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COHOMOLOGY OPERATIONS IN THE LOOP SPACE OF THE COMPACT EXCEPTIONAL GROUP F_4

Dedicated to Professor A. Komatu on his 70-th birthday

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1. Introduction

Let F_4 be the compact, simply connected, exceptional Lie group of rank 4. In [6] we have described the Hopf algebra structure of $H_*(\Omega F_4;Z)$. Using this thoroughly, we can compute the action of the mod p Steenrod algebra $\mathcal{A}_p$ on $H^*(\Omega F_4;Z_p)$ for every prime p. But here we deal with the cases $p=2$ (Theorem 4) and $p=3$ (Theorem 5) only, because in the other cases the result follows immediately from a spectral sequence argument for the path fibration $\Omega F_4 \to PF_4 \to F_4$.

Let $C(=C_s)=T^1\cdot Sp(3)$ in the notation of [3], which is a closed connected subgroup of F_4. Then in [6] the homogeneous space F_4/C has been found to be a *generating variety* for F_4. That is, there exists a map $f_s\colon F_4/C \to \Omega F_4$ such that the image of $f_{s*}\colon H_*(F_4/C;Z) \to H_*(\Omega F_4;Z)$ generates the Pontrjagin ring $H_*(\Omega F_4;Z)$. In this situation Bott [1, §6] asserted that the Steenrod operations in $H^*(\Omega F_4;Z_p)$ can be deduced from their effect on $H^*(F_4/C;Z_p)$. This is the motive of our work.

Throughout the paper X will always denote any connected space such that $H_*(X;Z)$ is of finite type.

2. The generating variety

In this section we shall compute the $\mathcal{A}_p$-module structure of $H^*(F_4/C;Z_p)$ for $p=2$ and 3.

First since C contains a maximal torus T of F_4, we have a commutative diagram

$$(2.1)\qquad \begin{array}{ccc} F_4/T & \xrightarrow{q} & F_4/C \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle j} \\ BT & \xrightarrow{\rho} & BC\,. \end{array}$$

We require the following notations and results (2.2)-(2.6), whose details can be found in [3, §4]:

(2.2) $H^*(BT; Z)=Z[t, y_1, y_2, y_3]$ *where* deg $t=$deg $y_i=2$ $(i=1, 2, 3)$.

Put $z_i=y_i(t-y_i)\in H^4(BT; Z)$ and let $q_i=\sigma_i(z_1, z_2, z_3)\in H^{4i}(BT; Z)$ for $i=1, 2, 3$ where σ_i denotes the i-th elementary symmetric function. Then we have

(2.3) $H^*(BC; Z)=Z[t, q_1, q_2, q_3]$ *where* deg $t=2$ *and* deg $q_i=4i$ $(i=1, 2, 3)$.

(2.4) $\rho^*(t)=t$ *and* $\rho^*(q_i)=q_i$ $(i=1, 2, 3)$.

On the other hand we have

(2.5) $H^*(F_4/C; Z)=Z[t, u, v, w]/(t^3-2u, u^2-3t^2v+2w, 3v^2-t^2w, v^3-w^2)$ *where* deg $t=2$, deg $u=6$, deg $v=8$ *and* deg $w=12$.

(2.6) $j^*(t)=t, j^*(q_1)=t^2, j^*(q_2)=3v$ *and* $j^*(q_3)=w$.

We shall say that an $\mathcal{A}_p$-action on $H^*(X; Z_p)$ is *non-trivial* if it does not follow directly from the axioms 1), 4) or 5) of [4, p.1 and p.76]. With these preliminaries we have

Proposition 1. *The non-trivial* $\mathcal{A}_2$*-action on*

$$H^*(F_4/C; Z_2)=Z_2[t, u, v, w]/(t^3, u^2-t^2v, v^2-t^2w, v^3-w^2)$$

is given by:

(1) $Sq^2(t) = t^2$.

(2) $Sq^2(u) = v$, $Sq^4(u) = tv$ *and* $Sq^6(u) = t^2v$.

(3) $Sq^2(v) = 0$, $Sq^4(v) = w$, $Sq^6(v) = tw$ *and* $Sq^8(v) = v^2$.

(4) $Sq^2(w) = tw$, $Sq^4(w) = 0$, $Sq^6(w) = 0$, $Sq^8(w) = vw$, $Sq^{10}(w) = tvw$ *and* $Sq^{12}(w) = w^2$.

Proof. (1) and the last equalities in (2), (3) and (4) are immediate from the axiom 3) of [4, p.1].

First we consider (3) and (4). Since $v=j^*(q_2)$ and $w=j^*(q_3)$ in $H^*(F_4/C; Z_2)$ by (2.6), it suffices to determine $Sq^i(q_2)$ and $Sq^i(q_3)$ in $H^*(BC; Z_2)$. To do so, by (2.4), it suffices to compute $Sq^i(q_2)$ and $Sq^i(q_3)$ in $H^*(BT; Z_2)$. But this is a direct calculation, for $H^*(BT; Z_2)$ is multiplicatively generated by the elements of degree 2 (see (2.2)).

Finally we show the remaining part of (2). By [3, Corollary 4.5] we may set

$$Sq^2(u) = k\cdot tu+l\cdot v \text{ and}$$
$$Sq^4(u) = m\cdot t^2u+n\cdot tv$$

for some $k, l, m, n\in Z_2$. Then from the Adem relations $Sq^2Sq^2=Sq^3Sq^1$, $Sq^2Sq^4=Sq^6+Sq^5Sq^1$ and $Sq^4Sq^4=Sq^7Sq^1$, it follows that $kl=0$, $lm+n=1$ and $l=n$ respectively. Hence $k=m=0$ and $l=n=1$, which proves (2).

Next we turn to the case $p=3$. To begin with we need some preparations. As in [6, §3], put $x=t/2$ and $x_i=x-y_i$ for $i=1, 2, 3$. Thus x, $x_i \in H^2(BT; Z[1/2])$ $(i=1, 2, 3)$. Furthermore put

(2.7) $t_1=-x_1+x_2$, $t_2=x_1+x_2$, $t_3=-x_3-x$ *and* $t_4=-x_3+x$.

Note that $t_i \in H^2(BT; Z)$ $(i=1, 2, 3, 4)$. For later convenience we introduce the notation:

$$c_i = \sigma_i(t_1, t_2, t_3, t_4) \in H^{2i}(BT; Z) ;$$
$$p_j = \sigma_j(x_1^2, x_2^2, x_3^2) \in H^{4j}(BT; Q) ,$$

where $1 \leq i \leq 4$ and $1 \leq j \leq 3$. A straightforward calculation using (2.7) yields:

(2.8) $c_1^2-2c_2=2(p_1+x^2)$;
$c_2^2-2c_3c_1+2c_4=(p_1+x^2)^2+2(p_2+p_1x^2-3x_1^2x_2^2-3x_3^2x^2)$.

We also need:

(2.9) $p_1=-q_1+3x^2$; $p_2=q_2-2q_1x^2+3x^4$.

This follows from [6, (3.6)].

Now put $\gamma_1=c_1/2$. From the discussion in [3, §4.2] we observe that

$$H^*(BT; Z) = Z[t_1, t_2, t_3, t_4, \gamma_1]/(c_1-2\gamma_1)$$

on which the Weyl group $\Phi(F_4)$ acts as follows:

	$\tilde{R}$	R_1	R_2	R_3	R_4
t_1			t_2	$-t_1$	$t_1-\gamma_1$
t_2		t_3	t_1		$t_2-\gamma_1$
t_3	t_4	t_2			$t_3-\gamma_1$
t_4	t_3				$t_4-\gamma_1$

This allows us to identify the t_i with that given in [5, §4(A)]. Then by [5, Theorem A] we have

(2.10) $H^*(F_4/T; Z)=Z[t_1, t_2, t_3, t_4, \gamma_1, \gamma_3, w_4]/(c_1-2\gamma_1, c_2-2\gamma_1^2, c_3-2\gamma_3, c_4-2c_3\gamma_1+2\gamma_1^4-3w_4, -c_4\gamma_1^2+\gamma_3^2, 3c_4\gamma_1^4-\gamma_1^8+3c_3\gamma_1w_4+3w_4^2, w_4^3)$ *where* $\deg t_i=\deg \gamma_1=2$ $(i=1, 2, 3, 4)$, $\deg \gamma_3=6$ *and* $\deg w_4=8$.

(By abuse of notation we have written t_i, c_i, etc. for their images under ι^*.)

It is well known that Spin(9) is a closed connected subgroup of F_4, and the homogeneous space $F_4/\mathrm{Spin}(9)$ can be identified with the Cayley projective plane Π, whose integral cohomology is given by:

(2.11) $H^*(\Pi; Z)=Z[w]/(w^3)$ *where* $\deg w=8$.

Since $T\subset \mathrm{Spin}\,(9)\subset F_4$, we have a natural map

$$F_4/T \xrightarrow{p} F_4/\mathrm{Spin}\,(9) = \Pi\ .$$

Then it follows from [5, (6.9)] that

(2.12) $p^*(w)=w_4$.

The following result may be of independent interest.

Lemma 2. $q^*(v)=w_4+x_1^2x_2^2+x_3^2x^2+2x^4$ *in* $H^8(F_4/T; Z)$.

Proof. From (2.6), the commutativity of (2.1), and (2.4) we see that $q^*(v)=q_2/3\in H^8(F_4/T; Z)$. (2.10), together with (2.8) and (2.9), gives:

$$\begin{aligned} w_4 &= \frac{1}{3}(c_4-c_3c_1+\frac{1}{2}c_2^2) \\ &= \frac{1}{3}(p_2+p_1x^2-3x_1^2x_2^2-3x_3^2x^2) \\ &= \frac{1}{3}q_2-x_1^2x_2^2-x_3^2x^2-2x^4\ . \end{aligned}$$

Combining these we get the result. (It is easy to verify that $x_1^2x_2^2+x_3^2x^2+2x^4$ is in fact an integral class.)

Proposition 3. *The non-trivial* $\mathcal{A}_3$*-action on*

$$H^*(F_4/C; Z_3) = Z[t,v]/(t^8, v^3)$$

is given by:

(1) $\mathcal{P}^1(t) = t^3$.

(2) $\mathcal{P}^1(v) = -t^6$, $\mathcal{P}^2(v) = 0$, $\mathcal{P}^3(v) = 0$ *and* $\mathcal{P}^4(v) = 0$.

Proof. (1) and the last equality in (2) are immediate from the axiom 3) of [4, p.76].

To show (2), we must compute $\mathcal{P}^i(v)$ for $i=1, 2, 3$. Since $q^*: H^*(F_4/C; Z)\rightarrow H^*(F_4/T; Z)$ is a split monomorphism (see [3, §3]) and its image is known with Z_3-coefficients (see Lemma 2), it suffices to determine $q^*(\mathcal{P}^i(v))$ in $H^*(F_4/T; Z_3)$. (2.10) and the same calculation as in (2.8) yield:

$$
\begin{aligned}
H^*(F_4/T; Z_3) &= Z_3[t_1, t_2, t_3, t_4, w_4]/(c_2+c_1^2, c_4-c_3c_1-c_1^4, \\
&\qquad c_3^2-c_4c_1^2, c_1^8, w_4^3) \\
&= Z_3[x_1, x_2, x_3, x, w_4]/(x_1^2+x_2^2+x_3^2+x^2, x_2^4+ \\
&\qquad x_2^2x_3^2+x_3^4+x_2^2x^2+x_3^2x^2+x^4, x_3^6+x_3^4x^2+x_3^2x^4 \\
&\qquad +x^6, x^8, w_4^3)\,.
\end{aligned}
$$

Moreover (2.12) and (2.11) imply that $\mathcal{P}^i(w_4)=0$ for all $i\geq 1$. Therefore we have

$$
\begin{aligned}
q^*(\mathcal{P}^1(v)) &= \mathcal{P}^1(q^*(v)) \\
&= \mathcal{P}^1(x_1^2x_2^2+x_3^2x^2-x^4) \\
&= -x_1^4x_2^2-x_1^2x_2^4-x_3^4x^2-x_3^2x^4-x^6 \\
&= -x^6\,.
\end{aligned}
$$

So since $q^*(t)=-x$ it follows that $\mathcal{P}^1(v)=-t^6$. Using the Adem relation $\mathcal{P}^2=-\mathcal{P}^1\mathcal{P}^1$, we also get $\mathcal{P}^2(v)=0$. Finally we consider $\mathcal{P}^3(v)$. By [3, Corollary 4.5] we may set

$$\mathcal{P}^3(v) = k\cdot t^6v+l\cdot t^2v^2$$

for some $k, l\in Z_3$. Then $q^*(\mathcal{P}^3(v))=k\cdot x^6w_4+l\cdot x^2w_4^2+\cdots$. On the other hand notice that $\mathcal{P}^3(q^*(v))$ does not involve x^6w_4 or $x^2w_4^2$, and they are linearly independent in $H^{20}(F_4/T; Z_3)$. This implies that $k=l=0$ as required.

3. Main results

As seen in [6], the algebraic description of $H_*(\Omega F_4; Z)$ is much easier than that of $H^*(\Omega F_4; Z)$. For this reason we shall treat the right $\mathcal{A}_p$-action on $H_*(X; Z_p)$ which dualizes to the usual left $\mathcal{A}_p$-action on $H^*(X; Z_p)$.

We first consider the case $p=2$ and follow the notation of [2]. For $i\geq 0$ let $(\ )Sq^i$ be the dual to $Sq^i(\ )$. Then these operations have the following properties (cf. [4, p. 1]):

(3.1) $(\ )Sq^i\colon H_n(X; Z_2)\to H_{n-i}(X; Z_2)$.

(3.2) *If* $\deg \alpha<2i$, $(\alpha)Sq^i=0$.

(3.3) *If* $\deg \alpha=2i$, $(\alpha)Sq^i=\sqrt{\alpha}$ *where* $\sqrt{\ }$ *is the dual of the squaring map for* Z_2*-algebras.*

(3.4) (*diagonal Cartan formula*) *Let* $\psi\colon H_*(X; Z_2)\to H_*(X; Z_2)\otimes H_*(X; Z_2)$ *be the coproduct* (*induced from the diagonal map* $\Delta\colon X\to X\times X$). *If* $\psi(\alpha)=\sum\alpha'\otimes\alpha''$, *then*

$$\psi((\alpha)Sq^k) = \sum_{i+j=k}(\alpha')Sq^i\otimes(\alpha'')Sq^j\,.$$

Suppose now that X is an H-space, and $\alpha\cdot\beta$ denotes the Pontrjagin product of α and β in $H_*(X; Z_2)$. Then one can readily check:

(3.5) (*internal Cartan formula*)

$$(\alpha\cdot\beta)Sq^k = \sum_{i+j=k}(\alpha)Sq^i\cdot(\beta)Sq^j .$$

We shall say that an $\mathcal{A}_2$-action on $H_*(X; Z_2)$ is *non-trivial* if it does not follow from (3.1), (3.2) or (3.5).

Let us now consider the case $X=\Omega F_4$. Hereafter we shall use the notations and results of [6] without specific reference.

First we have

(3.6) $H_*(\Omega F_4; Z_2)=Z_2[\sigma_1, \sigma_2, \sigma_5, \sigma_7, \sigma_{11}]/(\sigma_1^2)$ *where* $\deg\sigma_i=2i\,(i=1, 2, 5, 7, 11)$. *Moreover* σ_1, $\sigma_5'=\sigma_5+\sigma_2^2\sigma_1$, σ_7 *and* $\sigma_{11}'=\sigma_{11}+\sigma_5^2\sigma_1+\sigma_7\sigma_2^2$ *are primitive, and* $\tilde{\psi}(\sigma_2)=\sigma_1\otimes\sigma_1$.

Therefore (by (3.5)) we have only to determine the $(\ )Sq^i$ on the elements $\sigma_1, \sigma_2, \sigma_5', \sigma_7$ and σ_{11}'. On the other hand, (3.4) implies that for $i\geq 1$ $(\ )Sq^i$ sends a primitive element to another primitive element. In view of (3.6), the primitive elements of $H_*(\Omega F_4; Z_2)$ which appear in degrees ≤ 22 are:

deg	2	8	10	14	16	20	22
	σ_1	σ_2^2	σ_5'	σ_7	σ_2^4	$\sigma_5'^2$	σ_{11}' .

These, together with (3.1) and (3.2), show that possible non-zero operations (among non-trivial operations) are:

deg	2	8	10	14	16	20
	$(\sigma_2)Sq^2$	$(\sigma_5')Sq^2$	$(\sigma_7)Sq^4$	$(\sigma_{11}')Sq^8$	$(\sigma_{11}')Sq^6$	$(\sigma_{11}')Sq^2$
		$(\sigma_7)Sq^6$				

Let us compute these operations. First by (3.3) we have $(\sigma_2)Sq^2=\sigma_1$. Next we want to determine the coefficient $k\in Z_2$ in the equation $(\sigma_5')Sq^2=k\cdot\sigma_2^2$. By use of (3.5) we have $(\sigma_5')Sq^2=(\sigma_5)Sq^2+(\sigma_2^2\sigma_1)Sq^2=(\sigma_5)Sq^2$ and so $(\sigma_5)Sq^2=k\cdot\sigma_2^2$. Dualizing this gives $Sq^2(a_4)=k\cdot b_5+l\cdot a_5$ for some $l\in Z_2$. Since $f_s^*(Sq^2(a_4))=Sq^2(f_s^*(a_4))=Sq^2(tu)=t^2u+tv$ by use of (1) and (2) of Proposition 1, and since $f_s^*(b_5)=t^2u+tv$ and $f_s^*(a_5)=tv$, it follows that $k=1$ (and also $l=0$). Thus we obtain $(\sigma_5')Sq^2=\sigma_2^2$.

Instead of proceeding further, we state here a pattern of computation: The problem is to determine the coefficient $k'\in Z_2$ in the equation

$$(\alpha')Sq^i = k'\cdot\beta$$

where α' and β are primitive. In particular $\alpha'=\alpha+$decomposables and α is the image under the mod 2 reduction of an integral class which is indecomposable. Using (3.5) we get

$$(\alpha)Sq^i = k\cdot\beta + \cdots$$

where $k(\in Z_2)$ and k' determine each other. Dualizing this gives

$$(*) \qquad Sq^i(b) = k\cdot a + \cdots$$

where a and b are dual to α and β respectively. In particular a is the image under the mod 2 reduction of an integral class which is primitive. Since the composite

$$PH^*(\Omega F_4; Z) \xrightarrow{\subset} H^*(\Omega F_4; Z) \xrightarrow{f_s^*} H^*(F_4/C; Z)$$

is a split monomorphism, it is sufficient to consider (*) in $H^*(F_4/C; Z_2)$ via f_s^*. But in [6, §4] the cohomology ring $H^*(\Omega F_4; Z)$ and its image under f_s^* have been described, and by Proposition 1 we already know the $\mathcal{A}_2$-action on $H^*(F_4/C; Z_2)$. Thus k and hence k' are computable.

In this way routine computations yield

Theorem 4. *The non-trivial $\mathcal{A}_2$-action on*

$$H_*(\Omega F_4; Z_2) = Z_2[\sigma_1, \sigma_2, \sigma_5', \sigma_7, \sigma_{11}']/(\sigma_1^2)$$

is given by:

(1) $(\sigma_2)Sq^2 = \sigma_1$.

(2) $(\sigma_5')Sq^2 = \sigma_2^2$ *and* $(\sigma_5')Sq^4 = 0$.

(3) $(\sigma_7)Sq^2 = 0$, $(\sigma_7)Sq^4 = \sigma_5'$ *and* $(\sigma_7)Sq^6 = 0$.

(4) $(\sigma_{11}')Sq^2 = \sigma_5'^2$, $(\sigma_{11}')Sq^4 = 0$, $(\sigma_{11}')Sq^6 = \sigma_2^4$, $(\sigma_{11}')Sq^8 = \sigma_7$ *and* $(\sigma_{11}')Sq^{10} = 0$.

The argument for the case $p=3$ is similar (we have prepared Proposition 3 in place of Proposition 1) and so we only present the result.

Theorem 5. *The non-trivial $\mathcal{A}_3$-action on*

$$H_*(\Omega F_4; Z_3) = Z_3[\sigma_1, \sigma_3, \sigma_5', \sigma_7', \sigma_{11}']/(\sigma_1^3)$$

is given by:

(1) $(\sigma_3)\mathcal{P}^1 = \sigma_1$.

(2) $(\sigma_5')\mathcal{P}^1 = 0$.

(3) $(\sigma'_7)\mathcal{P}^1=\sigma'_5$ *and* $(\sigma'_7)\mathcal{P}^2=0$.

(4) $(\sigma'_{11})\mathcal{P}^1=\sigma_3^3, (\sigma'_{11})\mathcal{P}^2=0$ *and* $(\sigma'_{11})\mathcal{P}^3=0$.

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