Title	A note on the equivariant Whitehead groups of dihedral groups
Author(s)	Inoue, Tsuyoshi
Citation	Osaka Journal of Mathematics. 1990, 27(2), p. 421-430
Version Type	VoR
URL	https://doi.org/10.18910/10500
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Inoue, T.
Osaka J. Math.
27 (1990), 421–430

A NOTE ON THE EQUIVARIANT WHITEHEAD GROUPS OF DIHEDRAL GROUPS

Dedicated to Professor Shôrô Araki on his 60th birthday

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(Received March 23, 1989)

0. Introduction

This note is intended as "The equivariant Whitehead torsions of equivariant homotopy equivalence between the unit spheres of representations II". Therefore, we shall use the notations in [11]. In this note, restriction maps in Whitehead groups play an importnat role. To illustrate this, we begin with an example pointed out by M. Masuda. Let C_n and D_n be the cyclic group and dihedral group of order n and $2n$ respectively. As we remarked in [11], a generator of $Wh(C_5)$ appears as the reduced equivariant Whitehead torsion of any C_5-homptopy equivalence

$$f\colon S(V_3 \oplus V_2) \to S(V_1 \oplus V_1)\,.$$

where V_a $(a=1, 2, 3)$ denotes the complex C_5-module $\boldsymbol{C}$ with $g \in C_5$ acting as multiplication by $\exp 2\pi ia/5$ and $S(V)$ denotes the unit sphere of C_5-module V. Since the torsion does not depend on the choice of f, we can assume that f is the map due to T. Petrie (see §2). By the complex conjugation, C_5-modules V_a can be regarded as D_5-modules. Then the Petrie's map f turns out to be a D_5-homotopy equivalence. The reduced equivariant Whitehead torsion $\bar{\tau}_{D_5}(f)= p_*\tau_{D_5}(f)$ of f as a D_5-homotpoy equivalence lies in $Wh_{D_5}(*) \cong Wh(D_5)$ where $p_*\colon Wh_{D_5}(S(V_3 \oplus V_2)) \to Wh_{D_5}(*)$ is the induced map by the obvious map $p\colon S(V_3 \oplus V_2) \to *$. It is obvious that the restriction map from D_5 to C_5 sends the torsion to the generator of $Wh_{C_5}(*) \cong Wh(C_5)$. Therefore the restriction map induces an isomorphism of the Whitehead groups because $Wh(D_5)$ is a free abelian group of rank 1 (see [3], [21], [19], [20] and [17]). Moreover we see that the torsion is a generator of $Wh(D_5)$. Our main result (Theorem A) is a generalization of this observation.

Theorem A. *The restriction map induces an isomorphism*

$$\mathrm{Res}^{D_n}_{C_n}\colon Wh_{\mathrm{rep}}(D_n) \to Wh_{\mathrm{rep}}(C_n)\,,$$

where $Wh_{\text{rep}}(G)$ denotes the subgroups of $Wh_G(*)$ generated by the reduced torsions of G-homotpoy equivalences between the unit spheres of G modules.

By the Thorem A, the same conclusion as [11, Theorem C] holds for dihedral groups.

Corollary B. *$Wh_{\text{rep}}(D_n)$ is of finite index in $Wh_{D_n}(*)$ if and only if $n=$ 8, 9, 12, 16, 18, p or $2p$ for odd prime integers p.*

In §1, we discuss the restriction maps of Whitehead groups from dihedral groups to cyclic groups. We give a sufficient condition for the restriction map being an isomorphism. In §2, we investigate the C_n-homotopy equivalences between the unit spheres of C_n-modules due to T. Petrie. In §3, we state the main results and prove them. We also exhibit an example concerning generators of Whitehead groups of dihedral groups in §3.

The author owes to Professors Shôrô Araki and Mikiya Masuda by useful discussions and advices, and would like to express here his hearty thanks to them.

1. The restriction maps from dihedral group to cyclic group

In this section, we shall investigate the restriction map of Whitehead groups from a dihedral group to a cyclic group. First, we consider the standard involution on Whitehead groups. Let G be a finite group. The assignment "$g\mapsto g^{-1}$" in G induces a conjugation $^-$: $\boldsymbol{Z}[G]\to\boldsymbol{Z}[G]$. This conjugation induces the standard involution $^-$: $Wh(G)\to Wh(G)$. The following lemma is fundamental in our investigation.

Lemma 1.1. *Let G be an abelian group. Then, each element of $(\boldsymbol{Z}[G])^*/\pm G$ is represented by a unit $u\in(\boldsymbol{Z}[G])^*$ such that $u=\bar{u}$. In particular, if $Wh(G)$ is torsion free, each element of $Wh(G)$ is represented by a unit $u\in(\boldsymbol{Z}[G])^*$ such that $u=\bar{u}$.*

Proof. It is well known that the standard involution on $Wh'(G)=Wh(G)/$ torsion is trivial (see [24], [2] or [16]). According to the proof of [2] for this fact, for each $u\in(\boldsymbol{Z}[G])^*$, there exists $g_0\in G$ such that $u\cdot(\bar{u})^{-1}=\pm g_0$. Applying the augmention map $\boldsymbol{Z}[G]\to\boldsymbol{Z}$ to both sides of the identity, we see $u\cdot(\bar{u})^{-1}=g_0$. Here, we consider an involution θ: $G\to G$, $\theta(g)=g_0g^{-1}$. If we put $u=\sum a_gg$ ($a_g\in\boldsymbol{Z}$), the identity $u\cdot(\bar{u})^{-1}=g_0$ implies

$$a_g = a_{\theta(g)} \quad \text{for each} \quad g\in G\,.$$

Therefore, θ must have a fixed point because $\sum a_g=\pm 1$. The fixed point of θ, say $g\in G$, satisfies $g^2=g_0$. If we put $v=g^{-1}u$, v is a required element because $v=u$ in $(\boldsymbol{Z}[G])^*/\pm G$ and

$$v = g^{-1} u = g^{-1} g_0 \bar{u} = g \bar{u} = \bar{v} . \qquad \text{Q.E.D.}$$

NOTATION 1.2.

D_n: the dihedral group of order $2n$ generated by two elements s and t with relations $t^n = s^2 = 1$ and $sts = t^{-1}$.

C_n: the cyclic subgroup of D_n generated by t.

In later sections, we shall consider the equivariant Whitehead group of D_n (called the generalized Whitehead group of D_n by Rothenberg). Therefore, we shall treat the classical Whitehead groups and the equivariant Whitehead groups at the same time. To do this, we need the following lemma.

Lemma 1.3. $Wh_{D_n}(*) = \oplus_{d|n} Wh(D_d)$ *and the following diagram commutes*

$$\begin{array}{ccc} Wh_{D_n}(*) & \xrightarrow{\mathrm{Res}_{C_n}^{D_n}} & Wh_{C_n}(*) \\ \downarrow & & \downarrow \\ \oplus_{d|n} Wh(D_d) & \xrightarrow{\oplus \mathrm{Res}_{C_n}^{D_n}} & \oplus_{d|n} Wh(C_d) . \end{array}$$

Proof. For a subset A of D_n, we denote by $\langle A \rangle$ the subgroup generated by A. Since $\langle st^k, st^m \rangle = \langle t^{k-m}, st^m \rangle$ in D_n, any subgroup of D_n has a form $\langle t^k \rangle$ or $\langle t^k, st^m \rangle$. On the other hand,

$$\langle t^k, st^m \rangle \text{ is conjugate to } \begin{cases} \langle t^k, st \rangle & \text{if } m \text{ is odd,} \\ \langle t^k, s \rangle & \text{if } m \text{ is even.} \end{cases}$$

Moreover, if n is odd, $\langle t^k, st \rangle$ is conjugate to $\langle t^k, s \rangle$. Therefore, $C(D_n)$, the conjugacy classes of the subgroups of D_n, is

$$\begin{cases} \{ (\langle t^d \rangle), (\langle t^d, s \rangle) \mid d \mid n \} & \text{if } n \text{ is odd,} \\ \{ (\langle t^d \rangle), (\langle t^d, s \rangle), (\langle t^d, st \rangle) \mid d \mid n \} & \text{if } n \text{ is even.} \end{cases}$$

Moreover, we have

$$N\langle t^d \rangle = D_n , \qquad W\langle t^d \rangle = N\langle t^d \rangle / \langle t^d \rangle = D_d ,$$

$$N\langle t^d, s \rangle = \begin{cases} \langle t^d, s \rangle & \text{if } d \text{ is odd,} \\ \langle t^{d/2}, s \rangle & \text{if } d \text{ is even,} \end{cases} \qquad W\langle t^d, s \rangle = \begin{cases} 1 & \text{if } d \text{ is odd,} \\ C_2 & \text{if } d \text{ is even,} \end{cases}$$

$$N\langle t^d, st \rangle = \begin{cases} \langle t^d, st \rangle & \text{if } d \text{ is odd,} \\ \langle t^{d/2}, st \rangle & \text{if } d \text{ is even,} \end{cases} \qquad W\langle t^d, ts \rangle = \begin{cases} 1 & \text{if } d \text{ is odd,} \\ C_2 & \text{if } d \text{ is even,} \end{cases}$$

where NH denotes the normalizer of $H \subset D_n$ in D_n and WH denotes NH/H. Since $Wh(C_2) = 0$, we have

$$\begin{aligned} Wh_{D_n}(*) &\cong \oplus_{(H) \in C(D_n)} Wh_{D_n}(*, (H)) \cong \oplus_{(H) \in C(D_n)} Wh(WH) \\ &\cong \oplus_{d|n} Wh(W\langle t^d \rangle) \qquad \cong \oplus_{d|n} Wh(D_d) . \end{aligned}$$

By the definition of $\mathrm{Res}_{C_n}^{D_n}$: $Wh_{D_n}(*)\to Wh_{C_n}(*)$, we have the commutative diagram

$$\begin{array}{ccc} Wh_{D_n}(*) & \xrightarrow{\mathrm{Res}_{C_n}^{D_n}} & Wh_{C_n}(*) \\ \uparrow & & \uparrow \\ Wh_{D_n}(*, (\langle t^d\rangle)) & \longrightarrow & Wh_{C_n}(*, (\langle t^d\rangle)) \\ \downarrow\cong & & \downarrow\cong \\ Wh(D_d) & \xrightarrow{\mathrm{Res}_{C_d}^{D_d}} & Wh(C_d)\,. \end{array}$$

This completes the proof. Q.E.D.

Lemma 1.4. $\mathrm{Res}_{C_n}^{D_n}$: $Wh(D_n)\to Wh(C_n)$ *and* $\mathrm{Res}_{C_n}^{D_n}$: $Wh_{D_n}(*)\to Wh_{C_n}(*)$ *are monomorphisms.*

Proof. By Lemma 1.3, it is sufficient to show that $\mathrm{Res}_{C_n}^{D_n}$: $Wh(D_n)\to Wh(C_n)$ is a monomorphism. We note that $Wh(D_n)$ and $Wh(C_n)$ are free abelian groups of the same rank by [21], [19], [20] and [17]. Moreover

$$\mathrm{Res}_{C_n}^{D_n}\,\mathrm{Ind}_{C_n}^{D_n}\,y = y^2 \quad \text{for each} \quad y\in Wh(C_n)\,.$$

Therefore $\mathrm{Ind}_{C_n}^{D_n}$: $Wh(C_n)\to Wh(D_n)$ is a monomorphism and its image is a subgroup of finite index. So, for each $x\in Wh(D_n)$, there exixt $m\in \boldsymbol{Z}$ adn $y\in Wh(C_n)$ such that $x^m=\mathrm{Ind}_{C_n}^{D_n}\,y$. Suppose that $\mathrm{Res}_{C_n}^{D_n}\,x=1$, then

$$1 = (\mathrm{Res}_{C_n}^{D_n}\,x)^m = \mathrm{Res}_{C_n}^{D_n}\,x^m = \mathrm{Res}_{C_n}^{D_n}\,\mathrm{Ind}_{C_n}^{D_n}\,y = y^2\,.$$

Since $Wh(C_n)$ and $Wh(D_n)$ are torsion free, we have $y=1$ and $x=1$. This completes the proof. Q.E.D.

Now we shall observe the classical restriction homomorphism of the unit groups. The point of our observation is to consider C_{2n} and D_n parallelly. Let r be a generator of C_{2n}. Identifying $t=r^2$, we can regard C_n as a subgroup of C_{2n}. Because each element of $\boldsymbol{Z}[D_n]$ can be expressed by $a+sb$, $a, b\in \boldsymbol{Z}[C_n]$, we can define a homomorphism

$$\begin{array}{ccc} (\boldsymbol{Z}[D_n])^* & \to & (\boldsymbol{Z}[C_n])^* \\ a+sb & \mapsto & a\bar{a}-b\bar{b}\,. \end{array}$$

Similarly, we can define a homomorphism

$$\begin{array}{ccc} (\boldsymbol{Z}[C_{2n}])^* & \to & (\boldsymbol{Z}[C_n])^* \\ a+rb & \mapsto & a^2-tb^2\,. \end{array}$$

The above two homomorphisms are the classical restriction homomorphisms in the following sense.

Lemma 1.5. *The following diagrams commute.*

(1)
$$\begin{array}{ccc} (\boldsymbol{Z}[D_n])^* & \longrightarrow & (\boldsymbol{Z}[C_n])^* \\ \downarrow & & \downarrow \\ Wh(D_n) & \xrightarrow{\mathrm{Res}_{C_n}^{D_n}} & Wh(C_n)\,. \end{array}$$

(2)
$$\begin{array}{ccc} (\boldsymbol{Z}[C_{2n}])^* & \longrightarrow & (\boldsymbol{Z}[C_n])^* \\ \downarrow & & \downarrow \\ Wh(C_{2n}) & \xrightarrow{\mathrm{Res}_{C_n}^{C_{2n}}} & Wh(C_n)\,. \end{array}$$

Proof. If we regard $a+sb\in(\boldsymbol{Z}[D_n])^*$ as a $\boldsymbol{Z}[C_n]$-isomorphism $\boldsymbol{Z}[D_n]\to\boldsymbol{Z}[D_n]$ and take basis 1 and s of $\boldsymbol{Z}[D_n]$ as a $\boldsymbol{Z}[C_n]$-module, then $a+sb$ is expressed by a matrix

$$\begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}.$$

Since

$$\det\begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}=a\bar{a}-b\bar{b}\,,$$

we have the commutativity of (1) by the definition of $\mathrm{Res}_{C_n}^{D_n}$. By the same argument, we have the commutativity of (2). Q.E.D.

Using the above lemma, we have the following.

Proposition 1.6. If $\mathrm{Res}_{C_n}^{C_{n2}}$: $Wh(C_{2n})\to Wh(C_n)$ is an epimorphism, $\mathrm{Res}_{C_n}^{D_n}$: $Wh(D_n)\to Wh(C_n)$ is an isomorphism.

Proof. By lemma 1.4, it is sufficient to show that $\mathrm{Res}_{C_n}^{D_n}$ is an epimorphism, i.e., for each $x\in Wh(C_n)$, there exists $y\in Wh(D_n)$ such that $\mathrm{Res}_{C_n}^{D_n}\,y=x$. By the assumption, there exists a $y'\in Wh(C_{2n})$ such that $\mathrm{Res}_{C_n}^{C_{2n}}\,y'=x$. According to Lemma 1.1, y' is represented by a unit $a+rb\in(\boldsymbol{Z}[C_{2n}])^*$ such that $\overline{a+rb}=a+rb$. Since the condition $\overline{a+rb}=a+rb$ implies $\bar{a}=a$ and $\bar{b}=br^2=bt$, it is easy to see that $a+sb$ is a unit of $\boldsymbol{Z}[D_n]$. By lemma 1.5, $\mathrm{Res}_{C_n}^{D_n}$ sends $a+sb$ to $a\bar{a}-b\bar{b}=a^2-tb^2$ at the unit level. On the other hand $\mathrm{Res}_{C_n}^{C_{2n}}$ sends $a+rb$ to a^2-tb^2. Therefore $a+sb$ represents the required y. Q.E.D.

EXAMPLE 1.7. $\mathrm{Res}_{C_n}^{C_{2n}}$ $Wh(C_{2n})\to Wh(C_n)$ is an epimorphism in the following cases.

(1) n: *odd.*

(2) $n=8$ or 12.

But if $n=2^k(k\geqq 4)$, $\mathrm{Res}_{C_n}^{C_{2n}}$ is *not* an epimoephiam.

Corollary 1.8. *If n is odd or $n=8, 12$,* $\mathrm{Res}_{C_n}^{D_n}$: $Wh(D_n)\to Wh(C_n)$ *and* $\mathrm{Res}_{C_n}^{D_n}$: $Wh_{D_n}(*)\to Wh_{C_n}(*)$ *are isomorphisms.*

Proof of Example 1.7. In the case (2), since the generator of $Wh(C_n)$ is known (see [11]), a direct computation shows that $\mathrm{Res}^{C_{2n}}_{C_n}$ is an epimorphism. By the following Lemma 1.9, it follows from [5, Theorem 3] that $\mathrm{Res}^{C_{2n}}_{C_n}$ is an epimorphism if n is odd. The example that $\mathrm{Res}^{C_{2n}}_{C_n}$ is not an epimorphism is given by [9, Theorem 1.1]. Q.E.D.

Lemma 1.9. *The following are equivalent to each other*:
(1) $\mathrm{Res}^{C_{mn}}_{C_n}$: $Wh(C_{mn})\to Wh(C_n)$ *is an epimorphism.*
(2) $\widetilde{\mathrm{tr}}: (R_{C_{mn}})^*/\pm C_{mn}\to(R_{C_n})^*/\pm C_n$ *is an epimorphism where* $R_{C_n}=\boldsymbol{Z}[C_n]/(\sum_{g\in C_n}g)$ (*see* [5] *and* [9] *for the definition of* $\widetilde{\mathrm{tr}}$).
(3) *Any free* C_n*-action on* $S^{2k+1}(k\geqq 2)$ *extends to a free* C_{mn}*-action.*

Proof. [5, Theorem 4] shows that (2) and (3) are equivalent to each other. To show (1)$\Leftrightarrow$(2), we note that there exists a split extension

$$1\to Wh(C_n)\to(R_{C_n})^*/\pm C_n\xrightarrow{A}(\boldsymbol{Z}/n\boldsymbol{Z})^*/\pm 1\to 1$$

where $A: (R_{C_n})^*/\pm C_n\to(\boldsymbol{Z}/n\boldsymbol{Z})^*/\pm 1$ is induced by the augmentation. Moreover we have the commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & Wh(C_{mn}) & \to & (R_{C_{mn}})^*/\pm C_{mn} & \to & (\boldsymbol{Z}/mn\,\boldsymbol{Z})^*/\pm 1 & \to & 1 \\
 & & \downarrow \mathrm{Res}^{C_{mn}}_{C_n} & & \widetilde{\mathrm{tr}}\downarrow & & \downarrow & & \\
1 & \to & Wh(C_n) & \to & (R_{C_n})^*/\pm C_n & \to & (\boldsymbol{Z}/n\,\boldsymbol{Z})^*/\pm 1 & \to & 1
\end{array}$$

where $(\boldsymbol{Z}/mn\,\boldsymbol{Z})^*/\pm 1\to(\boldsymbol{Z}/n\,\boldsymbol{Z})^*/\pm 1$ is the natural map. A simple diagram chasing shows that (1) and (2) are equivalent to each other. Q.E.D.

2. The Petrie's maps

In this section, we shall discuss an interesting example of maps between C_n-modules due to T. Petrie.

NOTATION 2.1.

V_a: The complex C_n-module $\boldsymbol{C}$ with $g\in C_n$ acting as multiplication by exp $2\pi ia/n$.

Let a and b be integers which are relatively prime and prime to n. Choose integers p, q such that $-ap+bq=1$. It is well known that the Petrie's map

$$\begin{aligned} f\colon V_a\oplus V_b &\to V_1\oplus V_{ab} \\ (x,y) &\mapsto (x^p\,\bar{y}^q,\ x^b+y^a) \end{aligned}$$

is a C_a-homotopy equivalence. This induces a C_n-homotpoy equivalence

$$\begin{aligned} h\colon S(V_a\oplus V_b) &\to S(V_1\oplus V_{ab}) \\ (x,y) &\mapsto f(x,y)/||f(x,y)|| \end{aligned}$$

which will be also called Petrie's map.

Lemma 2.2. *Let V and V' be complex C_n-modules such that C_n acts freely on $S(V)$ and $S(V')$. If $S(V)$ and $S(V')$ are C_n-homotopy equivalent, then one can choose a C_n-homotopy equivalence as composition of suitable suspension of Petrie's maps, inverse of Petrie's maps, and a complex conjugation.*

Proof. Let $\bigoplus_{i=1}^{j} V_{a_i}$ be a direct sum decomposition of V to irreducible C_n-modules. Since C_n acts freely on $S(V)$, each a_i is prime to n. Relacing a_i with a_i+mn, we can assume a_i $(i=1, \cdots, j)$ are mutually distinct prime integers. Now we have a composition of Petrie's maps

$$\begin{aligned} f\colon S(V) = S(V_{a_1}\oplus V_{a_2}\oplus\cdots\oplus V_{a_j}) &\to S(V_1\oplus V_{a_1a_2}\oplus V_{a_3}\oplus\cdots\oplus V_{a_j}) \\ \to S(V_1\oplus V_1\oplus V_{a_1a_2a_3}\oplus\cdots\oplus V_{a_j}) &\to\cdots\to S(V_1\oplus\cdots\oplus V_1\oplus V_{a_1\cdots a_j})\,. \end{aligned}$$

Similarly for $V'=\bigoplus_{i=1}^{k} V_{b_i}$, we have a composition of Petrie's maps

$$f'\colon S(V') \to S(V_1\oplus\cdots\oplus V_1\oplus V_{b_1\cdots b_k})\,.$$

Since $S(V)$ and $S(V')$ are C_n-homotopy equivalent, we have

$$j=k \quad\text{and}\quad a_1\cdots a_j \equiv \pm b_1\cdots b_j \pmod n$$

In case $a_1\cdots a_j\equiv b_1\cdots b_j \pmod n$, $f'^{-1}\circ f$ is a required C_n-homotpoy equivalence. In case $a_1\cdots a_j\equiv -b_1\cdots b_j \pmod n$, $f'^{-1}\circ c\circ f$ is a required one where

$$\begin{array}{cccc} c\colon & S(V_1\oplus\cdots\oplus V_1\oplus V_{a_1\cdots a_j}) & \to & S(V_1\oplus\cdots\oplus V_1\oplus V_{b_1\cdots b_j}) \\ & (x_1, \cdots, x_j) & \mapsto & (x_1, \cdots, \bar{x}_j) \end{array}$$

is a suspension of a complex conjugation. Q.E.D.

Since $D_n=C_n\rtimes C_2$, V_a can be considered as a real D_n-module on which $s\in C_2$ acts by complex conjugation. The following lemma was pointed out by M. Masuda.

Lemma 2.3. *The Petrie's map*

$$h\colon S(V_a\oplus V_b) \to S(V_1\oplus V_{ab})$$

is a D_n-homotopy equivalence.

Proof. A direct computation shows that h is a D_n-map. Therefore it is sufficient to show that h is homotopy equivalence on the fixed point set of each subgroup H of D_n. We shall show that

$$\begin{array}{rcl} \bar{f}\colon \boldsymbol{R}^2 & \to & \boldsymbol{R}^2 \\ (x, y) & \mapsto & (x^p y^q, x^b+y^a) \end{array}$$

has degree ± 1. This is sufficient because h is C_n-homotopy equivalence. To calculate the degree of $\bar{f}$, we consider the image of $S^1=\{(\cos\theta, \sin\theta)\in \boldsymbol{R}^2 \mid 0\leqq\theta\leqq 2\pi\}$ by $\bar{f}$. We put $S^1_{\theta_1,\theta_2}=\{(\cos\theta, \sin\theta) \mid \theta_1\leqq\theta\leqq\theta_2\}$. Then $S^1=S^1_{0,\pi/2}\cup S^1_{\pi/2,\pi}\cup S^1_{\pi,3\pi/2}\cup S^1_{3\pi/2,2\pi}$. We shall distinguish the following four cases.

(1) a: odd, b: even, p: odd and q: even.
(2) a: even, b: odd, p: even and q: odd.
(3) a: odd, b: odd, p: even, and q: odd.
(4) a: odd, b: odd, p: odd, and q: odd.

We note that the other cases do not occur by the choice of a, b, p and q. Since the arguments for the cases (1), (2), (3) and (4) are similar, we shall only discuss the case (1). In this case,
$\bar{f}(S^1_{0,\pi/2})$ is a loop at (0, 1) in $\{(x, y) \mid x\geqq 0, y\geqq 0\}$,
$\bar{f}(S^1_{\pi/2,\pi})$ is a loop at (0, 1) in $\{(x, y) \mid x\leqq 0, y\geqq 0\}$,
$\bar{f}(S^1_{\pi,3\pi/2})$ is a path from (0, 1) to (0, -1) in $\{(x, y) \mid x\leqq 0\}$ and
$\bar{f}(S^1_{3\pi/2,2\pi})$ is a path from (0, -1) to (0, 1) in $\{(x, y) \mid x\geqq 0\}$.
Therefore $\bar{f}$ must have degree $+1$. Q.E.D.

Using the above lemma, we have

Propisition 2.4. *Let U and U' be real C_n-modules such that $S(U)$ and $S(U')$ are C_n-homotopy equivalent. Then there exist real D_n-modules V and V' such that*

(1) $$\mathrm{Res}^{D_n}_{C_n} V = U \quad \textit{and} \quad \mathrm{Res}^{D_n}_{C_n} V' = U' ,$$

(2) $$S(V) \quad \textit{and} \quad S(V') \quad \textit{are } D_n\textit{-homotopy equivalent.}$$

Proof. We write

$$U = \oplus_{H\subset C_n} U(H) \quad \text{and} \quad U' = \oplus_{H\subset C_n} U'(H)$$

where $U(H)$ and $U'(H)$ collects the irreducible submodules of U and U' respectively which have kernel H. It is well known that $S(U)$ is homotopy equivalent to $S(U')$ if and only if $S(U(H))$ is homotopy equivalent to $S(U'(H))$ for each $H\subset C_n$. Therefore, it is sufficient to show this lemma for each $U(H)$ and $U'(H)$. In case $H=C_n$ or the subgroup of index 2, it is obvious. Since C_n/H acts freely on $S(U(H))$, we may assume that C_n acts freely on $S(U)$ and $S(U')$. If we can choose a C_n-homotopy equivalence $S(U)\to S(U')$ as a Petrie's map (or its suitable suspension), the Petrie's map itself gives a D_n-homotopy equivalence by Lemma 2.3. Of course, the complex conjugation gives a D_n-homotopy equivalence. This together with Lemma 2.2 completes the proof. Q.E.D.

3. Main results

Finally, we state our main results which are easy consequences of previous sections.

Theorem A. $\mathrm{Res}_{C_n}^{D_n}$: *$Wh_{\mathrm{rep}}(D_n) \to Wh_{\mathrm{rep}}(C_n)$ is an isomorphism.*

Proof. Since $Wh_{\mathrm{rep}}(D_n)$ and $Wh_{\mathrm{1ep}}(C_n)$ are subgroups of $Wh_{D_n}(*)$ and $Wh_{C_n}(*)$ respectively, Lemma 1.4 shows the injectivity. On the other hand Proposition 2.4 shows the surjectivity because the reduced torsion depends only on G-modules if $Wh_G(*)$ is 2-torsion free. Q.E.D.

Using [11, Theorem C], we have a corollary to Theorem A.

Corollary B. *$Wh_{\mathrm{rep}}(D_n)$ is of finite index in $Wh_{D_n}(*)$ if and only if $n=$ 8, 9, 12, 16, 18, p or $2p$ for odd prime integers p.*

We shall conclude this note by referring the generators of Whitehead group of dihedral groups.

EXAMPLE. The generators of $Wh(D_5)$, $Wh(D_8)$ or $Wh(D_{12})$ are given by the reduced torsions of D_i-homotopy equivalences between the unit sphere of D_i-modules. The units which represent the generators of $Wh(D_5)$, $Wh(D_8)$ and $Wh(D_{12})$ are

(1) $1+(t+t^{-1})-(t^2+t^{-2})+s(-2+(t^2+t^{-2}))$ in case $Wh(D_5)$,

(2) $-1+(t^2+t^{-2})+s(t-t^3-t^4+t^{-2})$ in case $Wh(D_8)$,

(3) $4+2(t+t^{-1})-(t^2+t^{-2})-(t^4+t^{-4})-(t^5+t^{-5})-t^6$
$+s(3+t-t^2-t^3-t^4-t^5-t^6-t^{-5}-t^{-4}-t^{-3}+t^{-2}+3t^{-1})$ in case $Wh(D_{12})$.

Proof. We note that the generators of $Wh(C_5)$, $Wh(C_8)$ and $Wh(C_{12})$ appear as the reduced torsions of the Petrie's maps $S(V_2\oplus V_3)\to S(V_1\oplus V_1)$, $S(V_3\oplus V_5)\to S(V_1\oplus V_7)$ and $S(V_5\oplus V_7)\to S(V_1\oplus V_1)$ respectively (see [11]). Therefore the reduced torsions of the above Petrie's maps (as D_n-homotopy equivalences) represent each generator of $Wh(D_5)$, $Wh(D_8)$ and $Wh(D_{12})$. Using the method of Proposition 1.6, we can find the elements of (1), (2) and (3).

Q.E.D.

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