



Title	On small ring homomorphisms
Author(s)	Harada, Manabu
Citation	Osaka Journal of Mathematics. 1978, 15(2), p. 365-370
Version Type	VoR
URL	https://doi.org/10.18910/10519
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

ON SMALL RING HOMOMORPHISMS

MANABU HARADA

(Received February, 3, 1977)

(Revised May 11, 1977)

The author studied the total quotient ring of a commutative ring R from the point of view of small R -submodules [2]. In this note, we shall extend those methods to a ring extension of R . Let R and R' be commutative rings and $f: R \rightarrow R'$ a ring homomorphism. If $f(R)$ is a small R -submodule of R' , we say f being *small* or R being *small* in R' . In the first section, we shall give a criterion for R to be small in R' in terms of maximal ideals in R and R' and obtain fundamental properties of small homomorphisms. In the second section, we shall give a characterization of maximal ideals M by the multiplicative systems $R - M$ and small homomorphisms.

Throughout this note, we assume every ring R is a commutative ring with identity unless otherwise stated and every ring homomorphism is also unitary, i.e. $f(1)$ is the identity.

The author would like to express his thanks to his colleague Mr. T. Sumioka for his useful advice on Theorem 1.

1. Small homomorphisms

Let R be a (commutative) ring and let $M \supseteq N$ be R -modules. N is called a *small submodule* in M if it satisfies the following condition: the fact $M = N + T$ for some R -submodule T implies $T = M$. Let R' be commutative and $f: R \rightarrow R'$ a ring homomorphism. Then every R' -module may be regarded as an R -module via f . If $f(R)$ is a small R -submodule in R' , we say that f is *small* or R is *small* in R' . Let A and A' be ideals in R and R' , respectively. We put $f(A)R' = AR'$ and $f^{-1}(f(R) \cap A') = A' \cap R$. We shall denote the set of prime ideals by $\text{spec}(R)$ and the set of maximal ideals by $\text{Spec}(R)$. Then we have the induced map $f_*: \text{spec}(R') \rightarrow \text{spec}(R)$.

The following lemma is well known and the proofs are trivial.

Lemma 0. 1) Let $X \supseteq Y \supseteq Z$ be R -modules. If Z is a small R -submodule in Y , so is in X and if Y is small in X , so is Z . 2) Let W be an R -module and $f: X \rightarrow W$ an R -homomorphism. If Z is small in X , $f(Z)$ is small in W . 3) Furthermore, if U is a small submodule in W , $Z \oplus U$ is small in $X \oplus W$.

Theorem 1. *Let R and R' be commutative rings and $f: R \rightarrow R'$ a ring homomorphism. Then the following conditions are equivalent.*

- 1) f is a small homomorphism.
- 2) Every R -finitely generated submodule of R' is small in R' .
- 3) $f_*(\text{Spec}(R')) \cap \text{Spec}(R) = \emptyset$.

Proof. We may assume $R = f(R) \subseteq R'$.

1) \rightarrow 2). Let $N = \sum_{i=1}^l n_i R$ be a finitely generated R -submodule in R' . We consider a standard exact sequence: $F = \sum_{i=1}^l \oplus u_i R' \xrightarrow{h} \sum_{i=1}^l n_i R' \rightarrow 0$. Since R is small in R' , $\sum_{i=1}^l \oplus u_i R$ is small in F from Lemma 0. Hence, $N = h(\sum_{i=1}^l \oplus u_i R)$ is small in $\sum_{i=1}^l n_i R'$ and so in R' from Lemma 0.

2) \rightarrow 1). It is trivial.

1) \rightarrow 3). Let M' be a maximal ideal in R' and put $M = f_*(M') = R \cap M'$. If M is maximal, R/M is a subfield of R'/M' . Hence, there exists an R -submodule L in R' such that $L \supseteq M'$, $L \neq R'$ and $R' = R + L$, which is a contradiction.

3) \rightarrow 1). Let M be a maximal ideal in R . If $MR' \neq R'$, we can take a maximal ideal M' in R' containing MR' . Then $M = M' \cap R$. Hence, $MR' = R'$ for every $M \in \text{Spec}(R)$. Now, we assume $R' = R + T$ for an R -submodule T in R' . Then $R'_M = R'MR_M = R_M M + T_M$. Since R'_M/T_M is a finitely generated R_M -module, $R'_M = T_M$ from Nakayama's Lemma. Hence, $R' = T$.

REMARKS. 1. The condition 3) is equivalent to 3') $MR' = R'$ for $M \in \text{Spec}(R)$.

2. In case R' is a non-commutative ring but an R -algebra, Theorem 1 remains valid. We assume that R is a non-commutative ring with Jacobson radical J such that R/J is artinian. Then we obtain from the above proof that $f: R \rightarrow R'$ is small as a right R -module if and only if $R'J = R'$. Hence, if R is right perfect [1], then any ring extension f is never small. We note that the concept of small homomorphism as a right R -module is different from one as a left R -modules in case of non-commutative rings.

3. The following is also valid for non-commutative rings from Lemma 0, 2).

Let R , R' and R'' be rings and $f: R \rightarrow R'$, $g: R' \rightarrow R''$ ring homomorphisms. If f is small, then gf is small.

We shall give several fundamental properties of a small homomorphism as applications of Theorem 1.

Proposition 2. *Let K be a field and R a subring of K . Then R is small in K if and only if R is not a field.*

Let P be in $\text{spec}(R)$. By μ_P we shall denote the natural homomorphism of R to R_P .

Proposition 3. *Let $f: R \rightarrow R'$ be a ring homomorphism.*

1) *f is small if and only if $f_M: R_M \rightarrow R'_M$ is small for every M in $\text{Spec}(R)$.* 2) *For $P \in \text{spec}(R)$, f_P is small if and only if $P \notin \text{Im } f_*$. In this case $f_P \mu_P$ is also small.* 3) *For $P' \in \text{spec}(R')$ and $P = f_*(P')$, $f_{P'}: R_P \rightarrow R'_{P'}$ is never small, but $f_{P'} \mu_P$ is small if and only if μ_P is small, namely $P \notin \text{Spec}(R)$.*

Proof. 1) $MR' = R'$ for $M \in \text{Spec}(R)$ if and only if $(MR')_N = R'_N$ for every $N \in \text{Spec}(R)$. 2) It is clear from a commutative diagram

$$\begin{array}{ccc} \text{spec}(R_P) & \xleftarrow{f_{P^*}} & \text{spec}(R'_P) \\ \downarrow \cong & & \downarrow \cong \\ A = \{Q \in \text{spec}(R) \mid Q \subseteq P\} & \xleftarrow{f_*} f_*^{-1}(A) & (\subseteq \text{spec}(R')) \end{array}$$

3) It is clear that $f_{P'}(P'R'_P) = R_PP$. Hence, $f_{P'}$ is not small. Furthermore, from Theorem 1 $f_{P'} \mu_P$ is small if and only if P is not maximal.

Proposition 4. *Let $f: R \rightarrow R'$ be a ring homomorphism. Then the following are equivalent.*

- 1) *For any ring homomorphism $g: R' \rightarrow R''$, g is small if and only if gf is small.*
- 2) *$f_*^{-1}(\text{Spec}(R)) = \text{Spec}(R')$.*

Proof. 1) \rightarrow 2). Let M' be maximal in R' . Since $\mu'_M: R' \rightarrow R'_M$ is not small from Theorem 1, $\mu'_M f$ is not small. Hence, $f_*(M') = (\mu'_M f)_*(M'R'_M)$ is maximal. Let P' be in $\text{spec}(R') - \text{Spec}(R')$. Then $\mu_{P'}: R' \rightarrow R'_{P'}$ is small from Theorem 1. Hence, $\mu_{P'} f$ is small. Therefore, $f_*(P') = (f \mu_{P'})_*(P'R'_{P'})$ is not maximal by Theorem 1.

2) \rightarrow 1). We assume g is small. Then $g_*(M'')$ is in $\text{spec}(R') - \text{Spec}(R')$ for any maximal ideal M'' in R'' . Hence, $(gf)_*(M'')$ is not maximal from 2). Therefore, gf is small from Theorem 1. Conversely, we assume gf is small. Then $(gf)_*(M'')$ is not maximal and so $g_*(M'')$ is not maximal from 2). Therefore, g is small.

If $R' = R_M$ for a maximal ideal M , $R' = R(x)$ or R' is integral over R , then they satisfy the above conditions [3].

Let A be an ideal in R . By ρ_A we denote the natural epimorphism of R to R/A .

Proposition 5. *Let R and R' be rings and $f: R \rightarrow R'$ a ring homomorphism. Then the following statements are equivalent.*

- 1) *f is small.*
- 2) *$\rho_{M'} f$ is small for every M' in $\text{Spec}(R')$.*

- 3) $\rho_{MR'} f$ is small for every M in $\text{Spec}(R)$.
- 4) $\rho_{J'} f$ is small for the Jacobson radical J' of R' .
- 5) $\rho_{JR'} f$ is small for the Jacobson radical J of R .

Proof. 1) \leftrightarrow 2) and 1) \leftrightarrow 3) are clear from Remark 3, Proposition 2 and Theorem 1.

4) \rightarrow 1). Let M' be a maximal ideal in R' . Then $\rho_{M'} f = \rho_{M'/J'} \rho_{J'} f$ is small from Proposition 1. Hence, f is small by 2).

5) \rightarrow 1). We can prove it similarly to the above by using 3).

REMARK 4. If R' (resp. R) is local, we can replace 2) (resp. 3)) by $\rho_{A'} f$ (resp. $\rho_{AR'} f$) for some ideal A' (resp. A such that $AR' \neq R'$).

Proposition 6. Let $R \xrightarrow{f} R' \xrightarrow{g} R''$ be rings and ring homomorphisms. We assume that R' is local and gf is small, then either f or g is small, (see Example 2 below).

Proof. Let M' be the unique maximal ideal in R' . If $R \cap M'$ is maximal, $R'' = R''(R \cap M') = R''M'$ from Theorem 1.

2. Quotient rings

Let S be a multiplicative system in R . If $\mu_S: R \rightarrow R_S$ is small, S is called *large*. If S satisfies the following two conditions, we call S *critical*.

- 1) If $S \subseteq S'$, S' is large.
- 2) If $S \supseteq S'$, S' is not large,

where S' is a multiplicative system in R .

We obtain immediately from Theorem 1

Proposition 7 ([2]). Let S be a multiplicative system. Then the following are equivalent.

- 1) S is large.
- 2) $M \cap S \neq \emptyset$ for every M in $\text{Spec}(R)$.

Theorem 8. Let R be a commutative ring. Then there exists a one-to-one mapping between $\text{Spec}(R)$ and the set of critical multiplicative systems S in R as follows: $M = R - S$ and $S = R - M$, where $M \in \text{Spec}(R)$.

Proof. Let M be a maximal ideal and $S = R - M$. Then it is clear from Proposition 7 that S is critical. Conversely, let S be critical. Since S is not large, there exists a maximal ideal M' such that $M' \cap S = \emptyset$ from Proposition 7. Then we obtain again from Proposition 7 and the definition that $S = R - M'$.

Proposition 9. R is never small for any non-zero ring homomorphism $f: R \rightarrow$

R' if and only if MR_M is a nil ideal for every maximal ideal M in R .

Proof. "Only if" part. We may assume R is local from Proposition 3. If there exists m in M which is not nil, then $\{m_i\}_i$ is large from Proposition 7. Which is a contradiction. "If" part. If $R'M=R'$, $1=\sum_{i=1}^t r'_i m_i$; $r'_i \in R'$, $m_i \in M$. There exists s in $R-M$ such that $sm_i^n=0$ for all i and some n . Then $s=s(\sum_{i=1}^t r'_i m_i)^{tn}=0$.

Proposition 10. *Let R be an integral domain and K the field of quotients. Then R is local if and only if R is small in any subring T is K such that $T \supset R$ and there exists an element $a^{-1} \in T-R$, $a \in R$.*

Proof. Let R be a local and T as above. Then $\{a^i\}_i$ is large from Proposition 7. Hence, R is small in T by Remark 3. Conversely, let M be maximal. Then R is not small in R_M . Hence, $R=R_M$ from the assumption.

Proposition 11. *Let R be a domain with K quotient field. Then the following are equivalent.*

- 1) *Let R' be an over ring of R . If R is small in R' , $R'=K$.*
- 2) *Krull dim $R=1$ i.e. every non-zero prime is maximal in R .*

Proof. 1) \rightarrow 2). Let P be a non-zero prime ideal. Then $R_P=R$ or R is not small in R_P . Hence, P is maximal from Proposition 7.

2) \rightarrow 1). Let $K \supseteq R'$ be an over ring and R be small in R' . Then for every maximal ideal M' , $M' \cap R \neq 0$ is not maximal, which is a contradiction.

Proposition 12. *Let R be a Dedekind domain and L an R -submodule in K containing R . Then*

- 1) *R is small in L if and only if $L \supset \sum_P P^{-1}$*

where P runs through the set P of non-zero primes in R .

If L is a subring, then

- 2) *R is small in L if and only if $K=L$, and L is small in K as an R -module if and only if $L=R$.*

Proof. Since $K/R = \sum_P \oplus (\sum_n P^{-n}/R)$ and every R -submodule in $\sum_n P^{-n}/R$ is of P^{-m}/R , $L = \sum_P P^{-n(P)}$; $n(P) \geq 0$. First, we shall show R is small in $\sum_P P^{-1} = A$. If $A=R+T$, $A_P = P^{-1}R_P = R_P + T_P$. Let $R_P P = (p)$ and $R_P \cap T_P = (p^e)$. Then $p^{-1} = r + ts^{-1}$; $r \in R_P$, $t \in T$ and $s \in R-P$. Hence, $s(1-rp) = tp \in T_P \cap R_P$ and so $(1-rp) \in (p^e)$. Therefore, $e=1$ and $A_P = T_P$ for every P . Accordingly, $A=T$. Next, we consider a submodule $A(L) = \sum_{P \neq L} P^{-n(P)}$. Then we can show as above $A(L) = R + LA(L)$ and $A(L) \neq LA(L)$. Hence, R is not small in $A(L)$.

We have proved 1). 2) is clear from 1) and the structure of K/R .

EXAMPLES 1) Let K be a field and x an indeterminant. Then K is not small in $K(x)$, however $K[x]$ ($\supset K$) is small in $K(x)$ (cf. Lemma 0).

2) Let Z be the ring of integers with Q quotient field and p a prime. Then $Z_p[x]$ is not small in $Q[x]$, since $(px-1)$ is a maximal ideal in $Z_p[x]$ such that $Q[x](px-1) \neq Q[x]$. Hence, Proposition 6 is not true without the assumption "local."

3) Let $R=K[x,y]_{(x,y)}$. Then R is not small in $R[yx^{-1}]$ as an R -module and $R[yx^{-1}]$ does not contain any element a^{-1} as in Proposition 10.

4) $Z_p=Z_p/((xp-1) \cap Z_p)$ is small in $Q=Z_p[x]/(xp-1)$, but Z_p is not small in $Z_p[x]$ (cf. Proposition 5).

OSAKA CITY UNIVERSITY

References

- [1] H. Bass: *Finitistic dimension and a homological generalization of semiprimary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.
- [2] M. Harada: *On small submodules in the total quotient ring of a commutative ring*, Rev. Union Mat. Argentina **28** (1977), 99–102.
- [3] M. Nagata: *Local ring*, Interscience, New York, 1962.