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MARTINGALE EXPANSION FOR STOCHASTIC VOLATILITY

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ABSTRACT. The martingale expansion provides a refined approximation to the marginal distributions of martingales beyond the normal approximation implied by the martingale central limit theorem. We develop a martingale expansion framework specifically suited to continuous stochastic volatility models. Our approach accommodates both small volatility-of-volatility and fast mean-reversion models, yielding first-order perturbation expansions under essentially minimal conditions.

1. INTRODUCTION

Stochastic volatility (SV) models constitute a central class of continuous-time asset price models in which the instantaneous variance of returns is itself governed by an additional stochastic process. Unlike the classical Black–Scholes framework, which assumes constant volatility, SV models allow volatility to evolve randomly over time, thereby capturing a range of empirically observed features of financial markets, including volatility clustering, heavy-tailed return distributions, and pronounced implied-volatility smiles and skews.

Formally, an SV model specifies the joint dynamics of an asset price process S and its latent spot variance process V , typically through a system of coupled stochastic differential equations. The spot variance process V is often mean-reverting and may be correlated with the asset-price shocks, a feature that enables the model to reproduce leverage effects observed in equity markets. Prominent examples include the Heston model, in which V follows a square-root diffusion, and the SABR model, widely used in interest-rate markets due to a closed form approximation formula for the implied volatility.

The flexibility afforded by stochastic volatility has made SV models indispensable in modern derivative pricing and risk management. They provide a more realistic representation of market dynamics and yield option prices that align more closely with observed implied-volatility surfaces. At the same time, the introduction of a latent volatility factor complicates both analytical tractability and statistical inference, motivating a substantial literature on approximation methods, asymptotic expansions, and efficient numerical schemes.

Consider an abstract SV model with zero interest rates

$$(1) \quad \frac{dS_t}{S_t} = \sqrt{V_t} dB_t, \quad B_t = \rho W_t + \sqrt{1 - \rho^2} W_t^\perp$$

for an asset price process S , where $(W, W^\perp)$ is a 2-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\})$, $\rho \in (-1, 1)$, and V is a nonnegative cadlag process adapted to a smaller filtration $\{\mathcal{G}_t\}$ to which W is also

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adapted while $W^\perp$ is independent. Suppose that $V = V^\epsilon$ depends on a parameter $\epsilon > 0$ and as $\epsilon \rightarrow 0$,

$$\int_0^T V_t^\epsilon dt - v^\epsilon$$

converges to 0 in probability for a deterministic positive sequence v^ϵ with limit

$$(2) \quad v^0 := \lim_{\epsilon \rightarrow 0} v^\epsilon > 0.$$

Then $S = S^\epsilon$ also depends on ϵ and by the martingale central limit theorem, $\log S_T = \log S_T^\epsilon$ converges in law to the normal distribution with mean $-v^0/2$ and variance v^0 . The model (1) with small $\epsilon > 0$ is close to the Black-Scholes model in this sense.

Since SV models generally do not admit closed-form expressions for derivative prices or hedging strategies, asymptotic expansions with respect to ϵ for various models of V^ϵ have been extensively developed for both practical implementation and theoretical analysis. Examples include small volatility-of-volatility models, e.g. [14, 5, 1, 4, 3], where ϵ is the volatility parameter of V^ϵ , and fast-mean-reverting models, e.g. [8, 10, 13], where V^ϵ is ergodic and a certain negative power of ϵ is the time scale parameter of V^ϵ .

In [9], the author introduced a unified framework for computing and validating first-order asymptotic expansions, based on Yoshida's martingale expansion theory [17, 18] combined with a partial Malliavin calculus. The framework in [9] is even more abstract than (1), in particular incorporating jumps and time-dependent ρ . In the present paper, we focus on the basic model (1) to propose an alternative approach that is more direct and elementary, which enables us to avoid technical moment conditions required in [9], and to establish the validity of the first-order expansion under weaker and essentially minimal conditions.

The martingale expansion was first formulated by Mykland [15, 16] for twice continuously differentiable test functions using Itô's formula, and then by Yoshida [17, 18] for a general test function under a non-degeneracy condition on Malliavin covariances. In [9], we relied on the condition $|\rho| < 1$ that provides a smoothness of the distribution of S_T , admitting an effective application of the partial Malliavin calculus to ensure the required non-degeneracy. In the present paper, we directly utilize a smoothness property due to $|\rho| < 1$ to bypass the Malliavin calculus.

2. RESULTS

Here we state the main results of the paper. All the proofs are deferred to Section 4. Let $T > 0$ be fixed and

$$(X^\epsilon, Y^\epsilon) = \left(\frac{1}{\sqrt{v^\epsilon}} \int_0^T \frac{dS_t^\epsilon}{S_t^\epsilon}, \frac{1}{\epsilon} \left(\int_0^T V_t^\epsilon dt - v^\epsilon \right) \right)$$

for a given sequence v^ϵ with (2). Let ϕ denote the standard normal density function.

Theorem 2.1. *Assume (1) for $(S, V) = (S^\epsilon, V^\epsilon)$ and (2). If Y^ϵ is uniformly integrable and if (X^ϵ, Y^ϵ) converges in law as $\epsilon \rightarrow 0$ to such (X, Y) that $x \mapsto \mathbb{E}[Y|X = x]$ is twice continuously differentiable with*

$$(3) \quad \lim_{|x| \rightarrow \infty} \mathbb{E}[Y|X = x]\phi(x) = \lim_{|x| \rightarrow \infty} \frac{d}{dx}(\mathbb{E}[Y|X = x]\phi(x)) = 0$$

and that

$$\phi^\epsilon(x) := \phi(x) + \frac{\epsilon}{2\sqrt{v^\epsilon}} \frac{d}{dx} (\mathbb{E}[Y|X=x]\phi(x)) + \frac{\epsilon}{2v^\epsilon} \frac{d^2}{dx^2} (\mathbb{E}[Y|X=x]\phi(x))$$

is integrable on $\mathbb{R}$ for each $\epsilon > 0$, then for any bounded Borel function f ,

$$(4) \quad \mathbb{E}[f(S_T^\epsilon)] = \int f\left(S_0 \exp\left\{\sqrt{v^\epsilon}x - \frac{v^\epsilon}{2}\right\}\right) \phi^\epsilon(x) dx + o(\epsilon).$$

See Section 4.1 for the proof.

Remark 2.2. By the martingale central limit theorem, the martingale marginal X^ϵ converges to the standard normal distribution if $\epsilon Y^\epsilon \rightarrow 0$. Therefore, X is always a standard normal random variable under the assumption of Theorem 2.1. Although Y is not necessarily Gaussian in general, for both small volatility-of-volatility and fast-mean-reverting models, by suitably choosing v^ϵ , it is straightforward to observe that (X^ϵ, Y^ϵ) converges in law to a 2-dimensional centered normal distribution, as illustrated in [9] and Section 3 below. In such a case, Y is equal in law to $\mathbb{E}[XY|X] + Z$ with a centered normal random variable Z with $\mathbb{E}[XZ] = 0$. In particular, $\mathbb{E}[Y|X=x] = \mathbb{E}[XY|x]$, and (3) is then trivial.

Remark 2.3. Theorem 2.1 provides an asymptotic expansion of the European option price. From the expression of the log price

$$(5) \quad \log S_T^\epsilon = \log S_0 + \sqrt{v^\epsilon} X^\epsilon - \frac{1}{2}(v^\epsilon + \epsilon Y^\epsilon),$$

it is natural to have

$$(6) \quad \int f\left(S_0 \exp\left\{\sqrt{v^\epsilon}x - \frac{v^\epsilon}{2}\right\}\right) \phi(x) dx$$

as the leading term of the expansion (4), which corresponds to the Black-Scholes price of the payoff $f(S_T)$ with volatility $\sqrt{v^\epsilon}/T$. The additional term of $O(\epsilon)$ refines this Black-Scholes approximation.

Corollary 2.4. Under the assumption of Theorem 2.1, for any $K > 0$,

$$(7) \quad \begin{aligned} \mathbb{E}[(K - S_T^\epsilon)_+] &= p_K(S_0, v^\epsilon) + \frac{\epsilon}{2\sqrt{v^\epsilon}} K \mathbb{E}[Y|X = -d_-(S_0, v^\epsilon)] \phi(-d_-(S_0, v^\epsilon)) + o(\epsilon) \\ &= p_K(S_0, v^\epsilon) + \epsilon \frac{\partial p_K}{\partial t}(S_0, v^\epsilon) \mathbb{E}[Y|X = -d_-(S_0, v^\epsilon)] + o(\epsilon) \\ &= p_K(S_0, v^\epsilon + \epsilon \mathbb{E}[Y|X = -d_-(S_0, v^\epsilon)]) + o(\epsilon), \end{aligned}$$

where

$$(8) \quad \begin{aligned} p_K(s, t) &= K\Phi(-d_-(s, t)) - s\Phi(-d_+(s, t)), \\ d_\pm(s, t) &= \frac{1}{\sqrt{t}} \left(\log \frac{s}{K} \pm \frac{t}{2} \right) \end{aligned}$$

and Φ is the standard normal distribution function.

See Section 4.2 for the proof.

Remark 2.5. Note that $p_K(s, t)$ in (8) is the Black-Scholes put option price function with total variance t . The last expression of (7) implies that the Black-Scholes implied total variance $\hat{v}(K)$, which is defined by

$$(9) \quad \mathbb{E}[(K - S_T^\epsilon)_+] = p_K(S_0, \hat{v}(K)),$$

is expanded as

$$(10) \quad \hat{v}(K) = v^\epsilon + \epsilon \mathbb{E}[Y|X = -d_-(S_0, v^\epsilon)] + o(\epsilon).$$

Although this is valid for any $K > 0$ as an asymptotic formula, taking into account how it is derived, we can expect the $o(\epsilon)$ term in (10) to be actually small for a given $\epsilon > 0$ only when

$$\frac{\partial p_K}{\partial t}(S_0, v^\epsilon) = \frac{K}{2\sqrt{v^\epsilon}} \phi(-d_-(S_0, v^\epsilon)) = \frac{S_0}{2\sqrt{v^\epsilon}} \phi(-d_+(S_0, v^\epsilon))$$

is not too small, which is the case when K is near S_0 , i.e., around the at-the-money. Since (X, Y) is the limit of (X^ϵ, Y^ϵ) , and so, is also the limit of

$$\left(\frac{1}{\sqrt{v^\epsilon}} \left(\log \frac{S_T^\epsilon}{S_0} + \frac{v^\epsilon}{2} \right), Y^\epsilon \right)$$

due to (5), by formally replacing (X, Y) in (10) with the above, we reach a conceptually interesting formula

$$\hat{v}(K) \approx \mathbb{E} \left[\int_0^T V_u^\epsilon du \middle| S_T^\epsilon = K \right].$$

A rigorous validation of this formula is left for future research.

Remark 2.6. As is well-known, differentiating the defining equation (9) of the implied total variance by log moneyness, we reach a volatility skew formula

$$\frac{\partial}{\partial k} \sqrt{\hat{v}(S_0 e^k)} = \frac{\mathbb{P}[S_0 e^k > S_T^\epsilon] - \Phi(-d_-(S_0, \hat{v}(S_0 e^k)))}{\phi(-d_-(S_0, \hat{v}(S_0 e^k)))}.$$

Here, the derivative exists because under the current assumption of $|\rho| < 1$, $\log S_T^\epsilon$ follows a mixed normal distribution and in particular admits a density. We have asymptotic expansions for both $\mathbb{P}[K > S_T^\epsilon]$ and $\hat{v}(K)$ respectively from (4) with $f(s) = 1_{(-\infty, K)}(s)$ and (10). Substituting those, we obtain

$$\frac{\partial}{\partial k} \sqrt{\hat{v}(S_0 e^k)} = \frac{\epsilon}{2\sqrt{v^\epsilon}} \frac{\partial}{\partial k} \mathbb{E} \left[Y \middle| X = \frac{k}{\sqrt{v^\epsilon}} + \frac{\sqrt{v^\epsilon}}{2} \right] + o(\epsilon).$$

In the case $\mathbb{E}[Y|X = x] = \mathbb{E}[XY]x$ mentioned in Remark 2.2, we have

$$(11) \quad \frac{\partial}{\partial k} \sqrt{\hat{v}(S_0 e^k)} = \frac{\epsilon}{2v^\epsilon} \mathbb{E}[XY] + o(\epsilon).$$

Remark 2.7. In [9], a more abstract framework is given, which in particular incorporates jumps and time-dependent ρ . For a continuous model (1), it however requires the L^p boundedness of (X^ϵ, Y^ϵ) for all $p > 0$, which is far stronger than the uniform integrability of Y^ϵ assumed in Theorem 2.1.

3. EXAMPLE

In [9], we have already observed how the martingale expansion framework accommodates various perturbation models including small volatility-of-volatility and fast-mean-reverting models. Here, for an illustrative purpose, we take the small volatility-of-volatility expansion of a Bergomi-type model as an example.

Consider

$$V_t^\epsilon = V_0(t) \exp \left\{ \epsilon \sum_{i=1}^d \int_0^t k_i(t-s) dW_s^i - \frac{\epsilon^2}{2} \sum_{i=1}^d \int_0^t k_i(t-s)^2 ds \right\}$$

with a small volatility-of-volatility parameter $\epsilon > 0$, where $V_0(t)$ is a deterministic positive continuous function, $(W^1, \dots, W^d)$ is a d -dimensional standard Brownian motion correlated with B in (1) as

$$d\langle B, W^i \rangle_t = \rho_i dt, \quad \rho := \sqrt{\sum_{i=1}^d \rho_i^2} < 1$$

and k_i are locally square integrable functions on $[0, \infty)$.

The function $t \mapsto V_0(t)$ is called the forward variance curve (at time 0), due to $V_0(t) = \mathbb{E}[V_t^\epsilon]$. The case $k_i(t) = a_i e^{-b_i t}$, $a_i, b_i > 0$, describes the Bergomi model (see [6]). The case $d = 1$ with $k_1(t) = at^{H-1/2}$, $a > 0$, $H \in (0, 1/2)$, describes the rough Bergomi model proposed by [3]. In [7, 11, 12], short-time (non-small volatility-of-volatility) expansions of the implied volatility and skew for this type of model are given. We refer the reader to Section 8.3 of [2] for the difference between short-time and small volatility-of-volatility expansions.

Consistently to (1), we have a decomposition

$$B = \rho W + \sqrt{1 - \rho^2} W^\perp, \quad W = \frac{1}{\rho} \sum_{i=1}^d \rho_i W^i$$

with $W^\perp$ being a standard Brownian motion independent of $(W^1, \dots, W^d)$. Here, we interpret $B = W^\perp$ when $\rho = 0$. We take as $\{\mathcal{G}_t\}$ the natural filtration generated by $(W^1, \dots, W^d)$. By taking

$$v^\epsilon = \mathbb{E} \left[\int_0^T V_t^\epsilon dt \right] = \int_0^T V_0(t) dt,$$

we have

$$(X^\epsilon, Y^\epsilon) = \left(\frac{1}{\sqrt{v^\epsilon}} \int_0^T \sqrt{V_0(s)} dB_s, \int_0^T V_0(s) \int_0^s \sum_{i=1}^d k_i(s-u) dW_u^i ds \right) + o_p(1)$$

as $\epsilon \rightarrow 0$, where $o_p(1)$ refers to a term converging to 0 in probability. The limit

$$(X, Y) := \left(\frac{1}{\sqrt{v^\epsilon}} \int_0^T \sqrt{V_0(s)} dB_s, \int_0^T V_0(s) \int_0^s \sum_{i=1}^d k_i(s-u) dW_u^i ds \right)$$

is centered normal with covariance

$$\begin{aligned} \mathbb{E}[XY] &= \mathbb{E} \left[\frac{1}{\sqrt{v^\epsilon}} \int_0^T \sqrt{V_0(s)} dB_s \int_0^T V_0(s) \int_0^s \sum_{i=1}^d k_i(s-u) dW_u^i ds \right] \\ &= \frac{1}{\sqrt{v^\epsilon}} \sum_{i=1}^d \mathbb{E} \left[\int_0^T \sqrt{V_0(s)} dB_s \int_0^T \int_s^T V_0(u) k_i(u-s) du dW_s^i \right] \\ &= \frac{1}{\sqrt{v^\epsilon}} \int_0^T \sqrt{V_0(s)} \int_s^T V_0(u) \sum_{i=1}^d \rho_i k_i(u-s) du ds. \end{aligned}$$

The uniform integrability of Y^ϵ can be easily shown by observing its boundedness in L^2 . Since $E[Y|X = x] = E[XY]x$, all the assumptions of Theorem 2.1 are satisfied. We have in particular (11) with $E[XY]$ given above.

4. PROOFS

4.1. Proof of Theorem 2.1. First we illustrate the idea of the proof by assuming f to be smooth enough. We start with a well-known conditional pricing formula using the independence between $W^\perp$ and $\mathcal{G}_T$:

$$E[f(S_T^\epsilon)] = E[E[f(S_T^\epsilon)|\mathcal{G}_T]] = E\left[p\left(S_0\mathcal{E}_T\left(\rho\int_0^T\sqrt{V_u^\epsilon}dW_u\right), (1-\rho^2)\int_0^TV_u^\epsilon du\right)\right],$$

where $\mathcal{E}$ denotes the stochastic exponential and

$$(12) \quad p(s, t) = \int f\left(s \exp\left\{\sqrt{t}x - \frac{t}{2}\right\}\right)\phi(x)dx.$$

Noting that

$$(1-\rho^2)\int_0^TV_u^\epsilon du = v^\epsilon - \rho^2\int_0^TV_u^\epsilon du + \epsilon Y^\epsilon,$$

we should have

$$\begin{aligned} E[f(S_T^\epsilon)] &= E\left[p\left(S_0\mathcal{E}_T\left(\rho\int_0^T\sqrt{V_u^\epsilon}dW_u\right), v^\epsilon - \rho^2\int_0^TV_u^\epsilon du\right)\right] \\ &\quad + \epsilon E\left[Y^\epsilon \frac{\partial p}{\partial t}\left(S_0\mathcal{E}_T\left(\rho\int_0^T\sqrt{V_u^\epsilon}dW_u\right), v^\epsilon - \rho^2\int_0^TV_u^\epsilon du\right)\right] + o(\epsilon) \end{aligned}$$

by the Taylor expansion. Using that p solves the partial differential equation

$$(13) \quad \frac{\partial p}{\partial t} = \frac{1}{2}s^2\frac{\partial^2 p}{\partial s^2}, \quad p(s, 0) = f(s),$$

the first term is shown to be $p(S_0, v^\epsilon)$ by Itô's formula, which coincides with the leading term (6) of the expansion. On the other hand, by the assumed convergence $(X^\epsilon, Y^\epsilon) \rightarrow (X, Y)$ and the uniform integrability of Y^ϵ , we have

$$\begin{aligned} &E\left[Y^\epsilon \frac{\partial p}{\partial t}\left(S_0\mathcal{E}_T\left(\rho\int_0^T\sqrt{V_u^\epsilon}dW_u\right), v^\epsilon - \rho^2\int_0^TV_u^\epsilon du\right)\right] \\ &\rightarrow E\left[Y \frac{\partial p}{\partial t}\left(S_0 \exp\left\{\rho Z - \frac{\rho^2}{2}v^\epsilon\right\}, (1-\rho^2)v^0\right)\right] \end{aligned}$$

for a centered normal random variable Z with variance v^0 . Then integration-by-parts with the aid of (13) gives the result. In order to treat a non-smooth function f , below we have to take several localization procedures through steps from i) to viii).

i). Now, we start the proof. Take $\delta \in (0, 1 - \rho^2)$ and let

$$\tau = \inf\left\{t \geq 0; \rho^2 \int_0^t V_s^\epsilon ds \geq (1-\delta)v^\epsilon\right\}.$$

Here we are going to show

$$(14) \quad E[f(S_T^\epsilon)] = E[f(S_{T \wedge \tau}^\epsilon)] + o(\epsilon).$$

Since f is bounded,

$$|\mathbb{E}[f(S_T^\epsilon)] - \mathbb{E}[f(S_{T \wedge \tau}^\epsilon)]| \leq 2\|f\|_\infty \mathbb{P}[\tau \leq T].$$

Then, we obtain (14) from

$$\begin{aligned} \mathbb{P}[\tau \leq T] &= \mathbb{P}[\epsilon \rho^2 Y^\epsilon \geq (1 - \delta - \rho^2)v^\epsilon] \\ (15) \quad &\leq \frac{\epsilon \rho^2}{(1 - \delta - \rho^2)v^\epsilon} \mathbb{E} \left[|Y^\epsilon|; |Y^\epsilon| \geq \frac{(1 - \delta - \rho^2)v^\epsilon}{\epsilon \rho^2} \right], \end{aligned}$$

which is of $o(\epsilon)$ by the uniform integrability of Y^ϵ and (2).

ii). Here we decompose $\mathbb{E}[f(S_{T \wedge \tau}^\epsilon)]$ to extract its leading term (6). Since $W^\perp$ is independent of $\mathcal{G}_T$, we have

$$\mathbb{E}[f(S_{T \wedge \tau}^\epsilon)] = \mathbb{E}[\mathbb{E}[f(S_{T \wedge \tau}^\epsilon) | \mathcal{G}_T]] = \mathbb{E} \left[p \left(\hat{S}_T^\epsilon, (1 - \rho^2) \int_0^{T \wedge \tau} V_s^\epsilon ds \right) \right],$$

where p is defined by (12) and

$$\hat{S}_t^\epsilon = S_0 \exp \left\{ \rho Z_t^\epsilon - \frac{\rho^2}{2} \langle Z^\epsilon \rangle_t \right\}, \quad Z_t^\epsilon = \int_0^{t \wedge \tau} \sqrt{V_s^\epsilon} dW_s.$$

Putting

$$\Sigma_u^\epsilon = v^\epsilon - \rho^2 \int_0^{u \wedge \tau} V_t^\epsilon dt,$$

Itô's formula and (13) give

$$p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon) = p(S_0, v^\epsilon) + \int_0^T \frac{\partial p}{\partial s}(\hat{S}_u^\epsilon, \Sigma_u^\epsilon) d\hat{S}_u^\epsilon,$$

and so, noting $\Sigma_u^\epsilon \geq \delta v^\epsilon > 0$,

$$\mathbb{E}[p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon)] = p(S_0, v^\epsilon).$$

Therefore,

$$\begin{aligned} (16) \quad \mathbb{E}[f(S_{T \wedge \tau}^\epsilon)] &= p(S_0, v^\epsilon) + \mathbb{E} \left[p \left(\hat{S}_T^\epsilon, (1 - \rho^2) \int_0^{T \wedge \tau} V_u^\epsilon du \right) - p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon) \right] \\ &= p(S_0, v^\epsilon) + \mathbb{E} \left[p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon + \epsilon \hat{Y}^\epsilon) - p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon) \right], \end{aligned}$$

where

$$\hat{Y}^\epsilon = \frac{1}{\epsilon} \left(\int_0^{T \wedge \tau} V_u^\epsilon du - v^\epsilon \right) = Y^\epsilon 1_{\{\tau > T\}} + \frac{(1 - \delta - \rho^2)v^\epsilon}{\epsilon \rho^2} 1_{\{\tau \leq T\}}.$$

iii). Here we are going to show that $\hat{Y}^\epsilon$ is uniformly integrable. By (15),

$$\begin{aligned} &\sup_{\epsilon > 0} \mathbb{E} \left[\frac{(1 - \delta - \rho^2)v^\epsilon}{\epsilon \rho^2} 1_{\{\tau \leq T\}}; \frac{(1 - \delta - \rho^2)v^\epsilon}{\epsilon \rho^2} 1_{\{\tau \leq T\}} \geq K \right] \\ &\leq \sup \left\{ \frac{(1 - \delta - \rho^2)v^\epsilon}{\epsilon \rho^2} \mathbb{P}[\tau \leq T]; \frac{(1 - \delta - \rho^2)v^\epsilon}{\epsilon \rho^2} \geq K \right\} \\ &\leq \sup \left\{ \mathbb{E} \left[|Y^\epsilon|; |Y^\epsilon| \geq \frac{(1 - \delta - \rho^2)v^\epsilon}{\epsilon \rho^2} \right]; \frac{(1 - \delta - \rho^2)v^\epsilon}{\epsilon \rho^2} \geq K \right\} \\ &\leq \sup_{\epsilon > 0} \mathbb{E} [|Y^\epsilon|; |Y^\epsilon| \geq K], \end{aligned}$$

which converges to 0 as $K \rightarrow \infty$ by the uniform integrability of Y^ϵ . Therefore, $\hat{Y}^\epsilon$ also is uniformly integrable.

iv). Let

$$A = \left\{ (1 - \rho^2) \int_0^{T \wedge \tau} V_u^\epsilon du > \delta v^\epsilon \right\}.$$

We have

$$\mathbb{P}[A^c] = \mathbb{P} \left[\hat{Y}^\epsilon \leq -\frac{(1 - \delta - \rho^2)v^\epsilon}{\epsilon(1 - \rho^2)} \right] \leq \frac{\epsilon(1 - \rho^2)}{(1 - \delta - \rho^2)v^\epsilon} \mathbb{E} \left[|\hat{Y}^\epsilon|; |\hat{Y}^\epsilon| \geq \frac{(1 - \delta - \rho^2)v^\epsilon}{\epsilon(1 - \rho^2)} \right],$$

which is of $o(\epsilon)$ by the uniform integrability of $\hat{Y}^\epsilon$ and (2). Since f is bounded, so is p . This implies that

$$\mathbb{E} \left[p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon + \epsilon \hat{Y}^\epsilon) - p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon) \right] = \mathbb{E} \left[p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon + \epsilon \hat{Y}^\epsilon) - p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon); A \right] + o(\epsilon)$$

We have

$$p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon + \epsilon \hat{Y}^\epsilon) - p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon) = \epsilon \hat{Y}^\epsilon \int_0^1 \frac{\partial p}{\partial t}(\hat{S}_T^\epsilon, \Sigma_T^\epsilon + r\epsilon \hat{Y}^\epsilon) dr$$

and on the set A ,

$$\Sigma_T^\epsilon + r\epsilon \hat{Y}^\epsilon \geq \delta v^\epsilon$$

for all $r \in [0, 1]$. Combining with the above, we obtain

$$\begin{aligned} & \mathbb{E} \left[p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon + \epsilon \hat{Y}^\epsilon) - p(\hat{S}_T^\epsilon, \Sigma_T^\epsilon) \right] \\ (17) \quad &= \epsilon \mathbb{E} \left[\hat{Y}^\epsilon \int_0^1 \frac{\partial p}{\partial t}(\hat{S}_T^\epsilon, (\Sigma_T^\epsilon + r\epsilon \hat{Y}^\epsilon) \vee (\delta v^\epsilon)) dr; A \right] + o(\epsilon). \end{aligned}$$

This approximation is the first step to compute the $O(\epsilon)$ term.

v). Notice that

$$\frac{\partial p}{\partial t}(s, t) = \int f(se^w) \frac{\partial}{\partial t} \phi \left(\frac{w}{\sqrt{t}} + \frac{\sqrt{t}}{2} \right) dw$$

and because f is bounded,

$$\frac{\partial p}{\partial t}(s, u \vee (\delta v^\epsilon))$$

is bounded in s, u and sufficiently small ϵ under (2).

vi). Since $(X^\epsilon, Y^\epsilon) \rightarrow (X, Y)$ in law, with the aid of i), we have

$$\left(\frac{1}{\sqrt{v^\epsilon}} \int_0^{T \wedge \tau} \sqrt{V_t^\epsilon} dB_t, \hat{Y}^\epsilon \right) \rightarrow (X, Y)$$

in law as $\epsilon \rightarrow 0$. On the other hand,

$$\int_0^{T \wedge \tau} \sqrt{V_t^\epsilon} dB_t = \rho Z_T^\epsilon + \sqrt{(1 - \rho^2)(v^\epsilon + \epsilon \hat{Y}^\epsilon)} N^\epsilon,$$

where N^ϵ is a standard normal random variable independent of $(Z_T^\epsilon, \hat{Y}^\epsilon)$ for each $\epsilon > 0$. Therefore, the sequence of joint distribution

$$\left(\frac{1}{\sqrt{v^\epsilon}} \int_0^{T \wedge \tau} \sqrt{V_t^\epsilon} dB_t, \hat{Y}^\epsilon, Z_T^\epsilon, N^\epsilon \right)$$

is tight, of which an accumulation point is uniquely determined as (X, Y, Z, N) such that N is a standard normal random variable independent of (Y, Z) and

$$(18) \quad \rho Z + \sqrt{(1 - \rho^2)v^0}N = \sqrt{v^0}X.$$

Since X must be a standard normal random variable by the martingale central limit theorem, Z is a centered normal random variable with variance v^0 .

vii). From iii), iv), v) and vi), we have

$$(19) \quad \begin{aligned} & \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\hat{Y}^\epsilon \int_0^1 \frac{\partial p}{\partial t} \left(\hat{S}_T^\epsilon, (\Sigma_T^\epsilon + r\epsilon \hat{Y}^\epsilon) \vee (\delta v^\epsilon) \right) dr; A \right] \\ &= \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\hat{Y}^\epsilon \int_0^1 \frac{\partial p}{\partial t} \left(\hat{S}_T^\epsilon, (\Sigma_T^\epsilon + r\epsilon \hat{Y}^\epsilon) \vee (\delta v^\epsilon) \right) dr \right] \\ &= \mathbb{E} \left[Y \frac{\partial p}{\partial t} (\hat{S}^0, (1 - \rho^2)v^0) \right], \end{aligned}$$

where

$$\hat{S}^0 = S_0 \exp \left\{ \rho Z - \frac{\rho^2}{2} v^0 \right\}.$$

From (14), (16), (17) and (19), we have

$$(20) \quad \mathbb{E}[f(S_T^\epsilon)] = p(S_0, v^\epsilon) + \epsilon \mathbb{E} \left[Y \frac{\partial p}{\partial t} (\hat{S}^0, (1 - \rho^2)v^0) \right] + o(\epsilon).$$

viii). Let $\tilde{W}$ be a standard Brownian motion independent of (Y, Z) and

$$\check{S}_\tau = \hat{S}^0 \exp \left\{ \sqrt{(1 - \rho^2)v^0} \tilde{W}_\tau - \frac{1}{2} (1 - \rho^2)v^0 \tau \right\}, \quad \check{\Sigma}_\tau = (1 - \rho^2)v^0(1 - \tau) + \delta.$$

Then, by Itô's formula and (13),

$$\frac{\partial p}{\partial t}(\check{S}_1, \check{\Sigma}_1) = \frac{\partial p}{\partial t}(\check{S}_0, \check{\Sigma}_0) + \int_0^1 \frac{\partial^2 p}{\partial s \partial t}(\check{S}_\tau, \check{\Sigma}_\tau) d\check{S}_\tau,$$

which implies

$$(21) \quad \mathbb{E} \left[Y \frac{\partial p}{\partial t}(\check{S}_1, \check{\Sigma}_1) \right] = \mathbb{E} \left[Y \frac{\partial p}{\partial t}(\check{S}_0, \check{\Sigma}_0) \right] = \mathbb{E} \left[Y \frac{\partial p}{\partial t}(\hat{S}^0, (1 - \rho^2)v^0 + \delta) \right].$$

Notice that by (18), the joint law of $(\check{S}_1, Y)$ is identical to

$$\left(S_0 \exp \left\{ \sqrt{v^0} X - \frac{v^0}{2} \right\}, Y \right).$$

Therefore, using again (13),

$$\begin{aligned} \mathbb{E} \left[Y \frac{\partial p}{\partial t}(\check{S}_1, \check{\Sigma}_1) \right] &= \frac{1}{2} \mathbb{E} \left[Y \hat{S}_1^2 \frac{\partial^2 p}{\partial s^2}(\check{S}_1, \check{\Sigma}_1) \right] \\ &= \frac{1}{2} \int \mathbb{E}[Y|X = x] \left(S_0 e^{\sqrt{v^0}x - v^0/2} \right)^2 \frac{\partial^2 p}{\partial s^2} \left(S_0 e^{\sqrt{v^0}x - v^0/2}, \delta \right) \phi(x) dx \\ &= \frac{1}{2} \int \mathbb{E}[Y|X = x] \left(\frac{1}{v^0} \frac{\partial^2}{\partial x^2} - \frac{1}{\sqrt{v^0}} \frac{\partial}{\partial x} \right) p \left(S_0 e^{\sqrt{v^0}x - v^0/2}, \delta \right) \phi(x) dx \\ &= \frac{1}{2} \int p \left(S_0 e^{\sqrt{v^0}x - v^0/2}, \delta \right) \left(\frac{1}{v^0} \frac{\partial^2}{\partial x^2} + \frac{1}{\sqrt{v^0}} \frac{\partial}{\partial x} \right) (\mathbb{E}[Y|X = x] \phi(x)) dx. \end{aligned}$$

Here we have used (3). Then, by (21),

$$\begin{aligned}
\mathbb{E} \left[Y \frac{\partial p}{\partial t}(\hat{S}^0, (1 - \rho^2)v^0) \right] &= \lim_{\delta \rightarrow 0} \mathbb{E} \left[Y \frac{\partial p}{\partial t}(\hat{S}^0, (1 - \rho^2)v^0 + \delta) \right] \\
&= \lim_{\delta \rightarrow 0} \frac{1}{2} \int p \left(S_0 e^{\sqrt{v^0}x - v^0/2}, \delta \right) \left(\frac{1}{v^0} \frac{\partial^2}{\partial x^2} + \frac{1}{\sqrt{v^0}} \frac{\partial}{\partial x} \right) (\mathbb{E}[Y|X = x] \phi(x)) dx \\
&= \lim_{\delta \rightarrow 0} \frac{1}{2} \int \int f \left(S_0 e^{\sqrt{v^0}x - v^0/2 + \sqrt{\delta}z - \delta/2} \right) \left(\frac{1}{v^0} \frac{\partial^2}{\partial x^2} + \frac{1}{\sqrt{v^0}} \frac{\partial}{\partial x} \right) (\mathbb{E}[Y|X = x] \phi(x)) dx \phi(z) dz \\
&= \frac{1}{2} \int f \left(S_0 e^{\sqrt{v^0}x - v^0/2} \right) \left(\frac{1}{v^0} \frac{\partial^2}{\partial x^2} + \frac{1}{\sqrt{v^0}} \frac{\partial}{\partial x} \right) (\mathbb{E}[Y|X = x] \phi(x)) dx.
\end{aligned}$$

Here, the last equality follows from that

$$w \mapsto \int f \left(S_0 e^{\sqrt{v^0}x - v^0/2 + w} \right) \left(\frac{1}{v^0} \frac{\partial^2}{\partial x^2} + \frac{1}{\sqrt{v^0}} \frac{\partial}{\partial x} \right) (\mathbb{E}[Y|X = x] \phi(x)) dx$$

is bounded and continuous, which can be shown by the generalized dominated convergence theorem. Similarly, we have

$$\begin{aligned}
&\lim_{\epsilon \rightarrow 0} \int f \left(S_0 e^{\sqrt{v^\epsilon}x - v^\epsilon/2} \right) \left(\frac{1}{v^\epsilon} \frac{\partial^2}{\partial x^2} + \frac{1}{\sqrt{v^\epsilon}} \frac{\partial}{\partial x} \right) (\mathbb{E}[Y|X = x] \phi(x)) dx \\
&= \int f \left(S_0 e^{\sqrt{v^0}x - v^0/2} \right) \left(\frac{1}{v^0} \frac{\partial^2}{\partial x^2} + \frac{1}{\sqrt{v^0}} \frac{\partial}{\partial x} \right) (\mathbb{E}[Y|X = x] \phi(x)) dx,
\end{aligned}$$

which enables us to replace v^0 in the last expression by v^ϵ when combined with (20) towards the claimed formula. $\square$

4.2. Proof of Corollary 2.4. First note that

$$(22) \quad p_K(S_0, v^\epsilon) = \int f \left(S_0 \exp \left\{ \sqrt{v^\epsilon}x - \frac{v^\epsilon}{2} \right\} \right) \phi(x) dx$$

for $f(s) = (K - s)_+$. Let $g(x) = \mathbb{E}[Y|X = x] \phi(x)$ and $d_- = d_-(S_0, v^\epsilon)$. We have

$$\begin{aligned}
(23) \quad \int f \left(S_0 \exp \left\{ \sqrt{v^\epsilon}x - \frac{v^\epsilon}{2} \right\} \right) \frac{d}{dx} g(x) dx &= \int_{-\infty}^{-d_-} \left(K - S_0 \exp \left\{ \sqrt{v^\epsilon}x - \frac{v^\epsilon}{2} \right\} \right) \frac{d}{dx} g(x) dx \\
&= \sqrt{v^\epsilon} \int_{-\infty}^{-d_-} S_0 \exp \left\{ \sqrt{v^\epsilon}x - \frac{v^\epsilon}{2} \right\} g(x) dx
\end{aligned}$$

by (3) and similarly,

$$\begin{aligned}
&\int f \left(S_0 \exp \left\{ \sqrt{v^\epsilon}x - \frac{v^\epsilon}{2} \right\} \right) \frac{d^2}{dx^2} g(x) dx \\
(24) \quad &= \sqrt{v^\epsilon} \int_{-\infty}^{-d_-} S_0 \exp \left\{ \sqrt{v^\epsilon}x - \frac{v^\epsilon}{2} \right\} \frac{d}{dx} g(x) dx \\
&= \sqrt{v^\epsilon} \left[S_0 \exp \left\{ \sqrt{v^\epsilon}x - \frac{v^\epsilon}{2} \right\} g(x) \right]_{-\infty}^{-d_-} - v^\epsilon \int_{-\infty}^{-d_-} S_0 \exp \left\{ \sqrt{v^\epsilon}x - \frac{v^\epsilon}{2} \right\} g(x) dx.
\end{aligned}$$

The result then follows from Theorem 2.1. $\square$

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