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ON THE CHARACTERIZATION OF HARMONIC MAP OF TYPE BL AT THE MARTIN BOUNDARY POINTS OF A HARMONIC SPACE

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Introduction

In 1955, M. Heins [6] introduced a class of analytic mappings of Riemann surfaces. The maps in this class, called maps of type B1, can be seen as a natural generalization of the Seidel functions. They have some remarkable interior properties, e.g., the covering property. The boundary behaviour of a map of type B1 is also interesting in suitable compactifications. We can find fruitful results concerning this problem under the Martin compactification [1], [4]. The notion of a map of type B1 is extended by Constantinescu-Cornea to a harmonic map of harmonic spaces satisfying Brelot's axioms [3]. They defined a harmonic map of type B1 and discussed similar interior properties and investigated the boundary properties by using the Wiener compactifications of harmonic spaces. The results obtained there are very beautiful. The detailed investigation on the covering properties of harmonic maps of type B1 will be found in [10], [11].

The purpose of this paper is to study the boundary behaviour of a harmonic map of type B1 at the Martin boundary defined in [8]. Let φ be a harmonic map of type B1 from a harmonic space X into X', each of which satisfying adequate conditions. We consider the Martin compactification X^M of X and define the fine cluster set $\phi(x)$ of φ at a minimal Martin boundary point x of X^M, which plays an important role in this article. The hypotheses, notations and definitions which will be used in the following are stated in §1. §1 contains also some results concerning the quotient sheaves. In §2, we deal with the case of an arbitrary compactification of X' including the case where X' itself is compact. The boundary characterization of maps of type B1 is stated in Theorem 2. In §3, we restrict ourselves to the case where X' is non-compact and has a potential. Our next concern is the case where the compactification of X' is metrizable and resolutive. In this case we can derive a version of covering property which is stated in the Corollary to Theorem 7 in §4. In §5, we consider the Martin compactification of X' and obtain the theorem that if φ is a finite covering (i.e., for every $a'\in X'$ $\varphi^{-1}(a')$ consists of at most n elements, where n is independent of a'), then for every minimal positive harmonic function v' on X' $v'\circ\varphi$ is a linear combination of a finite number of minimal positive harmonic functions on X(Theorem 11). The last section is devoted to a short remark on a more restricted class of harmonic maps, i.e., maps of type $B1_1$. They can be seen as a generalization of analytic maps of type $B1_1$, defined by Heins [6] originally and considered by Constantinescu-Cornea [1] with the Martin compactification.

1. Preliminaries

Hypotheses. Let X be a harmonic space in the sense of Brelot; we further assume that

1) X is non-compact,
2) $X\in\mathcal{P}$, i.e., there exists a positive potential on X,
3) X has a countable base of open sets,
4) X satisfies the proportionality axiom, i.e., for every $a\in X$ potentials with single point support $\{a\}$ are all proportional,
5) $1\in\mathcal{W}(X)$, i.e., constant functions are Wiener functions (for the definition of Wiener function, we refer to [3]).

Next, let X' be another harmonic space in the sense of Brelot. For X' we assume only

1) $X'\in\mathcal{P}\cup\mathcal{H}$, i.e., there exists a positive superharmonic function on X',
2) X' has a countable base of open sets,
3) $1\in\mathcal{W}(X')$.

The Martin boundary. The Martin compactification X^M of X is defined as follows [8]: let S^+ be the set of all non-negative superharmonic functions and E be the set of all potentials with single point support. By introducing Herve's topology [7], we know that the positive cone S^+ is metrizable and has a compact base A, and $E\cap A$ is homeomorphic to X. X^M is defined to be the closure of $E\cap A$ in A with respect to Herve's topology. $\Delta^M=X^M\setminus X$ is termed the Martin boundary of X. To each $x\in\Delta^M$ there corresponds a positive harmonic function k_x on X. Δ_1^M is the set of all $x\in\Delta^M$ such that k_x is minimal. The Martin compactification is resolutive, and the harmonic measure on Δ^M will be denoted by ω^M.

DEFINITIONS. Let φ be a harmonic map of X into X' ([3], p. 20). For a function f defined on X and for an open subset U of X, we define

$$\bar{h}_f^{U,X}=\inf\left\{s;\ \begin{array}{l}\text{hyperharmonic function defined on } U \text{ with non-}\\ \text{positive subharmonic minorant, } s\geq f \text{ outside}\\ \text{of a compact subset of } X,\ \liminf_{a\to b} s(a)\geq 0 \text{ for}\\ \text{every } b\in\partial U \text{ (the relative boundary of } U)\end{array}\right\}$$

and

$$\bar{H}_f^{U,X} = \inf \left\{ s;\ \begin{array}{l} \text{hyperharmonic function defined and bounded} \\ \text{below on } U,\ s \geq 0 \text{ outside of a compact subset} \\ \text{of } X,\ \liminf_{a \to b} s(a) \geq f(b) \text{ for every } b \in \partial U \end{array} \right\}$$

We abbreviate $\bar{h}_f^{X,X}$ to $\bar{h}_f^X$.

φ is called *of type Bl* at $a' \in X'$ if there exists an open neighbourhood U' of a' such that $\bar{h}_1^{\varphi^{-1}(U'),X} = 0$. A map of type B1 is defined to be of type B1 at every $a' \in X'$. Recall that if φ is of type B1, then $p' \circ \varphi$ is a potential for every potential p' which is locally bounded on X', and $\bar{h}_{f' \circ \varphi}^{\varphi^{-1}(U'),X} \leq \bar{h}_{f'}^{U',X'} \circ \varphi$ for every $U' \in \mathcal{P}$ and locally bounded f'.

Quotient sheaves. Let u_0 be a positive harmonic function on X, $\mathcal{H}_{u_0}$ a sheaf of u_0-harmonic functions. A u_0-harmonic function is of form u/u_0, where u is harmonic. Obviously, constant functions are u_0-harmonic. u_0-regular domains, i.e., regular domains for $\mathcal{H}_{u_0}$ are identical with original regular domains. The u_0-superharmonic functions (resp. u_0-potentials) are the quotients of superharmonic functions (resp. potentials) by u_0. For positive u_0-superharmonic function f, u_0-reduced function $\mathcal{R}_f^E$ defined by

$$\mathcal{R}_f^E = \inf \{f_1;\ \text{a } u_0\text{-superharmonic function dominating } f \text{ on } E\}$$

is equal to $R_{fu_0}^E/u_0$.

From the construction of the Martin compactification stated above, it is evident that the Martin compactification X^{M,u_0} of X with respect to $\mathcal{H}_{u_0}$ is homeomorphic to X^M. We identify them. A set E of X is termed thin at $x \in \Delta_1^M$ if $R_{k_x}^E \not\equiv k_x$. Thinness of sets at x is unaltered when we consider the quotient sheaves.

A continuous function f on X is termed a u_0-Wiener function if $u_0 f \in \mathcal{W}(X)$, then we have

$$d_f = h_{fu_0}/u_0\,,$$

where d_f is the harmonization of f with respect to $\mathcal{H}_{u_0}$. In particular, 1 is a u_0-Wiener function and $d_1 = 1$. We know ([8], Th. 5, p. 261) that for $f \in \boldsymbol{C}(X^M)$

$$\mathcal{D}_{f,u_0} = d_f\,,$$

where $\mathcal{D}_{f,u_0}$ is the u_0-Dirichlet solution of f, i.e., the Dirichlet solution with respect to $\mathcal{H}_{u_0}$, which means that the Martin compactification is u_0-resolutive. If we write

$$\mathcal{D}_{f,u_0}(a) = \int f\, d\omega_a^{M,u_0}\,,$$

then

$$d\omega_a^{M,u_0}(x) = (k_x(a)/u_0(a))d\mu_0(x)\,,$$

where $u_0=\int k_x d\mu_0(x)$ (the canonical representation of u_0). In particular, if $u_0=k_{x_0}$ for $x_0\in\Delta_1^M$, then $d\omega_a^{M,u_0}=\varepsilon_{x_0}$ (The Dirac measure at x_0).

Let $X'\in\mathcal{P}$ and X' be non-compact.

If X'^* is an arbitrary resolutive compactification of X', then X'^* is a u_0'-resolutive compactification of X', where $u_0'=h_1^{X'}$. For, let f' be an arbitrary continuous function on $\Delta'=X'^*\setminus X'$ and F' be a continuous extension of f' onto X'^*. It is readily seen that $\bar{h}_{F'u_0'}^{X'}=(\bar{\mathcal{D}}_{f',u_0'})\cdot u_0'$ and $\underline{h}_{F'u_0'}^{X'}=(\underline{\mathcal{D}}_{f',u_0'})\cdot u_0'$. Since $1=u_0'+q'$ and $|q'|\leq p'$ for a potential p', $F'u_0'=F'\cdot 1-F'\cdot q'$. F' is a Wiener function and $F'\cdot q'$ is a Wiener potential, whence

$$\bar{h}_{F'u_0'}^{X'} = \underline{h}_{F'u_0'}^{X'} = h_{F'}^{X'}\,.$$

Thus f' is u_0'-resolutive.

From the above proposition, we have for every $a'\in X'$ there exists the Radon measure $\nu'_{a'}$ on Δ' called u_0'-harmonic measure such that

$$\mathcal{D}_{f',u_0'}(a') = \int f'd\nu'_{a'} \qquad \text{for } f\in \boldsymbol{C}(\Delta')\,.$$

The following proposition can be verified in a standard way ([2], 8).

Let X'^* be a metrizable and resolutive compactification of X'.

(1) *for every lower semi-continuous and bounded below function f' on Δ', we have*

$$\bar{\mathcal{D}}_{f',u_0'} = \int f'd\nu' = \sup\ \{\int g'd\nu'\,;\, g' \text{ is bounded, continuous and } g'\leq f'\}\,.$$

(2) *for an arbitrary function f_1' on Δ', we have*

$$\bar{\mathcal{D}}_{f_1',u_0'}= \overline{\int} f_1'd\nu' = \inf\ \{\int f'd\nu'\,;\, f' \text{ is lower semi-continuous, bounded below and } f'\geq f_1'\}\,.$$

In particular, a function f' on Δ' is u_0'-resolutive if and only if it is $d\nu'$-summable.

Let f be a continuous function on $\Delta=X^*\setminus X$ and F be a continuous extension of f onto X^*, where X^* is a metrizable and resolutive compactification of X. We have seen that $h_F^X=h_{Fu_0}^X$. Since

$$h_F^X(a) = \int f d\omega_a = H_f(a) \qquad \text{and} \quad h_{Fu_0}^X(a)/u_0(a) = \mathcal{D}_{f,u_0}(a) = \int f d\nu_a\,,$$

we have $d\omega_a=u_0(a)\,d\nu_a$. Thus we have:

Let X^* be a metrizable and resolutive compactification of X and $u_0=h_1^X$. The u_0-harmonic measures and the harmonic measures are mutually absolutely continuous. A function f on Δ is u_0-resolutive, if and only if it is resolutive and $\mathcal{D}_{f,u_0}=H_f/u_0$.

2. The case of arbitrary compactifications of X'

For a subset A of X we write

$$\mathcal{E}_A = \{x \in \Delta_1^M;\ A \text{ is thin at } x\}\ .$$

We recall that a set A is thin at $x \in \Delta_1^M$ if and only if $R_{k_x}^A \not\equiv k_x$.

Lemma 1. *Let U be an open subset of X. The following properties are equivalent*:

(i) $\bar{h}_1^{U,X} = 0$,
(ii) $\hat{R}_{u_0}^{X \setminus U} = u_0$, where $u_0 = h_1^X$,
(iii) $\omega^M(\mathcal{E}_{X \setminus U}) = 0$.

Proof. (i) ⇔ (ii): the equivalence in question is derived from the equalities:

$$\begin{aligned} u_0 = \bar{h}_{u_0}^{U,U} &= \bar{h}_{u_0}^{U,X} + \bar{H}_{u_0}^{U,X} && \text{in } U \text{ ([3], Lemma 2.7)} \\ &= \bar{h}_1^{U,X} + R_{u_0}^{X \setminus U} && \text{in } U \text{ ([3], Lemma 2.5 and Cor. 1.1).} \end{aligned}$$

(ii) ⇔ (iii): from the integral representation

$$u_0 = \int_{\Delta_1 M} k_x \, d\mu_0(x)\ ,$$

where μ_0 is a Borel measure on Δ^M, called the canonical measure of u_0, we have

$$\hat{R}_{u_0}^{X \setminus U}(a) = \int_{\Delta_1 M} \hat{R}_{k_x}^{X \setminus U}(a)\, d\mu_0(x) \qquad \text{([7], Th. 22.3).}$$

Here $\hat{R}_{k_x}^{X \setminus U} \not\equiv k_x$ if and only if $x \in \mathcal{E}_{X \setminus U}$, which proves the equivalence in question, since μ_0 and ω^M are mutually absolutely continuous ([8], p. 262).

Let X'^* be an arbitrary compactification; if X' is compact, we consider X'^* is X' itself, and let $\phi(x)$ be the fine cluster set at $x \in \Delta_1^M$:

$$\phi(x) = \cap\ \{\overline{\varphi(E)};\ X \setminus E \text{ is thin at } x\}\ ,$$

where the closure is taken in X'^*.

Lemma 2. *The following* a) *and* b) *are equivalent*:

a) $\omega^M(\{x \in \Delta_1^M;\ \phi(x) \subsetneqq X'\}) = 0$,

b) *for every relatively compact open subset G' of X' not everywhere dense in X' we have $\omega^M(\mathcal{E}_{X \setminus \varphi^{-1}(G')}) = 0$.*

Proof. a) ⇒ b): $x \in \mathcal{E}_{X \setminus \varphi^{-1}(G')}$ implies $\phi(x) \subset \overline{\varphi[\varphi^{-1}(G')]} = \overline{G'}$. Let G' be a relatively compact open subset of X' and $\bar{G}' \subsetneqq X'$. Then,

$$\mathcal{E}_{X \setminus \varphi^{-1}(G')} \subset \{x \in \Delta_1^M;\ \phi(x) \subsetneqq X'\}\ ,$$

which proves the assertion.

b) $\Rightarrow$a): let $\{U_n'\}$ be a countable base of open sets for X'. Since

$$\{x\in\Delta_1^M;\ \phi(x)\subsetneqq X'\} = \bigcup_{n=1}^{\infty}\{x\in\Delta_1^M;\ \phi(x)\subset X'\setminus\bar{U}_n'\}\ ,$$

if $\omega^M(\{x\in\Delta_1^M;\ \phi(x)\subsetneqq X'\})>0$, then there exists a U_n' such that

$$\omega^M(\{x\in\Delta_1^M;\ \phi(x)\subset X'\setminus\bar{U}_n'\})>0\,.$$

In the same way, we can find a relatively compact open subset G' of $X'\setminus\bar{U}_n'$ with $\omega^M(\{x\in\Delta_1^M;\ \phi(x)\subset G'\})>0$. Then, we have $\omega^M(\mathcal{E}_{X\setminus\varphi^{-1}(G')})>0$, since $\phi(x)\subset G'$ implies $x\in\mathcal{E}_{X\setminus\varphi^{-1}(G')}$.

Theorem 1. *A map φ is of type Bl if*

$$\omega^M(\{x\in\Delta_1^M;\ \phi(x)\subsetneqq X'\}) = 0\,.$$

Proof. Given any point $a'\in X'$ we can find $U'(a')$ which is relatively compact open neighbourhood of a', small enough not to be everywhere dense in X'. Then, by Lemma 2, $\omega^M(\mathcal{E}_{X\setminus\varphi^{-1}(U'(a'))})=0$ and, by Lemma 1, this means that φ is of type B1, q.e.d..

Theorem 2. *Let X'^* be an arbitrary compactification of X'.*

a) *when $X'\in\mathcal{H}\setminus\mathcal{P}$ (i.e., there exists a positive harmonic function on X' whereas X' has no potentials), φ is of type Bl if and only if $\phi(x)=X'^*$ (or equivalently, $\phi(x)\cap X'=X'$) $d\omega^M$—a.e. on Δ^M.*

b) *when $X'\in\mathcal{P}$, φ is of type Bl if and only if $\phi(x)\subset\Delta'$ (or equivalently, $\phi(x)\cap X'=\phi$) $d\omega^M$—a.e. on Δ^M, where $\Delta'=X'^*\setminus X'$.*

Proof. a) we shall assume that φ is of type B1, then 1 is harmonic on X' ([3], Th. 3.11) and for any open set $U'\in\mathcal{P}$ we have $\bar{h}_1^{U',X'}=0$ ([3], p. 11). Since $\bar{h}_1^{\varphi^{-1}(U'),X}\leq\bar{h}_1^{U',X'}\circ\varphi$ we have $\omega^M(\mathcal{E}_{X\setminus\varphi^{-1}(U')})=0$ for every open subset U' of X' with $U'\in\mathcal{P}$.

Let $\{U_n'\}$ be a base of open sets for X' with $U_n'\in\mathcal{P}$ and

$$A_n = \{x\in\Delta_1^M;\ \phi(x)\cap U_n' = \phi\}\ .$$

If $x\in\Delta_1^M\setminus\bigcup_{n=1}^{\infty}A_n$ then $\phi(x)\cap U_n'\neq\phi$, $n=1, 2, \cdots$. We shall show that $\omega^M(A_n)=0$. In fact, if V' is a relatively compact open set contained in U_n', then $X'\setminus\bar{V}'\in\mathcal{P}$ and $A_n\subset\mathcal{E}_{X\setminus\varphi^{-1}(X'\setminus\bar{V}')}$. Thus we can conclude that $\phi(x)=X'^*$ whenever $x\in\Delta_1^M\setminus\bigcup_{n=1}^{\infty}A_n$ and $\bigcup_{n=1}^{\infty}A_n$ is of harmonic measure zero.

The proof of the "if" part is derived from Theorem 1.

Proof of b). Let $\{U_n'\}$ be a base of relatively compact open subsets for X' and $u_0'=h_1^{X'}$. $\hat{R}_{u_0'}^{U_n'}$ is a locally bounded potential on X'. If φ is of type B1, then since $u_0=h_1^X=h_1^{X'}\circ\varphi=u_0'\circ\varphi$,

$$\mathrm{R}_{u_0'}^{U_n'} \circ \varphi \geq \mathrm{R}_{u_0}^{\varphi^{-1}(U_n')}$$

and $\hat{R}_{u_0}^{\varphi^{-1}(U_n')}$ is a potential on X. Therefore $\omega^M(\Delta_1^M \setminus \mathcal{E}_{\varphi^{-1}(U_n')})=0$ ([5], Cor. to Th. 1), i.e., $\varphi^{-1}(U_n')$ is thin $d\omega^M$ —a.e. on Δ^M. Thus, except a set of harmonic measure zero we have

$$\phi(x) \subset X'^* \setminus U_n'$$

and further $\phi(x) \subset \bigcap_{n=1}^{\infty} [X'^* \setminus U_n'] = \Delta'$.

The proof of the "if" part is also immediately derived from Theorem 1.

REMARK 1. If φ: $X \to X'$ is of type B1 and $X' \in \mathcal{P}$, then X' is not compact; for we have always $\phi(x) \neq \phi$, i.e., there exists no harmonic map of type B1 from X into X' with $X' \in \mathcal{P}$ and compact.

REMARK 2. A harmonic function defined on a harmonic sapce X in which constant functions are harmonic can be considered as a harmonic map if we endow $\boldsymbol{R}$ (the set of all real numbers) with the following harmonic structure. A harmonic function on $\boldsymbol{R}$ is defined to be a linear function; any relatively compact open set is regular. The harmonic space with this harmonic structure will supply a somewhat pathological example of harmonic space. For example, if we consider an open interval (a, b), which is a harmonic subspace of $\boldsymbol{R}$, each point of it is non-polar. A bounded harmonic function defined on X, considered as a hamonic map of X into (a, b) is of type B1 if and only if $f=\alpha u+\beta$, where u is a harmonic measure (i.e., H_{χ_A} for some characteristic function χ_A of a boundary set A of Δ^M or, equivalently, $u \wedge (1-u)=0$) and α, β are constants. This is easily derived from Theorem 2.

3. The case of $X' \in \mathcal{P}$

Let $X' \in \mathcal{P}$. We know the following relations among the properties:

(i) $\omega^M(\{x \in \Delta_1^M ; \phi(x) \subsetneq X'\})=0$,

(ii) *φ is of type Bl,*

(iii) $\phi(x) \subset \Delta'$ *$d\omega^M$—a.e.,*

(iv) *for every relatively compact open subset G' of X' with $\bar{G}' \neq X'$ we have* $\omega^M(\mathcal{E}_{X \setminus \varphi^{-1}(G')})=0$;

(i) ⇔ (iv)	(Lemma 2)
(i) ⇒ (ii)	(Theorem 1)
(ii) ⇔ (iii)	(Theorem 2, b))
(iii) ⇒ (iv)	(trivially derived)

Thus:

Theorem 3. *In the case where $X' \in \mathcal{P}$, above properties* (i), (ii), (iii) *and* (iv) *are equivalent.*

Theorem 4. *Let $X'\in\mathcal{P}$. A harmonic map φ: $X\to X'$ is of type Bl if and only if for every potential p' on X' we have*

$$\text{fine lim } p'\circ\varphi = 0 \qquad d\omega^M - a.e..$$

To prove the theorem, we require the lemmas:

Lemma 3. *Let φ: $X\to X'$ be of type Bl and let v be a positive harmonic function on X such that v/u_0 is bounded above, $u_0=h_1^X$. If there exists a non-negative superharmonic function s' on X' satisfying $s'\circ\varphi\geq v$, then the lower envelope of such functions is harmonic on X'.*

Proof. Let U' be a relatively compact open subset of X', $\bar{U}'\neq X'$ and $U=\varphi^{-1}(U')$. By Theorem 3, we have $\omega^M(\mathcal{E}_{X\setminus U})=0$. We shall show that if s'' is a positive superharmonic function on X' such that

$$s''\geq s' \qquad \text{on } X'\setminus U',$$

then $s''\circ\varphi\geq v$. For, $s=s''\circ\varphi-v$ is a superharmonic function on X, s/u_0 is bounded below, $X\setminus U$ is not thin at $d\omega^M$-almost every point of Δ^M and $s\geq s'\circ\varphi -v$ on $X\setminus U$. By the minimum principle ([5], Th. 5), $s\geq 0$ on X and $s''\circ\varphi\geq v$. Thus,

$$\{s;\ \textit{non-negative superharmonic on } X',\ s'\circ\varphi\geq v\}$$

forms the Perron family, which proves the lemma.

Lemma 4. *If fine lim $p'\circ\varphi=0$ $d\omega^M-a.e.$ on Δ^M for every potential p' on X', then $u_0=h_1^X\leq u_0'\circ\varphi$, where $u_0'=h_1^{X'}$.*

Proof. Since $u_0-p\leq 1$ and $1\leq u_0'+p'$ for some potentials p and p' on X and X', respectively, $u_0\leq u_0'\circ\varphi+p'\circ\varphi+p$. $(u_0'\circ\varphi+p'\circ\varphi+p)/u_0$ is a u_0-superharmonic function, bounded below and with fine limit inf ≥ 1 $d\omega^M$—a.e.. Thus fine lim inf $(u_0'\circ\varphi)/u_0\geq 1$ $d\omega^M-a.e.$, since fine lim $p'\circ\varphi/u_0$ and fine lim p/u_0 are both zero $d\omega^M-a.e.$. This implies $u_0'\circ\varphi/u_0\geq \mathcal{D}_{1,u_0}=1$, q.e.d..

Proof of Theorem 4. Suppose that φ is of type B1 and p' is a potential on X'. We shall show that the positive superharmonic function $p'\circ\varphi$ dominates no positive harmonic function v such that v/u_0 is a limit of an increasing sequence of bounded u_0-harmonic functions. For, if $p'\circ\varphi$ dominates such function, then it also dominates a positive harmonic function v with v/u_0 is bounded above. By Lemma 3, we conclude that

$$v' = \inf\ \{s';\ \textit{non-negative superharmonic on } X',\ s'\circ\varphi\geq v\}$$

is harmonic on X'. Since $p' \geq v' \geq 0$, we have $v'=0$, whence $v=0$. This means that $p' \circ \varphi$ is the sum of a potential and a u_0-singular harmonic function w, i.e., inf (w, u_0) is a potential, each of which has a fine limit zero $d\omega^M - a.e.$.

Next, supposing that fine lim $p' \circ \varphi = 0$ whenever p' is a potential on X', we shall show that $\omega^M(\mathcal{E}_{X \setminus \varphi^{-1}(G')}) = 0$ for every relatively compact open set G' of X' with $\bar{G}' \neq X'$. Consider a potential $R^{G'}_{u_0'}$, where $u_0' = h_1^{X'}$. By Lemma 4, $u_0 = h_1^X \leq u_0' \circ \varphi$, whence $R^{\varphi^{-1}(G')}_{u_0} \leq R^{G'}_{u_0'} \circ \varphi$. This implies that $R^{\varphi^{-1}(G')}_{u_0}$ has a fine limit zero at $d\omega^M - a.e.$ on Δ^M and thus $\omega^M(\mathcal{E}_{X \setminus \varphi^{-1}(G')}) = 0$; for otherwise, at each point x of a set of positive harmonic measure $\varphi^{-1}(G')$ is a trace of a fine neighbourhood of x, i.e., the intersection of a fine neighbourhood of x with X. Then fine $\lim_x R^{\varphi^{-1}(G')}_{u_0} =$ fine $\lim_x u_0$ and the latter fine limit is 1 on a set of positive measure, which is absurd, q.e.d..

Theorem 5. *Let φ: $X \to X'$ be a harmonic map, then the following propositions are equivalent*:

(i) *for every $a' \in X'$ we have* fine lim $[g'_{a'} \circ \varphi] = 0$ $d\omega^M - a.e.$, *where $g'_{a'}$ is a potential on X' with single point support* $\{a'\}$.

(ii) *φ is of type Bl,*

(iii) *for every potential p' of X' we have* fine lim $[p' \circ \varphi] = 0$ $d\omega^M - a.e.$.

Proof. From Theorem 4, it is sufficient to prove that (i) implies (ii). Suppose φ is not of type B1. By Theorem 2, there exists a boundary set $A \subset \Delta_1^M$ of positive harmonic measure such that $\hat{\varphi}(x) \cap X' \neq \phi$ for every $x \in A$. Let $\{G_n'\}$ be a covering of X' by relatively compact open sets. Denoting by

$$A_n = \{x \in A;\ \hat{\varphi}(x) \cap G_n' \neq \phi\}$$

we have $\bigcup_{n=1}^{\infty} A_n = A$. Thus $\omega^M(A_{n_0}) > 0$ for some n_0. $\varphi^{-1}(G'_{n_0})$ is not thin at every point x of A_{n_0}. Select a point $a' \in G'_{n_0}$ and form a potential $g'_{a'}$ with support $\{a'\}$. Since G'_{n_0} is relatively compact, $\inf_{G'_{n_0}} g'_{a'} > 0$, whence $g'_{a'} \circ \varphi$ has a positive fine lim sup at $d\omega^M - a.e.$ point of A_{n_0}, q.e.d..

4. The case of metrizable and resolutive compactifications of X'

In what follows we shall assume that

X'^* is a metrizable and resolutive compactification of X'.

In [4], Doob characterized the point at which an analytic map of a hyperbolic Riemann surface into another Riemann surface is of type B1 in terms of fine cluster sets. In our present case, we know that ([9], Th. 5)

$$\omega^M(\Delta^M \setminus \hat{P} \setminus \hat{F}) = 0\,,$$

where $\hat{F}=\{x\in\Delta_1^M;\ \hat{\varphi}(x)$ *is a single point of* $X'^*\}$ and $\hat{P}=\{x\in\Delta_1^M;\ \hat{\varphi}(x)=X'^*\}$. The *essential closed range* of $\hat{\varphi}$ is defined to be

$$\cap\ \{\overline{\hat{\varphi}(\hat{F}\setminus E)};\ E \text{ is of } d\omega^M\text{-measure zero}\}\ ,$$

where $\hat{\varphi}(\hat{F}\setminus E)=\{\hat{\varphi}(x);\ x\in\hat{F}\setminus E\}$. The following theorem is a generalization of the theorem given by Doob in the probabilistic language.

Theorem 6. *A harmonic map φ: $X\rightarrow X'$ is of type Bl at $a'\in X'$ if and only if a' does not belong to the essential closed range of $\hat{\varphi}$.*

Proof. Suppose, first, φ is not of type B1 at a'. Then, by the definition of type B1 and by Lemma 1, we have $\omega^M(\mathcal{E}_{X\setminus\varphi^{-1}(U')})>0$ for every open neighbourhood U' of a' such that $U'\in\mathcal{P}$. Since $x\in\mathcal{E}_{X\setminus\varphi^{-1}(U')}$ implies $\hat{\varphi}(x)\subset\bar{U}'$, we have $\mathcal{E}_{X\setminus\varphi^{-1}(U')}\subset\hat{F}\ d\omega^M-a.e.$, i.e., $\omega^M(\mathcal{E}_{X\setminus\varphi^{-1}(U')}\setminus\hat{F})=0$. For every E of $d\omega^M$-measure zero, there exists $x_E\in(\mathcal{E}_{X\setminus\varphi^{-1}(U')}\cap\hat{F})\setminus E$; this implies $\hat{\varphi}(x_E)\in\bar{U}'\cap\overline{\hat{\varphi}(\hat{F}\setminus E)}$, i.e., $\bar{U}'\cap\overline{\hat{\varphi}(\hat{F}\setminus E)}\neq\phi$. Since E is arbitrary, $\bar{U}'$ intersects with the essential closed range of $\hat{\varphi}$; whence a' belongs to the essential closed range of $\hat{\varphi}$ since U' may range over a base of open neighbourhoods of a'.

Next, suppose φ is of type B1 at a', then there exists an open neighbourhood U' of a' such that $U'\in\mathcal{P}$ and $\omega^M(\mathcal{E}_{X\setminus\varphi^{-1}(U')})=0$. If $\omega^M(\hat{F})=0$, then there is nothing to prove. Assuming that $\omega^M(\hat{F})>0$, we have $\hat{\varphi}(x)$ is a single point and $\hat{\varphi}(x)\cap(X'^*\setminus U')\neq\phi$ if $x\in\hat{F}\setminus\mathcal{E}_{X\setminus\varphi^{-1}(U')}$. This implies that $\hat{\varphi}(x)\notin U'$ and $a'\notin\overline{\hat{\varphi}(\hat{F}\setminus\mathcal{E}_{X-\varphi^{-1}(U')})}$, and *a fortiori* a' does not belong to the essential closed range of $\hat{\varphi}$.

In the following of this section we shall always suppose that $X'\in\mathcal{P}$.

Lemma 5. *Let φ: $X\rightarrow X'$ be of type Bl. If f' is a resolutive function on $\Delta'=X'^*\setminus X'$, then*

$$f(x)=\begin{cases} f'[\hat{\varphi}(x)] & \textit{whenever } \hat{\varphi}(x) \textit{ is defined and a point of } \Delta' \\ 0 & \textit{elsewhere}\end{cases}$$

is resolutive and

$$H_{f'}\circ\varphi=H_f\,.$$

Proof. First of all, we shall show that f is u_0-resolutive and

$$\mathcal{D}_{f',u_0'}\circ\varphi=\mathcal{D}_{f,u_0}\,,$$

where $u_0'=h_1^{X'}$ and $u_0=h_1^X$. By Theorem 2, $f(x)$ is defined $d\omega^M$-*a.e.* on Δ^M. We have ([5])

$$\overline{\mathcal{D}}_{f,u_0}=\inf\left\{w/u_0;\ \begin{array}{l} w/u_0 \text{ is a } u_0\text{-hyperharmonic function, bounded below,} \\ \text{fine lim inf } (w/u_0)\geq f \text{ on } \Delta^M \text{ except on a set of} \\ d\omega^M\text{-harmonic measure zero}\end{array}\right\}$$

Let v'/u_0' be a u_0'-hyperharmonic function on X' satisfying that v'/u_0' is bounded below and lim inf $(v'/u_0')\geq f'$ on Δ'. Since φ is of type B1, $u_0'\circ\varphi=u_0$ and $(v'\circ\varphi)/u_0$ is in the class defining the lower envelope $\overline{\mathcal{D}}_{f,u_0}$, whence $\overline{\mathcal{D}}_{f',u_0'}\circ\varphi\geq\overline{\mathcal{D}}_{f,u_0}$. Similarly, we have $\underline{\mathcal{D}}_{f',u_0}\circ\varphi\leq\underline{\mathcal{D}}_{f,u_0}$, which proves the assertion, since f' is u_0'-resolutive (§1).

Next, we proceed to the ordinary Dirichlet solutions. From the result stated in §1, we have

$$H_{f'}[\varphi(a)]=u_0'[\varphi(a)]\mathcal{D}_{f',u_0'}[\varphi(a)]=u_0(a)\mathcal{D}_{f,u_0}(a)=H_f(a)\,,\quad \text{q.e.d.}.$$

Theorem 7. *A harmonic map φ: $X\rightarrow X'$ is of type Bl if and only if for every Borel set A' of Δ' we have*

$$\omega'(A',\varphi(a))=\omega^M(\phi^{-1}(A'),a)\qquad \textit{and } h_1^{X'}\circ\varphi=h_1^X\,,$$

where $\phi^{-1}(A')=\{x\in\Delta_1^M;\ \phi(x)$ is a point of $A'\}$, and ω' is a harmonic measure on Δ'.

Proof. The proof of the "only if" part is an immediate consequence of the above lemma. We proceed to the proof of the "if" part. Denoting by $u_0'=h_1^{X'}$ and $u_0=h_1^X$, $\omega^M(\Delta^M,a)=u_0(a)=u_0'[\varphi(a)]=\omega'(\Delta',\varphi(a))=\omega^M(\phi^{-1}(\Delta'),a)$ implies that $\phi(x)$ is a point of Δ' $d\omega^M$-*a.e.*, whence φ is of type B1 by Theorem 2, b, q.e.d..

Corollary. *If φ: $X\rightarrow X'$is of type Bl and X'^* is a metrizable and resolutive compactification of X', then $\phi(x)$ is a point of $\Delta'=X'^*\setminus X'$ $d\omega^M$-a.e. on Δ^M and the set formed by these points $\phi(x)$ covers Δ' except on a set of harmonic measure zero.*

In a metrizable and resolutive compactification X'^* of X' a u_0'-quasi-bounded harmonic function u' (a harmonic function which is the limit of an increasing sequence of harmonic functions w' such that w'/u_0' is bounded) is expressed by some boundary function f' as $u'=H_{f'}$. In connection with this we have:

Theorem 8. *Let φ: $X\rightarrow X'$ be of type Bl. If s' is a positive superharmonic function on X', then $s'\circ\varphi$ has a fine limit $f'[\phi(x)]$ at $d\omega^M$-a.e. point of Δ^M, where f' is a boundary function expressing the u_0'-quasi-bounded harmonic part of s'.*

Proof. Let us decompose s' into $s'=p'+u_B'+u_S'$, where p' is a potential, u_B' is a u_0-quasi-bounded harmonic function and u_S' is a u_0-singular harmonic function, and $u_B'=H_{f'}$. By Theorem 4, fine lim $p'\circ\varphi=0$ $d\omega^M$-*a.e.*. Since inf (u_S',u_0') is also a potential, fine lim $[\inf(u_S',u_0')\circ\varphi]=0$ $d\omega^M$-*a.e.*, which implies that fine lim $s'\circ\varphi=f'[\phi(x)]\,d\omega^M$-*a.e.*, q.e.d..

Lemma 6. *Let φ: $X\rightarrow X'$ be of type Bl, v' a positive harmonic function on*

X' and $v=v'\circ\varphi$. φ is of type Bl of X into X' endowed with quotient harmonic structures by v and v', respectively.

Proof. Let U' be a relatively compact open neighbourhood of $a'\in X'$, $U=\varphi^{-1}(U')$ and $A=\sup_{U'} v'$. A is finite. If s is superharmonic on U, lim inf $s\geq 0$ on ∂U and $s\geq A$ outside of a compact set K, then lim inf $s/v\geq 0$ on ∂U and $s/v\geq 1$ outside K, thus $s/v\geq \bar{d}_1^{U,X}$, where $\bar{d}_1^{U,X}$ denotes the upper harmonization of 1 with respect to the quotient harmonic structure $\mathcal{H}_v$ by v (Cf. §1). Since φ is of type B1, $\bar{h}_1^{U,X}=0$. From this it follows that

$$0=(A\bar{h}_1^{U,X})/v\geq \bar{d}_1^{U,X}\geq 0\,,$$

which proves the lemma.

Theorem 9. *Assuming the proportionality axiom for X', if φ: $X\to X'$ is of type Bl and $a'\in X'\setminus\varphi(X)$, then there exists a point $x\in\Delta_1^M$ such that $\phi(x)=a'$.*

Proof. Denoting by $X_0'=X'\setminus\{a'\}$, φ is of type B1 from X into X_0'. Let v' be a potential on X' with support $\{a'\}$, and $v=v'\circ\varphi$. v' is a positive harmonic function on X_0'. We consider harmonic spaces $(X, \mathcal{H}_v)$ and $(X_0', \mathcal{H}_{v'})$ endowed with quotient harmonic structures by v and v', respectively. φ is of type B1 from $(X, \mathcal{H}_v)$ into $(X_0', \mathcal{H}_{v'})$. We shall apply our preceding result to these spaces. In a suitable metrizable and resolutive compactification of $(X_0', \mathcal{H}_{v'})$, the point a' is restored and the new boundary point a' yields a v'-harmonic function 1 as the Dirichlet solution, i.e., $\mathcal{D}_{\chi_{\{a'\}},v'}=1$. Thus $\{a'\}$ is of v'-harmonic measure positive and the theorem is derived from Corollary to Theorem 7 since the Martin space X^M is homeomorphic to $X^{M,v}$ (Cf. §1).

5. The case of the Martin compactification X'^M of X'

In the following, we shall suppose that X' satisfies the same conditions as X. Then, the Martin compactification X'^M of X' is defined.

Theorem 10. *Let φ: $X\to X'$ be of type Bl, v' a positive harmonic function on X' and $v=v'\circ\varphi=\int k_x d\mu_v(x)$. Then, for $d\mu_v$-a.e. point x $\phi(x)\in\Delta'^M=X'^M\setminus X'$.*

In particular, if v' is a minimal harmonic function corresponding to $x'\in\Delta_1'^M$, then there exists $x\in\Delta_1^M$ such that $\phi(x)=x'$.

In view of the quotient sheaves (§1), this is an immediate consequence of Corollary to Theorem 7. Combining this with Theorem 9 we have ([4], Th. 8.2)

Corollary. *Let φ: $X\to X'$ be of type Bl. If either $x'\in X'\setminus\varphi(X)$ or $x'\in\Delta_1'^M$, then there exists $x\in\Delta_1^M$ such that $\phi(x)=x'$.*

Lemma 7. *If φ: $X\to X'$ is of type Bl and U' is an open subset of X', then*

$$\mathrm{R}_{u_0'}^{X'\setminus U'}\circ\varphi = \mathrm{R}_{u_0'\circ\varphi}^{X\setminus\varphi^{-1}(U')} .$$

where $u_0' = h_1^{X'}$.

Proof. It is obvious that

$$\mathrm{R}_{u_0'}^{X'\setminus U'}\circ\varphi \geq \mathrm{R}_{u_0'\circ\varphi}^{X\setminus\varphi^{-1}(U')} .$$

To prove the converse inequality, we recall that

$$\bar{h}_{u_0'}^{U',U'} = u_0' = \bar{h}_{u_0'}^{U',X'} + \mathrm{H}_{u_0'}^{U',X'} = \bar{h}_{u_0'}^{U',X'} + \mathrm{R}_{u_0'}^{X'\setminus U'} \qquad \text{in } U'.$$

Since φ is of type B1, we have

$$u_0'\circ\varphi - \mathrm{R}_{u_0'}^{X'\setminus U'}\circ\varphi = \bar{h}_{u_0'}^{U',X'}\circ\varphi \geq \bar{h}_{u_0'\circ\varphi}^{\varphi^{-1}(U),X} = u_0'\circ\varphi - \mathrm{R}_{u_0'\circ\varphi}^{X\setminus\varphi^{-1}(U')} ,$$

which proves

$$\mathrm{R}_{u_0'\circ\varphi}^{X\setminus\varphi^{-1}(U')} \geq \mathrm{R}_{u_0'}^{X'\setminus U'}\circ\varphi .$$

Theorem 11. *Let* φ: $X\rightarrow X'$ *be of type Bl. If for each* $a'\in X'$, $\varphi^{-1}(a')$ *contains at most* $n(<+\infty)$ *points of* X', *then for every* $x'\in\Delta_1'^M$, *there exist at least one and at most n points* $x_1, x_2, \cdots, x_l \in \Delta_1^M$ *such that*

1) $\phi(x_i) = x' \qquad 1\leq i\leq l$,
2) φ *maps every open neighbourhood of* x_i *onto a deleted fine neighbourhood of* x',
3) $k_x'\circ\varphi = \sum_{i=1}^{l} c_i\, k_{x_i}$, *where* $c_i>0$, $1\leq i\leq l$.

Moreover, if x' *is of positive harmonic measure, then each harmonic measure of* $\{x_i\}$ *is positive.*

Proof. Consider quotient sheaves by $k'_{x'}$ and $k'_{x'}\circ\varphi$ on X' and X, respectively. φ is also of type B1 with respect to these new quotient sheaves. Then x' is of harmonic measure 1, hence $\phi^{-1}(x')$ is of harmonic measure 1. We write

$$k'_{x'}\circ\varphi = \int k_x\, d\tilde{\mu}$$

and define

$$A = \{x\in\Delta_1^M,\ x \text{ has an open neighbourhood } U(x) \text{ with } \tilde{\mu}(U(x)\cap\Delta_1^M) = 0\} .$$

We note that $\phi^{-1}(x')\setminus A \neq \phi$; this is an immediate consequence of the Lindelöf property of metric space and $\tilde{\mu}(\phi^{-1}(x'))>0$. Let $x\in\phi^{-1}(x')\setminus A$, $U(x)$ an open neighbourhood of x. Then $X'\setminus\varphi(U(x)\cap X)$ is thin at x'; in fact, otherwise we would have

$$\begin{aligned}\int k_x\, d\tilde{\mu} &= k'_{x'}\circ\varphi = \mathrm{R}_{k'_{x'}}^{X'\setminus\varphi(U(x)\cap X)}\circ\varphi \\ &= \mathrm{R}_{k'_{x'}\circ\varphi}^{X\setminus\varphi^{-1}[\varphi(U(x)\cap X)]} \text{(byLemma 7, since } \varphi \text{ is an open map [3], Th. 3.3)} \\ &\leq \mathrm{R}_{k'_{x'}\circ\varphi}^{X\setminus U(x)} = \int \mathrm{R}_{k_x}^{X\setminus U(x)}\, d\tilde{\mu}\,.\end{aligned}$$

This would imply that

$$\tilde{\mu}(\{x\in\Delta_1^M;\ X\setminus U(x) \text{ is thin at } x\}) = 0\,.$$

On the other hand, since $X\setminus U(x)$ is thin at each point of $U(x)\cap\Delta_1^M$, $\tilde{\mu}(U(x)\cap\Delta_1^M)=0$, which would contradict with $x\notin A$.

Next, we shall show that $\phi^{-1}(x')\setminus A$ contains at most n points; let $x_i\in\phi^{-1}(x')\setminus A$, $i=1, 2, \cdots, k$; $k>n$, and let $\{U_i\}$ be a system of neighbourhoods which are mutually disjoint and $x_i\in U_i$, $i=1, 2, \cdots, k$. Since

$$\bigcup_{i=1}^{k}[X'\setminus\varphi(U_i\cap X)] = X'\setminus\bigcap_{i=1}^{k}\varphi(U_i\cap X)$$

is thin at x', there exists a points $a'\in\bigcap_{i=1}^{k}\varphi(U_i\cap X)$. Thus $\varphi^{-1}(a')\cap U_i\neq\phi$, $1\leq i\leq k$, which conflicts the hypothesis of the theorem. From the above consideration we conclude that

$$\tilde{\mu} = \sum_{i=1}^{l} c_i\, \varepsilon_{x_i}\,,$$

which means also that

$$k'_{x'}\circ\varphi = \sum_{i=1}^{l} c_i\, k_{x_i}\,.$$

It remains to prove the last assertion of the theorem, but now it is easily derived from above argument, q.e.d. .

6. Mappings of type Bl_1

In case of an analytic mapping of Riemann surface, Heins [6] defined a more restricted class of maps called of type $\mathrm{B1}_1$. In this section, we concern ourselves with a generalization of this class in our present setting. In view of Theorem 3, we shall define that a harmonic map φ: $X\to X'$ is *of type* Bl_1 if for every relatively compact open set G' with $\bar{G}'\neq X'$ we have $\mathcal{E}_{X\setminus\varphi^{-1}(G')}=\phi$, i.e., $X\setminus\varphi^{-1}(G')$ is not thin at every point of Δ_1^M. A map of type $\mathrm{B1}_1$ is obviously of type B1. We have:

Theorem 12 ([1], Satz 27). *Let φ be a harmonic map of X into X', $X'\in\mathcal{P}$ and X'^* be an arbitrary compactification of X', then φ is of type Bl_1 if and only if $\phi(x)\cap\Delta'\neq\phi$ for every $x\in\Delta_1^M$.*

In particular, if $\phi(x)\subset\Delta'$ for every $x\in\Delta_1^M$, then φ is of type Bl_1.

The proof of this theorem is carried out easily. We shall omit it.

Finally, we shall prove the following theorem, which will justify our definition.

Theorem 13. *Let φ be a harmonic map of X into X' and $X' \in \mathcal{P}$. If φ is of type Bl_1, then $p' \circ \varphi$ is a potential on X for every potential p' on X'.*

Proof. Let G' be a relatively compact open set with $\bar{G}' \neq X'$ and p' be a potential on X'. We decompose $p' \circ \varphi$ into the harmonic part w and the potential part p:

$$p' \circ \varphi = w + p .$$

Since $X \setminus \varphi^{-1}(G')$ is not thin at every point x of Δ_1^M, we have

$$R_{k_x}^{X \setminus \varphi^{-1}(G')} = k_x$$

for every $x \in \Delta_1^M$, where k_x is a suitably normalized positive harmonic function corresponding to x. Then

$$R_w^{X \setminus \varphi^{-1}(G')} = \int R_{k_x}^{X \setminus \varphi^{-1}(G')}\, d\mu_w = \int k_x\, d\mu_w = w ,$$

where $w = \int k_x\, d\mu_w$ is the canonical integral representation of w. We have ([7], Prop. 10.1)

$$R_{p'}^{X' \setminus G'} \circ \varphi \geq R_{p' \circ \varphi}^{X \setminus \varphi^{-1}(G')} = R_p^{X \setminus \varphi^{-1}(G')} + R_w^{X \setminus \varphi^{-1}(G')} \geq w .$$

Since G' is an arbitrary relatively compact open set we have $w=0$, therefore $p' \circ \varphi$ is a potential, q.e.d. .

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