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A NOTE ON THE EQUIVARIANT WHITEHEAD GROUPS OF DIHEDRAL GROUPS

Dedicated to Professor Shôrô Araki on his 60th birthday

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0. Introduction

This note is intended as "The equivariant Whitehead torsions of equivariant homotopy equivalence between the unit spheres of representations II". Therefore, we shall use the notations in [11]. In this note, restriction maps in Whitehead groups play an importnat role. To illustrate this, we begin with an example pointed out by M. Masuda. Let C_n and D_n be the cyclic group and dihedral group of order n and $2n$ respectively. As we remarked in [11], a generator of $Wh(C_5)$ appears as the reduced equivariant Whitehead torsion of any C_5-homptopy equivalence

$$f\colon S(V_3 \oplus V_2) \rightarrow S(V_1 \oplus V_1)\,.$$

where V_a $(a=1, 2, 3)$ denotes the complex C_5-module $\boldsymbol{C}$ with $g \in C_5$ acting as multiplication by $\exp 2\pi ia/5$ and $S(V)$ denotes the unit sphere of C_5-module V. Since the torsion does not depend on the choice of f, we can assume that f is the map due to T. Petrie (see §2). By the complex conjugation, C_5-modules V_a can be regarded as D_5-modules. Then the Petrie's map f turns out to be a D_5-homotopy equivalence. The reduced equivariant Whitehead torsion $\bar{\tau}_{D_5}(f) = p_* \tau_{D_5}(f)$ of f as a D_5-homotpoy equivalence lies in $Wh_{D_5}(*) \cong Wh(D_5)$ where $p_*\colon Wh_{D_5}(S(V_3 \oplus V_2)) \rightarrow Wh_{D_5}(*)$ is the induced map by the obvious map $p\colon S(V_3 \oplus V_2) \rightarrow *$. It is obvious that the restriction map from D_5 to C_5 sends the torsion to the generator of $Wh_{C_5}(*) \cong Wh(C_5)$. Therefore the restriction map induces an isomorphism of the Whitehead groups because $Wh(D_5)$ is a free abelian group of rank 1 (see [3], [21], [19], [20] and [17]). Moreover we see that the torsion is a generator of $Wh(D_5)$. Our main result (Theorem A) is a generalization of this observation.

Theorem A. *The restriction map induces an isomorphism*

$$\mathrm{Res}^{D_n}_{C_n}\colon Wh_{\mathrm{rep}}(D_n) \rightarrow Wh_{\mathrm{rep}}(C_n)\,,$$

where $Wh_{\mathrm{rep}}(G)$ denotes the subgroups of $Wh_G(*)$ generated by the reduced torsions of G-homotpoy equivalences between the unit spheres of G modules.

By the Thorem A, the same conclusion as [11, Theorem C] holds for dihedral groups.

Corollary B. *$Wh_{\mathrm{rep}}(D_n)$ is of finite index in $Wh_{D_n}(*)$ if and only if $n=$ 8, 9, 12, 16, 18, p or $2p$ for odd prime integers p.*

In §1, we discuss the restriction maps of Whitehead groups from dihedral groups to cyclic groups. We give a sufficient condition for the restriction map being an isomorphism. In §2, we investigate the C_n-homotopy equivalences between the unit spheres of C_n-modules due to T. Petrie. In §3, we state the main results and prove them. We also exhibit an example concerning generators of Whitehead groups of dihedral groups in §3.

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1. The restriction maps from dihedral group to cyclic group

In this section, we shall investigate the restriction map of Whitehead groups from a dihedral group to a cyclic group. First, we consider the standard involution on Whitehead groups. Let G be a finite group. The assignment "$g\mapsto g^{-1}$" in G induces a conjugation $^-$: $\boldsymbol{Z}[G]\to\boldsymbol{Z}[G]$. This conjugation induces the standard involution $^-$: $Wh(G)\to Wh(G)$. The following lemma is fundamental in our investigation.

Lemma 1.1. *Let G be an abelian group. Then, each element of $(\boldsymbol{Z}[G])^*/\pm G$ is represented by a unit $u\in(\boldsymbol{Z}[G])^*$ such that $u=\bar{u}$. In particular, if $Wh(G)$ is torsion free, each element of $Wh(G)$ is represented by a unit $u\in(\boldsymbol{Z}[G])^*$ such that $u=\bar{u}$.*

Proof. It is well known that the standard involution on $Wh'(G)=Wh(G)/$ torsion is trivial (see [24], [2] or [16]). According to the proof of [2] for this fact, for each $u\in(\boldsymbol{Z}[G])^*$, there exists $g_0\in G$ such that $u\cdot(\bar{u})^{-1}=\pm g_0$. Applying the augmention map $\boldsymbol{Z}[G]\to\boldsymbol{Z}$ to both sides of the identity, we see $u\cdot(\bar{u})^{-1}=g_0$. Here, we consider an involution $\theta\colon G\to G$, $\theta(g)=g_0g^{-1}$. If we put $u=\sum a_g g$ $(a_g\in\boldsymbol{Z})$, the identity $u\cdot(\bar{u})^{-1}=g_0$ implies

$$a_g = a_{\theta(g)} \quad \text{for each} \quad g\in G\,.$$

Therefore, θ must have a fixed point because $\sum a_g=\pm 1$. The fixed point of θ, say $g\in G$, satisfies $g^2=g_0$. If we put $v=g^{-1}u$, v is a required element because $v=u$ in $(\boldsymbol{Z}[G])^*/\pm G$ and

$$v=g^{-1}u=g^{-1}g_0\bar{u}=g\bar{u}=\bar{v}\,. \qquad \text{Q.E.D.}$$

NOTATION 1.2.

D_n: the dihedral group of order $2n$ generated by two elements s and t with relations $t^n=s^2=1$ and $sts=t^{-1}$.

C_n: the cyclic subgroup of D_n generated by t.

In later sections, we shall consider the equivariant Whitehead group of D_n (called the generalized Whitehead group of D_n by Rothenberg). Therefore, we shall treat the classical Whitehead groups and the equivariant Whitehead groups at the same time. To do this, we need the following lemma.

Lemma 1.3. $Wh_{D_n}(*)=\oplus_{d|n} Wh(D_d)$ *and the following diagram commutes*

$$\begin{array}{ccc} Wh_{D_n}(*) & \xrightarrow{\mathrm{Res}^{D_n}_{C_n}} & Wh_{C_n}(*) \\ \downarrow & & \downarrow \\ \oplus_{d|n} Wh(D_d) & \xrightarrow{\oplus\mathrm{Res}^{D_n}_{C_n}} & \oplus_{d|n} Wh(C_d)\,. \end{array}$$

Proof. For a subset A of D_n, we denote by $\langle A\rangle$ the subgroup generated by A. Since $\langle st^k, st^m\rangle=\langle t^{k-m}, st^m\rangle$ in D_n, any subgroup of D_n has a form $\langle t^k\rangle$ or $\langle t^k, st^m\rangle$. On the other hand,

$$\langle t^k, st^m\rangle \text{ is conjugate to } \begin{cases} \langle t^k, st\rangle & \text{if } m \text{ is odd,} \\ \langle t^k, s\rangle & \text{if } m \text{ is even.} \end{cases}$$

Moreover, if n is odd, $\langle t^k, st\rangle$ is conjugate to $\langle t^k, s\rangle$. Therefore, $C(D_n)$, the conjugacy classes of the subgroups of D_n, is

$$\begin{cases} \{(\langle t^d\rangle), (\langle t^d, s\rangle) \mid d\,|\,n\} & \text{if } n \text{ is odd,} \\ \{(\langle t^d\rangle), (\langle t^d, s\rangle), (\langle t^d, st\rangle) \mid d\,|\,n\} & \text{if } n \text{ is even.} \end{cases}$$

Moreover, we have

$$N\langle t^d\rangle = D_n\,, \qquad W\langle t^d\rangle = N\langle t^d\rangle/\langle t^d\rangle = D_d\,,$$

$$N\langle t^d, s\rangle = \begin{cases} \langle t^d, s\rangle & \text{if } d \text{ is odd,} \\ \langle t^{d/2}, s\rangle & \text{if } d \text{ is even,} \end{cases} \qquad W\langle t^d, s\rangle = \begin{cases} 1 & \text{if } d \text{ is odd,} \\ C_2 & \text{if } d \text{ is even,} \end{cases}$$

$$N\langle t^d, st\rangle = \begin{cases} \langle t^d, st\rangle & \text{if } d \text{ is odd,} \\ \langle t^{d/2}, st\rangle & \text{if } d \text{ is even,} \end{cases} \qquad W\langle t^d, ts\rangle = \begin{cases} 1 & \text{if } d \text{ is odd,} \\ C_2 & \text{if } d \text{ is even,} \end{cases}$$

where NH denotes the normalizer of $H\subset D_n$ in D_n and WH denotes NH/H. Since $Wh(C_2)=0$, we have

$$\begin{aligned} Wh_{D_n}(*) &\cong \oplus_{(H)\in C(D_n)} Wh_{D_n}(*, (H)) \cong \oplus_{(H)\in C(D_n)} Wh(WH) \\ &\cong \oplus_{d|n} Wh(W\langle t^d\rangle) \qquad \cong \oplus_{d|n} Wh(D_d)\,. \end{aligned}$$

By the definition of $\mathrm{Res}_{C_n}^{D_n}$: $Wh_{D_n}(*)\to Wh_{C_n}(*)$, we have the commutative diagram

$$\begin{array}{ccc}
Wh_{D_n}(*) & \xrightarrow{\mathrm{Res}_{C_n}^{D_n}} & Wh_{C_n}(*) \\
\uparrow & & \uparrow \\
Wh_{D_n}(*, (\langle t^d\rangle)) & \longrightarrow & Wh_{C_n}(*, (\langle t^d\rangle)) \\
\downarrow\cong & & \downarrow\cong \\
Wh(D_d) & \xrightarrow{\mathrm{Res}_{C_d}^{D_d}} & Wh(C_d)\,.
\end{array}$$

This completes the proof. Q.E.D.

Lemma 1.4. $\mathrm{Res}_{C_n}^{D_n}$: $Wh(D_n)\to Wh(C_n)$ *and* $\mathrm{Res}_{C_n}^{D_n}$: $Wh_{D_n}(*)\to Wh_{C_n}(*)$ *are monomorphisms.*

Proof. By Lemma 1.3, it is sufficient to show that $\mathrm{Res}_{C_n}^{D_n}$: $Wh(D_n)\to Wh(C_n)$ is a monomorphism. We note that $Wh(D_n)$ and $Wh(C_n)$ are free abelian groups of the same rank by [21], [19], [20] and [17]. Moreover

$$\mathrm{Res}_{C_n}^{D_n}\,\mathrm{Ind}_{C_n}^{D_n}\,y = y^2 \quad \text{for each} \quad y\in Wh(C_n)\,.$$

Therefore $\mathrm{Ind}_{C_n}^{D_n}$: $Wh(C_n)\to Wh(D_n)$ is a monomorphism and its image is a subgroup of finite index. So, for each $x\in Wh(D_n)$, there exixt $m\in \boldsymbol{Z}$ adn $y\in Wh(C_n)$ such that $x^m=\mathrm{Ind}_{C_n}^{D_n}\,y$. Suppose that $\mathrm{Res}_{C_n}^{D_n}\,x=1$, then

$$1=(\mathrm{Res}_{C_n}^{D_n}\,x)^m=\mathrm{Res}_{C_n}^{D_n}\,x^m=\mathrm{Res}_{C_n}^{D_n}\,\mathrm{Ind}_{C_n}^{D_n}\,y=y^2\,.$$

Since $Wh(C_n)$ and $Wh(D_n)$ are torsion free, we have $y=1$ and $x=1$. This completes the proof. Q.E.D.

Now we shall observe the classical restriction homomorphism of the unit groups. The point of our observation is to consider C_{2n} and D_n parallelly. Let r be a generator of C_{2n}. Identifying $t=r^2$, we can regard C_n as a subgroup of C_{2n}. Because each element of $\boldsymbol{Z}[D_n]$ can be expressed by $a+sb$, $a, b\in \boldsymbol{Z}[C_n]$, we can define a homomorphism

$$\begin{array}{ccc}
(\boldsymbol{Z}[D_n])^* & \to & (\boldsymbol{Z}[C_n])^* \\
a+sb & \mapsto & a\bar{a}-b\bar{b}\,.
\end{array}$$

Similarly, we can define a homomorphism

$$\begin{array}{ccc}
(\boldsymbol{Z}[C_{2n}])^* & \to & (\boldsymbol{Z}[C_n])^* \\
a+rb & \mapsto & a^2-tb^2\,.
\end{array}$$

The above two homomorphisms are the classical restriction homomorphisms in the following sense.

Lemma 1.5. *The following diagrams commute.*

(1)
$$\begin{array}{ccc} (\boldsymbol{Z}[D_n])^* & \longrightarrow & (\boldsymbol{Z}[C_n])^* \\ \downarrow & & \downarrow \\ Wh(D_n) & \xrightarrow{\mathrm{Res}_{C_n}^{D_n}} & Wh(C_n)\,. \end{array}$$

(2)
$$\begin{array}{ccc} (\boldsymbol{Z}[C_{2n}])^* & \longrightarrow & (\boldsymbol{Z}[C_n])^* \\ \downarrow & & \downarrow \\ Wh(C_{2n}) & \xrightarrow{\mathrm{Res}_{C_n}^{C_{2n}}} & Wh(C_n)\,. \end{array}$$

Proof. If we regard $a+sb\in(\boldsymbol{Z}[D_n])^*$ as a $\boldsymbol{Z}[C_n]$-isomorphism $\boldsymbol{Z}[D_n]\rightarrow\boldsymbol{Z}[D_n]$ and take basis 1 and s of $\boldsymbol{Z}[D_n]$ as a $\boldsymbol{Z}[C_n]$-module, then $a+sb$ is expressed by a matrix

$$\begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}.$$

Since

$$\det\begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}=a\bar{a}-b\bar{b}\,,$$

we have the commutativity of (1) by the definition of $\mathrm{Res}_{C_n}^{D_n}$. By the same argument, we have the commutativity of (2). Q.E.D.

Using the above lemma, we have the following.

Proposition 1.6. If $\mathrm{Res}_{C_n}^{C_{n2}}$: $Wh(C_{2n})\rightarrow Wh(C_n)$ is an epimorphism, $\mathrm{Res}_{C_n}^{D_n}$: $Wh(D_n)\rightarrow Wh(C_n)$ is an isomorphism.

Proof. By lemma 1.4, it is sufficient to show that $\mathrm{Res}_{C_n}^{D_n}$ is an epimorphism, i.e., for each $x\in Wh(C_n)$, there exists $y\in Wh(D_n)$ such that $\mathrm{Res}_{C_n}^{D_n}\,y=x$. By the assumption, there exists a $y'\in Wh(C_{2n})$ such that $\mathrm{Res}_{C_n}^{C_{2n}}\,y'=x$. According to Lemma 1.1, y' is represented by a unit $a+rb\in(\boldsymbol{Z}[C_{2n}])^*$ such that $\overline{a+rb}=a+rb$. Since the condition $\overline{a+rb}=a+rb$ implies $\bar{a}=a$ and $\bar{b}=br^2=bt$, it is easy to see that $a+sb$ is a unit of $\boldsymbol{Z}[D_n]$. By lemma 1.5, $\mathrm{Res}_{C_n}^{D_n}$ sends $a+sb$ to $a\bar{a}-b\bar{b}=a^2-tb^2$ at the unit level. On the other hand $\mathrm{Res}_{C_n}^{C_{2n}}$ sends $a+rb$ to a^2-tb^2. Therefore $a+sb$ represents the required y. Q.E.D.

EXAMPLE 1.7. $\mathrm{Res}_{C_n}^{C_{2n}}$ $Wh(C_{2n})\rightarrow Wh(C_n)$ is an epimorphism in the following cases.

(1) n: *odd.*

(2) $n=8$ or 12.

But if $n=2^k(k\geqq 4)$, $\mathrm{Res}_{C_n}^{C_{2n}}$ is *not* an epimoephiam.

Corollary 1.8. *If n is odd or $n=8, 12$,* $\mathrm{Res}_{C_n}^{D_n}$: $Wh(D_n)\rightarrow Wh(C_n)$ *and* $\mathrm{Res}_{C_n}^{D_n}$: $Wh_{D_n}(*)\rightarrow Wh_{C_n}(*)$ *are isomorphisms.*

Proof of Example 1.7. In the case (2), since the generator of $Wh(C_n)$ is known (see [11]), a direct computation shows that $\mathrm{Res}_{C_n}^{C_{2n}}$ is an epimorphism. By the following Lemma 1.9, it follows from [5, Theorem 3] that $\mathrm{Res}_{C_n}^{C_{2n}}$ is an epimorphism if n is odd. The example that $\mathrm{Res}_{C_n}^{C_{2n}}$ is not an epimorphism is given by [9, Theorem 1.1]. Q.E.D.

Lemma 1.9. *The following are equivalent to each other*:

(1) $\mathrm{Res}_{C_n}^{C_{mn}}$: $Wh(C_{mn})\to Wh(C_n)$ *is an epimorphism.*

(2) $\widetilde{\mathrm{tr}}: (R_{C_{mn}})^*/\pm C_{mn}\to(R_{C_n})^*/\pm C_n$ *is an epimorphism where* $R_{C_n}=\boldsymbol{Z}[C_n]/(\sum_{g\in C_n}g)$ (*see* [5] *and* [9] *for the definition of* $\widetilde{\mathrm{tr}}$).

(3) *Any free* C_n*-action on* $S^{2k+1}(k\geqq 2)$ *extends to a free* C_{mn}*-action.*

Proof. [5, Theorem 4] shows that (2) and (3) are equivalent to each other. To show (1)$\Leftrightarrow$(2), we note that there exists a split extension

$$1\to Wh(C_n)\to(R_{C_n})^*/\pm C_n\xrightarrow{A}(\boldsymbol{Z}/n\boldsymbol{Z})^*/\pm 1\to 1$$

where A: $(R_{C_n})^*/\pm C_n\to(\boldsymbol{Z}/n\boldsymbol{Z})^*/\pm 1$ is induced by the augmentation. Moreover we have the commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & Wh(C_{mn}) & \to & (R_{C_{mn}})^*/\pm C_{mn} & \to & (\boldsymbol{Z}/mn\,\boldsymbol{Z})^*/\pm 1 & \to & 1\\
 & & \downarrow \mathrm{Res}_{C_n}^{C_{mn}} & & \widetilde{\mathrm{tr}}\downarrow & & \downarrow & & \\
1 & \to & Wh(C_n) & \to & (R_{C_n})^*/\pm C_n & \to & (\boldsymbol{Z}/n\,\boldsymbol{Z})^*/\pm 1 & \to & 1
\end{array}$$

where $(\boldsymbol{Z}/mn\,\boldsymbol{Z})^*/\pm 1\to(\boldsymbol{Z}/n\,\boldsymbol{Z})^*/\pm 1$ is the natural map. A simple diagram chasing shows that (1) and (2) are equivalent to each other. Q.E.D.

2. The Petrie's maps

In this section, we shall discuss an interesting example of maps between C_n-modules due to T. Petrie.

NOTATION 2.1.

V_a: The complex C_n-module $\boldsymbol{C}$ with $g\in C_n$ acting as multiplication by $\exp 2\pi ia/n$.

Let a and b be integers which are relatively prime and prime to n. Choose integers p, q such that $-ap+bq=1$. It is well known that the Petrie's map

$$\begin{aligned} f\colon V_a\oplus V_b &\to V_1\oplus V_{ab}\\ (x,y) &\mapsto (x^p\,\bar{y}^q,\, x^b+y^a)\end{aligned}$$

is a C_a-homotopy equivalence. This induces a C_n-homotpoy equivalence

$$\begin{aligned} h\colon S(V_a\oplus V_b) &\to S(V_1\oplus V_{ab})\\ (x,y) &\mapsto f(x,y)/||f(x,y)||\end{aligned}$$

which will be also called Petrie's map.

Lemma 2.2. *Let V and V' be complex C_n-modules such that C_n acts freely on $S(V)$ and $S(V')$. If $S(V)$ and $S(V')$ are C_n-homotopy equivalent, then one can choose a C_n-homotopy equivalence as composition of suitable suspension of Petrie's maps, inverse of Petrie's maps, and a complex conjugation.*

Proof. Let $\bigoplus_{i=1}^{j} V_{a_i}$ be a direct sum decomposition of V to irreducible C_n-modules. Since C_n acts freely on $S(V)$, each a_i is prime to n. Relacing a_i with a_i+mn, we can assume a_i $(i=1, \cdots, j)$ are mutually distinct prime integers. Now we have a composition of Petrie's maps

$$\begin{aligned} f\colon S(V) = S(V_{a_1}\oplus V_{a_2}\oplus\cdots\oplus V_{a_j}) &\to S(V_1\oplus V_{a_1a_2}\oplus V_{a_3}\oplus\cdots\oplus V_{a_j}) \\ \to S(V_1\oplus V_1\oplus V_{a_1a_2a_3}\oplus\cdots\oplus V_{a_j}) &\to\cdots\to S(V_1\oplus\cdots\oplus V_1\oplus V_{a_1\cdots a_j}) . \end{aligned}$$

Similarly for $V'=\bigoplus_{i=1}^{k} V_{b_i}$, we have a composition of Petrie's maps

$$f'\colon S(V') \to S(V_1\oplus\cdots\oplus V_1\oplus V_{b_1\cdots b_k}) .$$

Since $S(V)$ and $S(V')$ are C_n-homotopy equivalent, we have

$$j=k \quad \text{and} \quad a_1\cdots a_j \equiv \pm b_1\cdots b_j \pmod n$$

In case $a_1\cdots a_j\equiv b_1\cdots b_j \pmod n$, $f'^{-1}\circ f$ is a required C_n-homotpoy equivalence. In case $a_1\cdots a_j\equiv -b_1\cdots b_j \pmod n$, $f'^{-1}\circ c\circ f$ is a required one where

$$\begin{array}{ccc} c\colon S(V_1\oplus\cdots\oplus V_1\oplus V_{a_1\cdots a_j}) & \to & S(V_1\oplus\cdots\oplus V_1\oplus V_{b_1\cdots b_j}) \\ (x_1, \cdots, x_j) & \mapsto & (x_1, \cdots, \bar{x}_j) \end{array}$$

is a suspension of a complex conjugation. Q.E.D.

Since $D_n=C_n\rtimes C_2$, V_a can be considered as a real D_n-module on which $s\in C_2$ acts by complex conjugation. The following lemma was pointed out by M. Masuda.

Lemma 2.3. *The Petrie's map*

$$h\colon S(V_a\oplus V_b) \to S(V_1\oplus V_{ab})$$

is a D_n-homotopy equivalence.

Proof. A direct computation shows that h is a D_n-map. Therefore it is sufficient to show that h is homotopy equivalence on the fixed point set of each subgroup H of D_n. We shall show that

$$\begin{array}{rcl} \bar{f}\colon \boldsymbol{R}^2 & \to & \boldsymbol{R}^2 \\ (x, y) & \mapsto & (x^p y^q, x^b+y^a) \end{array}$$

has degree ± 1. This is sufficient because h is C_n-homotopy equivalence. To calculate the degree of $\bar{f}$, we consider the image of $S^1=\{(\cos\theta, \sin\theta)\in \boldsymbol{R}^2 | 0\leqq\theta\leqq 2\pi\}$ by $\bar{f}$. We put $S^1_{\theta_1,\theta_2}=\{(\cos\theta, \sin\theta) | \theta_1\leqq\theta\leqq\theta_2\}$. Then $S^1=S^1_{0,\pi/2}\cup S^1_{\pi/2,\pi}\cup S^1_{\pi,3\pi/2}\cup S^1_{3\pi/2,2\pi}$. We shall distinguish the following four cases.

(1) a: odd, b: even, p: odd and q: even.

(2) a: even, b: odd, p: even and q: odd.

(3) a: odd, b: odd, p: even, and q: odd.

(4) a: odd, b: odd, p: odd, and q: odd.

We note that the other cases do not occur by the choice of a, b, p and q. Since the arguments for the cases (1), (2), (3) and (4) are similar, we shall only discuss the case (1). In this case,

$\bar{f}(S^1_{0,\pi/2})$ is a loop at (0, 1) in $\{(x, y) | x\geqq 0, y\geqq 0\}$,

$\bar{f}(S^1_{\pi/2,\pi})$ is a loop at (0, 1) in $\{(x, y) | x\leqq 0, y\geqq 0\}$,

$\bar{f}(S^1_{\pi,3\pi/2})$ is a path from (0, 1) to (0, -1) in $\{(x, y) | x\leqq 0\}$ and

$\bar{f}(S^1_{3\pi/2,2\pi})$ is a path from (0, -1) to (0, 1) in $\{(x, y) | x\geqq 0\}$.

Therefore $\bar{f}$ must have degree $+1$. Q.E.D.

Using the above lemma, we have

Propisition 2.4. *Let U and U' be real C_n-modules such that $S(U)$ and $S(U')$ are C_n-homotopy equivalent. Then there exist real D_n-modules V and V' such that*

$$\text{(1)} \qquad \mathrm{Res}^{D_n}_{C_n} V = U \quad \textit{and} \quad \mathrm{Res}^{D_n}_{C_n} V' = U' ,$$

$$\text{(2)} \qquad S(V) \quad \textit{and} \quad S(V') \quad \textit{are } D_n\textit{-homotopy equivalent.}$$

Proof. We write

$$U = \oplus_{H\subset C_n} U(H) \quad \text{and} \quad U' = \oplus_{H\subset C_n} U'(H)$$

where $U(H)$ and $U'(H)$ collects the irreducible submodules of U and U' respectively which have kernel H. It is well known that $S(U)$ is homotopy equivalent to $S(U')$ if and only if $S(U(H))$ is homotopy equivalent to $S(U'(H))$ for each $H\subset C_n$. Therefore, it is sufficient to show this lemma for each $U(H)$ and $U'(H)$. In case $H=C_n$ or the subgroup of index 2, it is obvious. Since C_n/H acts freely on $S(U(H))$, we may assume that C_n acts freely on $S(U)$ and $S(U')$. If we can choose a C_n-homotopy equivalence $S(U)\to S(U')$ as a Petrie's map (or its suitable suspension), the Petrie's map itself gives a D_n-homotopy equivalence by Lemma 2.3. Of course, the complex conjugation gives a D_n-homotopy equivalence. This together with Lemma 2.2 completes the proof. Q.E.D.

3. Main results

Finally, we state our main results which are easy consequences of previous sections.

Theorem A. $\mathrm{Res}_{C_n}^{D_n}$: *$Wh_{\mathrm{rep}}(D_n) \to Wh_{\mathrm{rep}}(C_n)$ is an isomorphism.*

Proof. Since $Wh_{\mathrm{rep}}(D_n)$ and $Wh_{\mathrm{rep}}(C_n)$ are subgroups of $Wh_{D_n}(*)$ and $Wh_{C_n}(*)$ respectively, Lemma 1.4 shows the injectivity. On the other hand Proposition 2.4 shows the surjectivity because the reduced torsion depends only on G-modules if $Wh_G(*)$ is 2-torsion free. Q.E.D.

Using [11, Theorem C], we have a corollary to Theorem A.

Corollary B. *$Wh_{\mathrm{rep}}(D_n)$ is of finite index in $Wh_{D_n}(*)$ if and only if $n=$ 8, 9, 12, 16, 18, p or $2p$ for odd prime integers p.*

We shall conclude this note by referring the generators of Whitehead group of dihedral groups.

EXAMPLE. The generators of $Wh(D_5)$, $Wh(D_8)$ or $Wh(D_{12})$ are given by the reduced torsions of D_i-homotopy equivalences between the unit sphere of D_i-modules. The units which represent the generators of $Wh(D_5)$, $Wh(D_8)$ and $Wh(D_{12})$ are

(1) $1+(t+t^{-1})-(t^2+t^{-2})+s(-2+(t^2+t^{-2}))$ in case $Wh(D_5)$,

(2) $-1+(t^2+t^{-2})+s(t-t^3-t^4+t^{-2})$ in case $Wh(D_8)$,

(3) $4+2(t+t^{-1})-(t^2+t^{-2})-(t^4+t^{-4})-(t^5+t^{-5})-t^6$
$+s(3+t-t^2-t^3-t^4-t^5-t^6-t^{-5}-t^{-4}-t^{-3}+t^{-2}+3t^{-1})$ in case $Wh(D_{12})$.

Proof. We note that the generators of $Wh(C_5)$, $Wh(C_8)$ and $Wh(C_{12})$ appear as the reduced torsions of the Petrie's maps $S(V_2\oplus V_3)\to S(V_1\oplus V_1)$, $S(V_3\oplus V_5)\to S(V_1\oplus V_7)$ and $S(V_5\oplus V_7)\to S(V_1\oplus V_1)$ respectively (see [11]). Therefore the reduced torsions of the above Petrie's maps (as D_n-homotopy equivalences) represent each generator of $Wh(D_5)$, $Wh(D_8)$ and $Wh(D_{12})$. Using the method of Proposition 1.6, we can find the elements of (1), (2) and (3). Q.E.D.

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