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ON CONGRUENCES BETWEEN THE COEFFICIENTS OF TWO L-SERIES WHICH ARE RELATED TO A HYPERELLIPTIC CURVE OVER Q

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1. Introduction

Let $f(x)$ be a monic irreducible polynomial with rational integer coefficients and let p be a prime integer. Reducing the coefficients of $f(x)$ modulo p, we obtain the polynomial $f_p(x)$ with coefficients in $\mathbf{Z}/p\mathbf{Z}$. A rule of the factorization of $f_p(x)$ over $\mathbf{Z}/p\mathbf{Z}$ is called a reciprocity law for $f(x)$ (cf. Wyman [11]). For example, when $f(x)$ is of degree 2, a reciprocity law for $f(x)$ is given by the Legendre symbol (D_f/p) for the discriminant D_f of $f(x)$.

In the case that $f(x)$ is of degree 3, the minimal splitting field K of $f(x)$ over $\mathbf{Q}$ is the Galois extension generated by the coordinates of the two-division points of the elliptic curve $E : y^2 = f(x)$. A reciprocity law for $f(x)$ is given by the Legendre symbol (D_f/p) and the coefficients of the L-series of E over $\mathbf{Q}$, which is the Mellin transform of a modular form of weight two under the Taniyama-Shimura conjecture (the Wiles theorem). Furthermore, in the case that $f(x)$ is of degree 3 and $D_f < 0$, the inverse Mellin transform of the Artin L-function $L(\pi, K/\mathbf{Q}, s)$ attached to the two-dimensional irreducible representation π for the Galois group of K over $\mathbf{Q}$, is a modular form of weight one, by the Weil-Langlands theorem. Thus the Fourier coefficients of the modular form of weight one also gives a reciprocity law for $f(x)$.

In the latter case, we can associate two modular forms with E and the Galois extension generated by the coordinates of its two-division points. Koike [3] obtained congruences between the Fourier coefficients of two modular forms. His congruences describe the relation of the above two reciprocity laws. Naito [6] gave congruences between the coefficients of the L-series of E and those of an Artin L-series attached to the Galois extension generated by the coordinates of the three-division points of E.

In this paper we consider congruences modulo 2 between the coefficients of the L-series of the Jacobian variety of a hyperelliptic curve $y^2 = f(x)$ and those of an Artin L-series which is related to the Galois extension over $\mathbf{Q}$, generated by the coordinates of the two-division points of the same Jacobian variety.

Let $f(x)$ be a polynomial of degree n over $\mathbf{Q}$ with no multiple roots. Let C be a hyperelliptic curve defined by $y^2 = f(x)$. We denote by g the genus of C. We see that either $n = 2g + 1$ or $n = 2g + 2$ holds. We assume that $g \geq 1$ and C has at least one $\mathbf{Q}$-rational point. Then we can choose its Jacobian variety (J, φ) defined over $\mathbf{Q}$.

Let K be the Galois extension over $\mathbf{Q}$, generated by the coordinates of the two-division points of the Jacobian variety J and let G be its Galois group. We assume that $n \neq 1, 2, 4$. Then we can identify G with a suitable subgroup of the permutation group S_n of n letters (See Proposition 2.2). Let π be the restriction of the standard representation of S_n to G. Let ρ_2 be the 2-adic representation of the absolute Galois group of $\mathbf{Q}$ with respect to the 2-adic Tate module of J.

For each odd good prime p of J we put

$$\mathrm{P}_p(u) := \det(I_{n-1} - \pi(\sigma_{\mathfrak{p}})u) \tag{1.1}$$

and

$$\mathrm{Q}_p(u) := \det(I_{2g} - \rho_2(\sigma_{\mathfrak{P}})u), \tag{1.2}$$

where I_m is the unit matrix of size m, $\sigma_{\mathfrak{P}}$ is the Frobenius automorphism for a prime divisor $\mathfrak{P}$ in $\bar{\mathbf{Q}}$, and $\sigma_{\mathfrak{p}}$ is its restriction to K. Then $1/\mathrm{P}_p(p^{-s})$ (resp. $1/\mathrm{Q}_p(p^{-s})$) is the p-factor of Artin L-series $L(\pi, K/\mathbf{Q}, s)$ attached to π (resp. the L-series $L(J/\mathbf{Q}, s)$ of J).

Theorem. (i) *If n is odd and $n \neq 1$, the congruence* $\mathrm{P}_p(u) \equiv \mathrm{Q}_p(u) \bmod 2$ *holds for any odd good prime p of J.*
(ii) *If n is even and $n \neq 2, 4$, the congruence* $\mathrm{P}_p(u) \equiv (1 - u)\mathrm{Q}_p(u) \bmod 2$ *holds for any odd good prime p of J.*

In the case of $n = 3$, the theorem is that of Koike [3]. Thus our theorem is a generalization of Koike's theorem.

The organization of this paper is as follows. In §2, we construct the reduction $\rho_{2,1}$ of the 2-adic representation ρ_2 modulo 2 by matrices in $\mathrm{GL}(2g, \mathbf{Z}/2\mathbf{Z})$. In §3, we construct the standard representation π^{st} of S_n by matrices in $\mathrm{GL}(n-1, \mathbf{Z})$. By comparing two representations $\rho_{2,1}$ and the restriction π of π^{st}, we prove our theorem in §4. In §5, we give some examples of a reciprocity law for $f(x)$ by using our theorem.

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2. The field of two-division points of the Jacobian variety of a hyperelliptic curve over Q

Let $f(x)$ be a polynomial over $\mathbf{Q}$ of degree n with no multiple roots and let C be a hyperelliptic curve of genus g defined by $y^2 = f(x)$. We see that either $n = 2g + 1$ or $2g + 2$ holds. When n is even, the hyperelliptic curve C has two points P_∞, P'_∞ at infinity. When n is odd, the hyperelliptic curve C has one point P_∞ at infinity, which is ramified and $\mathbf{Q}$-rational. In the latter case we put $P'_\infty := P_\infty$.

We assume that the hyperelliptic curve C has at least one $\mathbf{Q}$-rational point. Then we can assume that the Jacobian variety (J, φ) is defined over $\mathbf{Q}$.

Let $\mathrm{Pic}^0(C)$ be the divisor class group of C. The canonical mapping φ induces the isomorphism

$$\bar{\varphi} : \mathrm{Pic}^0(C) \to J : \sum P \mapsto \sum \varphi(P). \tag{2.1}$$

The point corresponding to a $\mathbf{Q}$-rational divisor class is $\mathbf{Q}$-rational.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $f(x) = 0$ and put $P_i := (\alpha_i, 0) \in C$ for $i = 1, 2, \ldots, n$. We see that

$$\mathrm{div}(x - \alpha_i) = 2P_i - P_\infty - P'_\infty \quad \text{for} \quad i = 1, 2, \ldots, n, \tag{2.2}$$

and

$$\mathrm{div}(y) = \begin{cases} P_1 + \cdots + P_{2g+1} - (2g+1)P_\infty & \text{if } n \text{ is odd,} \\ P_1 + \cdots + P_{2g+2} - (g+1)(P_\infty + P'_\infty) & \text{if } n \text{ is even.} \end{cases} \tag{2.3}$$

Let $J[2]$ be the group of two-division points of J. By the equation (2.2) we have that

$$\bar{\varphi}(P_i - P_{2g+1}) \in J[2] \quad \text{for} \quad i = 1, 2, \ldots, 2g. \tag{2.4}$$

Proposition 2.1. $\{\bar{\varphi}(\mathrm{P}_i - \mathrm{P}_{2g+1})\}_{i=1}^{2g}$ *is a basis of* $J[2]$.

For a divisor D on C, we define the set $L(D)$ of rational functions on C over $\bar{\mathbf{Q}}$ by

$$L(D) := \{h : \text{a rational function on } C \,|\, \mathrm{div}(h) + D \text{ is effective.}\} \cup \{0\}. \tag{2.5}$$

$L(D)$ is a vector space over $\bar{\mathbf{Q}}$.

Proof. Since $J[2]$ is a $\mathbf{Z}/2\mathbf{Z}$-module of rank $2g$, it is enough to show that $\bar{\varphi}(\mathrm{P}_i - \mathrm{P}_{2g+1})$ $(i = 1, 2, \ldots, 2g)$ are linearly independent. Suppose

$$\sum_{i=1}^{2g} a_i \ \bar{\varphi}(P_i - P_{2g+1}) = 0 \quad \text{for} \quad a_1, \ldots, a_{2g} \in \{0, 1\}. \tag{2.6}$$

Then there exists a rational function h on C such that

$$\mathrm{div}(h) = \sum_{i=1}^{2g} a_i(P_i - P_{2g+1}). \tag{2.7}$$

We put $a_{2g+1} := a_1+\cdots+a_{2g}$. For the largest integer m less than or equal to $(a_{2g+1}+1)/2$, we put $h_1 := (x-\alpha_{2g+1})^m h$. We have

$$\operatorname{div}(h_1) = \sum_{i=1}^{2g} a_i P_i + (2m - a_{2g+1})P_{2g+1} - m(P_\infty + P'_\infty). \tag{2.8}$$

Since $a_{2g+1} = \sum_{i=1}^{2g+1} a_i \le 2g$, $m \le g$. Thus h_1 is contained in $L(g(P_\infty + P'_\infty))$. By the Riemann-Roch theorem, h_1 is a linear combination of $1, x, \ldots, x^g$. Together with the fact P_i is ramified for $i = 1, \ldots, 2g+1$, the order of h_1 at P_i is even for $i = 1, \ldots, 2g+1$. Since $a_1, \ldots, a_{2g} = 0, 1$, we have $a_1, \ldots, a_{2g} = 0$. Thus $a_{2g+1} = a_1 + \cdots + a_{2g} = 0$. This completes the proof. □

Let K be the Galois extension over $\mathbf{Q}$ generated by the coordinates of the points of $J[2]$. Since φ is a rational function defined over $\mathbf{Q}$, $\varphi(P_i)$ is defined over $\mathbf{Q}(\alpha_i)$ for each i. We note that the addition on J are also defined over $\mathbf{Q}$. Thus the point $\bar{\varphi}(P_i - P_{2g+1}) = \varphi(P_i) - \varphi(P_{2g+1})$ is defined over $\mathbf{Q}(\alpha_i, \alpha_{2g+1})$. Hence K is a subfield of the minimal splitting field $\mathbf{Q}(f) = \mathbf{Q}(\alpha_1, \ldots, \alpha_n)$ of f over $\mathbf{Q}$.

Proposition 2.2. (i) *If $n \ne 1, 2, 4$, then $K = \mathbf{Q}(f)$.*
(ii) *If $n = 4$, then K is the minimal splitting field of the decomposition cubic of f over $\mathbf{Q}$.*

For the proof of Proposition 2.2, we need the following two lemmas.

Lemma 2.3. *Assume that $n \ne 1, 2, 4$. If $\bar{\varphi}(P_i - P_j) = \bar{\varphi}(P_k - P_l)$ for $i \ne j$ and $k \ne l$, then $\{P_i, P_j\} = \{P_k, P_l\}$.*

Proof. Assume that $n = 3$. Then $g = 1$. We have

$$\bar{\varphi}(P_1 - P_2) = \bar{\varphi}(P_1 - P_3) + \bar{\varphi}(P_2 - P_3). \tag{2.9}$$

Since it follows from Proposition 2.1 that

$$\bar{\varphi}(P_1 - P_3),\ \bar{\varphi}(P_2 - P_3),\ \bar{\varphi}(P_1 - P_2) \tag{2.10}$$

are distinct, our assertion follows in this case.

We assume that $n \ge 5$. Then $g \ge 2$. Suppose that $\bar{\varphi}(P_i - P_j) = \bar{\varphi}(P_k - P_l)$. Then there exists a function h satisfying $\operatorname{div}(h)=P_i + P_j + P_k + P_l - 2(P_\infty + P'_\infty)$. Thus h is contained in $L(2(P_\infty + P'_\infty))$, which is spanned by $1, x, x^2$ by the Riemann-Roch theorem, and h has zero at P_i and P_j. Since $i \ne j$, h is equal to $(x-\alpha_i)(x-\alpha_j)$ up to a constant, that is, $\operatorname{div}(h) = 2P_i + 2P_j - 2(P_\infty + P'_\infty)$. Thus we have that $\{P_i, P_j\} = \{P_k, P_l\}$. □

Lemma 2.4. *When* $n = 4$,

$$\bar{\varphi}(P_1 - P_3) = \bar{\varphi}(P_2 - P_4), \ \bar{\varphi}(P_2 - P_3) = \bar{\varphi}(P_1 - P_4), \tag{2.11}$$

and

$$\bar{\varphi}(P_1 - P_3) + \bar{\varphi}(P_2 - P_3) = \bar{\varphi}(P_1 - P_2) = \bar{\varphi}(P_3 - P_4). \tag{2.12}$$

Proof. These equations follow from (2.2) and (2.3). □

Proof of Proposition 2.2. (i) Let σ be an element of the Galois group of $\mathbf{Q}(f)$ over $\mathbf{Q}$. Suppose that σ fixes all elements in K. Then $\sigma\bar{\varphi}(P_i - P_{2g+1}) = \bar{\varphi}(\sigma(P_i) - \sigma(P_{2g+1})) = \bar{\varphi}(P_i - P_{2g+1})$ for $i = 1, \dots, 2g$. By Lemma 2.3, we have that $\{\sigma(P_i), \sigma(P_{2g+1})\} = \{P_i, P_{2g+1}\}$ for $i = 1, \dots, 2g$. Thus we have $\sigma(P_i) = P_i$, that is, $\sigma(\alpha_i) = \alpha_i$ for $i = 1, 2, \dots, 2g+1$. Hence σ is the identity element. Thus our assertion (i) follows.
(ii) Suppose that $\sigma\bar{\varphi}(P_i - P_3) = \bar{\varphi}(P_i - P_3)$ for $i = 1, 2$. By Lemma 2.4 we have that $\{\sigma(P_i), \sigma(P_3)\} = \{P_i, P_3\}$, or $\{P_{3-i}, P_4\}$ for $i = 1, 2$. Equivalently, σ fixes $(\alpha_1+\alpha_3)(\alpha_2+\alpha_4)$, $(\alpha_2+\alpha_3)(\alpha_1+\alpha_4)$, and $(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)$, Since these 3 elements are all roots of the decomposition cubic of f over $\mathbf{Q}$, K is its minimal splitting field. □

In the following we always assume $n \neq 1, 2, 4$. Let S_n be the permutation group of n letters $\{1, 2, \dots, n\}$. The group S_n acts on the set $\{\alpha_i\}_{i=1}^n$ of the roots of $f(x) = 0$ by

$$\sigma\alpha_i = \alpha_{\sigma(i)} \text{ for } \ i = 1, 2, \dots, n. \tag{2.13}$$

The group S_n acts on $J[2]$ from the left hand side by

$$\sigma\bar{\varphi}(P_i - P_{2g+1}) = \bar{\varphi}(P_{\sigma(i)} - P_{\sigma(2g+1)}) \text{ for } i = 1, 2, \dots, 2g. \tag{2.14}$$

We take a basis $\{w_i\}_{i=1}^{2g}$ as follows:

$$w_i := \bar{\varphi}(P_i - P_{2g+1}) \ (1 \le i \le 2g). \tag{2.15}$$

For $i = 1, 2, \dots, n$, let $\sigma_j := (j, 2g+1)$ be the transposition.

Proposition 2.5. (i) *When* $n = 2g+1$ *and* $n \neq 1$,

$$\sigma_j w_i = \begin{cases} w_i & \textit{if either } j = 2g+1 \textit{ or } i = j, \\ w_i + w_j & \textit{if } j \neq 2g+1 \textit{ and } i \neq j. \end{cases} \tag{2.16}$$

(ii) *When $n = 2g+2$ and $n \neq 2,4$,*

$$\sigma_j w_i = \begin{cases} w_i & \textit{if } j = 2g+1, \\ & \textit{or if } j \neq 2g+1, 2g+2, \textit{ and } i = j, \\ w_i + w_j & \textit{if } j \neq 2g+1, 2g+2 \textit{ and } i \neq j, \\ w_1 + w_2 + \cdots + w_{2g} + w_i & \textit{if } j = 2g+2. \end{cases} \tag{2.17}$$

Let G be the Galois group of K over $\mathbf{Q}$. By Proposition 2.2, for any element $\sigma \in G$, there exists the unique element τ in S_n such that

$$\sigma(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\alpha_{\tau(1)}, \alpha_{\tau(2)}, \ldots, \alpha_{\tau(n)}). \tag{2.18}$$

We can identify G with a suitable subgroup of S_n through the inclusion $G \to S_n : \sigma \mapsto \tau$.

We define the representation $\rho_{2,1} : G \to \mathrm{GL}(2g, \mathbf{Z}/2\mathbf{Z})$ by

$$\sigma(w_1, w_2, \ldots, w_{2g}) = (w_1, w_2, \ldots, w_{2g})\rho_{2,1}(\sigma) \text{ for } \sigma \in G. \tag{2.19}$$

The representation $\rho_{2,1}$ is the restriction to G of the representation of S_n defined by (2.14).

Let $T_2(J)$ be the 2-adic Tate module of J. $T_2(J)$ is a free $\mathbf{Z}_2$-module of rank $2g$, where $\mathbf{Z}_2$ is the 2-adic integer ring. Taking a basis $T_2(J)$, we get a representation ρ_2 of the absolute Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ of $\mathbf{Q}$ by matrices in $\mathrm{GL}(2g, \mathbf{Z}_2)$. We can take a basis of $T_2(J)$ satisfying

$$\rho_{2,1}(\sigma') \equiv \rho_2(\sigma) \bmod 2 \text{ for all } \sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}), \tag{2.20}$$

where σ' is the restriction of σ to K. We call the representation ρ_2 is the *2-adic representation of* $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ *with respect to* $T_2(J)$ and we call the representation $\rho_{2,1}$ the *reduction modulo* 2 *of* ρ_2.

3. Standard representation of S_n

Let S_n be the permutation group of n letters $\{1, 2, \ldots, n\}$. Let V^{pr} be an n-dimensional vector space over $\mathbf{Q}$ with basis $\{\varepsilon_i\}_{i=1}^n$. The group S_n acts on the vector space V^{pr} from the left hand side by

$$\sigma\varepsilon_i := \varepsilon_{\sigma(i)} \text{ for } i = 1, 2, \ldots, n, \text{ and } \sigma \in S_n. \tag{3.1}$$

The vector space V^{pr} is called the *permutation representation* of S_n. The permutation representation V^{pr} of S_n is decomposed into the direct sum of two irreducible representations of S_n. Namely, the 1-dimensional subspace V^{tr} spanned by $\varepsilon_1 + \cdots + \varepsilon_n$ and the $(n-1)$-dimensional subspace V^{st} with basis $\{\varepsilon_i - \varepsilon_n\}_{i=1}^{n-1}$. The representations V^{tr}

and V^{st} are called the *trivial representation* and the *standard representation*, respectively.

In this section, we investigate the standard representation V^{st} of S_n. As a matter of convenience, we denote by g the largest integer less than or equal to $(n-1)/2$. Then either $n = 2g+1$ or $n = 2g+2$ holds.

We take a basis $\{v_i\}_{i=1}^{n-1}$ of V^{st} as follows:

When $n = 2g+1$,

$$v_i := \varepsilon_i - \varepsilon_{2g+1} \quad \text{if } 1 \le i \le 2g; \tag{3.2}$$

When $n = 2g+2$,

$$v_i := \begin{cases} \varepsilon_i - \varepsilon_{2g+1} & \text{if } 1 \le i \le 2g, \\ \varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \cdots + \varepsilon_{2g+1} - \varepsilon_{2g+2} & \text{if } i = 2g+1. \end{cases} \tag{3.3}$$

We define the matrix representation π^{st} of S_n by

$$\sigma(v_1, v_2, \ldots, v_{n-1}) = (v_1, v_2, \ldots, v_{n-1})\pi^{st}(\sigma). \tag{3.4}$$

For $j = 1, 2, \ldots, n$, let $\sigma_j := (j, 2g+1)$ be the transposition in S_n.

Proposition 3.1. (i) *When $n = 2g+1$, we have*

$$\sigma_j v_i = \begin{cases} v_i & \text{if } j = 2g+1, \\ -v_i & \text{if } i = j \text{ and } j \ne 2g+1, \\ v_i - v_j & \text{if } i \ne j \text{ and } j \ne 2g+1. \end{cases} \tag{3.5}$$

(ii) *When $n = 2g+2$, we have*

$$\sigma_j v_i = \begin{cases} v_i & \text{if } j = 2g+1, \\ -v_i & \text{if } i \ne 2g+1,\ j \ne 2g+1,\ 2g+2 \\ & \qquad \text{and } i = j, \\ v_i - v_j & \text{if } i \ne 2g+1,\ j \ne 2g+1,\ 2g+2 \\ & \qquad \text{and } i \ne j, \\ \displaystyle\sum_{m=1}^{2g} (-1)^m v_m + v_i + v_{2g+1} & \text{if } i \ne 2g+1 \text{ and } j = 2g+2, \\ v_{2g+1} & \text{if } i = 2g+1 \text{ and } j \text{ is odd}, \\ v_{2g+1} + 2v_j & \text{if } i = 2g+1 \text{ and } j \ne 2g+2 \text{ is even}, \\ -v_{2g+1} + 2\displaystyle\sum_{m=1}^{2g} (-1)^{m-1} v_m & \text{if } i = 2g+1 \text{ and } j = 2g+2. \end{cases} \tag{3.6}$$

Since σ_j's generate S_n, it follows from Proposition 3.1 that $\pi^{st}(\sigma)$ is a matrix in $\mathrm{GL}(n-1,\mathbf{Z})$. Thus we can consider the reduction of the representation π^{st} modulo 2.

Proposition 3.2. (i) *When $n = 2g+1$, we have*

$$\sigma_j v_i \equiv \begin{cases} v_i & \bmod 2 \text{ if either } j = 2g+1 \text{ or } i = j, \\ v_i + v_j & \bmod 2 \text{ if } i \neq j \text{ and } j \neq 2g+1. \end{cases} \tag{3.7}$$

(ii) *When $n = 2g+2$,*

$$\sigma_j v_i \equiv \begin{cases} v_i & \mathrm{mod} 2 \text{ if } i \neq 2g+1,\ j = 2g+1, \\ & \qquad \text{or if } i \neq 2g+1,\ i = j, \\ v_i + v_j & \bmod 2 \text{ if } i \neq 2g+1,\ j \neq 2g+1,\ 2g+2, \\ & \qquad \text{and } i \neq j, \\ \displaystyle\sum_{m=1}^{2g} v_m + v_i + v_{2g+1} & \bmod 2 \text{ if } i \neq 2g+1 \text{ and } j = 2g+2, \\ v_{2g+1} & \bmod 2 \text{ if } i = 2g+1. \end{cases} \tag{3.8}$$

Proof. Proposition 3.2 follows from (3.5) and (3.6). □

The conjugate classes of S_n correspond to partitions of n bijectively. We call an element σ in S_n of *type* $(n_1, n_2, \ldots, n_r)$ if σ belongs to the conjugacy class corresponding to the partition $(n_1, n_2, \ldots, n_r)$. The following is well-known.

Proposition 3.3. *Let σ be an element in S_n of type $(n_1, n_2, \ldots, n_r)$. Then the characteristic polynomial of σ in S_n for π^{st} is given by*

$$\det(\mathrm{I}_{n-1} - \pi^{st}(\sigma)u) = \frac{1}{1-u} \prod_{i=1}^{r} (1 - u^{n_i}), \tag{3.9}$$

where I_{n-1} is the unit matrix of size $n-1$

Proof. We note that $V^{pr} = V^{st} \oplus V^{tr}$. Our assertion follows from direct computations. □

4. Proof of Theorem

Let the notation be the same as in §1. We note that any odd good prime is unramified in K.

Let $\rho_{2,1} : G \to \mathrm{GL}(2g, \mathbf{Z}/2\mathbf{Z})$ be the representation defined by (2.19). It follows from (2.20) that

$$\mathrm{Q}_p(u) = \det(I_{2g} - \rho_2(\sigma_{\mathfrak{P}})u) \equiv \det(I_{2g} - \rho_{2,1}(\sigma_{\mathfrak{p}})u) \mod 2. \tag{4.1}$$

We can take $\pi^{st} : S_n \to \mathrm{GL}(n-1, \mathbf{Z})$ defined by (3.4) in §4 as the standard representation of S_n. Compared with (2.16), (2.17) and (3.7), (3.8), we have

$$\pi(\sigma) \equiv \rho_{2,1}(\sigma) \quad \left(\text{resp. } \pi(\sigma) \equiv \begin{pmatrix} \rho_{2,1}(\sigma) & 0 \\ * & 1 \end{pmatrix}\right) \mod 2 \qquad \text{for all } \sigma \in G, \tag{4.2}$$

if n is odd and $n \neq 1$ (resp. if n is even and $n \neq 2, 4$). Thus we have

$$\mathrm{P}_p(u) \equiv \mathrm{Q}_p(u) \ (\text{resp. } \mathrm{P}_p(u) \equiv (1-u)\mathrm{Q}_p(u)) \mod 2. \tag{4.3}$$

5. Numerical examples

Let the notation be the same as in §1. We assume that $f(x)$ is a monic polynomial with rational integer coefficients. We denote by $f_p(x)$ the reduction of $f(x)$ modulo p. The type of the factorization of $f_p(x)$ corresponds to that of the conjugate class of the Frobenius automorphism $\sigma_{\mathfrak{p}}$. By Proposition 3.3 and by our Theorem, we have:

Proposition 5.1. (i) *If $f_p(x) = g_1(x)g_2(x)\cdots g_r(x)$ in $\mathbf{Z}/p\mathbf{Z}[x]$ for some irreducible polynomials $g_i(x)$ of degree n_i, then*

$$\mathrm{Q}_p(u) \equiv \frac{1}{(1-u)^{\varepsilon}} \prod_{i=1}^{r} (1 - u^{n_i}) \mod 2, \tag{5.1}$$

where $\varepsilon = 1$ (resp. $\varepsilon = 2$) if n is odd and $n \neq 1$ (resp. if n is even and $n \neq 2, 4$).
(ii) *The signature of $\sigma_{\mathfrak{p}}$ in S_n is equal to the Legendre symbol (D_f/p).*

In the following we give three examples, which describe the law of decomposition of primes in terms of $\mathrm{Q}_p(u)$ mod 2 and (D_f/p), in the case of $g = 2$. We note that an odd prime integer q is a good prime of J if q is prime to the discriminant D_f of $f(x)$.

EXAMPLE 1. We put $f(x) := x^5 - x - 1$. Then $D_f = 2869 = 19 \cdot 151$ and $G = S_5$ (cf. [4], p. 121). For any $p \neq 2, 19, 151$ we have the following:

$Q_p(u) \bmod 2$	$\left(\frac{2869}{p}\right)$	degrees of irreducible factors of f_p	example of p
$(1-u)^4$	1	1, 1, 1, 1, 1	1973, 3769, 5101
		1, 2, 2	67, 71
	-1	1, 1, 1, 2	163, 193, 227
		1, 4	23, 29, 31, 61, 97
$(1-u)^2(1+u+u^2)$	1	1, 1, 3	17, 41, 43, 47, 53
	-1	2, 3	7, 13, 37, 59, 73, 83
$1+u+u^2+u^3+u^4$	1	5	3, 5, 11, 79, 89

EXAMPLE 2. We put $f(x) := x^6-4x^5-12x^4+2x^3+8x^2+8x-7$. Then $D_f = 2^{12}29^5$ and the hyperelliptic curve C is the modular curve $X_0(29)$ (cf. [5]). We can check that the endomorphism algebra of J is $\mathbf{Q}(\sqrt{2})$. By choosing suitable indices of roots of f, $G = \langle (1,2,3)(4,5,6),\ (1,2)(4,5),\ (1,4)(2,5)(3,6)\rangle$, which is isomorphic to the dihedral group of order 12 (cf. [8]). For any $p \neq 2, 29$ we have the following:

$Q_p(u) \bmod 2$	$\left(\frac{29}{p}\right)$	degrees of irreducible factors of f_p	example of p
$(1-u)^4$	1	1, 1, 1, 1, 1, 1	173, 197, 277
		1, 1, 2, 2	7, 23, 59, 67, 71, 83
	-1	2, 2, 2	17, 19, 37, 41, 61, 73, 89, 97
$(1+u+u^2)^2$	1	3, 3	5, 13, 53
	-1	6	3, 11, 31, 43, 47, 79

In this example, by using the fact that K contains $\mathbf{Q}(\sqrt{-1})$, we can distinguish the first row and the second row by the Legendre symbol $(-1/p)$. And also the fourth row and the fifth row.

EXAMPLE 3. We put $f(x) := x^6-4x^5+6x^4-6x^3+9x^2-14x+9$. Then $D_f = 2^{12}67^2$ and the hyperelliptic curve C is the modular curve $X_0^*(67)$ (cf. [5]). Then we can checked that the endomorphism algebra of J is $\mathbf{Q}(\sqrt{5})$. By choosing suitable indices of roots of f, $G = \langle (1,2,6)(3,5,4),\ (1,2,3,4,5),\ (2,5)(3,4)\rangle$, which is isomorphic to the alternative group of degree 5 (cf. [8]). For any $p \neq 2, 67$ we have the following:

$Q_p(u) \bmod 2$	degrees of irreducible factors of f_p	example of p
$(1-u)^4$	1, 1, 1, 1, 1, 1	311, 1163, 1453
	1, 1, 2, 2	17, 59, 73
$(1+u+u^2)^2$	3, 3	5, 11, 23
$1+u+u^2+u^3+u^4$	1, 5	3, 7, 13

In Example 2 and in Example 3, there exist modular forms h_1, h_2 of weight two with respect to a congruence subgroup such that $L(J/\mathbf{Q}, s)$ and the product $L(h_1, s)L(h_2, s)$ of their Mellin transforms are essentially same as in Shimura's sense (cf. [7]). Thus by our theorem, we can consider congruences between the coefficients of the Artin L-series $L(\pi, K/\mathbf{Q}, s)$ and the Fourier coefficients of the modular forms h_1, h_2 of weight two in those examples.

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