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<td>Watanabe, Takashi</td>
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Osaka University
1. Introduction

Let \( F_4 \) be the compact, simply connected, exceptional Lie group of rank 4. In [6] we have described the Hopf algebra structure of \( H_*(\Omega F_4;\mathbb{Z}) \). Using this thoroughly, we can compute the action of the mod \( p \) Steenrod algebra \( \mathcal{A}_p \) on \( H^*(\Omega F_4;\mathbb{Z}_p) \) for every prime \( p \). But here we deal with the cases \( p=2 \) (Theorem 4) and \( p=3 \) (Theorem 5) only, because in the other cases the result follows immediately from a spectral sequence argument for the path fibration \( \Omega F_4 \to PF_4 \to F_4 \).

Let \( C(=C_o) = T^1 \cdot Sp(3) \) in the notation of [3], which is a closed connected subgroup of \( F_4 \). Then in [6] the homogeneous space \( F_4/C \) has been found to be a *generating variety* for \( F_4 \). That is, there exists a map \( f: F_4/C \to \Omega F_4 \) such that the image of \( f_*: H_*(F_4/C;\mathbb{Z}_p) \) generates the Pontrjagin ring \( H_*(\Omega F_4;\mathbb{Z}_p) \). In this situation Bott [1, §6] asserted that the Steenrod operations in \( H^*(\Omega F_4;\mathbb{Z}_p) \) can be deduced from their effect on \( H^*(F_4/C;\mathbb{Z}_p) \). This is the motive of our work.

Throughout the paper \( X \) will always denote any connected space such that \( H_*(X;\mathbb{Z}) \) is of finite type.

2. The generating variety

In this section we shall compute the \( \mathcal{A}_p \)-module structure of \( H^*(F_4/C;\mathbb{Z}_p) \) for \( p=2 \) and 3.

First since \( C \) contains a maximal torus \( T \) of \( F_4 \), we have a commutative diagram

\[
\begin{array}{ccc}
F_4/T & \xrightarrow{q} & F_4/C \\
\downarrow{\iota} & & \downarrow{j} \\
BT & \xrightarrow{\rho} & BC.
\end{array}
\]

We require the following notations and results (2.2)-(2.6), whose details can be found in [3, §4]:
(2.2) \[ H^*(BT; Z) = Z[t, y_1, y_2, y_3] \] where \( \deg t = \deg y_i = 2 \) \((i = 1, 2, 3)\).

Put \( z_i = y_i(t - y_i) \in H^i(BT; Z) \) and let \( q_i = \sigma_i(z_1, z_2, z_3) \in H^i(BT; Z) \) for \( i = 1, 2, 3 \) where \( \sigma_i \) denotes the \( i \)-th elementary symmetric function. Then we have

(2.3) \[ H^*(BC; Z) = Z[t, q_1, q_2, q_3] \] where \( \deg t = 2 \) and \( \deg q_i = 4i \) \((i = 1, 2, 3)\).

(2.4) \[ \rho^*(t) = t \text{ and } \rho^*(q_i) = q_i \] \((i = 1, 2, 3)\).

On the other hand we have

(2.5) \[ H^*(F_4/C; Z) = Z[t, u, v, w]/(t^3 - 2u, u^2 - 3t^2v + 2aw, 3v^2 - t^2w, v^3 - w^2) \] where \( \deg t = 2, \deg u = 6, \deg v = 8 \) and \( \deg w = 12 \).

(2.6) \[ j^*(t) = t, j^*(q_1) = t^2, j^*(q_2) = 3v \text{ and } j^*(q_3) = w. \]

We shall say that an \( \mathcal{A}_2 \)-action on \( H^*(X; Z_2) \) is non-trivial if it does not follow directly from the axioms 1), 4) or 5) of \([4, \text{p.1 and p.76}]\). With these preliminaries we have

**Proposition 1.** The non-trivial \( \mathcal{A}_2 \)-action on

\[ H^*(F_4/C; Z_2) = Z[t, u, v, w]/(t^3 - 2u, u^2 - t^2v, v^3 - t^2w, v^3 - w^2) \]

is given by:

1. \( Sq^2(t) = t^2 \).
2. \( Sq^2(u) = v, Sq^4(u) = tv \) and \( Sq^6(u) = t^3v \).
3. \( Sq^2(v) = 0, Sq^4(v) = w, Sq^6(v) = tw \) and \( Sq^8(v) = v^3 \).
4. \( Sq^2(w) = tv, Sq^4(w) = 0, Sq^6(w) = 0, Sq^8(w) = tvw \) and \( Sq^{10}(w) = vtw \).

Proof. (1) and the last equalities in (2), (3) and (4) are immediate from the axiom 3) of \([4, \text{p.1}]\).

First we consider (3) and (4). Since \( v = j^*(q_2) \) and \( w = j^*(q_3) \) in \( H^*(F_4/C; Z_2) \) by (2.6), it suffices to determine \( Sq^6(q_2) \) and \( Sq^8(q_3) \) in \( H^*(BC; Z_2) \). To do so, by (2.4), it suffices to compute \( Sq^4(q_2) \) and \( Sq^6(q_3) \) in \( H^*(BT; Z_2) \). But this is a direct calculation, for \( H^*(BT; Z_2) \) is multiplicatively generated by the elements of degree 2 (see (2.2)).

Finally we show the remaining part of (2). By \([3, \text{Corollary 4.5}]\) we may set

\[ Sq^2(u) = k \cdot tu + l \cdot v \]
\[ Sq^4(u) = m \cdot t^3u + n \cdot tv \]

for some \( k, l, m, n \in Z_2 \). Then from the Adem relations \( Sq^2Sq^2 = Sq^4 \), \( Sq^2Sq^4 = Sq^6 + Sq^2Sq^2 \) and \( Sq^2Sq^6 = Sq^8 \), it follows that \( k = 0, m + n = 1 \) and \( l = n = 1 \) respectively. Hence \( k = m = 0 \) and \( l = n = 1 \), which proves (2).
Next we turn to the case $p=3$. To begin with we need some preparations. As in [6, §3], put $x=t/2$ and $x_i=x-y_i$ for $i=1, 2, 3$. Thus $x, x_i \in H^2(BT; Z[1/2])$ ($i=1, 2, 3$). Furthermore put

\[ t_i = -x_1 + x_2, \quad t_2 = x_1 + x_2, \quad t_3 = -x_3 - x \quad \text{and} \quad t_4 = -x_3 + x. \]

Note that $t_i \in H^2(BT; Z)$ ($i=1, 2, 3, 4$). For later convenience we introduce the notation:

\[ c_i = \sigma_i(t_1, t_2, t_3, t_4) \in H^{2i}(BT; Z); \]
\[ p_j = \sigma_j(x_1^2, x_2^2, x_3^2) \in H^{4j}(BT; Q), \]

where $1 \leq i \leq 4$ and $1 \leq j \leq 3$. A straightforward calculation using (2.7) yields:

\[ c_1^2 - 2c_2 = 2(p_1 + x^2); \]
\[ c_2^2 - 2c_3 c_1 + 2c_4 = (p_1 + x^2)^2 + 2(p_2 + p_3 x^2 - 3x_1^2 x_2^2 - 3x_3^2 x_4^2). \]

We also need:

\[ p_1 = -q_1 + 3x^2; \quad p_2 = q_2 - 2q_1 x^2 + 3x^4. \]

This follows from [6, (3.6)].

Now put $\gamma_1 = c_1/2$. From the discussion in [3, §4.2] we observe that

\[ H^*(BT; Z) = Z[t_1, t_2, t_3, t_4, \gamma_1]/(c_1 - 2\gamma_1) \]

on which the Weyl group $\Phi(F_4)$ acts as follows:

| \begin{array}{c|cccc} \hline R & R_1 & R_2 & R_3 & R_4 \\ \hline t_1 & t_2 & -t_1 & t_1 - \gamma_1 \\ t_2 & t_3 & t_1 & t_2 - \gamma_1 \\ t_3 & t_4 & t_2 & t_3 - \gamma_1 \\ t_4 & t_3 & & t_4 - \gamma_1 \\ \hline \end{array} |

This allows us to identify the $t_i$ with that given in [5, §4(A)]. Then by [5, Theorem A] we have

\[ H^*(F_4/T; Z) = Z[t_1, t_2, t_3, t_4, \gamma_1, \gamma_3, w_4]/(c_1 - 2\gamma_1, c_2 - 2\gamma_1^2, c_3 - 2\gamma_3, c_4 - 2c_3 \gamma_1 + 2\gamma_1^2 - 3w_4, -c_1 \gamma_1^2 + \gamma_3^2, 3c_4 \gamma_1^2 - \gamma_1^4 + 3c_3 \gamma_1 w_4 + 3w_3, w_4^2) \]

where $\deg t_i = \deg \gamma_1 = 2$ ($i=1, 2, 3, 4$), $\deg \gamma_3 = 6$ and $\deg w_4 = 8$.

(By abuse of notation we have written $t_i, c_i, \gamma_i$ etc. for their images under $\epsilon^*$.)

It is well known that Spin(9) is a closed connected subgroup of $F_4$, and
the homogeneous space $F_4/\text{Spin}(9)$ can be identified with the Cayley projective plane $\Pi$, whose integral cohomology is given by:

\[(2.11) \quad H^\ast(\Pi; \mathbb{Z}) = \mathbb{Z}[w]/(w^3) \text{ where } \deg w = 8.\]

Since $T \subset \text{Spin}(9) \subset F_4$, we have a natural map

\[F_4/T \xrightarrow{p} F_4/\text{Spin}(9) = \Pi.\]

Then it follows from [5, (6.9)] that

\[(2.12) \quad p^\ast(w) = w^4.\]

The following result may be of independent interest.

**Lemma 2.** $q^\ast(v) = w_4 + x_2^2 x_3^2 + x_4^2 x_3^2 + 2x_4^4$ in $H^\ast(F_4/T; \mathbb{Z})$.

Proof. From (2.6), the commutativity of (2.1), and (2.4) we see that $q^\ast(v) = q_2/3 \in H^\ast(F_4/T; \mathbb{Z})$. (2.10), together with (2.8) and (2.9), gives:

\[w_4 = \frac{1}{3}(c_4 - c_2 c_1 + \frac{1}{2} c_2^2)\]
\[= \frac{1}{3}(p_2 + p_1 x^2 - 3x_1^2 x_2^2 - 3x_3^2 x_2)\]
\[= \frac{1}{3} q_2 - x_2^2 x_3^2 - x_3^3 x_2 - 2x_4^4.\]

Combining these we get the result. (It is easy to verify that $x_2^2 x_3^2 + x_3^3 x_2^2 + 2x_4^4$ is in fact an integral class.)

**Proposition 3.** The non-trivial $A_3$-action on

\[H^\ast(F_4/C; \mathbb{Z}_3) = \mathbb{Z}[t, v]/(t^3, v^3)\]

is given by:

1. $\varphi^1(t) = t^3$.
2. $\varphi^1(v) = -t^6$, $\varphi^3(v) = 0$, $\varphi^3(v) = 0$ and $\varphi^4(v) = 0$.

Proof. (1) and the last equality in (2) are immediate from the axiom 3) of [4, p.76].

To show (2), we must compute $\varphi^i(v)$ for $i = 1, 2, 3$. Since $q^\ast : H^\ast(F_4/C; \mathbb{Z}) \to H^\ast(F_4/T; \mathbb{Z})$ is a split monomorphism (see [3, §3]) and its image is known with $\mathbb{Z}_3$-coefficients (see Lemma 2), it suffices to determine $q^\ast(\varphi^i(v))$ in $H^\ast(F_4/T; \mathbb{Z}_3)$. (2.10) and the same calculation as in (2.8) yield:
Moreover (2.12) and (2.11) imply that \( \varphi^i(w_k) = 0 \) for all \( i \geq 1 \). Therefore we have

\[
\varphi^i(\varphi^j(v)) = \varphi^j(\varphi^i(v))
\]

So since \( \varphi^j(t) = -x \) it follows that \( \varphi^j(v) = -t^6 \). Using the Adem relation \( \varphi^2 = -\varphi^1(\varphi^1) \), we also get \( \varphi^2(v) = 0 \). Finally we consider \( \varphi^3(v) \).

3. Main results

As seen in [6], the algebraic description of \( H_*(\Omega F_4; Z) \) is much easier than that of \( H_*(\Omega F_4; Z) \). For this reason we shall treat the right \( \mathcal{A}_p \)-action on \( H_*(X; Z_p) \) which dualizes to the usual left \( \mathcal{A}_p \)-action on \( H_*(X; Z_p) \).

We first consider the case \( p = 2 \) and follow the notation of [2]. For \( i \geq 0 \) let \((\ )Sq^i\) be the dual to \( Sq^i\). Then these operations have the following properties (cf. [4, p. 1]):

(3.1) \((\ )Sq^i\colon H_*(X; Z_2) \rightarrow H_{n-i}(X; Z_2)\).

(3.2) If \( \deg \alpha < 2i \), \((\alpha)Sq^i = 0\).

(3.3) If \( \deg \alpha = 2i \), \((\alpha)Sq^i = \sqrt{\alpha} \) where \( \sqrt{-} \) is the dual of the squaring map for \( Z_2 \)-algebras.

(3.4) (diagonal Cartan formula) Let \( \psi_0 \colon H_*(X; Z_2) \rightarrow H_*(X; Z_2) \otimes H_*(X; Z_2) \) be the coproduct (induced from the diagonal map \( \Delta_0 \colon X \rightarrow X \times X \)). If \( \psi(\alpha) = \sum \alpha' \otimes \alpha'' \), then

\[
\psi((\alpha)Sq^i) = \sum_{i+j=k} (\alpha')Sq^i \otimes (\alpha'')Sq^j.
\]
Suppose now that $X$ is an $H$-space, and $\alpha \cdot \beta$ denotes the Pontrjagin product of $\alpha$ and $\beta$ in $H_* (X; \mathbb{Z}_2)$. Then one can readily check:

\begin{equation}
(\alpha \cdot \beta) Sq^i = \sum_{j+k=i} (\alpha) Sq^j \cdot (\beta) Sq^k.
\end{equation}

We shall say that an $\mathcal{A}_2$-action on $H_* (X; \mathbb{Z}_2)$ is non-trivial if it does not follow from (3.1), (3.2) or (3.5).

Let us now consider the case $X = \Omega F_4$. Hereafter we shall use the notations and results of [6] without specific reference.

First we have

\begin{equation}
H_* (\Omega F_4; \mathbb{Z}_2) = \mathbb{Z}_2 [\sigma_1, \sigma_2, \sigma_5, \sigma_7, \sigma_{11}] / (\sigma_i^2)
\end{equation}

where $\deg \sigma_i = 2i$ ($i = 1, 2, 5, 7, 11$).

Moreover $\sigma_1^2 = \sigma_2^2, \sigma_2^2 = \sigma_3^2, \sigma_3^2 = \sigma_4^2, \sigma_5^2 = \sigma_6^2, \sigma_6^2 = \sigma_7^2, \sigma_7^2 = \sigma_8^2, \sigma_8^2 = \sigma_9^2, \sigma_9^2 = \sigma_{10}^2, \sigma_{10}^2 = \sigma_{11}^2$. Therefore (by (3.5)) we have only to determine the $(\ ) Sq^i$ on the elements $\sigma_1, \sigma_2, \sigma_5, \sigma_7$ and $\sigma_{11}$. On the other hand, (3.4) implies that for $i \geq 1$ $(\ ) Sq^i$ sends a primitive element to another primitive element. In view of (3.6), the primitive elements of $H_* (\Omega F_4; \mathbb{Z}_2)$ which appear in degrees $\leq 22$ are:

\[
\begin{array}{cccccc}
\text{deg} & 2 & 8 & 10 & 14 & 16 & 20 & 22 \\
\sigma_1 & \sigma_2^2 & \sigma_5^2 & \sigma_7 & \sigma_4^2 & \sigma_6^2 & \sigma_8^2 & \sigma_{11}.
\end{array}
\]

These, together with (3.1) and (3.2), show that possible non-zero operations (among non-trivial operations) are:

\[
\begin{array}{cccccc}
\text{deg} & 2 & 8 & 10 & 14 & 16 & 20 \\
(\sigma_2) Sq^2 & (\sigma_5^2) Sq^2 & (\sigma_7) Sq^4 & (\sigma_{11}) Sq^8 & (\sigma_{11}) Sq^8 & (\sigma_{11}) Sq^8 \\
(\sigma_7) Sq^6
\end{array}
\]

Let us compute these operations. First by (3.3) we have $(\sigma_2) Sq^2 = \sigma_1$. Next we want to determine the coefficient $k \in \mathbb{Z}_2$ in the equation $(\sigma_5^2) Sq^2 = k \cdot \sigma_2^2$. By use of (3.5) we have $(\sigma_5^2) Sq^2 = (\sigma_2) Sq^2 + (\sigma_4^2) Sq^2 = (\sigma_3) Sq^2$ and so $(\sigma_5) Sq^2 = k \cdot \sigma_5^2$. Dualizing this gives $Sq^2(a_i) = k \cdot b_i + l \cdot a_i$ for some $l \in \mathbb{Z}_2$. Since $f^*(Sq^2(a_i)) = Sq^2(f^*(a_i)) = Sq^2(tu) = tu + tv$ by use of (1) and (2) of Proposition 1, and since $f^*(b_i) = tu + tv$ and $f^*(a_i) = tv$, it follows that $k = 1$ (and also $l = 0$). Thus we obtain $(\sigma_5^2) Sq^2 = \sigma_2^2$.

Instead of proceeding further, we state here a pattern of computation: The problem is to determine the coefficient $k \in \mathbb{Z}_2$ in the equation

\[ (\alpha') Sq^i = k' \cdot \beta \]

where $\alpha'$ and $\beta$ are primitive. In particular $\alpha' = \alpha + \text{decomposables}$ and $\alpha$ is
the image under the mod 2 reduction of an integral class which is indecomposable. Using (3.5) we get

\[(\alpha)Sq^i = k \cdot \beta + \ldots\]

where \(k(\in \mathbb{Z}_2)\) and \(k'\) determine each other. Dualizing this gives

\[(*)\quad Sq^i(b) = k \cdot a + \ldots\]

where \(a\) and \(b\) are dual to \(\alpha\) and \(\beta\) respectively. In particular \(a\) is the image under the mod 2 reduction of an integral class which is primitive. Since the composite

\[PH^*(\Omega F_4; \mathbb{Z}) \xrightarrow{\subset} H^*(\Omega F_4; \mathbb{Z}) \xrightarrow{f^*} H^*(F_4/C; \mathbb{Z})\]

is a split monomorphism, it is sufficient to consider (*) in \(H^*(F_4/C; \mathbb{Z}_2)\) via \(f^*_i\). But in [6, §4] the cohomology ring \(H^*(\Omega F_4; \mathbb{Z})\) and its image under \(f^*_i\) have been described, and by Proposition 1 we already know the \(A_2\)-action on \(H^*(F_4/C; \mathbb{Z}_2)\). Thus \(k\) and hence \(k'\) are computable.

In this way routine computations yield

**Theorem 4.** The non-trivial \(A_2\)-action on

\[H_*(\Omega F_4; \mathbb{Z}_2) = \mathbb{Z}_2[\sigma_1, \sigma_2, \sigma_3, \sigma_7, \sigma_1](\sigma^2_1)\]

is given by:

1. \((\sigma_2)Sq^2 = \sigma_1\).
2. \((\sigma_3)Sq^2 = \sigma^2_3\) and \((\sigma_3)Sq^4 = 0\).
3. \((\sigma_7)Sq^2 = 0, (\sigma_7)Sq^4 = \sigma'\) and \((\sigma_7)Sq^8 = 0\).
4. \((\sigma_1)Sq^2 = \sigma^2_1, (\sigma_1)Sq^4 = 0, (\sigma_1)Sq^8 = (\sigma_1)Sq^{10} = 0, (\sigma_1)Sq^{10} = \sigma_7\) and \((\sigma_1)Sq^{10} = 0\).

The argument for the case \(p=3\) is similar (we have prepared Proposition 3 in place of Proposition 1) and so we only present the result.

**Theorem 5.** The non-trivial \(A_3\)-action on

\[H_*(\Omega F_4; \mathbb{Z}_3) = \mathbb{Z}_3[\sigma_1, \sigma_3, \sigma_5, \sigma_7, \sigma_1](\sigma^3_1)\]

is given by:

1. \((\sigma_3)\partial^1 = \sigma_1\).
2. \((\sigma_5)\partial^1 = 0\).
(3) \((\sigma'_1)\mathcal{O}^1 = \sigma'_1\) and \((\sigma'_1)\mathcal{O}^2 = 0\).

(4) \((\sigma'_1)\mathcal{O}^1 = \sigma'_3, (\sigma'_1)\mathcal{O}^2 = 0\) and \((\sigma'_1)\mathcal{O}^3 = 0\).

OSAKA CITY UNIVERSITY

References


