Title	On the homotopy classification of 4-manifolds having the fundamental group of an aspherical 4-manifold
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Citation	Osaka Journal of Mathematics. 2000, 37(4), p. 859-871
Version Type	VoR
URL	https://doi.org/10.18910/10516
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Cavicchioli, A. and Hegenbarth, F.
Osaka J. Math.
37 (2000), 859–871

ON THE HOMOTOPY CLASSIFICATION OF 4-MANIFOLDS HAVING THE FUNDAMENTAL GROUP OF AN ASPHERICAL 4-MANIFOLD

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(Received October 19, 1998)

1. Introduction.

In this paper we shall study the homotopy type of closed connected oriented topological 4-manifolds M^4 with fundamental group isomorphic to $\Pi_1(Q)$, where Q is a fixed closed oriented aspherical 4-manifold. A standard example of such a manifold is the connected sum $M = Q\#M'$, where M' is an arbitrary simply-connected closed 4-manifold. In general, we shall always assume that M and Q are provided with CW-structures (up to homotopy) such that $M^{(3)} = M\backslash \mathring{D}^4$ and $Q^{(3)} = Q\backslash \mathring{D}^4$ (see for example [16], Lemma 2.9). Here the symbol $X^{(q)}$ denotes the q-skeleton of a CW-complex X as usual.

There are long outstanding conjectures concerning the topological structure of aspherical 4-manifolds (see for example [5]). One of these states that the Whitehead group of $\Pi_1(Q)$ is zero. So we can not assume in our case that homotopy equivalences are automatically simple.

Let $\Lambda = \mathbb{Z}[\Pi_1(Q)]$ be the integral group ring of $\Pi_1(Q)$ and $\mathrm{Out}(\Pi_1(Q))$ the outer automorphism group of $\Pi_1(Q)$, i.e., automorphisms modulo inner automorphisms.

Let $f : M \to Q$ be the classifying map of the universal covering. For this we shall prove the following result (see Section 3).

Theorem 1.1. *If f is of degree 1, then there is a homotopy equivalence of $M^{(3)}$ with $(Q\#M')^{(3)}$ for some simply-connected closed topological 4-manifold M'.*

As a consequence, $H_2(M;\Lambda)$ is Λ-free. In Section 2 we show that the classifying map $f : M \to Q$ is of degree 1 if and only if the k-invariant $k^3_M \in H^3(B\Pi_1;\Pi_2(M))$

1991 *Mathematics Subject Classification* : 57 N 65, 57 R 67.

Key words and phrases. Homotopy type, aspherical manifolds, obstruction theory, torsion invariant, Whitehead's exact sequence, homology with local coefficients.

Work performed under the auspices of the G.N.S.A.G.A. of the C.N.R. (National Research Council) of Italy and partially supported by the Ministero per la Ricerca Scientifica e Tecnologica of Italy within the projects "Proprietà Geometriche delle Varietà Reali e Complesse" and "Geometria Algebrica".

of M vanishes. Observe that $B\Pi_1 = Q$.

For degree one maps we have split exact sequences

$$0 \longrightarrow K_2(f;\Lambda) \longrightarrow H_2(M;\Lambda) \xrightarrow{f_*} H_2(Q;\Lambda) \longrightarrow 0$$

and

$$0 \longrightarrow K_2(f;\mathbb{Z}) \longrightarrow H_2(M;\mathbb{Z}) \xrightarrow{f_*} H_2(Q;\mathbb{Z}) \longrightarrow 0.$$

Note that $H_2(Q;\Lambda) \cong 0$ in our case. The splittings preserve the intersection forms. By the result of Freedman (see [6] and [7]) there is a simply-connected closed topological 4-manifold M' which realizes the intersection form on $K_2(f;\mathbb{Z})$.

Using a result of [1] we are going to prove the main theorem of the present paper.

Theorem 1.2. *Let M^4 be a closed connected oriented topological 4-manifold with $\Pi_1(M) \cong \Pi_1(Q)$, where Q is a fixed closed connected oriented aspherical 4-manifold. Assume that $k^3_M = 0$. Then $M^{(3)}$ is homotopy equivalent to $(Q\#M')^{(3)}$. If the Λ-intersection form $\mu^\Lambda_M : H_2(M;\Lambda) \times H_2(M;\Lambda) \to \Lambda$ is extended from the $\mathbb{Z}$-intersection form $\mu^{\mathbb{Z}}_M : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$, then M is homotopy equivalent to $Q\#M'$. Moreover, there is an obstruction*

$$\tau(M) \in \mathrm{Wh}(\Pi_1(Q))/\,\mathrm{Out}(\Pi_1(Q))$$

for M being simple homotopy equivalent to $Q\#M'$.

Examples for the case when the Whitehead group vanishes are given by flat Riemannian (resp. hyperbolic) manifolds and manifolds M with $\Pi_1(M) \cong \Pi_1(Q)$ poly-$\mathbb{Z}$-group, i.e., a group having finite composition series whose factors are all infinite cyclic (see [2], [3], [4] and [5]).

The fact about the torsion invariant $\tau(M)$ follows from standard arguments of simple homotopy theory (see for example [15]).

2. The classifying map $f : M \to Q$.

We begin by choosing an isomorphism $\Pi_1(M) \underset{\cong}{\to} \Pi_1(Q)$. The homotopy equivalence to be constructed depends on this isomorphism. According to it there is a classifying map $f : M \to Q$. We observe that Q is homotopy equivalent (not simple homotopy equivalent) to a space K obtained from M by adjoining cells of dimension $q \geq 3$. Then we have $\Pi_q(Q, M) \cong \Pi_q(K, M) \cong 0$ for $q \leq 2$ and $H_q(Q, M; \mathcal{B}) \cong 0$ for $q \leq 2$, where $\mathcal{B}$ is an arbitrary local coefficient system.

Lemma 2.1. *If $k^3_M = 0$, then there is a map $j : Q^{(3)} \to M$ such that the composition*

$$Q^{(3)} \xrightarrow{j} M \xrightarrow{f} Q$$

is homotopic to the inclusion $Q^{(3)} \subset Q$.

Proof. Let $D \to B\Pi_1 = Q$ be the 2-stage Postnikov system classified by k^3_M. Then there is a 3-equivalence $\gamma : M \to D$ such that the composite map

$$M \xrightarrow{\gamma} D \longrightarrow Q$$

is homotopic to f. There exists a map $j' : Q^{(2)} \to M$ such that $f \circ j' : Q^{(2)} \to Q$ is homotopic to the inclusion. Hence we can consider $s = \gamma \circ j'$ as a section in $D \to Q$ over $Q^{(2)}$. Since $k^3_M = 0$, the map s extends over $Q^{(3)}$ (in fact, over Q since $\Pi_3(Q) \cong 0$). Because γ is a 3-equivalence, this gives us a map j with the desired property (since $\Pi_q(Q) \cong 0$ for any $q > 1$, $f \circ j$ is homotopic to the inclusion $Q^{(3)} \subset Q$). □

Lemma 2.2. *The map $f : M \to Q$ is of degree 1 by choosing appropriated orientations of M and Q.*

Proof. First we note that $\Pi_1(Q)$ is necessarily an infinite group (see [5]), hence $H_4(Q; \Lambda) \cong H_4(M; \Lambda) \cong 0$. Of course, $H_q(Q; \Lambda) \cong H_q(\widetilde{Q}; \mathbb{Z}) \cong 0$ for any $q > 0$. Moreover, by Poincaré duality it follows that

$$H_3(M; \Lambda) \cong H^1(M; \Lambda) \cong H^1(Q; \Lambda) \cong H_3(Q; \Lambda) \cong 0,$$

hence we have isomorphisms

$$\Lambda \cong H_4(M, M \backslash \mathring{D}^4; \Lambda) \underset{\cong}{\to} H_3(M \backslash \mathring{D}^4; \Lambda)$$

and

$$\Lambda \cong H_4(Q, Q \backslash \mathring{D}^4; \Lambda) \underset{\cong}{\to} H_3(Q \backslash \mathring{D}^4; \Lambda).$$

Let $f^{(3)} : M \backslash \mathring{D}^4 \to Q \backslash \mathring{D}^4$ denote a cellular approximation of $f : M \to Q$ restricted to the 3-skeletons. By Lemma 2.1, the composition map $f \circ j$ is homotopic to the inclusion $Q^{(3)} \subset Q$, hence

$$f^{(3)}_* : H_3(M \backslash \mathring{D}^4; \Lambda) \to H_3(Q \backslash \mathring{D}^4; \Lambda)$$

is surjective. It follows that

$$f_*^\Lambda : H_4(M, M\backslash \overset{\circ}{D}{}^4; \Lambda) \to H_4(Q, Q\backslash \overset{\circ}{D}{}^4; \Lambda)$$

is onto, and hence

$$\mathbb{Z} \cong H_4(M, M\backslash \overset{\circ}{D}{}^4; \Lambda) \otimes_\Lambda \mathbb{Z} \to H_4(Q, Q\backslash \overset{\circ}{D}{}^4; \Lambda) \otimes_\Lambda \mathbb{Z} \cong \mathbb{Z}$$

is onto too, i.e., an isomorphism. But we have an isomorphism

$$H_4(M, M\backslash \overset{\circ}{D}{}^4; \Lambda) \otimes_\Lambda \mathbb{Z} \cong H_4(M, M\backslash \overset{\circ}{D}{}^4; \mathbb{Z})$$

in a natural way. Hence the map $f : M \to Q$ induces an isomorphism

$$H_4(M, M\backslash \overset{\circ}{D}{}^4; \mathbb{Z}) \underset{\cong}{\to} H_4(Q, Q\backslash \overset{\circ}{D}{}^4; \mathbb{Z}).$$

Now the lemma follows from the diagram

$$\begin{array}{ccc} H_4(M;\mathbb{Z}) & \xrightarrow{\cong} & H_4(M, M\backslash \overset{\circ}{D}{}^4;\mathbb{Z}) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \cong} \\ H_4(Q;\mathbb{Z}) & \xrightarrow[\cong]{} & H_4(Q, Q\backslash \overset{\circ}{D}{}^4;\mathbb{Z}). \end{array}$$

The horizontal isomorphisms are given by the local orientations. □

If, conversely, $f : M \to Q$ is of degree 1, then we have

$$H^3(B\Pi_1; \Pi_2(M)) \cong H^3(Q; \Pi_2(M)) \cong H_1(Q; \Pi_2(M)) \cong \mathrm{Tor}_1^\Lambda(\mathbb{Z}, \Pi_2(M)) \cong 0$$

as $\Pi_2(M)$ is stably Λ-free. In particular, it follows that $k_M^3 = 0$.

Summarizing, we have proved the following

Proposition 2.3. *Let M^4 be a closed orientable 4-manifold with fundamental group isomorphic to that of an orientable aspherical 4-manifold Q. Then the classifying map $f : M \to Q$ is of degree 1 if and only if $k_M^3 = 0$.*

Let us denote

$$K_q(f; \mathcal{B}) = \mathrm{Ker}(f_* : H_q(M; \mathcal{B}) \to H_q(Q; \mathcal{B})),$$

where $\mathcal{B}$ is an arbitrary local coefficient system.

It follows that the exact sequence

$$0 \longrightarrow K_q(f;\mathcal{B}) \longrightarrow H_q(M;\mathcal{B}) \longrightarrow H_q(Q;\mathcal{B}) \longrightarrow 0$$

splits. Moreover, the restrictions of the intersection forms

$$\mu_M^\Lambda : H_2(M;\Lambda) \times H_2(M;\Lambda) \to \Lambda$$

and

$$\mu_M^\mathbb{Z} : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$$

to $K_2(f;\Lambda)$ and $K_2(f;\mathbb{Z})$, respectively, are non-degenerate (see for instance [16]).
In particular, we obtain the following consequence.

Corollary 2.4. $\mu_M^\mathbb{Z} \cong \mu_Q^\mathbb{Z} \oplus \mu_M^\mathbb{Z}|_{K_2(f;\mathbb{Z})}$

Lemma 2.5. *$K_2(f;\mathbb{Z})$ is isomorphic to $H_2(M;\Lambda)\otimes_\Lambda \mathbb{Z}$.*

Proof. First we note that $K_q(f;\mathcal{B})$ can be identified with $H_{q+1}(Q,M;\mathcal{B})$ for any local coefficient system $\mathcal{B}$. Then we consider the universal coefficient spectral sequence:

$$\mathrm{Tor}_p^\Lambda(H_{q+1}(Q,M;\Lambda),\mathbb{Z}) \Longrightarrow H_{p+q+1}(Q,M;\mathbb{Z}).$$

Since $H_q(Q;\Lambda)\cong 0$ for $q\neq 0$ and $H_q(M;\Lambda)\cong 0$ for $q\neq 0,2$, we have

$$H_{q+1}(Q,M;\Lambda) \cong \begin{cases} 0 & \text{if} \quad q\neq 2 \\ H_2(M;\Lambda) & \text{if} \quad q=2. \end{cases}$$

Therefore it follows that

$$\begin{aligned} K_2(f;\mathbb{Z}) \cong H_3(Q,M;\mathbb{Z}) &\cong \mathrm{Tor}_0^\Lambda(H_3(Q,M;\Lambda),\mathbb{Z}) \\ &\cong H_3(Q,M;\Lambda)\otimes_\Lambda \mathbb{Z} \cong H_2(M;\Lambda)\otimes_\Lambda \mathbb{Z} \end{aligned}$$

as claimed. □

Lemma 2.6. *The Λ-module $H_2(M;\Lambda)\cong K_2(f;\Lambda)$ is stably Λ-free. Moreover, $H_2(M;\Lambda)$ has a preferred s-base.*

Proof. This is the assertion of Lemma 2.3 (c) in [16]. Indeed, we have $K_q(f;\Lambda)\cong 0$ for $q\neq 2$, i.e. $H_q(f;\Lambda)\cong H_q(Q,M;\Lambda)\cong 0$ for $q\neq 3$. Moreover, we have

$$H^4(f;\mathcal{B}) \cong K^3(f;\mathcal{B}) \cong K_1(f;\mathcal{B}) \cong 0$$

for any local coefficient system $\mathcal{B}$. Therefore the hypothesis of Lemma 2.3 (c) in [16] are verified. This completes the proof. □

REMARK. The s-base of the Λ-module $K_2(f;\Lambda) \cong H_2(M;\Lambda)$ is determined by the CW-structure considered in M.

Since

$$H_q(Q,M;\Lambda) \cong \begin{cases} 0 & \text{if} \quad q \neq 3 \\ H_2(M;\Lambda) & \text{if} \quad q = 3 \end{cases}$$

and $H_2(M;\Lambda)$ is stably Λ-free, we obtain $H_3(Q,M;\Lambda)\otimes_\Lambda \mathbb{Z} \cong H_3(Q,M;\mathbb{Z})$ as proved in Lemma 2.5. In other words,

$$K_2(f;\mathbb{Z}) \cong H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z} \cong \Pi_2(M) \otimes_\Lambda \mathbb{Z}$$

is $\mathbb{Z}$-free, of rank r say, i.e., the restriction $\mu_M^{\mathbb{Z}}|_{K_2(f;\mathbb{Z})}$ is an unimodular symmetric non-degenerate form. By the fundamental result of Freedman (see [6] and [7]) there is a simply-connected closed topological 4-manifold M' such that

$$\mu_{M'}^{\mathbb{Z}} = \mu_M^{\mathbb{Z}}|_{K_2(f;\mathbb{Z})}.$$

Lemma 2.7. *There exists a map*

$$\psi : (Q\#M')\backslash \overset{\circ}{D}{}^4 \to M$$

which induces isomorphisms on Π_1 *and on* $H_2(\cdot\,;\mathbb{Z})$.

Proof. First we observe that $(Q\#M')\backslash \overset{\circ}{D}{}^4$ is homotopy equivalent to the wedge $(Q\backslash \overset{\circ}{D}{}^4) \vee (M'\backslash \overset{\circ}{D}{}^4)$. Now $M'\backslash \overset{\circ}{D}{}^4$ is homotopy equivalent to $\vee_r \mathbb{S}^2$ and by the above isomorphism

$$K_2(f;\mathbb{Z}) \cong \Pi_2(M) \otimes_\Lambda \mathbb{Z}$$

we can represent a basis of $H_2(M;\mathbb{Z})$ by a map $\varphi : \vee_r \mathbb{S}^2 \to M$.

Let us define

$$\psi = j \vee \varphi : (Q\#M')\backslash \overset{\circ}{D}{}^4 \to M.$$

Obviously, the induced homomorphism

$$\psi_* : \Pi_1((Q\#M')\backslash \overset{\circ}{D}{}^4) \to \Pi_1(M)$$

is bijective.

Let us consider the following diagram

$$\begin{array}{ccccccc} 0 \to & K_2(f;\mathbb{Z}) \to & H_2(M;\mathbb{Z}) & \xrightarrow{f_*} & H_2(Q;\mathbb{Z}) & \to 0 \\ & & \uparrow{\scriptstyle j_*\vee\varphi_*} & & \cong\uparrow{\scriptstyle i_*} & \\ & & H_2((Q\backslash \overset{\circ}{D}{}^4)\vee(\vee_r\mathbb{S}^2);\mathbb{Z}) & \underset{c_*}{\to} & H_2(Q\backslash \overset{\circ}{D}{}^4;\mathbb{Z}), & \end{array}$$

where $c : (Q\backslash \overset{\circ}{D}{}^4)\vee(\vee_r\mathbb{S}^2) \to Q\backslash \overset{\circ}{D}{}^4$ is the projection map. By construction we have $\operatorname{Ker} c_* \cong \operatorname{Ker} f_*$, hence $\operatorname{Ker}(j_*\vee\varphi_*) \cong 0$. Moreover,

$$\varphi_* : H_2(M'\backslash \overset{\circ}{D}{}^4;\mathbb{Z}) \underset{\cong}{\to} K_2(f;\mathbb{Z}),$$

hence φ_* is surjective. This completes the proof of the lemma. □

Let us denote $M_1 = Q\#M'$. In Section 3 we will show that the map ψ is a homotopy equivalence from $M_1^{(3)}$ to $M^{(3)}$. So it induces an isomorphism

$$\psi_* : \Pi_2(M_1^{(3)}) \cong H_2(M_1;\Lambda) \to \Pi_2(M^{(3)}) \cong H_2(M;\Lambda).$$

We can complete the proof of the first statement in Theorem 1.2 under this hypothesis by using a method described in [1]. By construction,

$$\psi_* : H_2(M_1^{(3)};\mathbb{Z}) \cong H_2(M_1;\mathbb{Z}) \to H_2(M^{(3)};\mathbb{Z}) \cong H_2(M;\mathbb{Z})$$

is an isomorphism of the $\mathbb{Z}$-intersection forms. Obviously, the Λ-intersection form

$$\mu^{\Lambda}_{M_1} : H_2(M_1;\Lambda)\times H_2(M_1;\Lambda) \to \Lambda$$

is extended from $\mu^{\mathbb{Z}}_{M_1} : H_2(M_1;\mathbb{Z})\times H_2(M_1;\mathbb{Z}) \to \mathbb{Z}$. By hypothesis, this holds also in M. Therefore, $\psi_* : H_2(M_1;\Lambda) \to H_2(M;\Lambda)$ is an isomorphism of the Λ-intersection forms. We can now apply the following result proved in [1] only for free fundamental groups (further information on closed 4-manifolds with free fundamental group or with infinite cyclic first homology can be found in [9], [11–13], [14] and [17]). Let X and Y be closed connected oriented 4-dimensional Poincaré spaces (in particular, 4-manifolds) with $\Pi_1(X) \cong \Pi_1(Y) \cong *_p\mathbb{Z}$ (free product of p factors $\mathbb{Z}$, $p \geq 1$). Then X is homotopy equivalent to Y if and only if the intersection pairings $(H_2(X;\Lambda), \mu^{\Lambda}_X)$ and $(H_2(Y;\Lambda), \mu^{\Lambda}_Y)$ are isomorphic, where Λ denotes here the group ring of $*_p\mathbb{Z}$. But one can verify that the proof of this result is based on the following facts: Π_1 is a finitely presentable torsion free infinite group, Π_2 is Λ-free (and whence on the use of the special Künneth formula), and the first k-invariant vanishes. These are all verified in our case. Observe that the fundamental group of an aspherical manifold is in fact torsion free since a $K(\mathbb{Z}_n, 1)$ can not be finite dimensional when $n > 1$ (see for example

[5]). Thus the result on the homotopy type holds also for fundamental groups $\Pi_1(Q)$ since $B\Pi_1 = Q$ and, in particular, $H_q(B\Pi_1) \cong 0$ for any $q \geq 5$. Moreover, $\Pi_1(Q)$ is a finitely presentable PD_4-group of type FF, i.e., the augmentation Λ-module $\mathbb{Z}$ has a finite resolution consisting of finitely generated free Λ-modules, where $\Lambda = \mathbb{Z}[\Pi_1(Q)]$ (see [10], Theorem 5). Thus the proof of the first statement in Theorem 1.2 has been completed.

3. Proof of Theorem 1.2.

We assume in this section that $k^3_M = 0$. Let $j : Q^{(3)} \to M$ be as in Section 2. In that section we have chosen a basis in $K_2(f;\mathbb{Z}) \cong H_2(M;\mathbb{Z}) \cong H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z}$ which defines a map

$$\varphi : \vee_1^r \mathbb{S}^2 \to M.$$

Let $(e_1, \dots, e_r)$ be this basis, and let $(e_1^*, \dots, e_r^*)$ be its dual in $\mathrm{Hom}_{\mathbb{Z}}(H_2(M;\mathbb{Z}), \mathbb{Z})$. We can represent each element e_i^* by a map

$$e_i^* : M \to K(\mathbb{Z}, 2) = \mathbb{C}P^\infty,$$

so we obtain a map

$$\prod_{i=1}^r e_i^* : M \to \prod_1^r \mathbb{C}P^\infty$$

which induces a map on the 2-skeleton

$$g : M^{(2)} \to \left(\prod_1^r \mathbb{C}P^\infty\right)^{(2)} = \vee_1^r \mathbb{S}^2.$$

By construction, the composite map

$$(M')^{(2)} = \vee_1^r \mathbb{S}^2 \xrightarrow{\varphi} M^{(2)} \xrightarrow{g} \vee_1^r \mathbb{S}^2 = (M')^{(2)}$$

is homotopic to the identity.

REMARK. It is not difficult to show that g extends to a degree 1 map from M to M'.

Let us consider the map

$$h = (f \times g)|_{M^{(2)}} : M^{(2)} \to (Q \times M')^{(2)} = Q^{(2)} \vee (M')^{(2)}.$$

Lemma 3.1. *The map h extends to a map $M^{(3)} \to M_1^{(3)}$, again denoted by h. Moreover, the induced homomorphism $h_* : \Pi_2(M^{(3)}) \to \Pi_2(M_1^{(3)})$ is bijective (here $M_1 = Q\#M'$ as usual).*

Proof. The obstruction for extending h over $M^{(3)}$ belongs to

$$H^3(M;\Pi_2(M_1^{(3)})) \cong H_1(M;\Pi_2(M_1^{(3)})) \cong 0$$

since $\Pi_2(M_1^{(3)}) \cong \Pi_2(M') \otimes_{\mathbb{Z}} \Lambda$ is Λ-free.

The composition

$$M_1^{(3)} \xrightarrow{j\vee\varphi} M^{(3)} \xrightarrow{h} M_1^{(3)}$$

induces an isomorphism on Π_2, hence the homomorphism

$$h_* : \Pi_2(M^{(3)}) \to \Pi_2(M_1^{(3)})$$

is surjective. Because $\Pi_2(M_1^{(3)})$ is Λ-free, we have

$$\Pi_2(M^{(3)}) \cong \Pi_2(M_1^{(3)}) \oplus \operatorname{Ker} h_*.$$

Observe that $\operatorname{Ker} h_*$ is stably Λ-free (since the Λ-module $\Pi_2(M^{(3)}) \cong K_2(f;\Lambda)$ is stably Λ-free). But tensoring with $\otimes_\Lambda \mathbb{Z}$ gives isomorphisms

$$\Pi_2(M^{(3)}) \otimes_\Lambda \mathbb{Z} \cong \oplus_1^r \mathbb{Z} \cong \Pi_2(M_1^{(3)}) \otimes_\Lambda \mathbb{Z},$$

hence $\operatorname{Ker} h_* \otimes_\Lambda \mathbb{Z} \cong 0$. By Kaplansky' s lemma (see for example [8] and [10]) we get $\operatorname{Ker} h_* \cong 0$. □

Corollary 3.2. *The map $\psi = j \vee \varphi : M_1^{(3)} \to M^{(3)}$ is a homotopy equivalence.*

Proof. It suffices to show that the induced homomorphism

$$\psi_* : \Pi_3(M_1^{(3)}) \to \Pi_3(M^{(3)})$$

is bijective. For this, it is convenient to recall the Whitehead certain exact sequence [18] for a CW-complex X:

$$H_4(X;\Lambda) \longrightarrow \Gamma(\Pi_2(X)) \longrightarrow \Pi_3(X) \longrightarrow H_3(X;\Lambda) \longrightarrow 0.$$

Here $\Gamma(A)$ is the quadratic functor on the category of abelian groups, and $\Pi_3(X) \to H_3(X;\Lambda)$ is the Hurewicz homomorphism. This sequence is natural, hence the map ψ induces a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\Pi_2(M_1^{(3)})) & \longrightarrow & \Pi_3(M_1^{(3)}) & \longrightarrow & H_3(M_1^{(3)};\Lambda) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Gamma(\psi_*)} & & \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \psi_*} & & \\
0 & \longrightarrow & \Gamma(\Pi_2(M^{(3)})) & \longrightarrow & \Pi_3(M^{(3)}) & \longrightarrow & H_3(M^{(3)};\Lambda) & \longrightarrow & 0.
\end{array}$$

It follows from Lemma 3.1 that $\Gamma(\psi_*)$ is an isomorphism. The claim is proved once we have $\psi_* : H_3(M_1^{(3)}; \Lambda) \underset{\cong}{\to} H_3(M^{(3)}; \Lambda)$. If we denote by

$$c : M_1^{(3)} \simeq Q^{(3)} \vee (M')^{(3)} \to Q^{(3)}$$

the projection map, then we can immediately see that $f \circ \psi \simeq c$. Observe that c is the restriction of the collapsing map $c : M_1 \to Q$. Now the result follows from the commutative diagram

$$\begin{array}{ccc} \Lambda \cong H_4(M, M^{(3)}; \Lambda) & \underset{\cong}{\longrightarrow} & H_3(M^{(3)}; \Lambda) \\ f_* \downarrow \cong & & \cong \downarrow f_* \\ \Lambda \cong H_4(Q, Q^{(3)}; \Lambda) & \underset{\cong}{\longrightarrow} & H_3(Q^{(3)}; \Lambda) \\ c_* \uparrow \cong & & \cong \uparrow c_* \\ \Lambda \cong H_4(M_1, M_1^{(3)}; \Lambda) & \underset{\cong}{\longrightarrow} & H_3(M_1^{(3)}; \Lambda). \end{array}$$

□

4. The torsion invariant.

Let us fix the manifolds M^4 and Q^4 with $\Pi_1(M) \cong \Pi_1(Q)$, where Q is aspherical and $H_2(M; \Lambda)$ is Λ-flat. Recall that M and Q are provided with CW-structures such that $M^{(3)} = M \backslash \mathring{D}^4$ and $Q^{(3)} = Q \backslash \mathring{D}^4$. Let $f : M \to Q$ be a classifying map.

We have proved in Section 2 that $H_3(f; \Lambda)$ is Λ-free. Moreover, $H_q(f; \Lambda) \cong 0$ for any $q \neq 3$. Under these conditions one can define the torsion

$$\tau(f) \in \mathrm{Wh}(\Pi_1(Q))$$

of the map f. Namely, it is the torsion of the cellular complex of the pair $((M \times I) \cup_f Q, M \times 0)$, where $(M \times I) \cup_f Q$ denotes the mapping cylinder of f, that is the quotient space

$$(M \times I) \cup_f Q = \frac{(M \times I) \cup Q}{\{(x, 1) \equiv f(x)\}},$$

$I = [0, 1]$. The torsion $\tau(f)$ is defined upon the choice of a Λ-basis of

$$H_3(f; \Lambda) \cong H_2(M; \Lambda).$$

Hence we shall denote it by $\tau_e(f)$, where $e = (e_1, e_2, \ldots, e_r)$ indicates a Λ-basis of $H_2(M; \Lambda)$ (see [15]).

Let us consider the particular case of the collapsing map $g : Q\#M' \to Q$, where M' is simply-connected. Then we have the following (non surprising) result.

Lemma 4.1. *The torsion $\tau_e(g)$ vanishes for any Λ-basis e of $H_3(g;\Lambda)$.*

Proof. Let us consider the following diagram of inclusions:

$$\begin{array}{ccc} X_2 = (Q\backslash \overset{\circ}{D}{}^4) \times I \cup_{g_1} Q & \longrightarrow & X = (Q\#M') \times I \cup_g Q \\ \uparrow & & \uparrow \\ X_1 = (Q\backslash \overset{\circ}{D}{}^4) \times 0 & \longrightarrow & X_3 = (Q\#M') \times 0, \end{array}$$

where $g_1 = g|_{Q\backslash \overset{\circ}{D}{}^4}$. Note that the torsions of each pair are defined because

$$H_2(X_2, X_1;\Lambda) \cong \Lambda \quad \text{and} \quad H_2(X_3, X_1;\Lambda) \cong H_2(M;\Lambda).$$

Moreover, $\tau_e(X, X_3) = \tau_e(g)$ by definition and formula

$$(*) \qquad \tau(X, X_2) + \tau(X_2, X_1) = \tau_e(g) + \tau(X_3, X_1)$$

holds. Since $g_1 : Q\backslash \overset{\circ}{D}{}^4 \to Q$ is the inclusion, it follows that $\tau(X_2, X_1) = 0$ for any choice of a generator in $H_2(X_2, X_1;\Lambda) \cong \Lambda$. Because $X\backslash X_2 = (M'\backslash \overset{\circ}{D}{}^4) \times [0, 1[$ and $X_3\backslash X_1 \simeq M'$ are simply-connected, Lemma 7.3 of [15] implies that $\tau(X, X_2)$ and $\tau(X_3, X_1)$ are both zero for any basis of $H_2(X, X_2;\Lambda)$ and $H_2(X_3, X_1;\Lambda)$, respectively. Note that the other condition in Lemma 7.3 of [15], concerning the universal covering space, is also satisfied. The result $\tau_e(g) = 0$ then follows from formula $(*)$. □

Let now $\psi : Q\#M' \to M$ be the homotopy equivalence constructed above. In particular, the composition map

$$Q\#M' \xrightarrow{\ \psi\ } M \xrightarrow{\ f\ } Q$$

is homotopic to g. Hence, by applying the standard formulae:

$$0 = \tau_e(g) = \tau(\psi) + \tau_e(f),$$

we obtain $\tau(\psi) = -\tau_e(f)$ for any basis e of $H_3(f;\Lambda) \cong H_2(M;\Lambda)$. Any other classifying map $f' : M \to Q$ can be written as $\alpha \circ f \circ \beta$, up to homotopy, where $\alpha : Q \to Q$ and $\beta : M \to M$ are homotopy equivalences. Let $\psi' : Q\#M' \to M$ be the resulting homotopy equivalence. Moreover, we can assume that

$$\beta_* = \mathrm{id} : \Pi_1(M) \to \Pi_1(M)$$

because the effect on Π_1 can be transmitted to α. Now we observe that the effect of β on $\tau(f')$ is restricted to $\beta_* : H_2(M;\Lambda) \underset{\cong}{\to} H_2(M;\Lambda)$, i.e., to a change of basis in

$H_2(M;\Lambda)$. But this has no influence on $\tau(\psi)$. On the other hand, any α changes $\tau(\psi)$. In fact, we have

$$\tau(\psi') = \tau(\psi) + \tau(\alpha).$$

Now we note that the group of homotopy classes of (orientation-preserving) homotopy self-equivalences of Q is isomorphic to $\mathrm{Out}(\Pi_1(Q))$. Hence we can define

$$\tau(M) = [\tau(\psi)] \in \mathrm{Wh}(\Pi_1(Q))/\,\mathrm{Out}(\Pi_1(Q)).$$

The following completes our main result.

Theorem 4.2. *The map* $\psi : Q\#M' \to M$ *is a simple homotopy equivalence if and only if* $\tau(M) = 0$.

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