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Title
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Citation Osaka Journal of Mathematics. 37(4) P.859-P.871
Issue Date 2000
Text Version publisher
URL https://doi.org/10.18910/10516
DOI 10.18910/10516
Note

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ON THE HOMOTOPY CLASSIFICATION
OF 4-MANIFOLDS HAVING THE FUNDAMENTAL GROUP
OF AN ASPHERICAL 4-MANIFOLD

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(Received October 19, 1998)

1. Introduction.

In this paper we shall study the homotopy type of closed connected oriented topological 4-manifolds $M^4$ with fundamental group isomorphic to $\Pi_1(Q)$, where $Q$ is a fixed closed oriented aspherical 4-manifold. A standard example of such a manifold is the connected sum $M = Q\#M'$, where $M'$ is an arbitrary simply-connected closed 4-manifold. In general, we shall always assume that $M$ and $Q$ are provided with CW-structures (up to homotopy) such that $M^{(3)} = M \setminus \hat{D}^4$ and $Q^{(3)} = Q \setminus \hat{D}^4$ (see for example [16], Lemma 2.9). Here the symbol $X^{(q)}$ denotes the $q$-skeleton of a CW-complex $X$ as usual.

There are long outstanding conjectures concerning the topological structure of aspherical 4-manifolds (see for example [5]). One of these states that the Whitehead group of $\Pi_i(Q)$ is zero. So we can not assume in our case that homotopy equivalences are automatically simple.

Let $A = \mathbb{Z}[\Pi_1(Q)]$ be the integral group ring of $\Pi_1(Q)$ and $\text{Out}(\Pi_1(Q))$ the outer automorphism group of $\Pi_1(Q)$, i.e., automorphisms modulo inner automorphisms.

Let $f : M \to Q$ be the classifying map of the universal covering. For this we shall prove the following result (see Section 3).

Theorem 1.1. If $f$ is of degree 1, then there is a homotopy equivalence of $M^{(3)}$ with $(Q\#M')^{(3)}$ for some simply-connected closed topological 4-manifold $M'$.

As a consequence, $H_2(M; \Lambda)$ is $\Lambda$-free. In Section 2 we show that the classifying map $f : M \to Q$ is of degree 1 if and only if the $k$-invariant $k^3_M \in H^3(B\Pi_1; \Pi_2(M))$
of \( M \) vanishes. Observe that \( B\pi_1 = Q \).

For degree one maps we have split exact sequences

\[
0 \rightarrow K_2(f; \Lambda) \rightarrow H_2(M; \Lambda) \xrightarrow{f_*} H_2(Q; \Lambda) \rightarrow 0
\]

and

\[
0 \rightarrow K_2(f; \mathbb{Z}) \rightarrow H_2(M; \mathbb{Z}) \xrightarrow{f_*} H_2(Q; \mathbb{Z}) \rightarrow 0.
\]

Note that \( H_2(Q; \Lambda) \cong 0 \) in our case. The splittings preserve the intersection forms. By the result of Freedman (see [6] and [7]) there is a simply-connected closed topological 4-manifold \( M' \) which realizes the intersection form on \( K_2(f; \mathbb{Z}) \).

Using a result of [1] we are going to prove the main theorem of the present paper.

**Theorem 1.2.** Let \( M^4 \) be a closed connected oriented topological 4-manifold with \( \pi_1(M) \cong \pi_1(Q) \), where \( Q \) is a fixed closed connected oriented aspherical 4-manifold. Assume that \( k^3_M = 0 \). Then \( M^{(3)} \) is homotopy equivalent to \((Q#M')^{(3)}\). If the \( \Lambda \)-intersection form \( \mu_M^\Lambda : H_2(M; \Lambda) \times H_2(M; \Lambda) \rightarrow \Lambda \) is extended from the \( \mathbb{Z} \)-intersection form \( \mu_M^\mathbb{Z} : H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \rightarrow \mathbb{Z} \), then \( M \) is homotopy equivalent to \( Q#M' \). Moreover, there is an obstruction

\[\tau(M) \in \text{Wh}(\pi_1(Q))/\text{Out}(\pi_1(Q))\]

for \( M \) being simple homotopy equivalent to \( Q#M' \).

Examples for the case when the Whitehead group vanishes are given by flat Riemannian (resp. hyperbolic) manifolds and manifolds \( M \) with \( \pi_1(M) \cong \pi_1(Q) \) poly-\( \mathbb{Z} \)-group, i.e., a group having finite composition series whose factors are all infinite cyclic (see [2], [3], [4] and [5]).

The fact about the torsion invariant \( \tau(M) \) follows from standard arguments of simple homotopy theory (see for example [15]).

2. The classifying map \( f : M \rightarrow Q \).

We begin by choosing an isomorphism \( \pi_1(M) \rightarrow \pi_1(Q) \). The homotopy equivalence to be constructed depends on this isomorphism. According to it there is a classifying map \( f : M \rightarrow Q \). We observe that \( Q \) is homotopy equivalent (not simple homotopy equivalent) to a space \( K \) obtained from \( M \) by adjoining cells of dimension \( q \geq 3 \). Then we have \( \pi_q(Q, M) \cong \pi_q(K, M) \cong 0 \) for \( q \leq 2 \) and \( H_q(Q, M; B) \cong 0 \) for \( q \leq 2 \), where \( B \) is an arbitrary local coefficient system.
Lemma 2.1. If $k^3_M = 0$, then there is a map $j : Q^{(3)} \to M$ such that the composition

$$Q^{(3)} \overset{j}{\to} M \overset{f}{\to} Q$$

is homotopic to the inclusion $Q^{(3)} \subset Q$.

Proof. Let $D \to B\Pi_1 = Q$ be the 2-stage Postnikov system classified by $k^3_M$. Then there is a 3-equivalence $\gamma : M \to D$ such that the composite map

$$M \overset{\gamma}{\to} D \to Q$$

is homotopic to $f$. There exists a map $j' : Q^{(2)} \to M$ such that $f \circ j' : Q^{(2)} \to Q$ is homotopic to the inclusion. Hence we can consider $s = \gamma \circ j'$ as a section in $D \to Q$ over $Q^{(2)}$. Since $k^3_M = 0$, the map $s$ extends over $Q^{(3)}$ (in fact, over $Q$ since $\Pi_3(Q) \cong 0$). Because $\gamma$ is a 3-equivalence, this gives us a map $j$ with the desired property (since $\Pi_q(Q) \cong 0$ for any $q > 1$, $f \circ j$ is homotopic to the inclusion $Q^{(3)} \subset Q$).

Lemma 2.2. The map $f : M \to Q$ is of degree 1 by choosing appropriated orientations of $M$ and $Q$.

Proof. First we note that $\Pi_1(Q)$ is necessarily an infinite group (see [5]), hence $H_4(Q; \Lambda) \cong H_4(M; \Lambda) \cong 0$. Of course, $H_q(Q; \Lambda) \cong H_q(Q; \mathbb{Z}) \cong 0$ for any $q > 0$. Moreover, by Poincaré duality it follows that

$$H_3(M; \Lambda) \cong H^1(M; \Lambda) \cong H^1(Q; \Lambda) \cong H_3(Q; \Lambda) \cong 0,$$

hence we have isomorphisms

$$\Lambda \cong H_4(M, M\setminus D^4; \Lambda) \overset{\cong}{\to} H_3(M\setminus D^4; \Lambda)$$

and

$$\Lambda \cong H_4(Q, Q\setminus D^4; \Lambda) \overset{\cong}{\to} H_3(Q\setminus D^4; \Lambda).$$

Let $f^{(3)} : M\setminus D^4 \to Q\setminus D^4$ denote a cellular approximation of $f : M \to Q$ restricted to the 3-skeletons. By Lemma 2.1, the composition map $f \circ j$ is homotopic to the inclusion $Q^{(3)} \subset Q$, hence

$$f^{(3)}_* : H_3(M\setminus D^4; \Lambda) \to H_3(Q\setminus D^4; \Lambda)$$
is surjective. It follows that

\[ f_*^\Lambda : H_4(M, M\setminus D^4; \Lambda) \to H_4(Q, Q\setminus D^4; \Lambda) \]

is onto, and hence

\[ Z \cong H_4(M, M\setminus D^4; \Lambda) \otimes_\Lambda Z \to H_4(Q, Q\setminus D^4; \Lambda) \otimes_\Lambda Z \cong Z \]

is onto too, i.e., an isomorphism. But we have an isomorphism

\[ H_4(M, M\setminus D^4; \Lambda) \otimes_\Lambda Z \cong H_4(M, M\setminus D^4; Z) \]

in a natural way. Hence the map \( f : M \to Q \) induces an isomorphism

\[ H_4(M, M\setminus D^4; Z) \to H_4(Q, Q\setminus D^4; Z). \]

Now the lemma follows from the diagram

\[
\begin{array}{ccc}
H_4(M; Z) & \cong & H_4(M, M\setminus D^4; Z) \\
\downarrow f_*^\Lambda & & \downarrow \cong \\
H_4(Q; Z) & \cong & H_4(Q, Q\setminus D^4; Z).
\end{array}
\]

The horizontal isomorphisms are given by the local orientations. \( \square \)

If, conversely, \( f : M \to Q \) is of degree 1, then we have

\[ H^3(B\Pi_1; \Pi_2(M)) \cong H^3(Q; \Pi_2(M)) \cong H_1(Q; \Pi_2(M)) \cong \text{Tor}^1_\Lambda(Z, \Pi_2(M)) \cong 0 \]

as \( \Pi_2(M) \) is stably \( \Lambda \)-free. In particular, it follows that \( k_3^M = 0 \).

Summarizing, we have proved the following

**Proposition 2.3.** Let \( M^4 \) be a closed orientable 4-manifold with fundamental group isomorphic to that of an orientable aspherical 4-manifold \( Q \). Then the classifying map \( f : M \to Q \) is of degree 1 if and only if \( k_3^M = 0 \).

Let us denote

\[ K_q(f; B) = \text{Ker}(f_* : H_q(M; B) \to H_q(Q; B)), \]

where \( B \) is an arbitrary local coefficient system.
It follows that the exact sequence

\[ 0 \longrightarrow K_q(f; \mathcal{B}) \longrightarrow H_q(M; \mathcal{B}) \longrightarrow H_q(Q; \mathcal{B}) \longrightarrow 0 \]

splits. Moreover, the restrictions of the intersection forms

\[ \mu^f_M : H_2(M; \Lambda) \times H_2(M; \Lambda) \to \Lambda \]

and

\[ \mu^f_M : H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \to \mathbb{Z} \]

to \(K_2(f; \Lambda)\) and \(K_2(f; \mathbb{Z})\), respectively, are non-degenerate (see for instance [16]).

In particular, we obtain the following consequence.

**Corollary 2.4.** \( \mu^f_M \cong \mu^f_Q \oplus \mu^f_M |_{K_2(f; \mathbb{Z})} \)

**Lemma 2.5.** \( K_2(f; \mathbb{Z}) \) is isomorphic to \( H_2(M; \Lambda) \otimes \Lambda \mathbb{Z} \).

**Proof.** First we note that \( K_q(f; \mathcal{B}) \) can be identified with \( H_{q+1}(Q, M; \mathcal{B}) \) for any local coefficient system \( \mathcal{B} \). Then we consider the universal coefficient spectral sequence:

\[ \text{Tor}^\Lambda_p(H_{q+1}(Q, M; \Lambda), \mathbb{Z}) \Rightarrow H_{p+q+1}(Q, M; \mathbb{Z}). \]

Since \( H_q(Q; \Lambda) \cong 0 \) for \( q \neq 0 \) and \( H_q(M; \Lambda) \cong 0 \) for \( q \neq 0, 2 \), we have

\[ H_{q+1}(Q, M; \Lambda) \cong \begin{cases} 0 & \text{if } q \neq 2 \\ H_2(M; \Lambda) & \text{if } q = 2. \end{cases} \]

Therefore it follows that

\[ K_2(f; \mathbb{Z}) \cong H_3(Q, M; \mathbb{Z}) \cong \text{Tor}^\Lambda_0(H_3(Q, M; \Lambda), \mathbb{Z}) \]

\[ \cong H_3(Q, M; \Lambda) \otimes \Lambda \mathbb{Z} \cong H_2(M; \Lambda) \otimes \Lambda \mathbb{Z} \]

as claimed. \( \Box \)

**Lemma 2.6.** The \( \Lambda \)-module \( H_2(M; \Lambda) \cong K_2(f; \Lambda) \) is stably \( \Lambda \)-free. Moreover, \( H_2(M; \Lambda) \) has a preferred \( s \)-base.

**Proof.** This is the assertion of Lemma 2.3 (c) in [16]. Indeed, we have \( K_q(f; \Lambda) \cong 0 \) for \( q \neq 2 \), i.e. \( H_4(f; \Lambda) \cong H_4(Q, M; \Lambda) \cong 0 \) for \( q \neq 3 \). Moreover, we have

\[ H^4(f; \mathcal{B}) \cong K^3(f; \mathcal{B}) \cong K_1(f; \mathcal{B}) \cong 0 \]
for any local coefficient system $B$. Therefore the hypothesis of Lemma 2.3 (c) in [16] are verified. This completes the proof. \hfill \Box

**Remark.** The $s$-base of the $\Lambda$-module $K_2(f; \Lambda) \cong H_2(M; \Lambda)$ is determined by the CW-structure considered in $M$.

Since

$$H_q(Q, M; \Lambda) \cong \begin{cases} 0 & \text{if } q \neq 3 \\ H_2(M; \Lambda) & \text{if } q = 3 \end{cases}$$

and $H_2(M; \Lambda)$ is stably $\Lambda$-free, we obtain $H_3(Q, M; \Lambda) \otimes \Lambda \cong H_3(Q, M; \mathbb{Z})$ as proved in Lemma 2.5. In other words,

$$K_2(f; \mathbb{Z}) \cong H_2(M; \Lambda) \otimes \Lambda \cong \Pi_2(M) \otimes \Lambda \mathbb{Z}$$

is $\mathbb{Z}$-free, of rank $r$ say, i.e., the restriction $\mu^\mathbb{Z}_M |_{K_2(f; \mathbb{Z})}$ is an unimodular symmetric non-degenerate form. By the fundamental result of Freedman (see [6] and [7]) there is a simply-connected closed topological 4-manifold $M'$ such that

$$\mu^\mathbb{Z}_M = \mu^\mathbb{Z}_{M'} |_{K_2(f; \mathbb{Z})}.$$ 

**Lemma 2.7.** There exists a map

$$\psi : (Q#M') \setminus \overset{\circ}{D}^4 \to M$$

which induces isomorphisms on $\Pi_1$ and on $H_2(\cdot; \mathbb{Z})$.

**Proof.** First we observe that $(Q#M') \setminus \overset{\circ}{D}^4$ is homotopy equivalent to the wedge $(Q \setminus \overset{\circ}{D}^4) \vee (M' \setminus \overset{\circ}{D}^4)$. Now $M' \setminus \overset{\circ}{D}^4$ is homotopy equivalent to $\vee r S^2$ and by the above isomorphism

$$K_2(f; \mathbb{Z}) \cong \Pi_2(M) \otimes \Lambda \mathbb{Z}$$

we can represent a basis of $H_2(M; \mathbb{Z})$ by a map $\varphi : \vee r S^2 \to M$.

Let us define

$$\psi = j \vee \varphi : (Q#M') \setminus \overset{\circ}{D}^4 \to M.$$ 

Obviously, the induced homomorphism

$$\psi_* : \Pi_1((Q#M') \setminus \overset{\circ}{D}^4) \to \Pi_1(M)$$

is bijective.
Let us consider the following diagram

\[
0 \rightarrow K_2(f; \mathbb{Z}) \rightarrow H_2(M; \mathbb{Z}) \xrightarrow{\varphi_*} H_2(Q; \mathbb{Z}) \rightarrow 0
\]

where \( c : (Q \setminus D^4) \vee (\vee, S^2) \rightarrow Q \setminus D^4 \) is the projection map. By construction we have \( \text{Ker} c_* \cong \text{Ker} f_* \), hence \( \text{Ker}(j_* \vee \varphi_*) \cong 0 \). Moreover,

\[
\varphi_* : H_2(M' \setminus \hat{D}^4; \mathbb{Z}) \rightarrow K_2(f; \mathbb{Z}),
\]

hence \( \varphi_* \) is surjective. This completes the proof of the lemma.

Let us denote \( M_1 = Q\# M' \). In Section 3 we will show that the map \( \psi \) is a homotopy equivalence from \( M_1^{(3)} \) to \( M^{(3)} \). So it induces an isomorphism

\[
\psi_* : \Pi_2(M_1^{(3)}) \cong H_2(M_1; \Lambda) \rightarrow \Pi_2(M^{(3)}) \cong H_2(M; \Lambda).
\]

We can complete the proof of the first statement in Theorem 1.2 under this hypothesis by using a method described in [1]. By construction,

\[
\psi_* : H_2(M_1^{(3)}; \mathbb{Z}) \cong H_2(M_1; \mathbb{Z}) \rightarrow H_2(M^{(3)}; \mathbb{Z}) \cong H_2(M; \mathbb{Z})
\]

is an isomorphism of the \( \mathbb{Z} \)-intersection forms. Obviously, the \( \Lambda \)-intersection form

\[
\mu_M^\Lambda : H_2(M_1; \Lambda) \times H_2(M_1; \Lambda) \rightarrow \Lambda
\]

is extended from \( \mu_M^\mathbb{Z} : H_2(M_1; \mathbb{Z}) \times H_2(M_1; \mathbb{Z}) \rightarrow \mathbb{Z} \). By hypothesis, this holds also in \( M \). Therefore, \( \psi_* : H_2(M_1; \Lambda) 
\rightarrow H_2(M; \Lambda) \) is an isomorphism of the \( \Lambda \)-intersection forms. We can now apply the following result proved in [1] only for free fundamental groups (further information on closed 4-manifolds with free fundamental group or with infinite cyclic first homology can be found in [9], [11–13], [14] and [17]). Let \( X \) and \( Y \) be closed connected oriented 4-dimensional Poincaré spaces (in particular, 4-manifolds) with \( \Pi_1(X) \cong \Pi_1(Y) \cong *_p \mathbb{Z} \) (free product of \( p \) factors \( \mathbb{Z} \), \( p \geq 1 \)). Then \( X \) is homotopy equivalent to \( Y \) if and only if the intersection pairings \( (H_2(X; \Lambda), \mu_X^\Lambda) \) and \( (H_2(Y; \Lambda), \mu_Y^\Lambda) \) are isomorphic, where \( \Lambda \) denotes here the group ring of \( *_p \mathbb{Z} \). But one can verify that the proof of this result is based on the following facts: \( \Pi_1 \) is a finitely presentable torsion free infinite group, \( \Pi_2 \) is \( \Lambda \)-free (and whence on the use of the special Küneth formula), and the first \( k \)-invariant vanishes. These are all verified in our case. Observe that the fundamental group of an aspherical manifold is in fact torsion free since a \( K(\mathbb{Z}_n, 1) \) can not be finite dimensional when \( n > 1 \) (see for example
Thus the result on the homotopy type holds also for fundamental groups $\Pi_1(Q)$ since $B\Pi_1 = Q$ and, in particular, $H_q(B\Pi_1) \cong 0$ for any $q \geq 5$. Moreover, $\Pi_1(Q)$ is a finitely presentable PD$_4$-group of type FF, i.e., the augmentation $\Lambda$-module $\mathbb{Z}$ has a finite resolution consisting of finitely generated free $\Lambda$-modules, where $\Lambda = \mathbb{Z}[\Pi_1(Q)]$ (see [10], Theorem 5). Thus the proof of the first statement in Theorem 1.2 has been completed.

3. Proof of Theorem 1.2.

We assume in this section that $k^2_M = 0$. Let $j : Q^{(3)} \to M$ be as in Section 2. In that section we have chosen a basis in $K_2(f; \mathbb{Z}) \cong H_2(M; \mathbb{Z}) \cong H_2(M; \Lambda) \otimes_\Lambda \mathbb{Z}$ which defines a map

$$\varphi : \vee_1^2 S^2 \to M.$$  

Let $(e_1, \ldots, e_r)$ be this basis, and let $(e_1^*, \ldots, e_r^*)$ be its dual in $\text{Hom}_\mathbb{Z}(H_2(M; \mathbb{Z}), \mathbb{Z})$. We can represent each element $e_i^*$ by a map

$$e_i^* : M \to K(\mathbb{Z}, 2) = \mathbb{C}P^\infty,$$

so we obtain a map

$$\prod_{i=1}^r e_i^* : M \to \prod_{i=1}^r \mathbb{C}P^\infty$$

which induces a map on the 2-skeleton

$$g : M^{(2)} \to \left( \prod_{i=1}^r \mathbb{C}P^\infty \right)^{(2)} = \vee_1^2 S^2.$$  

By construction, the composite map

$$(M')^{(2)} = \vee_1^2 S^2 \xrightarrow{\varphi} M^{(2)} \xrightarrow{g} \vee_1^2 S^2 = (M')^{(2)}$$

is homotopic to the identity.

Remark. It is not difficult to show that $g$ extends to a degree 1 map from $M$ to $M'$.

Let us consider the map

$$h = (f \times g)|_{M^{(3)}} : M^{(3)} \to (Q \times M')^{(2)} = Q^{(2)} \cup (M')^{(2)}.$$  

Lemma 3.1. The map $h$ extends to a map $M^{(3)} \to M^{(3)}_1$, again denoted by $h$. Moreover, the induced homomorphism $h_* : \Pi_2(M^{(3)}) \to \Pi_2(M^{(3)}_1)$ is bijective (here $M_1 = Q#M'$ as usual).
Proof. The obstruction for extending \( h \) over \( M^{(3)} \) belongs to

\[
H^3(M; \Pi_2(M^{(3)})) \cong H_1(M; \Pi_2(M^{(3)})) \cong 0
\]

since \( \Pi_2(M^{(3)}) \cong \Pi_2(M') \otimes \mathbb{Z} \Lambda \) is \( \Lambda \)-free.

The composition

\[
M_1^{(3)} \xrightarrow{j \vee \varphi} M^{(3)} \xrightarrow{h} M_1^{(3)}
\]

induces an isomorphism on \( \Pi_2 \), hence the homomorphism

\[
h_* : \Pi_2(M^{(3)}) \to \Pi_2(M_1^{(3)})
\]

is surjective. Because \( \Pi_2(M_1^{(3)}) \) is \( \Lambda \)-free, we have

\[
\Pi_2(M^{(3)}) \cong \Pi_2(M_1^{(3)}) \oplus \text{Ker} h_*.
\]

Observe that \( \text{Ker} h_* \) is stably \( \Lambda \)-free (since the \( \Lambda \)-module \( \Pi_2(M^{(3)}) \cong K_2(f; \Lambda) \) is stably \( \Lambda \)-free). But tensoring with \( \otimes \Lambda \mathbb{Z} \) gives isomorphisms

\[
\Pi_2(M^{(3)}) \otimes \Lambda \mathbb{Z} \cong \oplus_1 \mathbb{Z} \cong \Pi_2(M_1^{(3)}) \otimes \Lambda \mathbb{Z},
\]

hence \( \text{Ker} h_* \otimes \Lambda \mathbb{Z} \cong 0 \). By Kaplansky’ s lemma (see for example \([8]\) and \([10]\)) we get \( \text{Ker} h_* \cong 0 \).

**Corollary 3.2.** The map \( \psi = j \vee \varphi : M_1^{(3)} \to M^{(3)} \) is a homotopy equivalence.

Proof. It suffices to show that the induced homomorphism

\[
\psi_* : \Pi_3(M_1^{(3)}) \to \Pi_3(M^{(3)})
\]

is bijective. For this, it is convenient to recall the Whitehead certain exact sequence \([18]\) for a CW-complex \( X \):

\[
H_4(X; \Lambda) \longrightarrow \Gamma(\Pi_2(X)) \longrightarrow \Pi_3(X) \longrightarrow H_3(X; \Lambda) \longrightarrow 0.
\]

Here \( \Gamma(\Lambda) \) is the quadratic functor on the category of abelian groups, and \( \Pi_3(X) \to H_3(X; \Lambda) \) is the Hurewicz homomorphism. This sequence is natural, hence the map \( \psi \) induces a diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\Pi_2(M^{(3)})) & \longrightarrow & \Pi_3(M^{(3)}) & \longrightarrow & H_3(M^{(3)}; \Lambda) & \longrightarrow & 0 \\
\downarrow \Gamma(\psi_*) & & \downarrow \psi_* & & \downarrow \psi_* & & \downarrow \psi_* & & \\
0 & \longrightarrow & \Gamma(\Pi_2(M^{(3)})) & \longrightarrow & \Pi_3(M^{(3)}) & \longrightarrow & H_3(M^{(3)}; \Lambda) & \longrightarrow & 0.
\end{array}
\]
It follows from Lemma 3.1 that $\Gamma(\psi_*)$ is an isomorphism. The claim is proved once we have $\psi_* : H_3(M(\Lambda)) \rightarrow H_3(M ; \Lambda)$. If we denote by

$$c : M(\Lambda) \simeq Q(\Lambda) \setminus (M') (\Lambda) \rightarrow Q(\Lambda)$$

the projection map, then we can immediately see that $f \circ \psi \simeq c$. Observe that $c$ is the restriction of the collapsing map $c : M_1 \rightarrow Q$. Now the result follows from the commutative diagram

$$
\begin{array}{ccc}
\Lambda \cong H_4(M, M(\Lambda)) & \rightarrow & H_3(M(\Lambda)) \\
\cong \downarrow \quad & \cong \downarrow \quad & \cong \downarrow \\
\Lambda \cong H_4(Q, Q(\Lambda)) & \rightarrow & H_3(Q(\Lambda)) \\
\cong \uparrow \quad & \cong \uparrow \\
\Lambda \cong H_4(M_1, M(\Lambda)) & \rightarrow & H_3(M(\Lambda)).
\end{array}
$$

4. The torsion invariant.

Let us fix the manifolds $M^4$ and $Q^4$ with $\Pi_1(M) \cong \Pi_1(Q)$, where $Q$ is aspherical and $H_2(M ; \Lambda)$ is $\Lambda$-flat. Recall that $M$ and $Q$ are provided with CW-structures such that $M(\Lambda) = M \setminus D^4$ and $Q(\Lambda) = Q \setminus D^4$. Let $f : M \rightarrow Q$ be a classifying map.

We have proved in Section 2 that $H_3(f ; \Lambda)$ is $\Lambda$-free. Moreover, $H_q(f ; \Lambda) \cong 0$ for any $q \neq 3$. Under these conditions one can define the torsion

$$\tau(f) \in \text{Wh}(\Pi_1(Q))$$

of the map $f$. Namely, it is the torsion of the cellular complex of the pair $((M \times I) \cup_f Q, M \times 0)$, where $(M \times I) \cup_f Q$ denotes the mapping cylinder of $f$, that is the quotient space

$$(M \times I) \cup_f Q = \frac{(M \times I) \cup Q}{\{(x, 1) \equiv f(x)\}},$$

$I = [0, 1]$. The torsion $\tau(f)$ is defined upon the choice of a $\Lambda$-basis of

$$H_3(f ; \Lambda) \cong H_2(M ; \Lambda).$$

Hence we shall denote it by $\tau_e(f)$, where $e = (e_1, e_2, \ldots, e_r)$ indicates a $\Lambda$-basis of $H_2(M ; \Lambda)$ (see [15]).

Let us consider the particular case of the collapsing map $g : Q#M' \rightarrow Q$, where $M'$ is simply-connected. Then we have the following (non surprising) result.
Lemma 4.1. The torsion \( \tau_e(g) \) vanishes for any \( \Lambda \)-basis \( e \) of \( H_2(g; \Lambda) \).

Proof. Let us consider the following diagram of inclusions:

\[
\begin{array}{ccc}
X_2 = (Q \backslash D^4) \times I \cup g_I Q & \longrightarrow & X = (Q\#M') \times I \cup g Q \\
\uparrow & & \uparrow \\
X_1 = (Q \backslash D^4) \times 0 & \longrightarrow & X_3 = (Q\#M') \times 0,
\end{array}
\]

where \( g_1 = g|_{Q\backslash D^4} \). Note that the torsions of each pair are defined because

\[ H_2(X_2, X_1; \Lambda) \cong \Lambda \quad \text{and} \quad H_2(X_3, X_1; \Lambda) \cong H_2(M; \Lambda). \]

Moreover, \( \tau_e(X, X_3) = \tau_e(g) \) by definition and formula

\[
(*) \quad \tau(X, X_2) + \tau(X_2, X_1) = \tau_e(g) + \tau(X_3, X_1)
\]

holds. Since \( g_1 : Q\backslash D^4 \rightarrow Q \) is the inclusion, it follows that \( \tau(X_2, X_1) = 0 \) for any choice of a generator in \( H_2(X_2, X_1; \Lambda) \cong \Lambda \). Because \( X \backslash X_2 = (M' \backslash D^4) \times [0, 1] \) and \( X_3 \backslash X_1 \cong M' \) are simply-connected, Lemma 7.3 of [15] implies that \( \tau(X, X_2) \) and \( \tau(X_3, X_1) \) are both zero for any basis of \( H_2(X, X_2; \Lambda) \) and \( H_2(X_3, X_1; \Lambda) \), respectively. Note that the other condition in Lemma 7.3 of [15], concerning the universal covering space, is also satisfied. The result \( \tau_e(g) = 0 \) then follows from formula \((*)\).

Let now \( \psi : Q\#M' \rightarrow M \) be the homotopy equivalence constructed above. In particular, the composition map

\[
Q\#M' \xrightarrow{\psi} M \xrightarrow{f} Q
\]

is homotopic to \( g \). Hence, by applying the standard formulae:

\[ 0 = \tau_e(g) = \tau(\psi) + \tau_e(f), \]

we obtain \( \tau(\psi) = -\tau_e(f) \) for any basis \( e \) of \( H_3(f; \Lambda) \cong H_3(M; \Lambda) \). Any other classifying map \( f' : M \rightarrow Q \) can be written as \( \alpha \circ f \circ \beta \), up to homotopy, where \( \alpha : Q \rightarrow Q \) and \( \beta : M \rightarrow M \) are homotopy equivalences. Let \( \psi' : Q\#M' \rightarrow M \) be the resulting homotopy equivalence. Moreover, we can assume that

\[ \beta_* = \text{id} : \Pi_1(M) \rightarrow \Pi_1(M) \]

because the effect on \( \Pi_1 \) can be transmitted to \( \alpha \). Now we observe that the effect of \( \beta \) on \( \tau(f') \) is restricted to \( \beta_* : H_2(M; \Lambda) \rightarrow H_2(M; \Lambda) \), i.e., to a change of basis in
But this has no influence on $\tau(\psi)$. On the other hand, any $\alpha$ changes $\tau(\psi)$. In fact, we have

$$\tau(\psi') = \tau(\psi) + \tau(\alpha).$$

Now we note that the group of homotopy classes of (orientation-preserving) homotopy self-equivalences of $Q$ is isomorphic to $\text{Out}(\Pi_1(Q))$. Hence we can define

$$\tau(M) = [\tau(\psi)] \in \text{Wh}(\Pi_1(Q))/\text{Out}(\Pi_1(Q)).$$

The following completes our main result.

**Theorem 4.2.** The map $\psi : Q\#M' \to M$ is a simple homotopy equivalence if and only if $\tau(M) = 0$.

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