_i \otimes 1 \) and \( \psi(m) = 1 \otimes m + \sum m' \otimes m'' \). (\( \Delta \) denotes the coproduct of \( \Gamma \) and \( \psi \) denotes the coaction map of \( M \).) The element \( \gamma_1 \otimes \cdots \otimes \gamma_s \otimes m \) will be denoted by \( \gamma_1 | \cdots | \gamma_s | m \).

Then the following isomorphism holds.

**Theorem 2.6** ([2] A 1.2.12).

\[ \text{Ext}_A^t(A, \Gamma) \cong H^t(C_\Gamma(M)) \]

Finally we define a certain quotient complex of \( C_\Gamma(A) \) associated with a sequence of non negative integers \( (a_1, \cdots, a_t) \) if \( \Gamma = A[\gamma_1, \cdots, \gamma_m] \) (\( m \) may be
Let \( E = (e_1, e_2, \ldots) \) be a sequence of non-negative integers such that \( e_i = 0 \) for all but a finite number of \( i \). We introduce an order between such sequences by saying that \( E < F = (f_1, f_2, \ldots) \) iff there is a positive integer \( i \) such that \( e_j = f_j \) for \( j < i \) and \( e_i < f_i \). Let \( E + F \) denote a sequence \( (e_1 + f_1, e_2 + f_2, \ldots) \) and \( \gamma^E = \gamma_1 \cdots \gamma^s \in \Gamma \).

**Definition.**

\[
C_T((a_1, \ldots, a_l)) = \frac{C_T(A)}{\bigoplus_{i \geq 1} A \{ \gamma^E_1, \ldots, \gamma^E_s; E_1 + \cdots + E_s > (a_1, \ldots, a_l) \}}
\]

where \( A \{ \cdot \} \) denotes the submodule of \( C_T(A) \) generated by the indicated generators.

(Clearly \( C_T((a_1, \ldots, a_l)) \) depends on the choice of \( \gamma \), but we do not indicate the generators in our notation because our choice is always evident in this paper.)

Now we show that \( C_T((a_1, \ldots, a_l)) \) is a quotient complex of \( C_T(A) \). By our assumption

\[
\Delta(\gamma_i) = 1 \otimes \gamma_i + \sum \gamma'_i \otimes \gamma'_i + \gamma_i \otimes 1
\]

where \( \gamma'_i \in (\gamma_1, \ldots, \gamma_{i-1}) \) or \( \gamma'_i = (\gamma_{i+1}, \ldots, \gamma_{i-1}) \). Thus \( \Delta(\gamma_i) \in A \{ \gamma^F \otimes \gamma^G; F + G \geq (0, \ldots, 0, 1) \} \) and more generally we have \( \Delta(\gamma^E) \in A \{ \gamma^F \otimes \gamma^G; F + G \geq E \} \) since \( \Delta \) is an algebra homomorphism and \( \langle \gamma^F \otimes \gamma^G \rangle = \gamma^F + \gamma^G \). Therefore \( \bigoplus_{i \geq 1} A \{ \gamma^E_1, \ldots, \gamma^E_s; E_1 + \cdots + E_s > (a_1, \ldots, a_l) \} \) is a subcomplex of \( C_T(A) \) as desired.

### 3. Proof of Theorem 2.4

Let \( C(n, m) \) (resp. \( D(n, m) \)) denote

\[
C_{BP^*(BP)/J_{n,m}}((x_{m-2}^n+1, x_{m-3}^n, \ldots, x, 1))
\]

(resp. \( C_{BP^*(BP)/K_{n,m}}((x_{m-2}^n+1, x_{m-3}^n, \ldots, x, 1)) \))

where \( J_{n,m} = I_m^+ + I_{m+1}^++I_{n-1}^++I_1^+ \) and \( K_{n,m} = I_m^+ + I_{m+1}^++I_{n-1}^++I_1^+ \).

(Note that \( BP_*(BP) = BP_*[t_1, t_2, \ldots] \) and \( \Delta(t_i) \) has the form

\[
\Delta(t_i) = 1 \otimes t_i + t_i \otimes 1 \quad \text{in} \quad BP_*(BP) \otimes BP_*(BP)((t_1, \ldots, t_{i-1})
\]

for degree reasons.) It is obvious that the sequence

\[
0 \to C(n, m) \to D(n, m) \to C(n, m+1) \to 0
\]

is a short exact sequence of complexes and letting

\[
\delta_m: H^r(C(n, m+1)) \to H^{r+1}(C(n, m))
\]
denote the corresponding connecting homomorphism we have a commutative diagram

\[
\begin{align*}
\text{Ext}^t(BP_* / I_{m+1}) & \cong \text{Ext}^{t+1}(BP_* / I_m) \\
\psi_{m+1} & \downarrow \\
H^*(C(n, m+1)) & \cong H^{*+1}(C(n, m))
\end{align*}
\]

where \( \psi_m \) is the homomorphism induced by the natural projection \( C_{BP_*}(BP) \rightarrow C(n, m) \).

Thus it is sufficient to show 3.2 below for the proof of 2.4 since \( \psi_0 \) factors through

\[\varphi_n: \text{Ext}^t(BP_*) \rightarrow \text{Ext}^t(BP_*/(I_{n-1} + I_{n-1}^{p+1})).\]

**Proposition 3.2.** \( \hat{\alpha}^{(t)}_t = 0 \) in \( H^t(C(n, 0)) \) if \( 1 \leq t \leq p-1 \) where \( \hat{\alpha}^{(t)}_t \) denotes the element \( \psi_k(\alpha^{(t)}_t) = \delta \cdots \delta_{n-1} \psi_n(\alpha^{(t)}_t) \in H^t(C(n, 0)) \).

In order to prove 3.2 we begin with giving an explicit representative for \( \hat{\alpha}^{(t)}_t \) and this requires some formulas on \( \eta_k \) of \( BP \).

**Lemma 3.3** ([2] 4.3.21).

\[\eta_k(v_m) = v_m + v_{m-1} t_1^{m-1} - v_{m-1} t_1 \mod I_{m-1} \]

**Lemma 3.4.**

\[\eta_k(v_m) = \sum_{0 \leq i \leq m} v_i t_{m-i} \mod I_m \]

For the proof of 3.4 we first prove the following simple fact about the formal group law associated with \( BP \).

**Lemma 3.5.**

\[X + F Y = X + Y \quad \text{in} \quad BP_*[[X, Y]]/\langle X, Y \rangle.\]

**Proof.** Note that \( X + F Y \) has the form

\[X + Y + \sum_{i, j \geq 1} a_{i,j} X^i Y^j \quad \text{in} \quad BP_*[[X, Y]]\]

where \( a_{i,j} = a_{j,i} \in \text{BP}_{2(i+j-1)} \).

Considering the degree of \( a_{i,j} \), it is clear that \( a_{i,j} = 0 \) if \( i+j < p \) so we get the desired result.

**Proof of 3.4.** In the degree of \( \eta_k(v_m) \), the left hand side of 2.1 is congruent to \( \eta_k(v_m) \mod I_m \) by 2.3 (a) and 3.5.

The right hand side of 2.1 is congruent to \( \sum v_i t_{m-i} \mod I_m \) and the
result follows. □

We now describe a representative for $\bar{\alpha}_i^{(r)}$.

**Lemma 3.6.** $\bar{\alpha}_i^{(r)} \in H^n(C(n, 0))$ is represented by a cocycle

$$-t \frac{(p-1)!}{(p-n)!} v_i^{r-1} v_{n-1}^p t_{n-1} v_{n-2}^p \cdots |t_1^{n-2}| t_1 \in C^n(n, 0).$$

Proof. In $D(n, n-1)$,

$$d(v^i_n) = \eta_n(v^i_n) - v^i_n = (v^i_n + v^i_{n-1} t_1^{r-1} - v^i_{n-1} t_1^r) - v^i_n \quad \text{(by 3.3)}$$

$$= -tv^i_n v_{n-1}^p t_1.$$

So we have

$$\delta_{n-1}(v^i_n) = -tv^i_n v_{n-1}^p t_1 \in H^1(C(n, n-1)).$$

(We often abuse the same notation for a cocycle and its representing element in the cohomology.)

In $D(n, n-2)$,

$$d(v^i_n v_{n-1}^p t_1) = d(v^i_n v_{n-1}^p t_1^r) | t_1 \quad \text{(by 3.1)}$$

$$= \{\eta_n(v^i_n) v_{n-1}^p - v^i_n v_{n-1}^p t_1^r\} | t_1$$

$$= \{v^i_n v_{n-1}^p + v^i_{n-2} t_1^{r-1} - v^i_n v_{n-2}^p t_1^r\} | t_1 \quad \text{(by 3.4)}$$

$$= (p-1) v^i_n v_{n-1}^p v_{n-2}^p t_1^{n-2} | t_1$$

and thus

$$\delta_{n-2} \delta_{n-1}(v^i_n) = -t(p-1) v^i_n v_{n-1}^p v_{n-2}^p \cdots |t_1^{n-2} | t_1 \in H^2(C(n, n-2)).$$

More generally, by induction on $k$, we can easily show

$$\delta_{n-k} \cdots \delta_{n-1}(v^i_n) = -t \frac{(p-1)!}{(p-k)!} v^i_n v_{n-k}^p \cdots t_1^{n-k} | t_1 \in H^k(C(n, n-k)) \quad \text{for all} \quad k, 2 \leq k \leq n.$$  (3.7)

Let $k=n$ in 3.7 then we obtain the lemma. □

Next we define a subcomplex of $C(n, 0)$ which will be denoted by $C(n, 0)$.

Let $(P(v_{n-1}, v_n)/(v_{n-1}^{p-1})), P(v_{n-1}, v_n, t_1, \cdots, t_n)/(v_{n-1}^{p-1})$ be the sub-Hopf algebra of $(BP_*, F_0, BP_*(BP)/F_0)$ where $P(*)$ denotes the polynomial algebra which has the indicated generators over $F_p$. We define

$$C(n, 0) = C_p(v_{n-1}, v_n, t_1, \cdots, t_n)/(v_{n-1}^{p-1}) ((p^{n-2} + 1, p^{n-3}, \cdots, p, 1))$$

and let
\[ B(n, m) = C_{p(t_1, \ldots, t_m)} \left( (p^{n-2} + 1, p^{n-3}, \ldots, p, 1) \right) \]

where \( P(t_1, \ldots, t_m) \) is considered as a Hopf algebra over \( F_p \) whose coproduct is given by \( \Delta(t_i) = \sum_{j \leq i} t_j \otimes t_{i-j} \). Then the following isomorphism of differential graded algebras holds.

\[
C(n, 0) \cong P(v_{n-1}, v_n)/(v_{n-1}^{p+1}) \otimes B(n, n).
\]

This follows from 3.4 and the formulas on the coproduct of \( BP_*(BP) \) given by the next lemma.

**Lemma 3.9** ([2] 4.3.15). For \( m \geq 1 \)

\[
\Delta(t_m) = \sum_{0 \leq i \leq m} t_i \otimes t_{m-i} \quad \text{in} \quad BP_*(BP) \otimes BP_*(BP)/I_{m},
\]

and

\[
\Delta(t_{m+1}) = \sum_{0 \leq i \leq m+1} t_i \otimes t_{m+1-i} \quad \text{in} \quad BP_*(BP) \otimes BP_*(BP)/I_{m+1}.
\]

Now note that \(|a_i^m| < |v_{n+1}| \) for \( t \leq p-1 \) and \( C(n, 0) \) is equal to the subcomplex \( C(n, 0) \) defined above in the internal degree less than \(|v_{n+1}| \) and therefore 3.2 is equivalent to

**Proposition 3.10.** \( t_{n-1} | t_{n-2} | \cdots | t_1^{s-2} | t_1 \neq 0 \) in \( H^*(B(n, n-1)) \)

by 3.6 and 3.8 since \( B(n, n-1) = B(n, n) \) in the internal degree less than \(|t_s| \) (> \(|t_{n-1} | t_{n-2} | \cdots | t_1^{s-2} | t_1 | \) for \( p \geq n \)).

In order to show 3.10 we need the following lemma proved at the end of this section.

**Lemma 3.11.** There is a spectral sequence converging to \( H^*(B(n, m)) \) with

\[
E_2^{a,b} = H^b(C_{p(t_m)}(F_p)) \otimes H^a(B(n, m-1))/R_{a,b}
\]

and

\[
d_r: E_r^{a,b} \rightarrow E_r^{a+r,b-r+1}
\]

where \( P(t_m) \) is considered as a Hopf algebra over \( F_p \) with \( t_m \) primitive and

\[
R_{a,b} = F_p \{ x \otimes y \in H^a(C_{p(t_m)}(F_p)) \otimes H^b(B(n, m-1)); \text{ Both } x \text{ and } y \text{ have representative cocycles } \bar{x} \text{ and } \bar{y} \text{ such that } \bar{x}\bar{y} = 0 \text{ in } B^{a+b}(n, m) \}.
\]

Moreover this spectral sequence has the third grading induced by the internal degree in the cohomology which is preserved by all differentials.
Proof of 3.10. First note that
\[ t_1^{s-2} t_1 = 0 \text{ in } H^2(B(n, 1)) \]
by 3.12 below since \( B(n, 1) \) is a direct summand of \( C_p(t_1) (F_p) \) as a complex. (Recall our assumption \( n \geq 3 \) which assures \( p^{s-2} > 1 \).

**Lemma 3.12.**
\[ H^*(C_{p(t_1)} (F_p)) = E(h_{m,0}, h_{m,1}, \ldots) \otimes P(b_{m,0}, b_{m,1}, \ldots) \]
where \( h_{m,i} \) (resp. \( b_{m,i} \)) is represented by \( t_1^i \) (resp. \( \frac{1}{p} \sum_{P \in \mathbb{Z}_p} (p^{i-1}) t_1^{p^i \cdot (s-1)} \)) and \( E(\cdot) \) denotes the exterior algebra which has the indicated generators over \( F_p \).

Proof. This result is obtained by a routine calculation. □

Now suppose
\[ (3.13) \quad t_1^{s-2} \cdots t_1^{s-2} t_1 = 0 \text{ in } H^m(B(n, m-1)) \]
holds for some \( m, 1 < m \leq n-1 \). Then the element \( t_1^{s-2} \otimes t_1^{s-2} \cdots t_1^{s-2} t_1 \)
\((\in H^l(C_{p(t_1)} (F_p)) \otimes H^m(B(n, m-1)))\) defines a non-trivial element in the \( E_2 \)-term of the spectral sequence given by 3.11 which is clearly a permanent cycle and moreover there is no differential killing this element as observed below.

Let \( c_m \) denote the internal degree of the above element then
\[ (3.14) \quad c_m = 2(p-1) \{mp^{s-2} + (m-1)p^{s-3} + \cdots + 2p^{s-m} + p^{s-m-1} + 1 \} \]
and it is enough to prove \( E_r^{m-r, r, c_m} = 0 \) for all \( r \geq 2 \).

Using 3.11 and 3.12 we can identify the \( E_r \) with an appropriate subquotient of \( \otimes_{1 \leq i \leq m} (E(h_{i, i}) \otimes P(b_{i, i})) \) and let \( c_{i_1, i_2, \ldots, i_{2s}, i_{2s+1}, \ldots, i_{2s+1}} \) denote the internal degree of
\[ h_{i_1, i_2} \cdots h_{i_{2s}, i_{2s+1}} b_{i_1, i_2} \cdots b_{i_{2s}, i_{2s+1}} \]
\((1 \leq i_1 \leq \cdots \leq i_{2s}, 1 \leq i_{2s+1} \leq \cdots \leq i_{2s+1} \leq m)\) then
\[ (3.15) \quad c_{i_1, i_2, \ldots, i_{2s}, i_{2s+1}, \ldots, i_{2s+1}} = 2(p-1) \{ \sum_{1 \leq i \leq 2s} p^i (p^{i-1} + \cdots + p + 1) + \sum_{1 \leq i \leq 2s} p^{i+1} (p^{i-1} + \cdots + p + 1) \} . \]

Comparing 3.14 with 3.15 we see easily that \( c_m = c_{i_1, i_2, \ldots, i_{2s}, i_{2s+1}, \ldots, i_{2s+1}} \) does not hold for \( 2s+1 \leq m \) under our assumption \( m < n \leq p \) (\( \geq 3 \)) and consequently \( E_r^{m-r, r, c_m} = 0 \) for all \( r \geq 2 \). Therefore
\[ t_1^{s-2} \cdots t_1^{s-2} t_1 = 0 \text{ in } H^{m+1}(B(n, m)) \]
and by induction on \( m \) we have shown 3.13 for all \( m, 1 < m \leq n \).

Letting \( m = n \) in 3.13 we get 3.10 and thus complete the proof of 2.4 assuming 3.11. □
Proof of 3.11. We begin with recalling the construction of the Cartan-Eilenberg spectral sequence for the following cocentral Hopf algebra extension

\[ P(t_1, \ldots, t_{m-1}) \xrightarrow{f} P(t_1, \ldots, t_m) \xrightarrow{g} P(t_m). \]  


We first define a decreasing filtration on \( C_{p(t_1, \ldots, t_m)}(F_p) \) by

\[
\mathcal{F}_p^{a,b} = F^a_p \{ t^{E_1} | \cdots | t^{E_{a+b}} \} \text{ at least } a \text{ of the } t^{E_i} \text{ lie in kerg} \]

\[ \subset C^a_{p(t_1, \ldots, t_m)}(F_p) \]

and let \( \tilde{E}_r \) denote the \( E_r \)-term of the spectral sequence associated with this filtration.

Next define a homomorphism

\[
\tilde{h}_{a,b} : C^a_{p(t_m)}(F_p) \otimes C^a_{p(t_1, \ldots, t_{m-1})}(F_p) \rightarrow \tilde{E}_0^{a,b}
\]

which is given by

\[
\tilde{h}_{a,b}(t_1^{E_1} | \cdots | t_m^{E_1} | t_2^{E_2} | \cdots | t_m^{E_2} | t_3^{E_3} | \cdots | t_m^{E_3} \in \mathcal{F}_p^{a,b} | \mathcal{F}_p^{a+b-1} = \tilde{E}_0^{a,b}
\]

for \( t_1^{E_1} | \cdots | t_m^{E_1} \in C^a_{p(t_m)}(F_p) \) and \( t_1^{E_1} | \cdots | t_m^{E_2} \in C^a_{p(t_1, \ldots, t_{m-1})}(F_p) \). If we consider \( C^a_{p(t_m)}(F_p) \otimes C^a_{p(t_1, \ldots, t_{m-1})}(F_p) \) as a complex with its differential \( d \otimes 1 \) then \( \tilde{h}_{a,b} \) becomes a chain map and induces

\[
\tilde{h}_r^{a,b} : H^r(C^a_{p(t_m)}(F_p)) \otimes H^r(C^a_{p(t_1, \ldots, t_{m-1})}(F_p)) \rightarrow \tilde{E}_r^{a,b}.
\]

Moreover we can prove \( \tilde{h} \) is an isomorphism and if we consider \( H^r(C^a_{p(t_m)}(F_p)) \otimes C^a_{p(t_1, \ldots, t_{m-1})}(F_p) \) as a complex with its differential \( (-1)^b 1 \otimes d \) then \( \tilde{h}_r^{a,b} \) is also a chain map.

Hence we obtain an isomorphism

\[
\tilde{h}_r^{a,b} : H^r(C^a_{p(t_m)}(F_p)) \otimes H^r(C^a_{p(t_1, \ldots, t_{m-1})}(F_p)) \rightarrow \tilde{E}_r^{a,b}
\]

induced by \( \tilde{h}' \).

Therefore we have a spectral sequence converging to \( H^*(C^a_{p(t_1, \ldots, t_m)}(F_p)) \) whose \( E_2 \)-term is isomorphic to \( H^*(C^a_{p(t_m)}(F_p)) \otimes H^*(C^a_{p(t_1, \ldots, t_{m-1})}(F_p)) \). This spectral sequence is called the Cartan-Eilenberg spectral sequence.

We now turn to our case. It is trivial that \( \tilde{F}_p^{a,b} \) given by 3.16 also defines a decreasing filtration on \( B(n, m) \) naturally. Thus we obtain a spectral sequence \( E_r \) converging to \( H^*(B(n, m)) \) and a homomorphism

\[
h_r^{a,b} : H^*(C^a_{p(t_m)}(F_p)) \otimes H^*(B(n, m-1))/R_{a,b} \rightarrow E_1^{a,b}
\]

induced by a chain map

\[
h_{*,b} : H^*(C^a_{p(t_m)}(F_p)) \otimes B(n, m-1)/R_{a,b} \rightarrow E_1^{a,b}
\]
where
\[ R'_{a, b} = F_p \{ x \otimes y \in H^*(C_{p(t_m)}(F_p)) \otimes B^*(n, m-1); x \text{ has a representative cocycle } \tilde{x} \text{ such that } \tilde{x}y = 0 \text{ in } B^{*+k}(n, m) \} \]

and \( h' \) (resp. \( h'' \)) is the map induced by \( \tilde{h}' \) (resp. \( \tilde{h}'' \)) naturally. So we will show that \( h' \) is an isomorphism.

Let
\[ R'' = F_p \{ x \otimes y \in C_{p(t_m)}(F_p) \otimes B(n, m-1); xy = 0 \text{ in } B(n, m) \} . \]

It is easy to see that \( C_{p(t_m)}(F_p) \otimes B(n, m-1)/R'' \) (resp. \( E_0 \)) is a direct summand of \( C_{p(t_m)}(F_p) \otimes C_{p(t_1, \ldots, t_{m-1})}(F_p) \) (resp. \( E_0 \)) as a complex where \( C_{p(t_m)}(F_p) \otimes C_{p(t_1, \ldots, t_{m-1})}(F_p) \) is endowed with the differential \( d \otimes 1 \) and \( C_{p(t_m)}(F_p) \otimes B(n, m-1)/R'' \) with the induced one, and moreover through \( \tilde{h} \), \( C_{p(t_m)}(F_p) \otimes B(n, m-1)/R'' \) corresponds to \( E_0 \) and another summand of \( C_{p(t_m)}(F_p) \otimes C_{p(t_1, \ldots, t_{m-1})}(F_p) \) corresponds to another one of \( E_0 \).

Hence the fact \( \tilde{h}' \) is an isomorphism implies \( h' \) is also an isomorphism and we complete the proof of 3.11. □

References