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ON MULTIPLY TRANSITIVE GROUPS VIII

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1. Introduction

Let G be a 4-fold transitive group on $\Omega = \{1, 2, \dots, n\}$, and let P be a Sylow 2-subgroup of a stabilizer of four points in G . By a theorem of M. Hall [2, Theorem 5.8.1] and a lemma of E. Witt [7, Theorem 9.4], we have that P fixes exactly four, five, six, seven or eleven points and the normalizer of P in G restricted on the set of the fixed points of P is S_4 , S_5 , A_6 , A_7 or M_{11} . (cf. H. Nagao and T. Oyama [5], Lemma 1).

The purpose of this paper is to prove the following

Theorem. *Let G be a 4-fold transitive group. If a Sylow 2-subgroup of a stabilizer of four points in G fixes exactly six points, then G must be A_6 .*

The above theorem of M. Hall is that if a stabilizer of four points in G is of odd order then G must be one of the following groups: S_4 , S_5 , A_6 , A_7 or M_{11} . Therefore to prove our theorem we may assume that a Sylow 2-subgroup of a stabilizer of four points in G is not identity.

2. Definitions and notations

A permutation x is called semi-regular if there is no point fixed by x . A permutation group G is called semi-regular if every non-identity element of G is semi-regular on the points actually moved by G .

For a permutation group G on Ω the subgroup of G consisting of all the elements fixing the points i, j, \dots, k of Ω will be denoted by $G_{i,j,\dots,k}$, which we shall call the stabilizer of the points i, j, \dots, k . The totality of points left fixed by a subset X of G will be denoted by $I(X)$, and if a subset Δ of Ω is a fixed block of X , then the restriction of X on Δ will be denoted by X^Δ . A G -orbit of minimal length is called a minimal orbit of G .

For subsets X, Y of a group G , $\langle X, Y \rangle$ is the subgroup of G generated by the elements of X and Y , and $N_G(X)$ is the normalizer of X in G .

3. Proof of the theorem

In the following two lemmas we assume that G is a 4-fold transitive group

on $\Omega = \{1, 2, \dots, n\}$, and P is a Sylow 2-subgroup of G_{1234} . For a point t of a minimal orbit of P in $\Omega - I(P)$ let $P_t = Q$, $N_G(Q) = N$ and $I(Q) = \Delta$.

Lemma 1. *Let R be a Sylow 2-subgroup of N_{ijkl} for $\{i, j, k, l\} \subseteq \Delta$. Then R^Δ , which is a Sylow 2-subgroup of $(N^\Delta)_{ijkl}$, is semi-regular and $|I(R)| = |I(P)|$.*

Proof. Let φ be a natural homomorphism of N_{ijkl} onto $(N^\Delta)_{ijkl}$. Then $\varphi(R) = R^\Delta$. Since R is a Sylow 2-subgroup of N_{ijkl} , R^Δ is a Sylow 2-subgroup of $(N^\Delta)_{ijkl}$.

Let P' be a Sylow 2-subgroup of G_{ijkl} containing R . Since Q is a normal 2-subgroup of N_{ijkl} and R is a Sylow 2-subgroup of N_{ijkl} , $P' \geq R \geq Q$ and $I(P') \subseteq I(R) \subseteq \Delta$. Since G is 4-fold transitive, P and P' are conjugate, and $|I(P)| = |I(P')|$. Therefore $I(P')$ is a proper subset of Δ . For any point r of $\Delta - I(P')$, $P'_r \geq Q = P_t$. From the assumption that t belongs to a minimal P -orbit we have

$$|P' : P'_r| = |r^{P'}| \geq |t^P| = |P : P_t| = |P' : Q|.$$

Therefore $P'_r = Q$ and $R_r = Q$. Thus R^Δ is identity or semi-regular and $I(R^\Delta) = I(P')$. Since $\Delta = I(Q) \supseteq I(P')$ and $P'_\Delta = Q$, $N_{P'}(Q)^\Delta$ is a non-identity 2-group of $(N^\Delta)_{ijkl}$. Therefore R^Δ , which is a Sylow 2-subgroup of $(N^\Delta)_{ijkl}$, is semi-regular and $|I(R)| = |I(R^\Delta)| = |I(P')| = |I(P)|$.

Lemma 2. *In Lemma 1 if $|I(P)| = 6$, that is $N_G(P)^{I(P)} = A_6$, then R^Δ is an elementary abelian group, $N_N(R)^{I(R)} \leq A_6$ and $|\Delta| \geq 10$.*

Proof. If R has an element $x = (i' j' k' l') \dots$, where $\{i', j', k', l'\} \subset \Delta$, then x^Δ has no 2-cycle by Lemma 1. On the other hand, x normalizes $G_{i'j'k'l'}$ and Q . Therefore x normalizes some Sylow 2-subgroup P' of $G_{i'j'k'l'}$ containing Q . From the assumption $x^{I(P')} \in A_6$. Hence $x^{I(P')}$ has a 2-cycle. Since $I(P') \subset \Delta$, x^Δ has a 2-cycle, which is a contradiction. Thus R^Δ is elementary abelian.

From Lemma 1 $|I(R)| = 6$. If $N_N(R)^{I(R)} \not\leq A_6$, then $N_N(R)$ has a 2-element x such that $x^{I(R)}$ is an odd permutation. On the other hand x normalizes some Sylow 2-subgroup P'' of $G_{I(R)}$. Since $G_{I(R)}$ contains a Sylow 2-subgroup of G_{ijkl} , P'' is a Sylow 2-subgroup of some stabilizer of four points in G , and $I(P'') = I(R)$. Then $x^{I(R)} = x^{I(P'')} \in N_G(P'')^{I(P'')} = A_6$, which is a contradiction. Thus $N_N(R)^{I(R)} \leq A_6$.

Since $|I(R)| = 6$ and $\Delta \supseteq I(R)$, $|\Delta| \geq 8$. Suppose that $|\Delta| = 8$. Let $\Delta = \{i, j, k, l, r, s, u, v\}$ and $I(R) = \{i, j, k, l, r, s\}$. Then R has the following 2-element

$$a = (i)(j)(k)(l)(r)(s)(uv) \dots$$

Since a normalizes G_{ijuv} and Q , a normalizes a Sylow 2-subgroup P''' of G_{ijuv} containing Q . It follows from $I(P''') \subset \Delta$ that $a^{I(P''')}$ is a transposition. This is a contradiction since $N_G(P''')^{I(P''')} = A_6$. Therefore $|\Delta| \geq 10$.

From now on we consider a permutation group G on $\Omega = \{1, 2, \dots, n\}$, which is not necessarily 4-fold transitive. In the following two lemmas we assume that G satisfies the following condition.

- (*) *Let P be a Sylow 2-subgroup of any stabilizer of four points in G . Then*
 - (i) *P is semi-regular and elementary abelian,*
 - (ii) *$|\Omega| \geq 10$ and $|I(P)| = 6$,*
 - (iii) *$N_G(P)^{I(P)} \leq A_6$.*

By Lemma 1 and Lemma 2, N^Δ satisfies the assumption (*). In Lemma 4 we shall show that there is no group satisfying (*). Thus if G is a 4-fold transitive group as in Theorem, then G_{1234} is of odd order, and hence G must be A_6 .

Lemma 3. *G is a doubly transitive group and $N_G(P)^{I(P)}$ is A_6 or A_6^* , where A_6^* is a doubly transitive group of degree 6 isomorphic to A_6 .*

Proof. For any two points i_1 and i_2 let P' be a Sylow 2-subgroup of $G_{i_1 i_2 i_3 i_4}$, where $\{i_3, i_4\} \subset \Omega - \{i_1, i_2\}$. Then we have an involution a of P' , which has the following form

$$a = (i_1)(i_2)(i_3)(i_4)(i_5)(i_6)(j_1 j_2) \dots$$

where $I(P') = \{i_1, i_2, \dots, i_6\}$. Since a normalizes $G_{i_1 i_2 j_1 j_2}$, a normalizes a Sylow 2-subgroup P'' of $G_{i_1 i_2 j_1 j_2}$. From (iii)

$$a^{I(P'')} = (i_1)(i_2)(j_1 j_2)(j_3 j_4),$$

where $I(P'') = \{i_1, i_2, j_1, j_2, j_3, j_4\}$. Hence $\langle a, P'' \rangle$ is a 2-group and fixes exactly two points i_1 and i_2 . From a lemma of D. Livingstone and A. Wagner [4, Lemma 6] G is a doubly transitive group on Ω .

Since G is doubly transitive, for any two points u and v of $I(P)$ there is a Sylow 2-subgroup S of G_{uv} , which contains P and fixes only two points u and v . Let $T = N_S(P)$. Then $T \cong P$ and $I(S) \subseteq I(T) \subseteq I(P)$. Suppose that $I(S) \subsetneq I(T)$, then $|I(T)| = 4$ or 6 . By (iii) $|I(T)| = 6$. Therefore $I(T) = I(P)$. Since P is a Sylow 2-subgroup of $G_{I(P)}$, $T = P$, which is a contradiction. Therefore $I(T) = I(S) = \{u, v\}$. This shows that for any two points u, v of $I(P)$ $N_G(P)^{I(P)}$ contains a 2-group, which fixes only two points u and v . Using also Lemma 6 of [3] we have that $N_G(P)^{I(P)}$ is doubly transitive. Therefore, by assumption (iii), $N_G(P)^{I(P)}$ is either A_6 or A_6^* , where A_6^* is a doubly transitive group of degree 6 isomorphic to A_5 (see B. Huppert [3] II, 4.7 Satz).

To prove the next lemma we need the following result in [6]: Let G be a 4-fold transitive group. If a Sylow 2-subgroup P of a stabilizer of four points in G is semi-regular and not identity, then $|I(P)| \neq 6$.

Lemma 4. *There is no group satisfying (*).*

Proof. Assume that G satisfies (*). We may assume that P is a Sylow

2-subgroup of G_{1234} and $I(P)=\{1, 2, \dots, 6\}$. In the case $N_G(P)^{I(P)}=A_6^*$, we may assume that A_6^* is generated by $\{(3\ 4)(5\ 6), (1\ 2\ 3)(4\ 5\ 6), (1\ 3\ 5\ 6\ 4)\}$ (see W. Burnside [1] §166).

For any point i of $\Omega-\{1, 2, 3\}$ let P' be a Sylow 2-subgroup of G_{123i} . By (ii) P' fixes six points, say $1, 2, 3, i, j$ and k . Let Q be a Sylow 2-subgroup of G_{123} containing P' . Since $|\Omega|$ is even, $|I(Q)|$ is also even. Therefore Q fixes at least four points, and hence Q fixes six points by (ii), which are the points of $I(P')$. Thus $Q=P'$ is a Sylow 2-subgroup of G_{123} , and any point in $\Omega-\{1, 2, 3\}$ belongs to a G_{123} -orbit of odd length. On the other hand P and P' are conjugate in G_{123} , hence G_{123} has an element taking $\{4, 5, 6\}$ into $\{i, j, k\}$. Thus G_{123} has exactly one or three orbits in $\Omega-\{1, 2, 3\}$. If G_{123} has three orbits in $\Omega-\{1, 2, 3\}$, then three points 4, 5 and 6 belong to different G_{123} -orbits, say T_4, T_5 and T_6 respectively.

Suppose that G_{123} is transitive on $\Omega-\{1, 2, 3\}$. Since a Sylow 2-subgroup of G_{12} fixes only two points 1 and 2, G_{12} has an element taking 3 into some point of $\Omega-\{1, 2, 3\}$. Therefore G_{12} is transitive on $\Omega-\{1, 2\}$. It follows from Lemma 3 that G is 4-fold transitive on Ω . But this contradicts the theorem in [6].

From now on we assume that G_{123} has three orbits T_4, T_5 and T_6 in $\Omega-\{1, 2, 3\}$. Suppose that $N_G(P)^{I(P)}=A_6$. Then $N_G(P)$ contains an element x of the form

$$x = (1)(2)(3)(4\ 5\ 6)\dots$$

Since $x \in G_{123}$, 4, 5 and 6 belong to the same G_{123} -orbit, which is a contradiction.

Thus we have that $N_G(P')^{I(P')}=A_6^*$, for any Sylow 2-subgroup P' of an arbitrary stabilizer of four points in G .

Suppose that P has two involutions x and y . Since P is elementary abelian by (i), we may assume by (ii) that x and y are of the following forms

$$\begin{aligned} x &= (1)(2)\dots(6)(ij)(kl)\dots, \\ y &= (1)(2)\dots(6)(ik)(jl)\dots. \end{aligned}$$

$\langle x, y \rangle$ normalizes some Sylow 2-subgroup of G_{ijkl} . Hence the restriction of $\langle x, y \rangle$ on the set of the points fixed by this Sylow 2-subgroup is a four group and fixes two points. But a stabilizer of two points in A_6^* is of order 2, which is a contradiction. Therefore P is of order 2, and any Sylow 2-subgroup of a stabilizer of four points in G is also of order 2.

Let a be an involution of P . We may assume by (ii) that a is of the form

$$a = (1)(2)\dots(6)(ij)\dots$$

Then a normalizes a Sylow 2-subgroup of G_{12ij} , and hence a commutes with some involution b in G_{12ij} . Since b fixes only six points, $b^{I(a)}$ is not identity. Since $N_G(P)^{I(P)}=A_6^*$, b must be of the form

$$b = (1) (2) (3\ 4) (5\ 6) (i) (j) (k) (l) \dots,$$

and then a have a 2-cycle $(k\ l)$. If a and b have two 2-cycles $(i'\ j')$ $(k'\ l')$ and $(i'\ k')$ $(j'\ l')$ respectively, then $\langle a, b \rangle$ normalizes some Sylow 2-subgroup of $G_{i'j'k'l'}$. Using the same argument as above we have a contradiction. Therefore if a has 2-cycles in $\Omega - \{1, 2, \dots, 6, i, j, k, l\}$, then b has the same 2-cycles. Since a commutes with b , ab is also an involution, and fixes two points 1 and 2. Therefore $|I(ab)| = 2$ or 6 by (ii). If $|I(ab)| = 2$, then

$$\begin{aligned} a &= (1) (2) \dots (6) (i\ j) (k\ l), \\ b &= (1) (2) (3\ 4) (5\ 6) (i) (j) (k) (l), \end{aligned}$$

and hence $|\Omega| = 10$. If $|I(ab)| = 6$, then

$$\begin{aligned} a &= (1) (2) \dots (6) (i\ j) (k\ l) (i'\ j') (k'\ l'), \\ b &= (1) (2) (3\ 4) (5\ 6) (i) (j) (k) (l) (i'\ j') (k'\ l'), \end{aligned}$$

and hence $|\Omega| = 14$. On the other hand from the assumption that A_6^* has an element $(1\ 2\ 3) (4\ 5\ 6)$, there is an element

$$c = (1\ 2\ 3) (4\ 5\ 6) \dots$$

in $N_G(P)$. Then c normalizes G_{123} , and since the G_{123} -orbits in $\Omega - \{1, 2, 3\}$ are T_4, T_5 and T_6 , c takes T_4 into T_5 , T_5 into T_6 and T_6 into T_4 . Therefore T_4, T_5 and T_6 are of the same length and $|\Omega - \{1, 2, 3\}|$ is divisible by 3. But $|\Omega| - 3 = 7$ or 11, which is not divisible by 3. This contradiction arises from the first assumption that there is a group which satisfies the conditions (i), (ii) and (iii).

Now by Lemma 2 and 4 we have that, for a 4-fold transitive group G as in the theorem, the stabilizer of four points in G is of odd order. Therefore G must be A_6 .

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