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ON CONGRUENCES BETWEEN THE COEFFICIENTS OF TWO L-SERIES WHICH ARE RELATED TO A HYPERELLIPTIC CURVE OVER \mathbf{Q}

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1. Introduction

Let $f(x)$ be a monic irreducible polynomial with rational integer coefficients and let p be a prime integer. Reducing the coefficients of $f(x)$ modulo p , we obtain the polynomial $f_p(x)$ with coefficients in $\mathbf{Z}/p\mathbf{Z}$. A rule of the factorization of $f_p(x)$ over $\mathbf{Z}/p\mathbf{Z}$ is called a reciprocity law for $f(x)$ (cf. Wyman [11]). For example, when $f(x)$ is of degree 2, a reciprocity law for $f(x)$ is given by the Legendre symbol (D_f/p) for the discriminant D_f of $f(x)$.

In the case that $f(x)$ is of degree 3, the minimal splitting field K of $f(x)$ over \mathbf{Q} is the Galois extension generated by the coordinates of the two-division points of the elliptic curve $E : y^2 = f(x)$. A reciprocity law for $f(x)$ is given by the Legendre symbol (D_f/p) and the coefficients of the L-series of E over \mathbf{Q} , which is the Mellin transform of a modular form of weight two under the Taniyama-Shimura conjecture (the Wiles theorem). Furthermore, in the case that $f(x)$ is of degree 3 and $D_f < 0$, the inverse Mellin transform of the Artin L-function $L(\pi, K/\mathbf{Q}, s)$ attached to the two-dimensional irreducible representation π for the Galois group of K over \mathbf{Q} , is a modular form of weight one, by the Weil-Langlands theorem. Thus the Fourier coefficients of the modular form of weight one also gives a reciprocity law for $f(x)$.

In the latter case, we can associate two modular forms with E and the Galois extension generated by the coordinates of its two-division points. Koike [3] obtained congruences between the Fourier coefficients of two modular forms. His congruences describe the relation of the above two reciprocity laws. Naito [6] gave congruences between the coefficients of the L-series of E and those of an Artin L-series attached to the Galois extension generated by the coordinates of the three-division points of E .

In this paper we consider congruences modulo 2 between the coefficients of the L-series of the Jacobian variety of a hyperelliptic curve $y^2 = f(x)$ and those of an Artin L-series which is related to the Galois extension over \mathbf{Q} , generated by the coordinates of the two-division points of the same Jacobian variety.

Let $f(x)$ be a polynomial of degree n over \mathbf{Q} with no multiple roots. Let C be a hyperelliptic curve defined by $y^2 = f(x)$. We denote by g the genus of C . We see that

either $n = 2g + 1$ or $n = 2g + 2$ holds. We assume that $g \geq 1$ and C has at least one \mathbf{Q} -rational point. Then we can choose its Jacobian variety (J, φ) defined over \mathbf{Q} .

Let K be the Galois extension over \mathbf{Q} , generated by the coordinates of the two-division points of the Jacobian variety J and let G be its Galois group. We assume that $n \neq 1, 2, 4$. Then we can identify G with a suitable subgroup of the permutation group S_n of n letters (See Proposition 2.2). Let π be the restriction of the standard representation of S_n to G . Let ρ_2 be the 2-adic representation of the absolute Galois group of \mathbf{Q} with respect to the 2-adic Tate module of J .

For each odd good prime p of J we put

$$(1.1) \quad P_p(u) := \det(I_{n-1} - \pi(\sigma_p)u)$$

and

$$(1.2) \quad Q_p(u) := \det(I_{2g} - \rho_2(\sigma_{\mathfrak{P}})u),$$

where I_m is the unit matrix of size m , $\sigma_{\mathfrak{P}}$ is the Frobenius automorphism for a prime divisor \mathfrak{P} in $\bar{\mathbf{Q}}$, and σ_p is its restriction to K . Then $1/P_p(p^{-s})$ (resp. $1/Q_p(p^{-s})$) is the p -factor of Artin L-series $L(\pi, K/\mathbf{Q}, s)$ attached to π (resp. the L-series $L(J/\mathbf{Q}, s)$ of J).

Theorem. (i) *If n is odd and $n \neq 1$, the congruence $P_p(u) \equiv Q_p(u) \pmod{2}$ holds for any odd good prime p of J .*

(ii) *If n is even and $n \neq 2, 4$, the congruence $P_p(u) \equiv (1 - u)Q_p(u) \pmod{2}$ holds for any odd good prime p of J .*

In the case of $n = 3$, the theorem is that of Koike [3]. Thus our theorem is a generalization of Koike's theorem.

The organization of this paper is as follows. In §2, we construct the reduction $\rho_{2,1}$ of the 2-adic representation ρ_2 modulo 2 by matrices in $\mathrm{GL}(2g, \mathbf{Z}/2\mathbf{Z})$. In §3, we construct the standard representation π^{st} of S_n by matrices in $\mathrm{GL}(n-1, \mathbf{Z})$. By comparing two representations $\rho_{2,1}$ and the restriction π of π^{st} , we prove our theorem in §4. In §5, we give some examples of a reciprocity law for $f(x)$ by using our theorem.

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2. The field of two-division points of the Jacobian variety of a hyperelliptic curve over \mathbf{Q}

Let $f(x)$ be a polynomial over \mathbf{Q} of degree n with no multiple roots and let C be a hyperelliptic curve of genus g defined by $y^2 = f(x)$. We see that either $n = 2g + 1$ or $2g + 2$ holds. When n is even, the hyperelliptic curve C has two points P_∞, P'_∞ at

infinity. When n is odd, the hyperelliptic curve C has one point P_∞ at infinity, which is ramified and \mathbf{Q} -rational. In the latter case we put $P'_\infty := P_\infty$.

We assume that the hyperelliptic curve C has at least one \mathbf{Q} -rational point. Then we can assume that the Jacobian variety (J, φ) is defined over \mathbf{Q} .

Let $\text{Pic}^0(C)$ be the divisor class group of C . The canonical mapping φ induces the isomorphism

$$(2.1) \quad \bar{\varphi} : \text{Pic}^0(C) \rightarrow J : \sum P \mapsto \sum \varphi(P).$$

The point corresponding to a \mathbf{Q} -rational divisor class is \mathbf{Q} -rational.

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the roots of the equation $f(x) = 0$ and put $P_i := (\alpha_i, 0) \in C$ for $i = 1, 2, \dots, n$. We see that

$$(2.2) \quad \text{div}(x - \alpha_i) = 2P_i - P_\infty - P'_\infty \quad \text{for } i = 1, 2, \dots, n,$$

and

$$(2.3) \quad \text{div}(y) = \begin{cases} P_1 + \dots + P_{2g+1} - (2g+1)P_\infty & \text{if } n \text{ is odd,} \\ P_1 + \dots + P_{2g+2} - (g+1)(P_\infty + P'_\infty) & \text{if } n \text{ is even.} \end{cases}$$

Let $J[2]$ be the group of two-division points of J . By the equation (2.2) we have that

$$(2.4) \quad \bar{\varphi}(P_i - P_{2g+1}) \in J[2] \quad \text{for } i = 1, 2, \dots, 2g.$$

Proposition 2.1. $\{\bar{\varphi}(P_i - P_{2g+1})\}_{i=1}^{2g}$ is a basis of $J[2]$.

For a divisor D on C , we define the set $L(D)$ of rational functions on C over $\bar{\mathbf{Q}}$ by

$$(2.5) \quad L(D) := \{h : \text{a rational function on } C \mid \text{div}(h) + D \text{ is effective.}\} \cup \{0\}.$$

$L(D)$ is a vector space over $\bar{\mathbf{Q}}$.

Proof. Since $J[2]$ is a $\mathbf{Z}/2\mathbf{Z}$ -module of rank $2g$, it is enough to show that $\bar{\varphi}(P_i - P_{2g+1})$ ($i = 1, 2, \dots, 2g$) are linearly independent. Suppose

$$(2.6) \quad \sum_{i=1}^{2g} a_i \bar{\varphi}(P_i - P_{2g+1}) = 0 \quad \text{for } a_1, \dots, a_{2g} \in \{0, 1\}.$$

Then there exists a rational function h on C such that

$$(2.7) \quad \text{div}(h) = \sum_{i=1}^{2g} a_i (P_i - P_{2g+1}).$$

We put $a_{2g+1} := a_1 + \dots + a_{2g}$. For the largest integer m less than or equal to $(a_{2g+1} + 1)/2$, we put $h_1 := (x - \alpha_{2g+1})^m h$. We have

$$(2.8) \quad \text{div}(h_1) = \sum_{i=1}^{2g} a_i P_i + (2m - a_{2g+1})P_{2g+1} - m(P_\infty + P'_\infty).$$

Since $a_{2g+1} = \sum_{i=1}^{2g+1} a_i \leq 2g$, $m \leq g$. Thus h_1 is contained in $L(g(P_\infty + P'_\infty))$. By the Riemann-Roch theorem, h_1 is a linear combination of $1, x, \dots, x^g$. Together with the fact P_i is ramified for $i = 1, \dots, 2g+1$, the order of h_1 at P_i is even for $i = 1, \dots, 2g+1$. Since $a_1, \dots, a_{2g} = 0, 1$, we have $a_1, \dots, a_{2g} = 0$. Thus $a_{2g+1} = a_1 + \dots + a_{2g} = 0$. This completes the proof. \square

Let K be the Galois extension over \mathbf{Q} generated by the coordinates of the points of $J[2]$. Since φ is a rational function defined over \mathbf{Q} , $\varphi(P_i)$ is defined over $\mathbf{Q}(\alpha_i)$ for each i . We note that the addition on J are also defined over \mathbf{Q} . Thus the point $\bar{\varphi}(P_i - P_{2g+1}) = \varphi(P_i) - \varphi(P_{2g+1})$ is defined over $\mathbf{Q}(\alpha_i, \alpha_{2g+1})$. Hence K is a subfield of the minimal splitting field $\mathbf{Q}(f) = \mathbf{Q}(\alpha_1, \dots, \alpha_n)$ of f over \mathbf{Q} .

- Proposition 2.2.** (i) *If $n \neq 1, 2, 4$, then $K = \mathbf{Q}(f)$.*
 (ii) *If $n = 4$, then K is the minimal splitting field of the decomposition cubic of f over \mathbf{Q} .*

For the proof of Proposition 2.2, we need the following two lemmas.

Lemma 2.3. *Assume that $n \neq 1, 2, 4$. If $\bar{\varphi}(P_i - P_j) = \bar{\varphi}(P_k - P_l)$ for $i \neq j$ and $k \neq l$, then $\{P_i, P_j\} = \{P_k, P_l\}$.*

Proof. Assume that $n = 3$. Then $g = 1$. We have

$$(2.9) \quad \bar{\varphi}(P_1 - P_2) = \bar{\varphi}(P_1 - P_3) + \bar{\varphi}(P_2 - P_3).$$

Since it follows from Proposition 2.1 that

$$(2.10) \quad \bar{\varphi}(P_1 - P_3), \bar{\varphi}(P_2 - P_3), \bar{\varphi}(P_1 - P_2)$$

are distinct, our assertion follows in this case.

We assume that $n \geq 5$. Then $g \geq 2$. Suppose that $\bar{\varphi}(P_i - P_j) = \bar{\varphi}(P_k - P_l)$. Then there exists a function h satisfying $\text{div}(h) = P_i + P_j + P_k + P_l - 2(P_\infty + P'_\infty)$. Thus h is contained in $L(2(P_\infty + P'_\infty))$, which is spanned by $1, x, x^2$ by the Riemann-Roch theorem, and h has zero at P_i and P_j . Since $i \neq j$, h is equal to $(x - \alpha_i)(x - \alpha_j)$ up to a constant, that is, $\text{div}(h) = 2P_i + 2P_j - 2(P_\infty + P'_\infty)$. Thus we have that $\{P_i, P_j\} = \{P_k, P_l\}$. \square

Lemma 2.4. *When $n = 4$,*

$$(2.11) \quad \bar{\varphi}(P_1 - P_3) = \bar{\varphi}(P_2 - P_4), \quad \bar{\varphi}(P_2 - P_3) = \bar{\varphi}(P_1 - P_4),$$

and

$$(2.12) \quad \bar{\varphi}(P_1 - P_3) + \bar{\varphi}(P_2 - P_3) = \bar{\varphi}(P_1 - P_2) = \bar{\varphi}(P_3 - P_4).$$

Proof. These equations follow from (2.2) and (2.3). □

Proof of Proposition 2.2. (i) Let σ be an element of the Galois group of $\mathbf{Q}(f)$ over \mathbf{Q} . Suppose that σ fixes all elements in K . Then $\sigma\bar{\varphi}(P_i - P_{2g+1}) = \bar{\varphi}(\sigma(P_i) - \sigma(P_{2g+1})) = \bar{\varphi}(P_i - P_{2g+1})$ for $i = 1, \dots, 2g$. By Lemma 2.3, we have that $\{\sigma(P_i), \sigma(P_{2g+1})\} = \{P_i, P_{2g+1}\}$ for $i = 1, \dots, 2g$. Thus we have $\sigma(P_i) = P_i$, that is, $\sigma(\alpha_i) = \alpha_i$ for $i = 1, 2, \dots, 2g + 1$. Hence σ is the identity element. Thus our assertion (i) follows.

(ii) Suppose that $\sigma\bar{\varphi}(P_i - P_3) = \bar{\varphi}(P_i - P_3)$ for $i = 1, 2$. By Lemma 2.4 we have that $\{\sigma(P_i), \sigma(P_3)\} = \{P_i, P_3\}$, or $\{P_{3-i}, P_4\}$ for $i = 1, 2$. Equivalently, σ fixes $(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4)$, $(\alpha_2 + \alpha_3)(\alpha_1 + \alpha_4)$, and $(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)$. Since these 3 elements are all roots of the decomposition cubic of f over \mathbf{Q} , K is its minimal splitting field. □

In the following we always assume $n \neq 1, 2, 4$. Let S_n be the permutation group of n letters $\{1, 2, \dots, n\}$. The group S_n acts on the set $\{\alpha_i\}_{i=1}^n$ of the roots of $f(x) = 0$ by

$$(2.13) \quad \sigma\alpha_i = \alpha_{\sigma(i)} \text{ for } i = 1, 2, \dots, n.$$

The group S_n acts on $J[2]$ from the left hand side by

$$(2.14) \quad \sigma\bar{\varphi}(P_i - P_{2g+1}) = \bar{\varphi}(P_{\sigma(i)} - P_{\sigma(2g+1)}) \text{ for } i = 1, 2, \dots, 2g.$$

We take a basis $\{w_i\}_{i=1}^{2g}$ as follows:

$$(2.15) \quad w_i := \bar{\varphi}(P_i - P_{2g+1}) \text{ (} 1 \leq i \leq 2g \text{)}.$$

For $i = 1, 2, \dots, n$, let $\sigma_j := (j, 2g + 1)$ be the transposition.

Proposition 2.5. (i) *When $n = 2g + 1$ and $n \neq 1$,*

$$(2.16) \quad \sigma_j w_i = \begin{cases} w_i & \text{if either } j = 2g + 1 \text{ or } i = j, \\ w_i + w_j & \text{if } j \neq 2g + 1 \text{ and } i \neq j. \end{cases}$$

(ii) When $n = 2g + 2$ and $n \neq 2, 4$,

$$(2.17) \quad \sigma_j w_i = \begin{cases} w_i & \text{if } j = 2g + 1, \\ & \text{or if } j \neq 2g + 1, 2g + 2, \text{ and } i = j, \\ w_i + w_j & \text{if } j \neq 2g + 1, 2g + 2 \text{ and } i \neq j, \\ w_1 + w_2 + \dots + w_{2g} + w_i & \text{if } j = 2g + 2. \end{cases}$$

Let G be the Galois group of K over \mathbf{Q} . By Proposition 2.2, for any element $\sigma \in G$, there exists the unique element τ in S_n such that

$$(2.18) \quad \sigma(\alpha_1, \alpha_2, \dots, \alpha_n) = (\alpha_{\tau(1)}, \alpha_{\tau(2)}, \dots, \alpha_{\tau(n)}).$$

We can identify G with a suitable subgroup of S_n through the inclusion $G \rightarrow S_n : \sigma \mapsto \tau$.

We define the representation $\rho_{2,1} : G \rightarrow \text{GL}(2g, \mathbf{Z}/2\mathbf{Z})$ by

$$(2.19) \quad \sigma(w_1, w_2, \dots, w_{2g}) = (w_1, w_2, \dots, w_{2g})\rho_{2,1}(\sigma) \text{ for } \sigma \in G.$$

The representation $\rho_{2,1}$ is the restriction to G of the representation of S_n defined by (2.14).

Let $T_2(J)$ be the 2-adic Tate module of J . $T_2(J)$ is a free \mathbf{Z}_2 -module of rank $2g$, where \mathbf{Z}_2 is the 2-adic integer ring. Taking a basis $T_2(J)$, we get a representation ρ_2 of the absolute Galois group $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ of \mathbf{Q} by matrices in $\text{GL}(2g, \mathbf{Z}_2)$. We can take a basis of $T_2(J)$ satisfying

$$(2.20) \quad \rho_{2,1}(\sigma') \equiv \rho_2(\sigma) \pmod{2} \text{ for all } \sigma \in \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}),$$

where σ' is the restriction of σ to K . We call the representation ρ_2 is the 2-adic representation of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ with respect to $T_2(J)$ and we call the representation $\rho_{2,1}$ the reduction modulo 2 of ρ_2 .

3. Standard representation of S_n

Let S_n be the permutation group of n letters $\{1, 2, \dots, n\}$. Let V^{pr} be an n -dimensional vector space over \mathbf{Q} with basis $\{\varepsilon_i\}_{i=1}^n$. The group S_n acts on the vector space V^{pr} from the left hand side by

$$(3.1) \quad \sigma\varepsilon_i := \varepsilon_{\sigma(i)} \text{ for } i = 1, 2, \dots, n, \text{ and } \sigma \in S_n.$$

The vector space V^{pr} is called the permutation representation of S_n . The permutation representation V^{pr} of S_n is decomposed into the direct sum of two irreducible representations of S_n . Namely, the 1-dimensional subspace V^{tr} spanned by $\varepsilon_1 + \dots + \varepsilon_n$ and the $(n - 1)$ -dimensional subspace V^{st} with basis $\{\varepsilon_i - \varepsilon_n\}_{i=1}^{n-1}$. The representations V^{tr}

and V^{st} are called the *trivial representation* and the *standard representation*, respectively.

In this section, we investigate the standard representation V^{st} of S_n . As a matter of convenience, we denote by g the largest integer less than or equal to $(n - 1)/2$. Then either $n = 2g + 1$ or $n = 2g + 2$ holds.

We take a basis $\{v_i\}_{i=1}^{n-1}$ of V^{st} as follows:

When $n = 2g + 1$,

$$(3.2) \quad v_i := \varepsilon_i - \varepsilon_{2g+1} \quad \text{if } 1 \leq i \leq 2g;$$

When $n = 2g + 2$,

$$(3.3) \quad v_i := \begin{cases} \varepsilon_i - \varepsilon_{2g+1} & \text{if } 1 \leq i \leq 2g, \\ \varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \cdots + \varepsilon_{2g+1} - \varepsilon_{2g+2} & \text{if } i = 2g + 1. \end{cases}$$

We define the matrix representation π^{st} of S_n by

$$(3.4) \quad \sigma(v_1, v_2, \dots, v_{n-1}) = (v_1, v_2, \dots, v_{n-1})\pi^{st}(\sigma).$$

For $j = 1, 2, \dots, n$, let $\sigma_j := (j, 2g + 1)$ be the transposition in S_n .

Proposition 3.1. (i) *When $n = 2g + 1$, we have*

$$(3.5) \quad \sigma_j v_i = \begin{cases} v_i & \text{if } j = 2g + 1, \\ -v_i & \text{if } i = j \text{ and } j \neq 2g + 1, \\ v_i - v_j & \text{if } i \neq j \text{ and } j \neq 2g + 1. \end{cases}$$

(ii) *When $n = 2g + 2$, we have*

$$(3.6) \quad \sigma_j v_i = \begin{cases} v_i & \text{if } j = 2g + 1, \\ -v_i & \text{if } i \neq 2g + 1, j \neq 2g + 1, 2g + 2 \\ & \text{and } i = j, \\ v_i - v_j & \text{if } i \neq 2g + 1, j \neq 2g + 1, 2g + 2 \\ & \text{and } i \neq j, \\ \sum_{m=1}^{2g} (-1)^m v_m + v_i + v_{2g+1} & \text{if } i \neq 2g + 1 \text{ and } j = 2g + 2, \\ v_{2g+1} & \text{if } i = 2g + 1 \text{ and } j \text{ is odd,} \\ v_{2g+1} + 2v_j & \text{if } i = 2g + 1 \text{ and } j \neq 2g + 2 \text{ is even,} \\ -v_{2g+1} + 2 \sum_{m=1}^{2g} (-1)^{m-1} v_m & \text{if } i = 2g + 1 \text{ and } j = 2g + 2. \end{cases}$$

Since σ_j 's generate S_n , it follows from Proposition 3.1 that $\pi^{st}(\sigma)$ is a matrix in $GL(n - 1, \mathbf{Z})$. Thus we can consider the reduction of the representation π^{st} modulo 2.

Proposition 3.2. (i) *When $n = 2g + 1$, we have*

$$(3.7) \quad \sigma_j v_i \equiv \begin{cases} v_i & \text{mod 2 if either } j = 2g + 1 \text{ or } i = j, \\ v_i + v_j & \text{mod 2 if } i \neq j \text{ and } j \neq 2g + 1. \end{cases}$$

(ii) *When $n = 2g + 2$,*

$$(3.8) \quad \sigma_j v_i \equiv \begin{cases} v_i & \text{mod 2 if } i \neq 2g + 1, j = 2g + 1, \\ & \text{or if } i \neq 2g + 1, i = j, \\ v_i + v_j & \text{mod 2 if } i \neq 2g + 1, j \neq 2g + 1, 2g + 2, \\ & \text{and } i \neq j, \\ \sum_{m=1}^{2g} v_m + v_i + v_{2g+1} & \text{mod 2 if } i \neq 2g + 1 \text{ and } j = 2g + 2, \\ v_{2g+1} & \text{mod 2 if } i = 2g + 1. \end{cases}$$

Proof. Proposition 3.2 follows from (3.5) and (3.6). □

The conjugate classes of S_n correspond to partitions of n bijectively. We call an element σ in S_n of type (n_1, n_2, \dots, n_r) if σ belongs to the conjugacy class corresponding to the partition (n_1, n_2, \dots, n_r) . The following is well-known.

Proposition 3.3. *Let σ be an element in S_n of type (n_1, n_2, \dots, n_r) . Then the characteristic polynomial of σ in S_n for π^{st} is given by*

$$(3.9) \quad \det(I_{n-1} - \pi^{st}(\sigma)u) = \frac{1}{1-u} \prod_{i=1}^r (1 - u^{n_i}),$$

where I_{n-1} is the unit matrix of size $n - 1$

Proof. We note that $V^{pr} = V^{st} \oplus V^{tr}$. Our assertion follows from direct computations. □

4. Proof of Theorem

Let the notation be the same as in §1. We note that any odd good prime is unramified in K .

Let $\rho_{2,1} : G \rightarrow GL(2g, \mathbf{Z}/2\mathbf{Z})$ be the representation defined by (2.19). It follows

from (2.20) that

$$(4.1) \quad Q_p(u) = \det(I_{2g} - \rho_2(\sigma_{\mathfrak{P}})u) \equiv \det(I_{2g} - \rho_{2,1}(\sigma_{\mathfrak{P}})u) \pmod 2.$$

We can take $\pi^{st} : S_n \rightarrow \text{GL}(n - 1, \mathbf{Z})$ defined by (3.4) in §4 as the standard representation of S_n . Compared with (2.16), (2.17) and (3.7), (3.8), we have

$$(4.2) \quad \pi(\sigma) \equiv \rho_{2,1}(\sigma) \left(\text{resp. } \pi(\sigma) \equiv \begin{pmatrix} \rho_{2,1}(\sigma) & 0 \\ * & 1 \end{pmatrix} \right) \pmod 2 \quad \text{for all } \sigma \in G,$$

if n is odd and $n \neq 1$ (resp. if n is even and $n \neq 2, 4$). Thus we have

$$(4.3) \quad P_p(u) \equiv Q_p(u) \text{ (resp. } P_p(u) \equiv (1 - u)Q_p(u)) \pmod 2.$$

5. Numerical examples

Let the notation be the same as in §1. We assume that $f(x)$ is a monic polynomial with rational integer coefficients. We denote by $f_p(x)$ the reduction of $f(x)$ modulo p . The type of the factorization of $f_p(x)$ corresponds to that of the conjugate class of the Frobenius automorphism $\sigma_{\mathfrak{p}}$. By Proposition 3.3 and by our Theorem, we have:

Proposition 5.1. (i) *If $f_p(x) = g_1(x)g_2(x) \cdots g_r(x)$ in $\mathbf{Z}/p\mathbf{Z}[x]$ for some irreducible polynomials $g_i(x)$ of degree n_i , then*

$$(5.1) \quad Q_p(u) \equiv \frac{1}{(1 - u)^\varepsilon} \prod_{i=1}^r (1 - u^{n_i}) \pmod 2,$$

where $\varepsilon = 1$ (resp. $\varepsilon = 2$) if n is odd and $n \neq 1$ (resp. if n is even and $n \neq 2, 4$).

(ii) *The signature of $\sigma_{\mathfrak{p}}$ in S_n is equal to the Legendre symbol (D_f/p) .*

In the following we give three examples, which describe the law of decomposition of primes in terms of $Q_p(u) \pmod 2$ and (D_f/p) , in the case of $g = 2$. We note that an odd prime integer q is a good prime of J if q is prime to the discriminant D_f of $f(x)$.

EXAMPLE 1. We put $f(x) := x^5 - x - 1$. Then $D_f = 2869 = 19 \cdot 151$ and $G = S_5$ (cf. [4], p. 121). For any $p \neq 2, 19, 151$ we have the following:

$\mathbb{Q}_p(u) \bmod 2$	$\left(\frac{2869}{p}\right)$	degrees of irreducible factors of f_p	example of p
$(1-u)^4$	1	1, 1, 1, 1, 1	1973, 3769, 5101
		1, 2, 2	67, 71
	-1	1, 1, 1, 2	163, 193, 227
		1, 4	23, 29, 31, 61, 97
$(1-u)^2(1+u+u^2)$	1	1, 1, 3	17, 41, 43, 47, 53
	-1	2, 3	7, 13, 37, 59, 73, 83
$1+u+u^2+u^3+u^4$	1	5	3, 5, 11, 79, 89

EXAMPLE 2. We put $f(x) := x^6 - 4x^5 - 12x^4 + 2x^3 + 8x^2 + 8x - 7$. Then $D_f = 2^{12}29^5$ and the hyperelliptic curve C is the modular curve $X_0(29)$ (cf. [5]). We can check that the endomorphism algebra of J is $\mathbb{Q}(\sqrt{2})$. By choosing suitable indices of roots of f , $G = \langle (1, 2, 3)(4, 5, 6), (1, 2)(4, 5), (1, 4)(2, 5)(3, 6) \rangle$, which is isomorphic to the dihedral group of order 12 (cf. [8]). For any $p \neq 2, 29$ we have the following:

$\mathbb{Q}_p(u) \bmod 2$	$\left(\frac{29}{p}\right)$	degrees of irreducible factors of f_p	example of p
$(1-u)^4$	1	1, 1, 1, 1, 1, 1	173, 197, 277
		1, 1, 2, 2	7, 23, 59, 67, 71, 83
	-1	2, 2, 2	17, 19, 37, 41, 61, 73, 89, 97
$(1+u+u^2)^2$	1	3, 3	5, 13, 53
	-1	6	3, 11, 31, 43, 47, 79

In this example, by using the fact that K contains $\mathbb{Q}(\sqrt{-1})$, we can distinguish the first row and the second row by the Legendre symbol $(-1/p)$. And also the fourth row and the fifth row.

EXAMPLE 3. We put $f(x) := x^6 - 4x^5 + 6x^4 - 6x^3 + 9x^2 - 14x + 9$. Then $D_f = 2^{12}67^2$ and the hyperelliptic curve C is the modular curve $X_0^*(67)$ (cf. [5]). Then we can check that the endomorphism algebra of J is $\mathbb{Q}(\sqrt{5})$. By choosing suitable indices of roots of f , $G = \langle (1, 2, 6)(3, 5, 4), (1, 2, 3, 4, 5), (2, 5)(3, 4) \rangle$, which is isomorphic to the alternative group of degree 5 (cf. [8]). For any $p \neq 2, 67$ we have the following:

$\mathbb{Q}_p(u) \bmod 2$	degrees of irreducible factors of f_p	example of p
$(1-u)^4$	1, 1, 1, 1, 1, 1	311, 1163, 1453
	1, 1, 2, 2	17, 59, 73
$(1+u+u^2)^2$	3, 3	5, 11, 23
$1+u+u^2+u^3+u^4$	1, 5	3, 7, 13

In Example 2 and in Example 3, there exist modular forms h_1, h_2 of weight two with respect to a congruence subgroup such that $L(J/\mathbb{Q}, s)$ and the product $L(h_1, s)L(h_2, s)$ of their Mellin transforms are essentially same as in Shimura's sense (cf. [7]). Thus by our theorem, we can consider congruences between the coefficients

of the Artin L-series $L(\pi, K/\mathbf{Q}, s)$ and the Fourier coefficients of the modular forms h_1, h_2 of weight two in those examples.

References

- [1] P. Deligne and J. -P. Serre: *Formes modulaires de poids 1*, Ann. Sci. E.N.S. **4** (1974), 507–530.
- [2] W. Fulton and J. Harris: *Representation theory*, Springer-Verlag, New-York, 1991.
- [3] M. Koike: *Higher reciprocity law, modular forms of weight 1 and elliptic curves*, Nagoya Math. J. **98** (1985), 109–115.
- [4] S. Lang: *Algebraic Number Theory*, Springer-Verlag, 1986.
- [5] N. Murabayashi: *On normal forms of modular curves of genus 2*, Osaka J. Math. **29** (1992), 405–418.
- [6] H. Naito: *A congruence between the coefficients of the L-series which are related to an elliptic curve and algebraic number field generated by its 3-division points*, Mem. Fac. Edu. Kagawa Univ. **37** (1987), 43–45.
- [7] G. Shimura: *Introduction to the arithmetic theory of automorphic functions*, Iwanami Shoten, Publishers and Princeton Univ. Press, Tokyo, 1971.
- [8] R.P. Stauduhar: *The determination of Galois groups*, Math. Comp. **124** (1973), 981–996.
- [9] A. Weil: *Variété abéliennes et courbes algébriques*, Hermann, Paris, 1948.
- [10] A. Wiles: *Modular elliptic curves and Fermat’s Last Theorem*, Ann. of Math. **141** (1995), 443–551.
- [11] B.F. Wyman: *What is a reciprocity law?*, Amer. Math. Monthly, **79** (1972), 571–586.

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