<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>A note on the equivariant Whitehead groups of dihedral groups</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Inoue, Tsuyoshi</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 27(2) P.421–P.430</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1990</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/10500">https://doi.org/10.18910/10500</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/10500</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
</tbody>
</table>

*Osaka University Knowledge Archive : OUKA*

https://ir.library.osaka-u.ac.jp/repo/ouka/all/

Osaka University
A NOTE ON THE EQUIVARIANT WHITEHEAD GROUPS OF DIHEDRAL GROUPS

Dedicated to Professor Shōrō Araki on his 60th birthday

TSUYOSHI INOUE

(Received March 23, 1989)

0. Introduction

This note is intended as "The equivariant Whitehead torsions of equivariant homotopy equivalence between the unit spheres of representations II". Therefore, we shall use the notations in [11]. In this note, restriction maps in Whitehead groups play an important role. To illustrate this, we begin with an example pointed out by M. Masuda. Let $C_n$ and $D_n$ be the cyclic group and dihedral group of order $n$ and $2n$ respectively. As we remarked in [11], a generator of $Wh(C_5)$ appears as the reduced equivariant Whitehead torsion of any $C_5$-homotopy equivalence

$$f: S(V_3 \oplus V_2) \to S(V_3 \oplus V_1).$$

where $V_a$ ($a=1, 2, 3$) denotes the complex $C_5$-module $C$ with $g \in C_5$ acting as multiplication by $\exp(2\pi ia/5)$ and $S(V)$ denotes the unit sphere of $C_5$-module $V$. Since the torsion does not depend on the choice of $f$, we can assume that $f$ is the map due to T. Petrie (see §2). By the complex conjugation, $C_5$-modules $V_a$ can be regarded as $C_5$-modules. Then the Petrie's map $f$ turns out to be a $D_5$-homotopy equivalence. The reduced equivariant Whitehead torsion $\tau_{D_5}(f) = p_\ast \tau_{C_5}(f)$ of $f$ as a $D_5$-homotopy equivalence lies in $Wh_{D_5}(\ast) \approx Wh(D_5)$ where $p_\ast: Wh_{D_5}(S(V_3 \oplus V_2)) \to Wh_{D_5}(\ast)$ is the induced map by the obvious map $p: S(V_3 \oplus V_2) \to \ast$. It is obvious that the restriction map from $D_5$ to $C_5$ sends the torsion to the generator of $Wh_{C_5}(\ast) \approx Wh(C_5)$. Therefore the restriction map induces an isomorphism of the Whitehead groups because $Wh(D_5)$ is a free abelian group of rank 1 (see [3], [21], [19], [20] and [17]). Moreover we see that the torsion is a generator of $Wh(D_5)$. Our main result (Theorem A) is a generalization of this observation.

Theorem A. The restriction map induces an isomorphism

$$\text{Res}_{C_5}^{D_5}: Wh_{\text{rep}}(D_5) \to Wh_{\text{rep}}(C_5),$$
where $Wh_{\text{rep}}(G)$ denotes the subgroups of $Wh_G(*)$ generated by the reduced torsions of $G$-homotopy equivalences between the unit spheres of $G$ modules.

By the Theorem A, the same conclusion as [11, Theorem C] holds for dihedral groups.

**Corollary B.** $Wh_{\text{rep}}(D_n)$ is of finite index in $Wh_{D_n}(*)$ if and only if $n = 8, 9, 12, 16, 18, p$ or $2p$ for odd prime integers $p$.

In §1, we discuss the restriction maps of Whitehead groups from dihedral groups to cyclic groups. We give a sufficient condition for the restriction map being an isomorphism. In §2, we investigate the $C_n$-homotopy equivalences between the unit spheres of $C_n$-modules due to T. Petrie. In §3, we state the main results and prove them. We also exhibit an example concerning generators of Whitehead groups of dihedral groups in §3.

The author owes to Professors Shōrō Araki and Mikiya Masuda by useful discussions and advices, and would like to express here his hearty thanks to them.

1. The restriction maps from dihedral group to cyclic group

In this section, we shall investigate the restriction map of Whitehead groups from a dihedral group to a cyclic group. First, we consider the standard involution on Whitehead groups. Let $G$ be a finite group. The assignment $g \mapsto g^{-1}$ in $G$ induces a conjugation $\theta: Z[G] \to Z[G]$. This conjugation induces the standard involution $\theta: Wh(G) \to Wh(G)$. The following lemma is fundamental in our investigation.

**Lemma 1.1.** Let $G$ be an abelian group. Then, each element of $(Z[G])^*/\pm G$ is represented by a unit $u \in (Z[G])^*$ such that $u = u$. In particular, if $Wh(G)$ is torsion free, each element of $Wh(G)$ is represented by a unit $u \in (Z[G])^*$ such that $u = u$.

Proof. It is well known that the standard involution on $Wh(G) = Wh(G)/\text{torsion}$ is trivial (see [24], [2] or [16]). According to the proof of [2] for this fact, for each $u \in (Z[G])^*$, there exists $g_0 \in G$ such that $u \cdot (u)^{-1} = \pm g_0$. Applying the augmentation map $Z[G] \to Z$ to both sides of the identity, we see $u \cdot (u)^{-1} = g_0$. Here, we consider an involution $\theta: G \to G$, $\theta(g) = g g^{-1}$. If we put $u = \sum a_g g$ $(a_g \in Z)$, the identity $u \cdot (u)^{-1} = g_0$ implies

$$a_g = a_{\theta(g)} \quad \text{for each} \quad g \in G.$$

Therefore, $\theta$ must have a fixed point because $\sum a_g = \pm 1$. The fixed point of $\theta$, say $g \in G$, satisfies $g^2 = g_0$. If we put $v = g^{-1} u$, $v$ is a required element because $v = u$ in $(Z[G])^*/\pm G$ and
EQUIVARIANT WHITEHEAD GROUPS

\[ v = g^{-1} u = g^{-1} g_o u = g u = v . \]

Q.E.D.

**Notation 1.2.**

- **\( D_n \):** the dihedral group of order \( 2n \) generated by two elements \( s \) and \( t \) with relations \( t^n = s^2 = 1 \) and \( st = ts^{-1} \).
- **\( C_n \):** the cyclic subgroup of \( D_n \) generated by \( t \).

In later sections, we shall consider the equivariant Whitehead group of \( D_n \) (called the generalized Whitehead group of \( D_n \) by Rothenberg). Therefore, we shall treat the classical Whitehead groups and the equivariant Whitehead groups at the same time. To do this, we need the following lemma.

**Lemma 1.3.** \( \text{Wh}_{D_n}(\ast) = \bigoplus_{d|n} \text{Wh}(D_d) \)

and the following diagram commutes

\[
\begin{array}{ccc}
\text{Wh}_{D_n}(\ast) & \xrightarrow{\text{Res}_{C_n}^D} & \text{Wh}_{C_n}(\ast) \\
\downarrow & & \downarrow \\
\bigoplus_{d|n} \text{Wh}(D_d) & \xrightarrow{\bigoplus_{d|n} \text{Res}_{C_n}^D} & \bigoplus_{d|n} \text{Wh}(C_d) .
\end{array}
\]

**Proof.** For a subset \( A \) of \( D_n \), we denote by \( \langle A \rangle \) the subgroup generated by \( A \). Since \( \langle s^m, t^m \rangle = \langle t^{m-n}, s^m \rangle \) in \( D_n \), any subgroup of \( D_n \) has a form \( \langle t^k \rangle \) or \( \langle t^k, s^m \rangle \). On the other hand,

\[ \langle t^k, s^m \rangle \text{ is conjugate to } \begin{cases} \langle t^k, s \rangle & \text{if } m \text{ is odd,} \\
\langle t^k, s \rangle & \text{if } m \text{ is even.} \end{cases} \]

Moreover, if \( n \) is odd, \( \langle t^k, s \rangle \) is conjugate to \( \langle t^k, s \rangle \). Therefore, \( C(D_n) \), the conjugacy classes of the subgroups of \( D_n \), is

\[ \begin{cases} \{ \langle t^k, s \rangle, \langle t^k, s \rangle \mid d \mid n \} & \text{if } n \text{ is odd,} \\
\{ \langle t^k, s \rangle, \langle t^k, s \rangle, \langle t^k, s \rangle \mid d \mid n \} & \text{if } n \text{ is even.} \end{cases} \]

Moreover, we have

\[ N\langle t^d \rangle = D_n , \quad W\langle t^d \rangle = N\langle t^d \rangle \langle t^d \rangle = D_d , \]

\[ N\langle t^d, s \rangle = \begin{cases} \langle t^d, s \rangle & \text{if } d \text{ is odd,} \\
\langle t^{d/2}, s \rangle & \text{if } d \text{ is even,} \end{cases} \quad W\langle t^d, s \rangle = \begin{cases} 1 & \text{if } d \text{ is odd,} \\
C_2 & \text{if } d \text{ is even,} \end{cases} \]

\[ N\langle t^d, st \rangle = \begin{cases} \langle t^d, st \rangle & \text{if } d \text{ is odd,} \\
\langle t^{d/2}, st \rangle & \text{if } d \text{ is even,} \end{cases} \quad W\langle t^d, ts \rangle = \begin{cases} 1 & \text{if } d \text{ is odd,} \\
C_2 & \text{if } d \text{ is even,} \end{cases} \]

where \( NH \) denotes the normalizer of \( H \subset D_n \) in \( D_n \) and \( WH \) denotes \( NH/H \). Since \( \text{Wh}(C_2) = 0 \), we have

\[ \text{Wh}_{D_n}(\ast) \simeq \bigoplus_{\langle H \rangle \in \text{Cl}(D_n)} \text{Wh}_{D_n}(\ast, \langle H \rangle) \simeq \bigoplus_{\langle H \rangle \in \text{Cl}(D_n)} \text{Wh}(WH) \]

\[ \simeq \bigoplus_{d|n} \text{Wh}(W\langle t^d \rangle) \simeq \bigoplus_{d|n} \text{Wh}(D_d) . \]
By the definition of $\text{Res}_{2n}^n: Wh_{D_n}(\ast) \to Wh_{C_n}(\ast)$, we have the commutative diagram

\[
\begin{array}{ccc}
Wh_{D_n}(\ast) & \xrightarrow{\text{Res}_{2n}^n} & Wh_{C_n}(\ast) \\
\uparrow & & \uparrow \\
Wh_{D_n}(\ast, \langle t^d \rangle) & \xrightarrow{=} & Wh_{C_n}(\ast, \langle t^d \rangle) \\
\downarrow & & \downarrow \\
Wh(D) & \xrightarrow{\text{Res}_{2n}^n} & Wh(C).
\end{array}
\]

This completes the proof. Q.E.D.

**Lemma 1.4.** $\text{Res}_{2n}^n: Wh(D_n) \to Wh(C_n)$ and $\text{Res}_{2n}^n: Wh_{D_n}(\ast) \to Wh_{C_n}(\ast)$ are monomorphisms.

**Proof.** By Lemma 1.3, it is sufficient to show that $\text{Res}_{2n}^n: Wh(D_n) \to Wh(C_n)$ is a monomorphism. We note that $Wh(D_n)$ and $Wh(C_n)$ are free abelian groups of the same rank by [21], [19], [20] and [17]. Moreover

\[
\text{Res}_{2n}^n \text{ Ind}_{2n}^n y = y^2 \quad \text{for each} \quad y \in Wh(C_n).
\]

Therefore $\text{Ind}_{2n}^n: Wh(C_n) \to Wh(D_n)$ is a monomorphism and its image is a subgroup of finite index. So, for each $x \in Wh(D_n)$, there exist $m \in \mathbb{Z}$ and $y \in Wh(C_n)$ such that $x^m = \text{Ind}_{2n}^n y$. Suppose that $\text{Res}_{2n}^n x = 1$, then

\[
1 = (\text{Res}_{2n}^n x)^m = \text{Res}_{2n}^n x^m = \text{Res}_{2n}^n \text{ Ind}_{2n}^n y = y^2.
\]

Since $Wh(C_n)$ and $Wh(D_n)$ are torsion free, we have $y = 1$ and $x = 1$. This completes the proof. Q.E.D.

Now we shall observe the classical restriction homomorphism of the unit groups. The point of our observation is to consider $C_{2n}$ and $D_n$ parallelly. Let $r$ be a generator of $C_{2n}$. Identifying $t = r^2$, we can regard $C_n$ as a subgroup of $C_{2n}$. Because each element of $\mathbb{Z}[D_n]$ can be expressed by $a+sb$, $a, b \in \mathbb{Z}[C_n]$, we can define a homomorphism

\[
(\mathbb{Z}[D_n])^* \to (\mathbb{Z}[C_n])^*
\quad a+sb \mapsto aa - bb.
\]

Similarly, we can define a homomorphism

\[
(\mathbb{Z}[C_{2n}])^* \to (\mathbb{Z}[C_n])^*
\quad a+rb \mapsto a^2 - trb^2.
\]

The above two homomorphisms are the classical restriction homomorphisms in the following sense.

**Lemma 1.5.** The following diagrams commute.
(Z[DN]* \to (Z[CN])* \\
\text{Res}_{2n}^{Dn} \downarrow \text{Res}_{2n}^{Cn} \downarrow \\
Wh(Dn) \to Wh(Cn). \\
(2) \\
(Z[C_{2n}])* \to (Z[CN])* \\
\text{Res}_{2n}^{C_{2n}} \downarrow \text{Res}_{2n}^{C_{2n}} \downarrow \\
Wh(C_{2n}) \to Wh(Cn). \\

Proof. If we regard $a+sb \in (Z[DN])^{*}$ as a $Z[CN]$-isomorphism $Z[DN] \to Z[DN]$ and take basis 1 and $s$ of $Z[DN]$ as a $Z[CN]$-module, then $a+sb$ is expressed by a matrix 

$$
\begin{pmatrix}
a & b \\
b & a
\end{pmatrix}.
$$

Since

$$
\det \begin{pmatrix}
a & b \\
b & a
\end{pmatrix} = aa - bb,
$$

we have the commutativity of (1) by the definition of $Res_{2n}^{Dn}$. By the same argument, we have the commutativity of (2). Q.E.D.

Using the above lemma, we have the following.

**Proposition 1.6.** If $Res_{2n}^{C_{2n}}: Wh(C_{2n}) \to Wh(Cn)$ is an epimorphism, $Res_{2n}^{Dn}: Wh(Dn) \to Wh(Cn)$ is an isomorphism.

Proof. By lemma 1.4, it is sufficient to show that $Res_{2n}^{Dn}$ is an epimorphism, i.e., for each $x \in Wh(Cn)$, there exists $y \in Wh(Dn)$ such that $Res_{2n}^{Dn} y = x$. By the assumption, there exists a $y' \in Wh(C_{2n})$ such that $Res_{2n}^{C_{2n}} y' = x$. According to Lemma 1.1, $y'$ is represented by a unit $a+rb \in (Z[C_{2n}])^{*}$ such that $a+rb = a+rb$. Since the condition $a+rb = a+rb$ implies $a=a$ and $b=br^2=bt$, it is easy to see that $a+sb$ is a unit of $Z[DN]$. By lemma 1.5, $Res_{2n}^{Dn}$ sends $a+sb$ to $aa-bb = a^2 - tb^2$ at the unit level. On the other hand $Res_{2n}^{C_{2n}}$ sends $a+rb$ to $a^2 - tb^2$. Therefore $a+sb$ represents the required $y$. Q.E.D.

**Example 1.7.** $Res_{2n}^{C_{2n}}: Wh(C_{2n}) \to Wh(Cn)$ is an epimorphism in the following cases.

(1) $n$: odd.

(2) $n=8$ or 12.

But if $n=2^k (k \geq 4)$, $Res_{2n}^{C_{2n}}$ is not an epimorphism.

**Corollary 1.8.** If $n$ is odd or $n=8, 12$, $Res_{2n}^{Dn}: Wh(Dn) \to Wh(Cn)$ and $Res_{2n}^{Dn}: Wh_{Dn}(*) \to Wh_{Cn}(*)$ are isomorphisms.

Proof of Example 1.7. In the case (2), since the generator of $Wh(Cn)$ is
known (see [11]), a direct computation shows that Res$^\infty_n$ is an epimorphism.
By the following Lemma 1.9, it follows from [5, Theorem 3] that Res$^\infty_n$ is an epimorphism if $n$ is odd. The example that Res$^\infty_n$ is not an epimorphism is given by [9, Theorem 1.1]. Q.E.D.

**Lemma 1.9.** The following are equivalent to each other:

1. Res$^\infty_n$: Wh($C_{mn}$) $\rightarrow$ Wh($C_n$) is an epimorphism.
2. $\tilde{\tau}$: $(R_{C_{mn}})^*/\pm C_{mn} \rightarrow (R_{C_n})^*/\pm C_n$ is an epimorphism where $R_{C_n} = \mathbb{Z}[C_n]/(\sum_{g \in C_n} g)$ (see [5] and [9] for the definition of $\tilde{\tau}$).
3. Any free $C_n$-action on $S^{2k+1}$ ($k \geq 2$) extends to a free $C_{mn}$-action.

Proof. [5, Theorem 4] shows that (2) and (3) are equivalent to each other. To show (1)$\iff$(2), we note that there exists a split extension

$$1 \rightarrow \text{Wh}(C_n) \rightarrow (R_{C_n})^*/\pm C_n \xrightarrow{A} (\mathbb{Z}/n\mathbb{Z})^*/\pm 1 \rightarrow 1$$

where $A: (R_{C_n})^*/\pm C_n \rightarrow (\mathbb{Z}/n\mathbb{Z})^*/\pm 1$ is induced by the augmentation. Moreover we have the commutative diagram

$$\begin{array}{ccc}
1 & \rightarrow & \text{Wh}(C_{mn}) \\
\downarrow & & \downarrow \text{Res}^\infty_n \\
1 & \rightarrow & \text{Wh}(C_n) \\
\downarrow & & \downarrow \\
1 & \rightarrow & (R_{C_n})^*/\pm C_n \\
\downarrow & & \downarrow \\
1 & \rightarrow & (\mathbb{Z}/mn\mathbb{Z})^*/\pm 1 \rightarrow (\mathbb{Z}/n\mathbb{Z})^*/\pm 1 \rightarrow 1
\end{array}$$

where $(\mathbb{Z}/mn\mathbb{Z})^*/\pm 1 \rightarrow (\mathbb{Z}/n\mathbb{Z})^*/\pm 1$ is the natural map. A simple diagram chasing shows that (1) and (2) are equivalent to each other. Q.E.D.

2. The Petrie's maps

In this section, we shall discuss an interesting example of maps between $C_n$-modules due to T. Petrie.

**Notation 2.1.**

$V_a$: The complex $C_n$-module $C$ with $g \in C_n$ acting as multiplication by $\exp 2\pi ia/n$.

Let $a$ and $b$ be integers which are relatively prime and prime to $n$. Choose integers $p, q$ such that $-ap+bq=1$. It is well known that the Petrie's map

$$f: V_a \oplus V_b \rightarrow V_1 \oplus V_{ab}$$

$$(x, y) \mapsto (x^p y^q, x^q y^p)$$

is a $C_n$-homotopy equivalence. This induces a $C_n$-homotopy equivalence

$$h: S(V_a \oplus V_b) \rightarrow S(V_1 \oplus V_{ab})$$

$$(x, y) \mapsto f(x, y)/\|f(x, y)\|$$
Lemma 2.2. Let $V$ and $V'$ be complex $C_n$-modules such that $C_n$ acts freely on $S(V)$ and $S(V')$. If $S(V)$ and $S(V')$ are $C_n$-homotopy equivalent, then one can choose a $C_n$-homotopy equivalence as composition of suitable suspension of Petrie's maps, inverse of Petrie's maps, and a complex conjugation.

Proof. Let $\bigoplus_{i=1}^s V_{a_i}$ be a direct sum decomposition of $V$ to irreducible $C_n$-modules. Since $C_n$ acts freely on $S(V)$, each $a_i$ is prime to $n$. Relacing $a_i$ with $a_i + mn$, we can assume $a_i (i=1, \ldots, j)$ are mutually distinct prime integers. Now we have a composition of Petrie's maps

$$f: S(V) = S(V_{a_1} \oplus V_{a_2} \oplus \cdots \oplus V_{a_j}) \rightarrow S(V_{a_1,a_2} \oplus V_{a_2} \oplus \cdots \oplus V_{a_j}) \rightarrow S(V_{a_2} \oplus V_{a_2,a_3} \oplus \cdots \oplus V_{a_j}) \rightarrow \cdots \rightarrow S(V_1 \oplus \cdots \oplus V_1 \oplus V_{a_1,a_j}).$$

Similarly for $V' = \bigoplus_{i=1}^r V_{b_i}$, we have a composition of Petrie's maps

$$f': S(V') \rightarrow S(V_1 \oplus \cdots \oplus V_1 \oplus V_{b_1\cdots b_s}).$$

Since $S(V)$ and $S(V')$ are $C_n$-homotopy equivalent, we have

$$j = k \quad \text{and} \quad a_1 \cdots a_j \equiv \pm b_1 \cdots b_j \quad (\text{mod} \ n)$$

In case $a_1 \cdots a_j = b_1 \cdots b_j \quad (\text{mod} \ n)$, $f'^{-1} \circ f$ is a required $C_n$-homotopy equivalence.

In case $a_1 \cdots a_j = -b_1 \cdots b_j \quad (\text{mod} \ n)$, $f'^{-1} \circ f$ is a required one where

$$c: S(V_1 \oplus \cdots \oplus V_1 \oplus V_{a_1\cdots a_j}) \rightarrow S(V_1 \oplus \cdots \oplus V_1 \oplus V_{b_1\cdots b_j})$$

$$(x_1, \ldots, x_j) \mapsto (x_1, \ldots, x_j)$$

is a suspension of a complex conjugation. Q.E.D.

Since $D_n = C_n \times \langle C_2 \rangle$, $V_s$ can be considered as a real $D_n$-module on which $s \in C_2$ acts by complex conjugation. The following lemma was pointed out by M. Masuda.

Lemma 2.3. The Petrie's map

$$h: S(V_s \oplus V_s) \rightarrow S(V_s \oplus V_{s'}$$

is a $D_n$-homotopy equivalence.

Proof. A direct computation shows that $h$ is a $D_n$-map. Therefore it is sufficient to show that $h$ is homotopy equivalence on the fixed point set of each subgroup $H$ of $D_n$. We shall show that

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto (x^s y^s, x^s + y^s)$$
has degree $\pm 1$. This is sufficient because $h$ is $C_n$-homotopy equivalence. To calculate the degree of $f$, we consider the image of $S^1 = \{(\cos \theta, \sin \theta) \in \mathbb{R}^2 | 0 \leq \theta \leq 2\pi\}$ by $f$. We put $S^1_{\theta_1, \theta_2} = \{(\cos \theta, \sin \theta) | \theta_1 \leq \theta \leq \theta_2\}$. Then $S^1 = S^1_{0, \pi/2} \cup S^1_{\pi/2, \pi} \cup S^1_{\pi, 3\pi/2} \cup S^1_{3\pi/2, 2\pi}$. We shall distinguish the following four cases.

1. $a$: odd, $b$: even, $p$: odd and $q$: even.
2. $a$: even, $b$: odd, $p$: even and $q$: odd.
3. $a$: odd, $b$: odd, $p$: even, and $q$: odd.
4. $a$: odd, $b$: odd, $p$: odd, and $q$: odd.

We note that the other cases do not occur by the choice of $a$, $b$, $p$ and $q$. Since the arguments for the cases (1), (2), (3) and (4) are similar, we shall only discuss the case (1). In this case, $f(S^1_{\theta_1, \theta_2})$ is a loop at $(0, 1)$ in $\{(x, y) | x \geq 0, y \geq 0\}$, $f(S^1_{\pi/2, \pi})$ is a loop at $(0, 1)$ in $\{(x, y) | x \leq 0, y \geq 0\}$, $f(S^1_{\pi, 3\pi/2})$ is a path from $(0, 1)$ to $(0, -1)$ in $\{(x, y) | x \leq 0\}$ and $f(S^1_{3\pi/2, 2\pi})$ is a path from $(0, -1)$ to $(0, 1)$ in $\{(x, y) | x \geq 0\}$. Therefore $f$ must have degree $+1$. Q.E.D.

Using the above lemma, we have

**Proposition 2.4.** Let $U$ and $U'$ be real $C_n$-modules such that $S(U)$ and $S(U')$ are $C_n$-homotopy equivalent. Then there exist real $D_n$-modules $V$ and $V'$ such that

1. $\text{Res}_C^D V = U$ and $\text{Res}_C^D V' = U'$,
2. $S(V)$ and $S(V')$ are $D_n$-homotopy equivalent.

**Proof.** We write

$$U = \bigoplus_{H < C_n} U(H) \quad \text{and} \quad U' = \bigoplus_{H < C_n} U'(H)$$

where $U(H)$ and $U'(H)$ collects the irreducible submodules of $U$ and $U'$ respectively which have kernel $H$. It is well known that $S(U)$ is homotopy equivalent to $S(U')$ if and only if $S(U(H))$ is homotopy equivalent to $S(U'(H))$ for each $H < C_n$. Therefore, it is sufficient to show this lemma for each $U(H)$ and $U'(H)$. In case $H = C_n$ or the subgroup of index 2, it is obvious. Since $C_n/H$ acts freely on $S(U(H))$, we may assume that $C_n$ acts freely on $S(U)$ and $S(U')$. If we can choose a $C_n$-homotopy equivalence $S(U) \to S(U')$ as a Petrie's map (or its suitable suspension), the Petrie's map itself gives a $D_n$-homotopy equivalence by Lemma 2.3. Of course, the complex conjugation gives a $D_n$-homotopy equivalence. This together with Lemma 2.2 completes the proof. Q.E.D.

### 3. Main results

Finally, we state our main results which are easy consequences of previous
sections.

**Theorem A.** \(\text{Res}_{G}^{D_n}: \text{Wh}_{\text{rep}}(D_n) \rightarrow \text{Wh}_{\text{rep}}(C_n)\) is an isomorphism.

Proof. Since \(\text{Wh}_{\text{rep}}(D_n)\) and \(\text{Wh}_{\text{rep}}(C_n)\) are subgroups of \(\text{Wh}_{D_n}(\ast)\) and \(\text{Wh}_{C_n}(\ast)\) respectively, Lemma 1.4 shows the injectivity. On the other hand Proposition 2.4 shows the surjectivity because the reduced torsion depends only on \(G\)-modules if \(\text{Wh}_{G}(\ast)\) is 2-torsion free. Q.E.D.

Using [11, Theorem C], we have a corollary to Theorem A.

**Corollary B.** \(\text{Wh}_{\text{rep}}(D_n)\) is of finite index in \(\text{Wh}_{D_n}(\ast)\) if and only if \(n=8, 9, 12, 16, 18, p \text{ or } 2p\) for odd prime integers \(p\).

We shall conclude this note by referring the generators of Whitehead group of dihedral group of dihedral groups.

**Example.** The generators of \(\text{Wh}(D_3), \text{Wh}(D_6)\) or \(\text{Wh}(D_{12})\) are given by the reduced torsions of \(D_r\)-homotopy equivalences between the unit sphere of \(D_r\)-modules. The units which represent the generators of \(\text{Wh}(D_3), \text{Wh}(D_6)\) and \(\text{Wh}(D_{12})\) are

1. \(1+(t+t^{-1})-(t^2+t^{-2})+s(-2+(t^2+t^{-2}))\) in case \(\text{Wh}(D_3),\)
2. \(-1+(t^2+t^{-2})+s(t-t^2-t^2+t^{-2})\) in case \(\text{Wh}(D_6),\)
3. \(4+2(t+t^{-1})-(t^2+t^{-2})-(t^4+t^{-4})-(t^5+t^{-5})-t^6\)
   \[+s(3+t-t^2-t^3-t^4-t^5-t^6)-t^{-3}+t^{-2}+3t^{-1})\] in case \(\text{Wh}(D_{12}).\)

Proof. We note that the generators of \(\text{Wh}(C_3), \text{Wh}(C_6)\) and \(\text{Wh}(C_{12})\) appear as the reduced torsions of the Petrie’s maps \(S(V_3\oplus V_3)\rightarrow S(V_1\oplus V_1), S(V_3\oplus V_3)\rightarrow S(V_2\oplus V_2)\) and \(S(V_3\oplus V_3)\rightarrow S(V_1\oplus V_1)\) respectively (see [11]). Therefore the reduced torsions of the above Petrie’s maps (as \(D_n\)-homotopy equivalences) represent each generator of \(\text{Wh}(D_3), \text{Wh}(D_6)\) and \(\text{Wh}(D_{12})\). Using the method of Proposition 1.6, we can find the elements of (1), (2) and (3).

Q.E.D.

**References**


Department of Mathematics
Osaka City University
Sugimoto-3, Sumiyoshi-ku
Osaka 558, Japan

and

Intelligent Wave Inc.
C. Itoh Bldg. 8F 2–5–1,
Kita-Aoyama, Minato-ku,
Tokyo 107, Japan