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Author(s)	Htun, Thant Zin; Suzuki, Hiroyoshi; García-Vallejo, Daniel
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Dynamic Modeling of a Radially Multilayered Tether Cable for a Remotely-operated Underwater Vehicle (ROV) based on the Absolute Nodal Coordinate Formulation (ANCF)

Thant Zin Htun^{1*}, Hiroyoshi Suzuki¹, Daniel García-Vallejo²

¹*Dept. of Naval Architecture and Ocean Engineering, Graduate School of Engineering, Osaka University,
2-1, Yamadaoka, Suita, Osaka, 565-0871, Japan*

²*Department of Mechanical and Materials Engineering, Universidad de Sevilla, Camino de los descubrimientos s/n, 41092 Seville, Spain*

Abstract

This research is aimed at the accurate modeling of an ROV's tether cable with a radially multilayered circular cross-section. In the coupled motion analysis of an ROV-tether system, dynamic modeling of a numerical tether continuum, which can accurately reproduce the nonlinear motion behavior of a flexible cable, is crucial since the tether motion greatly affects the ROV's stability. To this end, a new tether cable element based on the multilayered modeling approach is developed. To deal with the inconsistency of the tether's cross-section, the integration domain on the cross-section is divided into subdomains to account for different material properties in each layer and an accurate representation of the tether's cross-section. A Kelvin-Voigt model is employed to account for the viscoelastic behavior of the inner matrix, while nearly incompressible Mooney-Rivlin model is applied for the outer sheath. Structural weakening parameters are applied to modify the beam's strain energy. A new approach for analytical determination of dissipation factors for dilatational and deviatoric responses is proposed. Through validations against the numerical studies and experiment, the effectiveness of the proposed tether cable element is verified.

Keywords: Dynamic modeling, Tether cable, ANCF, Viscoelastic, Multilayer beam

1. Introduction

Wires and fiber-optic cables are parts of many machines and are widely used in onshore, offshore and underwater applications. They come in various configurations and compositions with different geometrical arrangements of wire strands. Due to its inextricably intertwined interaction with other bodies and influence on the system's overall behavior, modeling dynamics of cables in multibody systems (MBS) has recently drawn special attention to researchers and engineers. Meanwhile, the subsea cable systems in offshore and underwater applications provide the vital power supply and the data telemetry between the operators/station and the vehicle/equipment. Also, the tether-operated underwater systems have proven extremely valuable in deep-sea explorations and are still widely used in both deep and shallow waters. In ROV operations, the tether can provide the operator not only the virtual telepresence but also the real-time data telemetry, *vice versa*, it serves as a physical-link feeding the power to the ROV. On the other hand, the tether cable brings about the undesired drag forces and nonlinear internal forces on ROV [1]. These forces significantly affect the vehicle and maneuverability, complicating the job of a human pilot.

As the cables are composed of composite polymers and multilayers, consequently the cable exhibits both material and geometric nonlinearities in practice. Subsea cables are usually composed by polymer materials such that rubber-like ether-

* Corresponding author. Tel.: +81-6-6879-7589; fax: +81-6-6879-7594; e-mail: tzhtun@naoe.eng.osaka-u.ac.jp

based elastomers are used for cable sheathing, and polypropylene resins, for insulation among the bundles of copper wires and outer cable jackets. When the cable undergoes large deformation, interlayer slip may occur between the adjacent layers if they are not perfectly bonding. This interface slipping occurs when the fluctuation of loads exceeds the binding force between the interlayers, consequently, the relative motion between the layers may complicate the dynamic modeling of cables. Besides, the cable usually has a quite low bending stiffness due to its large length-to-diameter ratio while its axial stiffness and torsional rigidity are relatively high. Therefore, accurate modeling of tether cable, which can cover all aspects of modeling, needs intensive study and experimental works. In the case of ROV's tether cables, they are very long and stored winding around a cable drum since after being manufactured from the factory and when not in use. Therefore, the residual stresses induced by the winding are also locked into the tether cable. When the ROV is launched from the cage, the tether cable is paid out by reeving through the sheave of the cable drum, and the tether configuration may not be straight due to the winding-induced curvatures. Besides, the tether cable is slack and low tensioned mostly during the ROV operations. Thus, bending forces and moments produced by the tether cable are dominant on the ROV's motion. To the best of the authors' knowledge, a numerical tether model of ROV, which can represent the geometrical compositions and can predict static/dynamic behaviors accurately, has not been well documented and published in the literature. Obtaining a general algorithm of a numerical tether model applicable to a wide range of operating conditions, therefore, remains a challenging topic.

Our main focus is to construct a reliable and accurate modeling of an ROV-TMS (Tether Management System) by investigating a coupled motion analysis of these rigid-flexible multi-bodies. To embody this goal, coupled motion analyses were performed in previous works [1, 2] based on the computational fluid dynamics (CFD) and finite element method (FEM). In the previous studies, the tether continuum, which was modeled by using lower-order ANCF beam elements, could not deliver accurate prediction of bending behavior of the tether. On the other hand, when the snap-through loading occurs, i.e., when the slack cable becomes taut with sudden changes in tension variation, it was found that the coupled interaction between the ROV and the tether seemed inaccurate and impractical. To this end, a new tether cable element based on the multilayered modeling approach is developed. In this modeling, a higher-order ANCF beam element, which employs quadratic approximations in transverse directions, is used to capture the dynamic behavior of the tether more accurately. To deal with the inconsistency of the tether's cross-section, the integration domain on the cross-section is divided into subdomains to account for different material properties in each layer and an accurate representation of the tether's cross-section.

In the literature, the ANCF method has been used in the multibody dynamics for the last two decades. This method imposes no restrictions on the amount of deformation or rotation and, therefore, it can be used for the linear as well as the nonlinear problems. It is, in general, more suited for large deformations and large rotation problems, and it can describe the exact description of rigid body motions including finite rotations [3, 4]. This formulation leads to a constant mass matrix and general constitutive models can be readily implemented in the case of the beam, plate, and shell problems. In addition, this method has a sturdy framework for the modeling of cables and ropes. Therefore, ANCF method is utilized in our modeling of tether cable. Based on the ANCF framework, wires and ropes were modeled using the cable elements, which usually lack one slope vector for transverse direction. Berzeri and Shabana [5], firstly introduced a planar beam element in ANCF and provided different elastic models based on continuum mechanics approach for two-dimensional beams. Sugiyama et al. [6] developed a spatial cable beam element based on the general continuum mechanics approach and the simulation results showed good agreement with the analytical procedure based on the classical cable beam theory. Gerstmayr and Shabana [7] provided a higher-order ANCF beam element for thin beams and cable problems by introducing additional shape functions to a conventional ANCF beam element (denoted by LO24 hereafter), which was originally presented in [3], and a lower-order cable element, which has only one position vector and a gradient vector w.r.t. the beam's axial coordinate. Though this cable element performed very efficiently in bending predominant applications, the beam could not describe cross-sectional deformation and torsion due to the incomplete sets of slope vectors. Wang et al. [8] used ANCF cable elements to simulate the frictional contact dynamics of thin beams with circular cross-sections based on the penalty method and the ANCF. Zhang et al. [9] proposed a computational strategy for the adaptive ANCF (AANCF) method, and planar AANCF cable elements were used in his study. However, the use of the above-mentioned cable elements has some shortcomings in applications. First, the cable element lacks the complete set of slope vectors, which can describe the deformation regarding the cross-section such as torsion, shear, and coupled flexural deformation modes. Second, as the tether cable is, in fact, a typical multilayered composite structure, the use of the cable elements fails to represent the material coordinates radially across the cross-section.

In other works regarding multilayer modeling based on the ANCF framework, the following works can be found in the literature. Liu et al. [10] presented a method to model the composite laminated plates by dividing the plate's thickness into subintervals to account for different material properties. Patel et al. [11] proposed a computational procedure to model the

complex geometry and modeling of tires using ANCF. The basic idea is to divide the integration domain into subdomains which represent different material properties within a single element. Orzechowski and Frączek [12] developed a multi-layered beam model with rectangular cross-sections, where each layer shares the same height of the beam and has the same width. To enforce the layer connectivity at the interface, kinematic constraints are imposed on the position and slope levels. In recent work by Lan et al. [13], the LO24 beam element was used in the modeling of a transmission wire and provided a detailed review of the integration technique for circular cross-sections, which was originally presented in [14, 15].

Regarding the marine and subsea applications, some mathematical models of cables can be found in the literature [16, 17, 18 and 19]. Their works are briefly reviewed herein. Ajayi et al. [16] presented a mathematical model to simulate the dynamics of a subsea umbilical cable and investigated the effect of axial tension on cable's frequency of oscillation. Cheepsakul et al. [17] used a variational model formulation in which the finite element method and shooting-optimization method were employed for simulating the steady-state behavior of subsea cables with specified top tension. Kamman and Huston [18] presented a towed cable model using the lumped mass method in which the cable was modeled as a series of rigid links connected with spherical joints and hydrodynamic forces of the surrounding water were incorporated at the joints. Radanovic and Driscoll [19] proposed an underwater cable model using a finite element method with linear elements. In their study, the cable was assumed to be mostly taut, and bending stiffness was neglected. Nevertheless, the aforementioned subsea cable models for marine applications usually neglected the bending stiffness and could not account for the geometric and material nonlinearities. Besides, these models mesh the umbilical cables, which usually have the inhomogeneous cross-sections and anisotropic behaviors, as an isotropic beam with circular solid cross-section. Consequently, they could not express the accurate geometrical description of the cable's cross-section and fail to capture the dynamics of the cable accurately. Therefore, the cable configurations may not be realistic when the cable is mostly slack and undergoes large relative translational motion and rotations. The formulation used in this research automatically eliminates such limitations in numerical modeling of subsea cables and meets the requirements for accurate modeling of underwater tether cables.

There are operating conditions of tether cable systems, like for instance the tether unwinding operation, where an arbitrary Lagrangian-Eulerian finite element description might lead to very efficient models as reported in the literature for tethered-satellite deployment dynamic simulation [20] or in the simulation of reeving systems [21]. However, since the present manuscript considers situations where the undeformed length of the cable is constant, the Lagrangian description adopted for the cable model development seems to be a reasonable modeling approach.

Apart from that, a good estimation of internal damping is also important in the modeling of cables and wires. In marine applications, the damping caused by the surrounding fluid on the cable's motion is known to be predominant. Meanwhile, in the ROV's operations in deep-water, the loads from marine-environments, the motion of the mother vessel at the surface, motion of the ROV's cage by ocean currents and motion of the ROV can cause the tension variations in tether cables. Also, they may tend to be easily excited due to their low weight in water and relatively high axial stiffness. Therefore when the tether is exposed to the ocean currents in the deep-water, it may vibrate due to the transverse oscillating forces caused by the vortex shedding. In some cases, it may cause severe damage to the system and therefore some investigations on the effect of internal damping in fatigue analysis of the subsea cables due to VIV (Vortex-Induced Vibration) can be found in the literature [22, 23]. Therefore, it is necessary to incorporate internal damping which is, in the case of subsea cables, the synergetic effect of material damping (viscoelasticity) and the structural damping corresponding to the interlayer slips, in the modeling. However, the present study assumes that there are no interlayer slips between layers so that each layer shares the same nodal coordinates and nodal variables to achieve an efficient formulation.

With the problems discussed above and to be consistent with our goal, a new tether cable element based on the multilayered modeling approach is presented. The main novelty of this study is attributed to (i) modeling of a radially multi-layered and inhomogeneous beam with a circular cross-section considering the mechanical properties of each layer, (ii) a systematic approach to incorporate material damping in the modeling and (iii) modification of strain energy for the tether cable by utilizing structural weakening coefficients. In addition, the proposed tether model fulfills the following fundamental requirements when modeling the dynamic tether continuum of ROV. Firstly, the tether model must be able to describe the exact modeling of rigid-body motion with zero strains. This is because the tether cable undergoes large relative translational motion and rotations under different types of combined loadings. Secondly, the beam used in modeling the numerical tether must employ sufficient polynomials in both axial and transverse directions, i.e., quadratic polynomials, at least, should be used for transverse directions to capture the bending modes more accurately. Besides, the use of quadratic approximations contributes to suppressing volumetric and Poisson locking problems, which are common issues in conventional fully parameterized lower-order ANCF beam elements. Thirdly, with such reasonably large numbers of nodal coordinates, the tether model should be able to execute highly nonlinear deformations when applying different constitutive models for each layer on the cross-section. Not least of all, the last essential requirement is the performance of the proposed beam for accurate prediction of the dynamic behavior of the tether at the reasonable expense of computational loads. This is because the computational efficiency is also one of the major factors that guarantee its performance in dynamic analysis.

1.1. Scope of the paper

The main purpose of this study is devoted to modeling of subsea tether cables used for ROV's operations. This paper proposes a numerical tether model based on the multilayered modeling approach utilizing a nonlinear finite element method, ANCF. The tether is treated as a slender beam with a circular cross-section, which is radially inhomogeneous and composed by different material properties. To account for the material inhomogeneity of the cross-section, the element volume is radially divided into subdomains. Inertia properties and internal forces regarding the different material properties are evaluated for each subdomain within a single element. The beam's cross-section is divided into two layers; inner-core wires-resin matrix layer and outer sheath layer. The inner-core layer is modeled as a viscoelastic material, and nonlinear Green-Lagrange strain-displacement relationship, which automatically accounts for the geometric nonlinearities, is used in the evaluation of viscoelastic forces. Structural weakening parameters, which allow one to modulate the axial, bending and torsional stiffnesses, are introduced to modify the strain energy of the beam. Meanwhile, a two-parameter Mooney-Rivlin material model is applied for nearly incompressible hyperelastic *polyurethane* elastomer. The performance of the proposed model is investigated through well-known static and dynamic numerical examples and experimental results.

The contents of the paper are structured as follows: Section 1 presents the introduction to modeling of the subsea cables, and the physical properties of the tether cable are given in Section 2. Section 3 provides a brief review of the higher-order ANCF beam element. Section 4 provides a detailed discussion on the efficient formulation of the viscoelastic forces and an analytical determination of dissipation factors is presented. The formulation of hyperelastic forces and its Jacobian based on the incompressible Mooney-Rivlin model is presented in Section 5. Section 6 discusses the dynamic modeling of the tether cable and solving the constrained motion equations in MBS. Section 7 presents the performance of the proposed tether model through numerical tests and experimental results. Section 8 summarizes the contents of the paper and draws some conclusions based on the results.

2. Description of tether cable

A blue tether cable used in this study is a type of subsea Ethernet cable with a *polyurethane* outer jacket and shielding made of aluminum/polyester tape and tinned copper braids. In the inner matrix layer, each copper conductor is insulated with TPE (thermoplastic elastomers) and the matrix is filled with *polypropylene* filling compound. The cross-section of the tether cable is shown in Fig. 1. This cable is tethered to a small-scale working class ROV, named CU-ROV, which was developed by JAMSTEC (*Japan Agency for Marine-Earth Science and Technology*).

Table 1
Parameters of the tether cable.

Description	inner matrix (Layer I)		outer sheath (Layer II)	
	symbol	values	symbol	values
Beam length (m)	l_1	0.9	l_2	0.9
Density (kg/m ³)	ρ_1	2082.653	ρ_2	1350
Young's modulus (MPa)	E_1	24	E_2	513.796
Outermost diameter (m)	d_1	0.007	d_2	0.01
Thickness of outer sheath (m)			t_2	0.003
Area moment of inertia (m ⁴)	I_1	1.179×10^{-10}	I_2	3.730×10^{-10}
Poisson's ratio	ν_1	0.35	ν_2	0.4997

The physical properties for each layer of the tether cable are given in Table 1. The Young's modulus of each layer is evaluated according to the rule of mixtures based on the effective Young's modulus (278 MPa) of the tether cable, which is obtained from the cantilever bending tests conducted in [2].

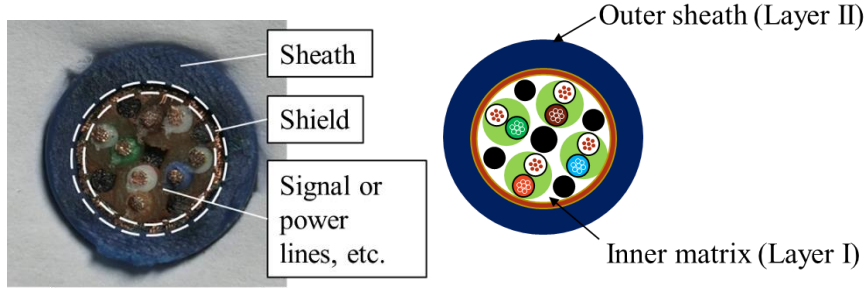


Fig. 1. Cross-section of the tether cable sample

3. Kinematics of a higher-order ANCF beam element

A three-dimensional fully parameterized higher-order ANCF beam element, which has forty two nodal coordinates and employs full quadratic approximation in transverse directions, is used to discretize the tether continuum. This beam element (denoted by HO42 hereafter) was initially introduced in [24] and further investigated in [25]. Despite the fact that this beam is computationally quite expensive due to the use of a large number of nodal coordinates and needs higher integration points compared to the LO24 ANCF beam, it provides the quality for suppressing volumetric and Poisson locking problems and can give reasonable results when modeling nonlinear material models [26] as well. Nonetheless, the computational loads can be dramatically reduced by applying reduced integration, and invariant matrices introduced in [27], where integrations of shape functions and its derivatives can be evaluated and stored systematically at the preprocessing stage. This will be further described in detail in the following section. It is worth mentioning here that the nodes of the element and their nodal variables will be used to describe the dynamics of both the inner matrix layer and the outer sheath.

A continuum mechanic approach is used for defining nonlinear internal forces of the tether. Using this approach, the evaluation of the viscoelastic and hyperelastic forces and their Jacobians can be implemented in a straightforward manner. In the ANCF framework, the global position vector $\mathbf{r}$ of an arbitrary point on a three-dimensional element can be described using the global shape function $\mathbf{S}$ and the absolute nodal coordinates $\mathbf{e}$ as

$$\mathbf{r} = \mathbf{S}(x, y, z) \mathbf{e} \quad (1)$$

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} a_0 + a_1x + a_2y + a_3z + a_4xy + a_5xz + a_6x^2 + a_7x^3 + a_8yz + a_9y^2 + a_{10}z^2 + a_{11}xyz + a_{12}xy^2 + a_{13}xz^2 \\ b_0 + b_1x + b_2y + b_3z + b_4xy + b_5xz + b_6x^2 + b_7x^3 + b_8yz + b_9y^2 + b_{10}z^2 + b_{11}xyz + b_{12}xy^2 + b_{13}xz^2 \\ c_0 + c_1x + c_2y + c_3z + c_4xy + c_5xz + c_6x^2 + c_7x^3 + c_8yz + c_9y^2 + c_{10}z^2 + c_{11}xyz + c_{12}xy^2 + c_{13}xz^2 \end{bmatrix} \quad (2)$$

where r_1, r_2 and r_3 are the components of the global position vector $\mathbf{r}$ of the HO42 beam element, which includes forty two polynomial coefficients. The element shape function matrix of the HO42 beam element is given by

$$\mathbf{S} = [\mathbf{S}_1 \mathbf{I} \quad \mathbf{S}_2 \mathbf{I} \quad \mathbf{S}_3 \mathbf{I} \quad \mathbf{S}_4 \mathbf{I} \quad \mathbf{S}_5 \mathbf{I} \quad \mathbf{S}_6 \mathbf{I} \quad \mathbf{S}_7 \mathbf{I} \quad \mathbf{S}_8 \mathbf{I} \quad \mathbf{S}_9 \mathbf{I} \quad \mathbf{S}_{10} \mathbf{I} \quad \mathbf{S}_{11} \mathbf{I} \quad \mathbf{S}_{12} \mathbf{I} \quad \mathbf{S}_{13} \mathbf{I} \quad \mathbf{S}_{14} \mathbf{I}] \quad (3)$$

where $\mathbf{I}$ is the 3×3 identity matrix. The shape functions for the HO42 beam element are written as

$$\begin{cases} S_1(x) = 1 - 3\xi^2 + 2\xi^3, \quad S_2(x) = l(\xi - 2\xi^2 + \xi^3), \quad S_3(x, y) = l\eta(1 - \xi), \quad S_4(x, z) = l\zeta(1 - \xi), \\ S_5(x, y, z) = l^2\eta\zeta(1 - \xi), \quad S_6(x, y) = \frac{l^2\eta^2}{2}(1 - \xi), \quad S_7(x, z) = \frac{l^2\zeta^2}{2}(1 - \xi), \quad S_8(x) = 3\xi^2 - 2\xi^3, \\ S_9(x) = l(\xi^3 - \xi^2), \quad S_{10}(x, y) = l\eta\xi, \quad S_{11}(x, z) = l\zeta\xi, \quad S_{12}(x, y, z) = l^2\eta\zeta\xi, \\ S_{13}(x, y) = \frac{l^2\eta^2}{2}\xi, \quad S_{14}(x, z) = \frac{l^2\zeta^2}{2}\xi, \end{cases} \quad (4)$$

where $\xi = x/l$, $\eta = y/l$, $\zeta = z/l$, l represents the initial undeformed length of the beam in the reference configuration. The nodal coordinate vector $\mathbf{e}$ of two-node HO42 beam element with node i and $i+1$ can be defined as

$$\mathbf{e} = [\mathbf{r}_i^T \mathbf{r}_{i,x}^T \mathbf{r}_{i,y}^T \mathbf{r}_{i,z}^T \mathbf{r}_{i,yz}^T \mathbf{r}_{i,yy}^T \mathbf{r}_{i,zz}^T \mathbf{r}_{i+1}^T \mathbf{r}_{i+1,x}^T \mathbf{r}_{i+1,y}^T \mathbf{r}_{i+1,z}^T \mathbf{r}_{i+1,yz}^T \mathbf{r}_{i+1,yy}^T \mathbf{r}_{i+1,zz}^T]^T \quad (5)$$

where $\mathbf{r}_{i,x}$, $\mathbf{r}_{i,y}$ and $\mathbf{r}_{i,z}$ are position gradient vectors at node i . The derivative $\partial^2 \mathbf{r}_i / \partial y \partial z$ can be geometrically interpreted as a curvature vector, which can describe the cross-sectional shrinkage. Thereby, it contributes to alleviate Poisson locking. $\partial^2 \mathbf{r}_i / \partial y^2$ and $\partial^2 \mathbf{r}_i / \partial z^2$ can be geometrically interpreted as curvature vectors, which vary linearly on the beam's longitudinal coordinate. Due to the quadratic approximations over the transverse directions, the HO42 beam element can predict bending and cross-section deformations more accurately compared to the LO24 beam element. A description of the deformations of the tether cable element based on the ANCF is given in Fig. 2.

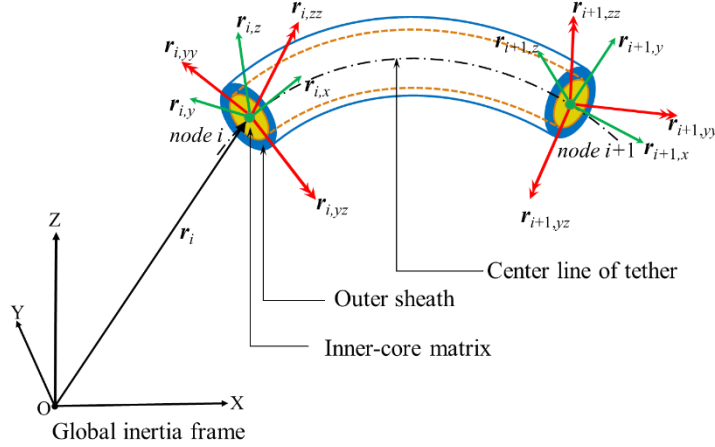


Fig. 2. Deformation of tether cable element in global inertia frame

4. Viscoelastic forces of a three-dimensional ANCF beam element

4.1. Constitutive equations for linear viscoelasticity based on ANCF

In the MBS framework based on the ANCF, different viscoelastic constitutive models for isotropic and linear-elastic materials were proposed in [28, 29, 30 and 31]. Ma Chao et al. [28] proposed an ANCF solid element with internal viscous damping based on the Kelvin-Voigt model. Mohamed and Shabana [29] proposed a nonlinear objective viscoelastic constitutive model as a general form of modified Kelvin-Voigt model. In their model, the damping forces depend on the strains nonlinearly and linearly on the strain rates. However, their models need to use the QR decomposition and beam's cross-section tangent frames to determine the deformation gradient tensor. Therefore, this model lacks the uniqueness for the problems concerned in our work due to the following facts; (i) the strain-dependent matrix in their model depends on the type of decomposition used, and (ii) the damping factors related to dilatational and deviatoric stresses remain as unknowns without the experimental determination. Zhang et al. [30] proposed a linear viscoelastic model using a fractional derivative method, which replaces the first time-derivative of the strain in the Kelvin-Voigt model with a fractional operator. Then, the fractional operator can be approximated by adopting the truncated *Grünwald* series. In this model, however, the material response strongly depends on the selection of the fractional index and the truncation number, meanwhile, there is no proven mean presented for choosing those parameters.

García-Vallejo et al. [31] proposed an internal damping model for the ANCF beam elements, based on the Rayleigh functions. Their developed model accounts for the linear viscoelastic relations to multiaxial stress states by distinguishing the material response to deviatoric and dilatational excitations. Yet, the Poisson's effect was neglected in both elastic and viscous terms to avoid Poisson locking issues. Based on their Rayleigh damping model, a Kelvin-Voigt model which can characterize the material responses to the bulk and deviatoric stresses can be developed. When the Poisson's effect is taken into account, the following constitutive equations, then, can be written for isotropic homogeneous viscoelastic material;

$$\begin{cases} \mathbf{S} = \mathbf{E}_d : \boldsymbol{\varepsilon}_d + \mathbf{D}_d : \dot{\boldsymbol{\varepsilon}}_d = (2\mu(1+\nu)\mathbf{I}) : \boldsymbol{\varepsilon}_d + (2\mu(1+\nu)\gamma_d\mathbf{I}) : \dot{\boldsymbol{\varepsilon}}_d \\ p = E_t \varepsilon_t + D_t \dot{\varepsilon}_t = 3K(1-2\nu)\varepsilon_t + 3K(1-2\nu)\gamma_s \dot{\varepsilon}_t \end{cases} \quad (6)$$

where $\mathbf{S}$ and $\boldsymbol{\varepsilon}_d$ are deviatoric stress and strain tensors respectively. $\mathbf{E}_d$ and $\mathbf{D}_d$ are the tensors of elastic and damping coefficients corresponding to the deviatoric stresses. p and ε_t are, respectively, the dilatational stress and strain. E_t and D_t are the elastic and damping coefficients corresponding to the dilatational stresses respectively. $K = E/3(1-2\nu)$ is the bulk modulus of the material, $\mu = E/2(1+\nu)$ is the shear modulus, and ν is the Poisson's ratio. Hence, the stress components of the isotropic homogeneous viscoelastic material can be defined as

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} + \mathbf{D}\dot{\boldsymbol{\varepsilon}} \quad (7)$$

where $\mathbf{E}$ and $\mathbf{D}$ are the matrices of elastic and damping coefficients respectively. $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$ are the Green-Lagrange Strain tensor and its time-derivative respectively. The elasticity matrix $\mathbf{E}$ can be expressed in terms of Lamé's constants, λ and μ .

$$\mathbf{E} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix}, \quad (8)$$

where $\lambda = E\nu/[(1+\nu)(1-2\nu)]$. Similarly, the damping coefficients matrix $\mathbf{D}$ can be defined in terms of the bulk modulus K and shear modulus μ as follow;

$$\mathbf{D} = \begin{bmatrix} K\gamma_s + \frac{4}{3}\mu\gamma_d & K\gamma_s & K\gamma_s & 0 & 0 & 0 \\ K\gamma_s & K\gamma_s + \frac{4}{3}\mu\gamma_d & K\gamma_s & 0 & 0 & 0 \\ K\gamma_s & K\gamma_s & K\gamma_s + \frac{4}{3}\mu\gamma_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu\gamma_d & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu\gamma_d & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu\gamma_d \end{bmatrix}, \quad (9)$$

where γ_d and γ_s are the dissipation factors associated to the deviatoric and dilatational stresses, respectively. γ_d and γ_s can be estimated by utilizing an analytical method [31]. In the following subsection, the method for determining the dissipation factors are briefly reviewed and a new approach, which relaxes the assumptions made in the original procedure, is used to analytically determine the value of γ_s .

4.2. Determination of dissipation factors

In the literature [31], it was assumed that the dilatational damping factor γ_d can be calculated as a fraction ξ of its critical value for a pure deviatoric (torsion) test case. In that case, γ_d can be approximately determined as follows;

$$\gamma_d = \xi \gamma_d^{\text{crit}}, \quad \gamma_d^{\text{crit}} = \frac{2}{\omega_n^d}, \quad (10)$$

where ω_n^d is the corresponding torsional natural frequency of the test problem. The first torsion natural frequency can be calculated as

$$\omega_n^d = \frac{\alpha}{l} \sqrt{\frac{C_t \mu}{\rho I_p}}, \quad (11)$$

where $\alpha = \frac{\pi}{2}$, ρ is the material density, μ is the shear modulus, I_p is the polar moment of the cross-section, and l is the length of the beam. The torsion factor C_t can be written as

$$C_t = \begin{cases} \frac{wh^3}{16} \left[\frac{16}{3} - 3.36 \frac{h}{w} \left(1 - \frac{1}{12} \frac{h^4}{w^4} \right) \right], & \text{for rectangular cross-section} \\ \frac{\pi R^4}{12}, & \text{for circular cross-section} \end{cases} \quad (12)$$

On the other hand, a volumetric dissipation factor γ_s can be determined from a pure bending or longitudinal test case, which can give rise to the axial stress only.

$$\sigma_{xx} \neq 0, \quad \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0, \quad (13)$$

where the axial stress produced in a viscoelastic material can be expressed according to [31] as

$$\sigma_{xx} = E\varepsilon_{xx} + D\dot{\varepsilon}_{xx} = E\varepsilon_{xx} + E\gamma_{xx}\dot{\varepsilon}_{xx} \quad (14)$$

If one compares equations (6), (9) and (14), γ_{xx} holds a relationship between two dissipation factors (γ_d and γ_s), bulk modulus K , and shear modulus μ . Therefore, it is more relevant to seek an equivalent expression of γ_{xx} in terms of those parameters. A new approach is presented here to determine the relation between γ_{xx} and γ_s without imposing any limitation on the value of ξ . First, γ_{xx} is estimated as

$$\gamma_{xx} = \xi \gamma_{xx}^{\text{crit}}, \quad \gamma_{xx}^{\text{crit}} = \frac{2}{\omega_n^b} \quad (15)$$

$$\omega_n^b = \frac{\beta}{l} \sqrt{\frac{EI}{ml}} \quad (16)$$

where β is a constant corresponding to the n^{th} mode of vibration and boundary condition. E is Young's modulus, I is the second moment of area, and m and l represent the beam's mass and length respectively. Using equations (8), (9) and (14), the expression for σ_{xx} can be given as

$$\sigma_{xx} = (\lambda + 2\mu) \varepsilon_{xx} + \lambda (\varepsilon_{yy} + \varepsilon_{zz}) + \left(K\gamma_s + \frac{4\mu\gamma_d}{3} \right) \dot{\varepsilon}_{xx} + \left(K\gamma_s - \frac{2\mu\gamma_d}{3} \right) (\dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz}) \quad (17)$$

Note that $\lambda = K - \frac{2\mu}{3}$ and σ_{xx} can be expressed in terms of K , μ , γ_d and γ_s ;

$$\sigma_{xx} = \left(K + \frac{4\mu}{3} \right) \varepsilon_{xx} + \left(K - \frac{2\mu}{3} \right) (\varepsilon_{yy} + \varepsilon_{zz}) + \left(K\gamma_s + \frac{4\mu\gamma_d}{3} \right) \dot{\varepsilon}_{xx} + \left(K\gamma_s - \frac{2\mu\gamma_d}{3} \right) (\dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz}) \quad (18)$$

Similarly, the expressions for σ_{yy} and σ_{zz} can be obtained.

$$\sigma_{yy} = \left(K + \frac{4\mu}{3} \right) \varepsilon_{yy} + \left(K - \frac{2\mu}{3} \right) (\varepsilon_{xx} + \varepsilon_{zz}) + \left(K\gamma_s + \frac{4\mu\gamma_d}{3} \right) \dot{\varepsilon}_{yy} + \left(K\gamma_s - \frac{2\mu\gamma_d}{3} \right) (\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{zz}) = 0 \quad (19)$$

$$\sigma_{zz} = \left(K + \frac{4\mu}{3} \right) \varepsilon_{zz} + \left(K - \frac{2\mu}{3} \right) (\varepsilon_{xx} + \varepsilon_{yy}) + \left(K\gamma_s + \frac{4\mu\gamma_d}{3} \right) \dot{\varepsilon}_{zz} + \left(K\gamma_s - \frac{2\mu\gamma_d}{3} \right) (\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy}) = 0 \quad (20)$$

Differentiating (19) and (20) w.r.t. time, and summing them gives the following relation

$$(\dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz}) = -\frac{3K - 2\mu}{3K + \mu} \dot{\varepsilon}_{xx} \quad (21)$$

Note that the only assumption made here is the constant strain rates, i.e., second time-derivatives of strains are assumed to be negligibly small. This assumption is reasonable for a linear viscoelastic model and if the influence of temperature is not considered. By substituting (21) into (19) and (20), the following relation can be obtained.

$$(\varepsilon_{yy} + \varepsilon_{zz}) = -\frac{3K - 2\mu}{3K + \mu} \varepsilon_{xx} - \frac{9K\mu(\gamma_s - \gamma_d)}{(3K + \mu)^2} \dot{\varepsilon}_{xx} \quad (22)$$

From (22), one can see that the second term on the right-hand side cannot be neglected unless the material responds to dilatational and deviatoric stresses equally, i.e., $\gamma_s = \gamma_d$, and it holds the relationship between two dissipation factors (γ_d and γ_s), K and μ . Substituting (21) and (22) into (18) gives

$$\sigma_{xx} = (K + \frac{4\mu}{3}) \varepsilon_{xx} + (K - \frac{2\mu}{3}) \left(-\frac{3K - 2\mu}{3K + \mu} \varepsilon_{xx} - \frac{9K\mu(\gamma_s - \gamma_d)}{(3K + \mu)^2} \dot{\varepsilon}_{xx} \right) + (K\gamma_s + \frac{4\mu\gamma_d}{3}) \dot{\varepsilon}_{xx} + (K\gamma_s - \frac{2\mu\gamma_d}{3}) \left(-\frac{3K - 2\mu}{3K + \mu} \right) \dot{\varepsilon}_{xx} \quad (23)$$

$$\sigma_{xx} = \frac{9K\mu}{(3K + \mu)} \varepsilon_{xx} + \frac{9K\mu}{(3K + \mu)} \left(\frac{3K\gamma_d + \mu\gamma_s}{3K + \mu} \right) \dot{\varepsilon}_{xx} \quad (24)$$

By comparing (24) with (14), the final expression, which holds the mutual relationship between the dissipation factors γ_d and γ_s , can be obtained;

$$\gamma_{xx} = \frac{3K\gamma_d + \mu\gamma_s}{3K + \mu} \quad (25)$$

Moreover, this expression can be simply reduced to a closed expression for γ_{xx} if one neglects the Poisson's effect, i.e., Young's modulus of an isotropic linear material is given as $E = 2\mu = 3K$. Hence, it can also be transformed into

$$\gamma_{xx} = \frac{E\gamma_d + \frac{E}{2}\gamma_s}{3E/2} = \frac{2\gamma_d + \gamma_s}{3}, \quad (26)$$

which is identical to the expression presented in [31] except the fact that in this context, no assumptions are imposed on the values of damping ratio and system's operating frequency. A comparison between (25) and (26) portrays the influence of Poisson's ratio in the evaluation of dissipation factors.

4.3. Viscoelastic forces

According to the principle of virtual work, the virtual work done by the j^{th} viscoelastic element δW_{ve}^j can be defined as

$$\delta W_{ve}^j = - \int_{\Omega^j} \boldsymbol{\sigma}_{ve}^j : \delta \boldsymbol{\varepsilon}^j d\Omega^j = - \int_{V^j} \boldsymbol{\sigma}_{ve}^j : \delta \boldsymbol{\varepsilon}^j | \mathbf{J} | dV^j = - \int_{V^j} [\underbrace{(\mathbf{E} \boldsymbol{\varepsilon}^j : \delta \boldsymbol{\varepsilon}^j)}_{\text{Elastic}} + \underbrace{(\mathbf{D} \dot{\boldsymbol{\varepsilon}}^j : \delta \dot{\boldsymbol{\varepsilon}}^j)}_{\text{Viscous}}] | \mathbf{J} | dV^j \quad (27)$$

where Ω^j is the volume of j^{th} element in the current configuration, V^j is the volume of j^{th} element in the reference configuration. $| \mathbf{J} |$ is the Jacobian of the mapping between the reference and current configurations. The mapping Jacobian allows the evaluation of integrand at current configuration over the reference integral volume. The matrix $\mathbf{J}$ can be defined using the gradient slope vectors as $\mathbf{J} = [\mathbf{r}_{,x} \quad \mathbf{r}_{,y} \quad \mathbf{r}_{,z}]$. Hence, the elastic strain energy U_e can be expressed using the second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}_{p2}$ as

$$U_e = \frac{1}{2} \int_{V^j} \boldsymbol{\sigma}_{p2}^j : \boldsymbol{\varepsilon}^j | \mathbf{J} | dV^j = \frac{1}{2} \int_{V^j} (\mathbf{E} \boldsymbol{\varepsilon}^j) : \boldsymbol{\varepsilon}^j | \mathbf{J} | dV^j \quad (28)$$

The dissipated power can be calculated in the same way as the elastic strain energy.

$$P_{vis} = \frac{1}{2} \int_{V^j} \boldsymbol{\sigma}_{vis}^j : \dot{\boldsymbol{\epsilon}}^j | \mathbf{J} | dV^j = \frac{1}{2} \int_{V^j} (\mathbf{D} \dot{\boldsymbol{\epsilon}}^j) : \dot{\boldsymbol{\epsilon}}^j | \mathbf{J} | dV^j \quad (29)$$

To this end, the generalized viscoelastic force vector $\mathbf{Q}_{ve}^j$ of j^{th} element can be obtained by partial differentiating the components of virtual work w.r.t. nodal coordinates $\mathbf{e}$ and time-derivatives of nodal coordinates $\dot{\mathbf{e}}$ respectively.

$$\mathbf{Q}_{ve}^j = - \int_{V^j} \left[\left(\frac{\partial \boldsymbol{\epsilon}^j}{\partial \mathbf{e}^j} \right)^T \mathbf{E} \boldsymbol{\epsilon}^j + \left(\frac{\partial \dot{\boldsymbol{\epsilon}}^j}{\partial \dot{\mathbf{e}}^j} \right)^T \mathbf{D} \dot{\boldsymbol{\epsilon}}^j \right] | \mathbf{J} | dV^j, \quad (30)$$

where the components of the partial derivatives of the strain tensor and strain rate tensor can be written respectively as

$$\frac{\partial \epsilon_{mn}}{\partial \mathbf{e}} = \frac{1}{2} (\mathbf{S}_{,m}^T \mathbf{S}_{,n} + \mathbf{S}_{,n}^T \mathbf{S}_{,m}) \mathbf{e}, \quad \frac{\partial \dot{\epsilon}_{mn}}{\partial \dot{\mathbf{e}}} = \frac{1}{2} (\mathbf{S}_{,m}^T \mathbf{S}_{,n} + \mathbf{S}_{,n}^T \mathbf{S}_{,m}) \dot{\mathbf{e}} \quad \text{for } m, n = 1, 2, 3 \text{ and } n \geq m \quad (31)$$

4.4. Modified viscoelastic forces of the inner-core composite layer

For computational efficiency and simplicity of the algorithm, the inner-core matrix layer (composed by conductors, polypropylene resin insulation, and cable filling compound) and a shielding cylindrical shell are assumed to be bonding perfectly and the binding layer between them, is assumed to be infinitesimally thin, i.e., there is no relative motion between them. Therefore, the inner-core layer (Layer I, see Fig. 1) is regarded as a single composite layer instead. As a result, the cross-section will exhibit different axial, bending and torsional stiffnesses compared to the solid cross-section with the same radius. Thus, in the present paper, the structural weakening coefficients k_{s1} and k_{s2} are utilized to breach the isotropy of the inner-core layer to be suitable for the inhomogeneous and anisotropic cross-section.

$$U_{Layer I} = \frac{1}{2} \int_{V_I} \boldsymbol{\sigma}_{p2} : \boldsymbol{\epsilon} | \mathbf{J} | dV_I = \frac{1}{2} \int_{V_I} [k_{s1} \sigma_{11} \epsilon_{11} + k_{s2} (\sigma_{22} \epsilon_{22} + \sigma_{33} \epsilon_{33} + \sigma_{12} \epsilon_{12} + \sigma_{13} \epsilon_{13}) + k_{s1} \sigma_{23} \epsilon_{23}] | \mathbf{J} | dV_I \quad (32)$$

Kerckänen et al. [32] used similar parameters to modify Young's modulus of two-dimensional shear deformable ANCF beam elements in the modeling of belt-drives. The coefficients, $0 < k_{s1}, k_{s2} \leq 1$, can be physically interpreted, in general, as the structural weakening parameters due to lamination and gaps formed by the fraying of wires under flexural loads. They allow one to modulate the axial, bending and torsional stiffnesses of the cross-section. In the case of a pure axial strength state, i.e., a pure bending or longitudinal deformation case, the parts of the strain energy due to axial and transverse normal strains are affected by the coefficients k_{s1} and k_{s2} respectively. In a pure deviatoric strength state, the parts of the strain energy due to torsion and shear deformations are affected by the coefficients k_{s1} and k_{s2} respectively. In the case of small strains, those coefficients k_{s1} and k_{s2} linearly affect the strain energy. Hence, the generalized viscoelastic force vector for the inner-core matrix layer $\mathbf{Q}_{Layer I}$ can be calculated as

$$\begin{aligned} \mathbf{Q}_{Layer I} = - \int_{V_I} & \left[(\lambda + 2\mu) \left(k_{s1} \epsilon_{11} \frac{\partial \epsilon_{11}}{\partial \mathbf{e}} + k_{s2} (\epsilon_{22} \frac{\partial \epsilon_{22}}{\partial \mathbf{e}} + \epsilon_{33} \frac{\partial \epsilon_{33}}{\partial \mathbf{e}}) \right) + 2\mu \left(k_{s2} (\epsilon_{12} \frac{\partial \epsilon_{12}}{\partial \mathbf{e}} + \epsilon_{13} \frac{\partial \epsilon_{13}}{\partial \mathbf{e}}) + k_{s1} \epsilon_{23} \frac{\partial \epsilon_{23}}{\partial \mathbf{e}} \right) \right. \\ & + \lambda k_{s2} \left(\epsilon_{11} (\frac{\partial \epsilon_{22}}{\partial \mathbf{e}} + \frac{\partial \epsilon_{33}}{\partial \mathbf{e}}) + \epsilon_{22} (\frac{\partial \epsilon_{11}}{\partial \mathbf{e}} + \frac{\partial \epsilon_{22}}{\partial \mathbf{e}}) + \epsilon_{33} (\frac{\partial \epsilon_{11}}{\partial \mathbf{e}} + \frac{\partial \epsilon_{22}}{\partial \mathbf{e}}) \right) \\ & + (K\gamma_s + \frac{4\mu\gamma_d}{3}) \left(\dot{\epsilon}_{11} \frac{\partial \dot{\epsilon}_{11}}{\partial \dot{\mathbf{e}}} + \dot{\epsilon}_{22} \frac{\partial \dot{\epsilon}_{22}}{\partial \dot{\mathbf{e}}} + \dot{\epsilon}_{33} \frac{\partial \dot{\epsilon}_{33}}{\partial \dot{\mathbf{e}}} \right) + 2\mu\gamma_d \left(2\dot{\epsilon}_{12} \frac{\partial \dot{\epsilon}_{12}}{\partial \dot{\mathbf{e}}} + \dot{\epsilon}_{13} \frac{\partial \dot{\epsilon}_{13}}{\partial \dot{\mathbf{e}}} + \dot{\epsilon}_{23} \frac{\partial \dot{\epsilon}_{23}}{\partial \dot{\mathbf{e}}} \right) \\ & \left. + (K\gamma_s - \frac{2\mu\gamma_d}{3}) \left(\dot{\epsilon}_{11} (\frac{\partial \dot{\epsilon}_{22}}{\partial \dot{\mathbf{e}}} + \frac{\partial \dot{\epsilon}_{33}}{\partial \dot{\mathbf{e}}}) + \dot{\epsilon}_{22} (\frac{\partial \dot{\epsilon}_{11}}{\partial \dot{\mathbf{e}}} + \frac{\partial \dot{\epsilon}_{22}}{\partial \dot{\mathbf{e}}}) + \dot{\epsilon}_{33} (\frac{\partial \dot{\epsilon}_{11}}{\partial \dot{\mathbf{e}}} + \frac{\partial \dot{\epsilon}_{22}}{\partial \dot{\mathbf{e}}}) \right) \right] | \mathbf{J} | dV_I \end{aligned} \quad (33)$$

In the above equation, since the Green-Lagrange strain tensor is used in the constitutive equations and the viscoelastic forces are evaluated without introducing any assumption such as small deformations or small rotations, geometric nonlinearity is preserved. On the other hand, evaluating the viscoelastic forces can be computationally expensive when using the higher-order beams with large DOFs. Besides, one must evaluate the Jacobian, which is usually a computationally cumbersome task, when using implicit integrators. Nonetheless, it is always advantageous to deduce the Jacobian by analytical differentiation because it enables a fast convergence rate of Newton's iterations. In ANCF, it is possible to evaluate the Jacobian of the viscoelastic forces analytically in a straight forward manner. García-Vallejo et al. [27] introduced an efficient method to evaluate the elastic forces and its Jacobian using invariant matrices, by which the algebraic operations can be done for shape function derivatives matrices at the preprocessing stage and can be stored in a systematic manner. Hence, the viscoelastic forces of Layer I can be written as a product of a coordinate-dependent stiffness matrix $\mathbf{K}(\mathbf{e})$ and a vector of nodal coordinates $\mathbf{e}$.

$$\mathbf{Q}_{Layer I} = -\mathbf{K}(\mathbf{e})\mathbf{e} = -(\mathbf{K}_{e1} + \mathbf{K}_{e2}(\mathbf{e}))\mathbf{e} - \mathbf{K}_v(\mathbf{e})\dot{\mathbf{e}}, \quad (34)$$

$$\mathbf{K}_{e1} = -\left(\frac{3\lambda}{2} + \mu\right) \sum_{\alpha=1}^3 k_{s,i} \int_V S_{,\alpha}^T S_{,\alpha} dV_i, i=1, 2 \quad (35)$$

$$\mathbf{K}_{e2}(\mathbf{e}) = \int_{V_i} \left\{ k_{s2} \left(\frac{\lambda}{2} + \mu \right) \sum_{\alpha=1}^3 (S_{,\alpha}^T S_{,\alpha} \mathbf{e} \mathbf{e}^T S_{,\alpha}^T S_{,\alpha}) + \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \alpha \neq \beta}}^3 \left[k_{s2} \frac{\lambda}{2} (S_{,\alpha}^T S_{,\alpha} \mathbf{e} \mathbf{e}^T S_{,\beta}^T S_{,\beta}) + k_{s,i} \mu (S_{,\alpha}^T S_{,\beta} \mathbf{e} \mathbf{e}^T S_{,\alpha}^T S_{,\beta}) \right] \right\} dV_i, i=1, 2 \quad (36)$$

$$\begin{aligned} \mathbf{K}_v(\mathbf{e}) = \int_{V_i} \left\{ \left(K\gamma_s + \frac{4\mu\gamma_d}{3} \right) \sum_{\alpha=1}^3 S_{,\alpha}^T S_{,\alpha} \mathbf{e} \mathbf{e}^T S_{,\alpha}^T S_{,\alpha} + \left(K\gamma_s - \frac{2\mu\gamma_d}{3} \right) \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \alpha \neq \beta}}^3 S_{,\alpha}^T S_{,\alpha} \mathbf{e} \mathbf{e}^T S_{,\beta}^T S_{,\beta} \right. \\ \left. + 2\mu\gamma_d \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \alpha \neq \beta}}^3 [(S_{,\alpha}^T S_{,\beta} \mathbf{e} \mathbf{e}^T S_{,\alpha}^T S_{,\beta}) + \mu (S_{,\alpha}^T S_{,\beta} \mathbf{e} \mathbf{e}^T S_{,\beta}^T S_{,\alpha})] \right\} dV_i, \end{aligned} \quad (37)$$

where coordinate-independent matrix $\mathbf{K}_{e1}$ and coordinate-dependent matrix $\mathbf{K}_{e2}(\mathbf{e})$ are square matrices, which reflect the elastic behavior while coordinate-dependent square matrix $\mathbf{K}_v(\mathbf{e})$ reflects the internal viscous-damping. As can be seen in (35), $\mathbf{K}_{e1}$ matrix, which corresponds to normal strains, is a constant matrix and does not depend on nodal coordinates. To evaluate $\mathbf{K}_{e2}(\mathbf{e})$ and $\mathbf{K}_v(\mathbf{e})$, one can use the matrix manipulation technique introduced in [27]. Let the square matrix $[\mathbf{D}]_{42 \times 42}$ represent each term of matrices $\mathbf{K}_{e2}(\mathbf{e})$ and $\mathbf{K}_v(\mathbf{e})$, then it can be stated as

$$[\mathbf{D}(\mathbf{e})]_{ij} = (\mathbf{A} \mathbf{e} \mathbf{e}^T \mathbf{B})_{ij} = \sum_k \sum_l A_k e_k e_l B_{lj} = \mathbf{e}^T [\mathbf{C}]_{ij} \mathbf{e}, \quad (38)$$

so the square invariant matrix $[\mathbf{C}]_{ij}$, which has the same dimension as matrix $\mathbf{D}$, can be expressed as

$$[\mathbf{C}]_{ij} = (\mathbf{A}_i)^T (\mathbf{B}^T)_j, \quad (39)$$

Using the expression in (39), constant terms in equations (36) and (37) can be written respectively as

$$[\mathbf{C}_{K_{e2}}]_{ij} = \int_{V_i} \left[k_{s2} \left(\frac{\lambda}{2} + \mu \right) \sum_{\alpha=1}^3 (S_{,\alpha}^T S_{,\alpha})_i^T (S_{,\alpha}^T S_{,\alpha})_j + k_{s2} \frac{\lambda}{2} \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \alpha \neq \beta}}^3 (S_{,\alpha}^T S_{,\alpha})_i^T (S_{,\beta}^T S_{,\beta})_j + k_{s,i} \mu \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \alpha \neq \beta}}^3 (S_{,\alpha}^T S_{,\beta})_i^T (S_{,\beta}^T S_{,\alpha})_j \right] dV_i, i=1, 2 \quad (40)$$

$$\begin{aligned} [\mathbf{C}_{K_v}]_{ij} = \int_{V_i} \left\{ \left(K\gamma_s + \frac{4\mu\gamma_d}{3} \right) \sum_{\alpha=1}^3 (S_{,\alpha}^T S_{,\alpha})_i^T (S_{,\alpha}^T S_{,\alpha})_j + \left(K\gamma_s - \frac{2\mu\gamma_d}{3} \right) \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \alpha \neq \beta}}^3 (S_{,\alpha}^T S_{,\alpha})_i^T (S_{,\beta}^T S_{,\beta})_j \right. \\ \left. + 2\mu\gamma_d \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \alpha \neq \beta}}^3 [(S_{,\alpha}^T S_{,\beta})_i^T (S_{,\beta}^T S_{,\alpha})_j + (S_{,\alpha}^T S_{,\beta})_i^T (S_{,\alpha}^T S_{,\beta})_j] \right\} dV_i \end{aligned} \quad (41)$$

To this end, the Jacobian of viscoelastic forces can be calculated in the same manner as in [27, 31].

$$\left(\frac{\partial \mathbf{Q}_{Layer 1}}{\partial \mathbf{e}} \right)_{ik} + \left(\frac{\partial \mathbf{Q}_{Layer 1}}{\partial \dot{\mathbf{e}}} \right)_{ik} = -[\mathbf{K}_{e1}]_{ik} - \mathbf{e}^T [\mathbf{C}_{K_{e2}}]_{ik} \mathbf{e} - \mathbf{e}^T [\mathbf{C}_{K_v}]_{ik} \mathbf{e} - \sum_{j=1}^{N_e} \sum_{m=1}^{N_e} \left[e_m [(C_{K_{e2}})^{ij}_{mk} + (C_{K_{e2}})^{ij}_{km}] e_j + e_m [(C_{K_v})^{ij}_{mk} + (C_{K_v})^{ij}_{km}] \dot{e}_j \right], \quad (42)$$

where $[C_K]_{mk}^{ij}$ means a component located at m^{th} row and k^{th} column of invariant matrix $\mathbf{C}$ corresponding to the position ij of matrix $\mathbf{D}(\mathbf{e})$ in (38). It is noteworthy that the invariant matrices, which are used for efficient evaluation of viscoelastic forces and Jacobians for the HO42 beam element, are presented for the first time in this paper.

Note that as pointed out by Liu et al. [10], the formulation of the viscoelastic forces described above might be highly demanding in terms of memory allocation in the case of having a mesh with a large number of finite elements having different cross-sectional dimensions, lengths and/or mechanical properties. Nevertheless, a uniform mesh, with all the elements having the same material properties, cross-section and length, will be used for the proposed tether cable model. Therefore, the invariant matrices of all elements are identical, and the memory allocation required to store a single set of invariant matrices will be small.

4.5. Locking phenomena in ANCF beam elements

It should be noted that when using the fully parameterized ANCF beam elements under the viscoelasticity regime, the Poisson's ratio should not be simply set to zero to avoid the Poisson locking problem as recommended by Sopanen et al. [33] for linear elastic material. This is because Poisson's ratio of the viscoelastic materials influences on dissipating the system's elastic energy as discussed in the preceding section. Therefore, Poisson's coupling effect may have a significant impact on the material's behavior. This can be explained by using equation (6). When the Poisson's ratio becomes close to 0.5, the bulk modulus K , which is associated with the dilatation or volumetric change, becomes very large. This causes the material to exhibit high stiffness to resist any volume change. When the general continuum mechanics approach is used to formulate the viscoelastic forces, the beam may predict the bending deformations poorly, showing overly stiff behaviors and leading to spurious results: which is referred to as Poisson locking.

The same problem may be addressed for nearly incompressible, hyperelastic materials, whose Poisson's ratios are nearly 0.5. The closer the Poisson's ratio to 0.5, the greater the impact by Poisson locking in fully parameterized ANCF beam elements. Therefore, an appropriate locking-alleviation technique as presented in [12] should be applied. However, this may not be the problem in this work because the HO42 beam, which employs quadratic approximation in the transverse directions and can also account for cross-sectional deformation, can suppress the Poisson locking and volumetric locking. In addition, its good performance in the case of nearly incompressible materials can be found in [12, 26]. On the other hand, the HO42 has slow convergence problems due to the presence of the in-slope vector, which is referred to as shear locking. This problem might be overcome by enriching the polynomials for slope vectors $\mathbf{r}_{i,y}$ and $\mathbf{r}_{i,z}$ concerning the beam's longitudinal coordinate as in [39]. However, this is not within the scope of this work.

5. Incompressible Mooney-Rivlin material

Underwater subsea cables and ROV tether cables are normally sheathed by ether-based polymers such as *polyurethane* or *polypropylene* for insulation, anti-corrosion, and anti-abrasion purposes. These elastomers usually have the Poisson's ratio close to 0.5 and thus exhibit a nearly incompressible and hyperelastic response. A general Hookean material model, therefore, cannot be applied. Several nonlinear constitutive models for hyperelastic, nearly incompressible, and isotropic materials developed within the framework of the ANCF beam elements can be found in [26, 35 and 36]. In the case of incompressible Mooney-Rivlin material, a general form of the strain energy can be written in terms of the invariant tensors.

$$U = U(I_1, I_2, I_3) \quad (43)$$

where I_1, I_2 and I_3 are first, second and third invariants of the right Cauchy-Green deformation tensor $\mathbf{C}_r$ [36], and for incompressible materials, $I_3 = 1$.

$$\mathbf{C}_r = \mathbf{J}^T \mathbf{J}, \quad I_1 = \text{tr}(\mathbf{C}_r), \quad I_2 = \frac{1}{2} \left[(\text{tr}(\mathbf{C}_r))^2 - \text{tr}(\mathbf{C}_r^2) \right], \quad I_3 = \det(\mathbf{C}_r) = |\mathbf{C}_r| = 1 \quad (44)$$

The strain energy function, which only depends on I_1 and I_2 can be given by [37]

$$U = U(I_1, I_2) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \mu_{rs} (I_1 - 3)^r (I_2 - 3)^s, \quad \mu_{00} = 0, \quad (45)$$

where μ_{rs} are material constants. Therefore, in the case of the two-parameter Mooney-Rivlin material model, the modified strain energy density function $\bar{U}$, which satisfies the incompressibility condition, i.e., $I_3 = 1$, can be written in terms of $\mathbf{C}_r$ as

$$\bar{U} = U + U_p = \int_V \left(\mu_{10} (\text{tr}(\mathbf{C}_r) - 3) + \frac{1}{2} \mu_{01} [(\text{tr}(\mathbf{C}_r))^2 - \text{tr}(\mathbf{C}_r^2) - 6] + k_1 \ln(|\mathbf{C}_r|) + k_2 [\ln(|\mathbf{C}_r|)]^2 \right) dV \quad (46)$$

In this equation, μ_{10} and μ_{01} are Mooney-Rivlin material constants, and U_p is the penalty term for strain energy density function, which can be defined as [38]

$$U_p = k_1 \ln(|\mathbf{C}_r|) + k_2 [\ln(|\mathbf{C}_r|)]^2, \quad (47)$$

where $k_1 = -(\mu_{10} + 2\mu_{01})$, $k_2 = \max(\mu_{10}, \mu_{01}) \times (10^3 \sim 10^7)$. In the case of small strains, the following relations can be expected to hold; shear modulus $\mu = E/3$, $\mu_{10} + \mu_{01} = \mu/2$, and $\mu_{10} = 4\mu_{01}$ [39]. To this end, from (46), the vector of generalized nonlinear elastic forces for outer layer *polyurethane* sheath can be calculated as

$$\mathbf{Q}_{\text{Layer II}} = - \int_{V_{II}} \left(\mu_{10} \frac{\partial(\text{tr}(\mathbf{C}_r))}{\partial \mathbf{e}} + \frac{1}{2} \mu_{01} \left(2 \text{tr}(\mathbf{C}_r) \frac{\partial(\text{tr}(\mathbf{C}_r))}{\partial \mathbf{e}} - \frac{\partial(\text{tr}(\mathbf{C}_r^2))}{\partial \mathbf{e}} \right) + \frac{k_1 + 2k_2}{|\mathbf{C}_r|} \frac{\partial(|\mathbf{C}_r|)}{\partial \mathbf{e}} \right) dV_{II}. \quad (48)$$

When using the implicit integrator to solve the nonlinear equation of motion, its Jacobian matrix can be calculated as

$$\begin{aligned} \frac{\partial \mathbf{Q}_{\text{Layer II}}}{\partial \mathbf{e}} = - \int_{V_{II}} & \left[\mu_{10} \frac{\partial^2(\text{tr}(\mathbf{C}_r))}{\partial \mathbf{e}^2} + \frac{1}{2} \mu_{01} \left(2 \left(\frac{\partial(\text{tr}(\mathbf{C}_r))}{\partial \mathbf{e}} \right)^T \frac{\partial(\text{tr}(\mathbf{C}_r))}{\partial \mathbf{e}} + 2 \text{tr}(\mathbf{C}_r) \frac{\partial^2(\text{tr}(\mathbf{C}_r))}{\partial \mathbf{e}^2} - \frac{\partial^2(\text{tr}(\mathbf{C}_r^2))}{\partial \mathbf{e}^2} \right) \right. \\ & \left. + \frac{k_1 + 2k_2}{|\mathbf{C}_r|} \frac{\partial^2(|\mathbf{C}_r|)}{\partial \mathbf{e}^2} - \frac{k_1 + 2k_2}{(|\mathbf{C}_r|)^2} \left(\frac{\partial(|\mathbf{C}_r|)}{\partial \mathbf{e}} \right)^T \frac{\partial(|\mathbf{C}_r|)}{\partial \mathbf{e}} \right] dV_{II}. \end{aligned} \quad (49)$$

The [second](#) Piola-Kirchhoff [stress](#) tensor can be calculated from the strain energy function as

$$\boldsymbol{\sigma}_{p2} = \frac{\partial}{\partial \boldsymbol{\varepsilon}} (U + U_p) = 2 \frac{\partial}{\partial \mathbf{C}_r} (U + U_p) = 2 \mu_{10} \mathbf{I} + 2 \mu_{01} (I_1 \mathbf{I} - \mathbf{C}_r) + 2 [k_1 + 2k_2 \ln(I_3)] \mathbf{C}_r^{-1}, \quad (50)$$

where the partial derivatives of the principal invariants can be given [36] as $\frac{\partial I_1}{\partial \mathbf{C}_r} = \mathbf{I}$, $\frac{\partial I_2}{\partial \mathbf{C}_r} = I_1 \mathbf{I} - \mathbf{C}_r$. Then the Cauchy [stress](#) tensor can be obtained using the deformation gradient tensor $\mathbf{F}$ as $\boldsymbol{\sigma} = \mathbf{J}^{-1} \mathbf{F} \boldsymbol{\sigma}_{p2} \mathbf{F}^T$.

6. Dynamic modeling of motion equation of tether cable

According to the principle of virtual work, the virtual work done δW_{int}^j by internal forces developed in the j^{th} element of the tether continuum can be defined as

$$\delta W_{int}^j = \int_{\Omega^j} \mathbf{F}_{int}^{jT} \delta \mathbf{r}^j d\Omega^j = - \int_{V^j} \mathbf{F}_{int}^{jT} \delta \mathbf{r}^j |J| dV^j = \int_{V^j} (\mathbf{Q}_{\text{Layer I}} + \mathbf{Q}_{\text{Layer II}}) \delta \mathbf{e}^j dV^j \quad (51)$$

Note that the combined effect of both the inner matrix layer and outer layer of the tether cable element follows naturally from the previous equation. The virtual work by the inertia forces can also be defined as follow;

$$\delta W_{inertia}^j = \int_{\Omega^j} \rho \dot{\mathbf{r}}^{jT} \delta \mathbf{r}^j d\Omega^j = \int_{V^j} \rho \dot{\mathbf{r}}^{jT} \delta \mathbf{r}^j | \mathbf{J} | dV^j = \int_{V^j} \dot{\mathbf{e}}^j \mathbf{M}^j \delta \mathbf{e}^j | \mathbf{J} | dV^j \quad (52)$$

where the consistent mass matrix of j^{th} element is expressed as;

$$\mathbf{M}^j = \int_{V^j} \rho S^{jT} S^j | \mathbf{J} | dV^j \quad (53)$$

On the other hand, the virtual work of the external forces applied to j^{th} element can be written as follow;

$$\delta W_{ext}^j = \int_{\Omega^j} \mathbf{F}_{ext}^{jT} \delta \mathbf{r}^j d\Omega^j = \int_{V^j} \mathbf{F}_{ext}^{jT} \delta \mathbf{r}^j | \mathbf{J} | dV^j = \int_{V^j} \mathbf{Q}_f^j \delta \mathbf{e}^j | \mathbf{J} | dV^j \quad (54)$$

where the generalized external force vector for j^{th} element is defined as

$$\mathbf{Q}_f^j = \int_{V^j} \mathbf{F}_{ext}^{jT} S^j | \mathbf{J} | dV^j \quad (55)$$

To this end, the weak form of the equations of motion for j^{th} element can be given by D' Alembert's principle,

$$\delta W_{inertia}^j + \delta W_{int}^j - \delta W_{ext}^j = 0 \quad (56)$$

Then, the dynamic motion equation of index-3 DAE for a general constrained multibody system can be written as

$$\begin{cases} \mathbf{M} \ddot{\mathbf{q}} + \Phi_q^T \boldsymbol{\lambda} = \mathbf{Q}_q \\ \Phi(\mathbf{q}, t) = \mathbf{0} \end{cases} \quad (57)$$

where $\mathbf{q}$ represents the vector of generalized coordinates in the multibody system, Φ_q is the Jacobian matrix of the constraints Φ , and $\mathbf{Q}_q$ is the generalized force vector which includes the external forces and internal forces. $\boldsymbol{\lambda}$ represents the vector of Lagrange multipliers. The DAE system in (57) is solved by applying index reduction to transform the index-3 DAE into index-1 DAE and numerically integrated with a second-order Adams-Moulton method. *Baumgarte's stabilization technique* [40] is, in addition, applied to keep the constraint violations under control. The system of equations in (57) can be written by utilizing the *Baumgarte's stabilization technique* for kinematic constraints as

$$\begin{bmatrix} \mathbf{M} & \Phi_q^T \\ \Phi_q & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix}, \quad (58)$$

where the values for *Baumgarte's* parameters are selected as 5 for both. Finally, the system of equations in (58) can be solved by updating the acceleration vector $\ddot{\mathbf{q}}$ and vector of Lagrange multipliers $\boldsymbol{\lambda}$ at every time step.

7. Numerical examples

The proposed formulation is validated in this section by numerical benchmark tests for static and dynamic analysis, and experimental results. The effectiveness of the proposed beam in modeling a radially multilayered beam is confirmed by some static and dynamic tests at every stage of modeling. In the computation of elastic forces and its Jacobian in the case of HO42, the exact integration can be obtained using 5 quadrature points for each ξ , η and ζ directions, which gives 125 evaluations of the integrand in total. However, the computation time can be tremendously sped up by reducing the

integration points by one for each direction without apparent loss of accuracy [15, 25 and 34]. Therefore, the integration is performed using 4 quadrature points for each ξ , η and ζ , which gives only 64 evaluations in total.

7.1. Static problems

7.1.1. Full circle shape bending test

In the first test, a bending moment, which can bend the beam into a full circle shape, is applied at the free end to ensure the performance of the proposed tether model in highly nonlinear cases. This bending test was also performed in [15, 41]. Through this test, not only the performance of the HO42 in highly nonlinear cases can be investigated but the effectiveness of integration schemes applied for hollow circular cross-section can also be confirmed. Two cases are selected for beam's cross-sections such as (i) homogeneous cross-section and (ii) inhomogeneous cross-section. Both of the beams have a length l of 1.0 m and are discretized using 20 elements. Poisson's ratios are set to zero. The parameters of the beams are shown in Table 2. The external moment M is applied using a quasi-static loading with a fixed load increment step $t = 0.1$ ms. The parameters k_{s1} and k_{s2} are both set to one in this example. The deformations of the beams for each 0.1 s are shown in Fig. 3.

$$M_y(t) = \frac{M}{2} \left[1 - \cos\left(\frac{\pi t}{T_s}\right) \right], 0 \leq t \leq T_s, T_s = 1s \quad (59)$$

$$\mathbf{Q}_f = M_y(t) \begin{bmatrix} 0 & 0 & 0 & \frac{-e_4}{d_{xz}} & 0 & \frac{e_6}{d_{xz}} & 0_{\times 15} \end{bmatrix}, d_{xz} = \left(\frac{\partial r_1}{\partial x} \right)^2 + \left(\frac{\partial r_3}{\partial x} \right)^2. \quad (60)$$

Table 2.
Parameters of the beams.

Description	homogeneous beam	inhomogeneous beam
Radius (m)	$R = 0.015$	$R_1 = 0.01$ (inner), $R_2 = 0.015$ (outer)
Young's modulus (N/m ²)	$E = 2 \times 10^{11}$	$E_1 = 1.5 \times 10^{11}$, $E_2 = 2.4 \times 10^{11}$
Second moment of area (m ⁴)	$I = \pi R^4/4$	$I_1 = \pi R_1^4/4$, $I_2 = \pi(R_2^4 - R_1^4)/4$
Applied bending moment (Nm)	$M = 2\pi EI/l$	$M = 2\pi(E_1 I_1 + E_2 I_2)/l$

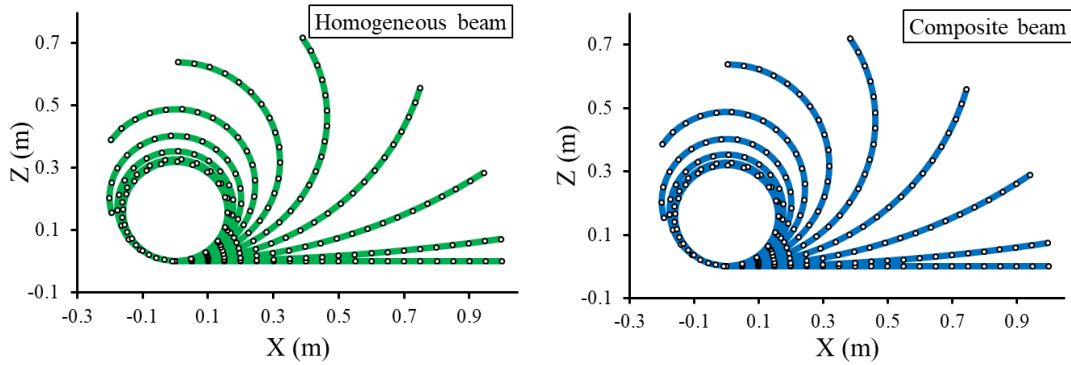


Fig. 3. Full circle shape bending test (for (i) homogeneous and (ii) inhomogeneous circular cross-section)

The beams took the almost perfect circular shape as can be seen in Fig. 3. The tip ends were offset by about 5.172 mm in the homogeneous beam case and 5.854 mm in a two-layered composite beam case respectively. The results show that the proposed multi-layered modeling method using the HO42 beam can deliver reliable results for a moderately slender beam. However, as in the case of standard beam elements with rectangular cross-section, when the beam becomes slender, the deformation results delivered by fully parameterized ANCF beam elements could not agree well with the linear Euler–Bernoulli beam theory. A similar problem was also reported in [15]. Nonetheless, all the following tests were performed using the HO42 beam elements.

7.1.2. Torsion test

Another static example of a **cantilever** beam with a twisting moment at the free end is studied to verify the performance of the proposed method in modeling a multilayered composite beam. The small deformation problem is chosen in order to compare the results with the theoretical value. The same composite beam in the previous example is used. The exact twisting moment, which can rotate the beam to 1 degree of rotation, can be obtained as

$$T = (G_1 J_1 + G_2 J_2) \frac{\theta}{l} \quad (61)$$

where GJ is the torsional rigidity of the beam. The simulation results obtained by using a different number of elements are shown in Table 3. The solution converges to the analytical result as the number of elements increases. Fig. 4 shows the torsional shear stress distributions in each layer of the beam's cross-section in cylindrical coordinates. It can be confirmed the symmetrical distribution of torsional shear stresses in cylindrical coordinates. The torsional shear stresses distribute radially across the cross-section with the maximum values which occur at the circumference of each layer.

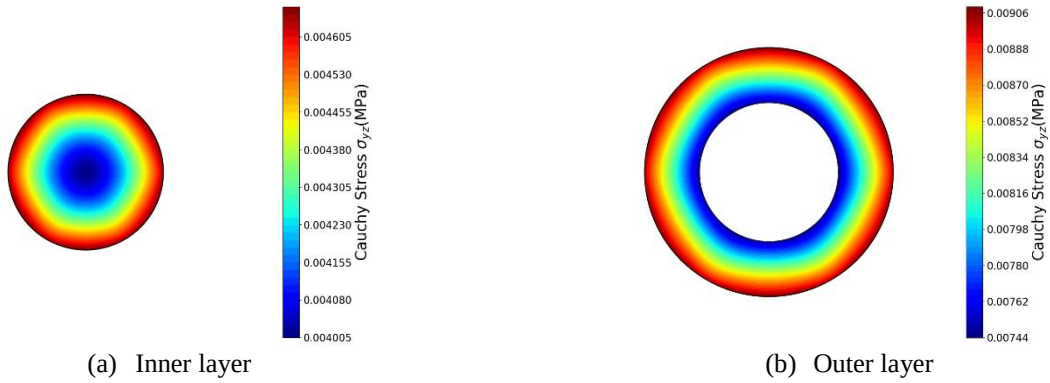


Fig. 4. Torsional shear stress distributions for each layer's cross-section (shown in cylindrical coordinates)

Table 3.

Comparison of the simulation results and theoretical value for the torsion test.

No. of elements	simulation result (rad)	analytical result (rad)	absolute error (rad)	relative error (%)
1	1.65522×10^{-2}	1.74533×10^{-2}	9.01100×10^{-4}	5.16292
2	1.74090×10^{-2}		4.43000×10^{-5}	0.25382
4	1.74527×10^{-2}		6.00000×10^{-7}	0.00344
6	1.74528×10^{-2}		5.00000×10^{-7}	0.00286

7.2. Dynamic tests

7.2.1. Torsional free vibration test

Another example of a **cantilever** beam subjected to an initial twist angle θ is studied to examine the distribution of twist angle along the beam length and strain energy resulted from torsional deformation in each layer of the multilayered composite beam. The same inhomogeneous beam in the previous examples is used and it is discretized with 16 elements. The torsional deformations along the beam length are shown in Fig. 5(a) and the balance of the energy is shown in Fig. 5(b).

$$\theta = \frac{Tl}{(G_1 J_1 + G_2 J_2)} = \frac{\pi}{180} \text{ rad} \quad (62)$$

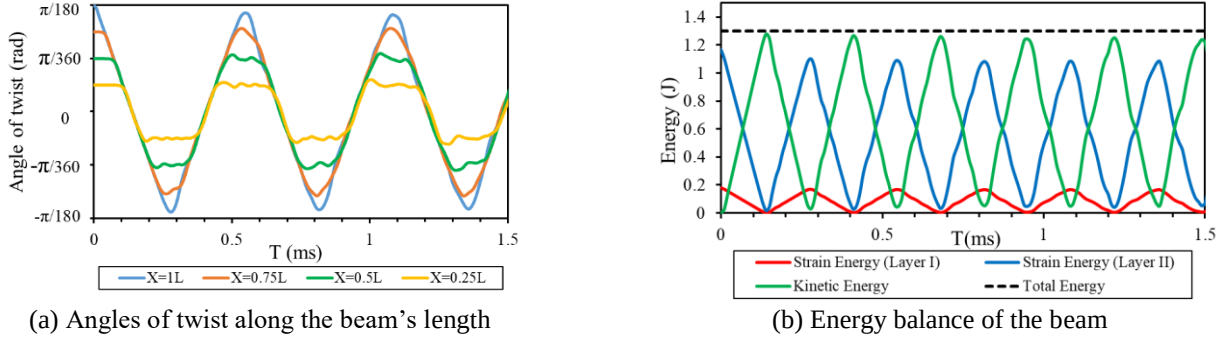


Fig. 5. Torsional free vibration test

As can be seen in Fig. 5(a), the angles of twist distribute proportionally along the beam's length as in [42]. As expected, the elastic energy in the outer layer, which has a higher torsional rigidity, is larger than the inner layer, and the beam follows the conservation of total energy.

7.2.2. Spin-up cantilever beam

A spin-up cantilever beam with an applied moment at the tip end is studied in this example. The beam has the same geometric properties and boundary conditions as in [31]. The moment of 5000 Nm was applied up to 2 s and after 2 s, the applied moment was removed. The damping ratio ζ is also set to 0.5. The fundamental torsional and bending frequencies for the hinge-free beam are calculated with $\alpha = \pi/2$ and $\beta = 3.93^2$ in equations (11) and (16), respectively.

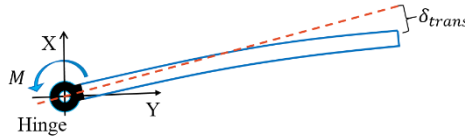


Fig. 6. Spin-up cantilever beam subjected to an applied moment

The parameters k_{s1} and k_{s2} are both set to one to maintain the comparability with the results from [31]. The beam is discretized with 16 elements for all cases with different Poisson's ratios. The longitudinal and transverse deflections at the free end of the beam are shown in Fig. 7. As expected, when the applied moment is removed, elastic deformation of the beam starts to diminish and eventually becomes zero. The comparisons show that the results are almost identical to the results in [31] when the Poisson's ratio is set to zero despite the different ANCF beams used, i.e., the LO24 beam was used in [31] while the present study uses HO42 beam. The results show the effectiveness of internal viscous damping in the proposed viscoelastic beam. Also, the influence of Poisson's ratio on the dissipating behavior in both longitudinal and transverse directions can be observed. A conclusion can be drawn here that the damping behavior of the system is sensitive to the values of Poisson's ratio and the system with a higher Poisson's ratio dissipates the elastic energy more quickly.

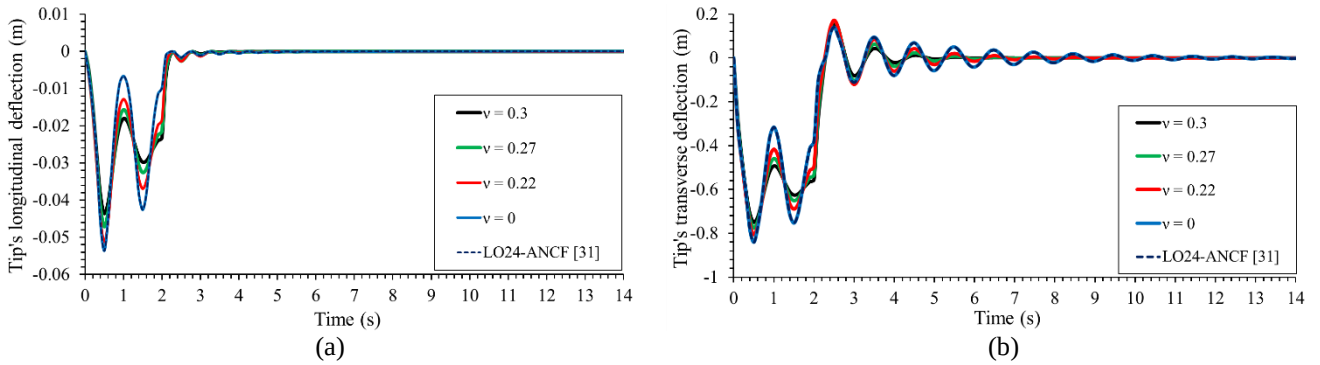


Fig. 7. (a) Longitudinal deflections of the cantilever beam's tip, (b) transverse deflections of the cantilever beam's tip

7.2.3. Influence of damping ratio on damping behavior of tether cable

The following test is chosen to investigate the influence of damping ratios on the inner-core viscoelastic layer of tether cable. In this test, a **cantilever** tether cable with applied load at the free end is studied with different damping ratios by varying ξ from 5% to 20%. The length of the tether is set to 0.3 m and a vertical load of 2 N is applied at the free end up to 1 s. After 1 s, the load is removed. The fundamental torsional and bending frequencies for cantilever beam are calculated with $\alpha = \pi/2$ and $\beta = 1.875^2$ in equations (11) and (16), respectively. The parameters k_{s1} and k_{s2} are both set to one. The beam is discretized with 16 elements for all cases of different damping ratios.

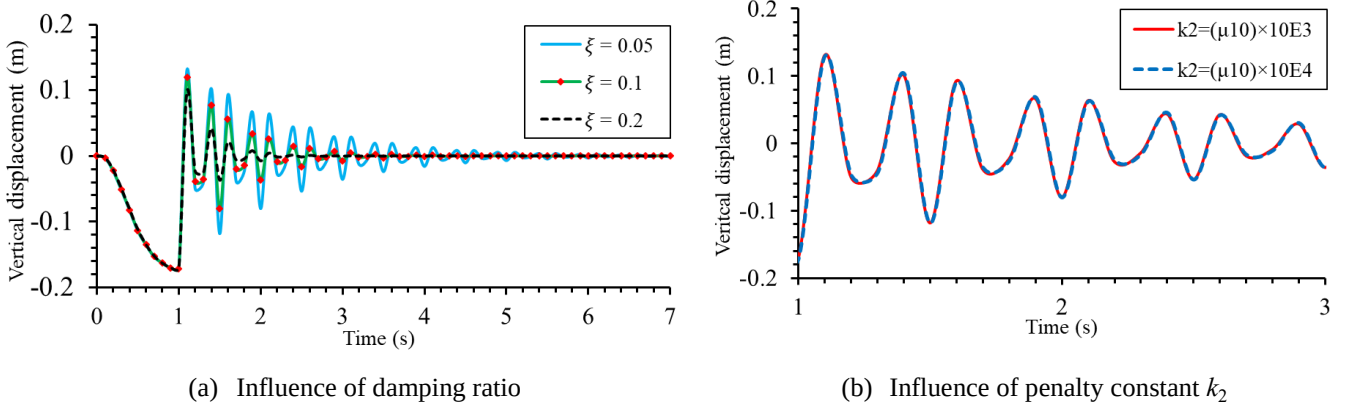


Fig. 8. Deflections of **cantilever** tether cable at the free end

As can be seen in Fig. 8, the amplitudes show the nonlinear responses of the tether cable in all cases. The vibrational frequency decreases as the damping ratio increases and the elastic energy is dissipated faster with a higher damping ratio.

7.2.4. Influence of penalty constants on cable motion in dynamic analysis

The tether cable used in the previous cantilever example with $\xi = 20\%$ is used to study the influence of penalty constants k_1 and k_2 on the tether cable's motion in dynamic analysis. The simulations are carried out by varying the penalty constant k_2 while k_1 remains constant according to $k_1 = -(\mu_{10} + 2\mu_{01})$. The numerical experiments showed that the case where $k_2 = \max(\mu_{10}, \mu_{01}) \times 10^3$ was found to be about two times faster than the case where $k_2 = \max(\mu_{10}, \mu_{01}) \times 10^4$. The displacement results of the free end of the beam showed very small differences for both k_2 values. Nevertheless, to compare the influence of penalty constant k_2 , which guarantees the incompressibility condition in the outer layer of the tether cable, the vertical displacements of the tip's end are shown as a function of time in Fig. 8(b). It was also observed that the numerical ill-conditioning occurs for the values above $k_2 = \max(\mu_{10}, \mu_{01}) \times 10^5$.

7.2.5. Energy balance of physical pendulum test

The freefall pendulum motion was studied to examine the tether cable element's fulfillment of energy conservation and dynamic response in large rotation and deformation. The tether cable of 1 m length is placed initially horizontal along the x -axis and falls under the effect of gravity with the spherical joint at the upper end. The damping ratio of tether cable ξ is obtained from the logarithmic decrement based on the theory of vibration: $\xi = (\delta/2\pi) / \sqrt{1 + (\delta/2\pi)^2}$, in which the logarithmic decrement δ can be calculated from experimental results.

$$\delta = \frac{1}{n} \ln \left| \frac{X_n}{X_{n+1}} \right|, \quad (63)$$

where X_n and X_{n+1} are the amplitudes of oscillations at n cycles apart. The calculated damping ratio is 0.1975, which is approximately 0.2. The fundamental torsional and bending frequencies were calculated with $\alpha = \pi/2$ and $\beta = 3.93^2$ in equations (11) and (16), respectively. The parameters k_{s1} and k_{s2} are both set to one. The tether model is discretized with 20 elements. The configurations of the tether model at every 0.05 s for 0 ~ 1 s are shown in Fig. 9. The displacements of the free end of the tether model are shown in Fig. 10(a) while Fig. 10(b) shows the energy conservation of the tether cable in a

pendulum motion. It can be seen that the displacement in the y -direction is zero throughout the entire motion since the external gravity force is acting in the xz -plane, and the system is damped out eventually over time. The results in Fig. 10(b) show that the total energy of the system is preserved and energy balance is achieved.

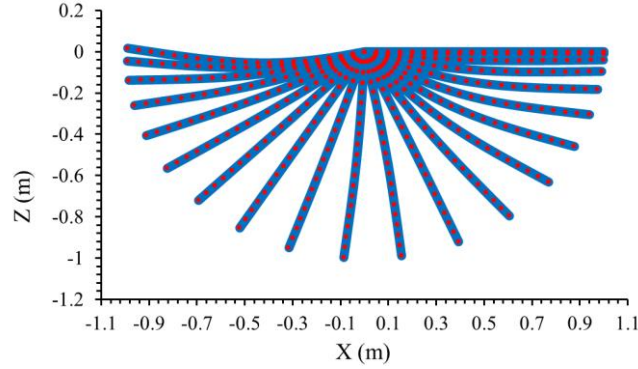


Fig. 9. Configurations of tether cable for pendulum motion

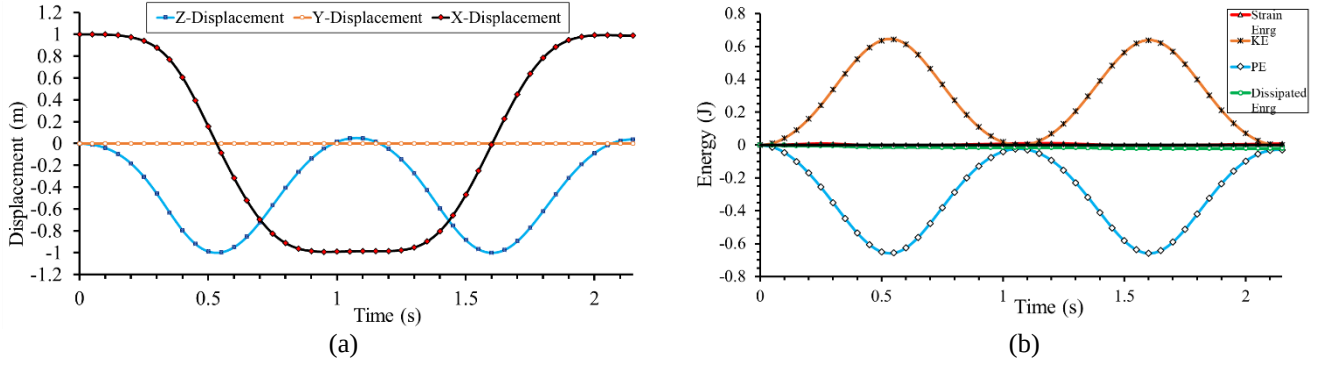
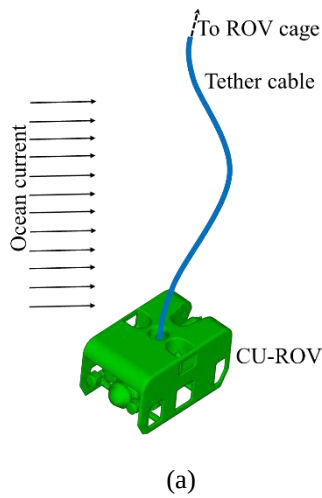


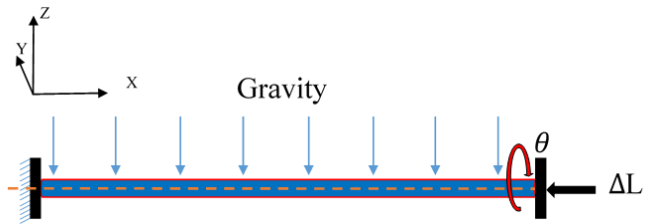
Fig. 10. (a) Displacements of tether cable's free end, (b) energy balance of tether cable

7.2.6. Twisted-slack tether cable

Another spatial deformation problem is studied to examine the performance of our proposed cable element in modeling the tether cable sample of ROV (see Fig. 15).



(a)



$$\theta = 180^\circ \text{ (clockwise)}$$

$$\Delta L(t) = \frac{l/4}{2} \left[1 - \cos\left(\frac{\pi t}{T_s}\right) \right], 0 \leq t \leq T_s, T_s = 1\text{s}$$

(b)

Fig. 11. (a) Deformations of tether cable when the ROV rotates about yaw axis and moves upwards along the yaw axis, (b) twisted-slack tether cable simulation

To mimic one of the likely scenarios of tether's deformations (depicted in Fig. 11(a)) during ROV's lurching, operation and recovery processes, a twisted-slack cable simulation is studied. First, the tether, which is fixed at both ends, is subjected to a certain twist angle θ at one end. After that, a certain amount of slack $\Delta L(t)$ as a function of time is given and let the cable buckle under the uniformly distributed transverse load, which in this case, gravity field is used. The tether cable, then, starts to develop lateral torsional buckling under the gravity field. The deformations of the tether cable are shown for certain time steps in Fig. 12.

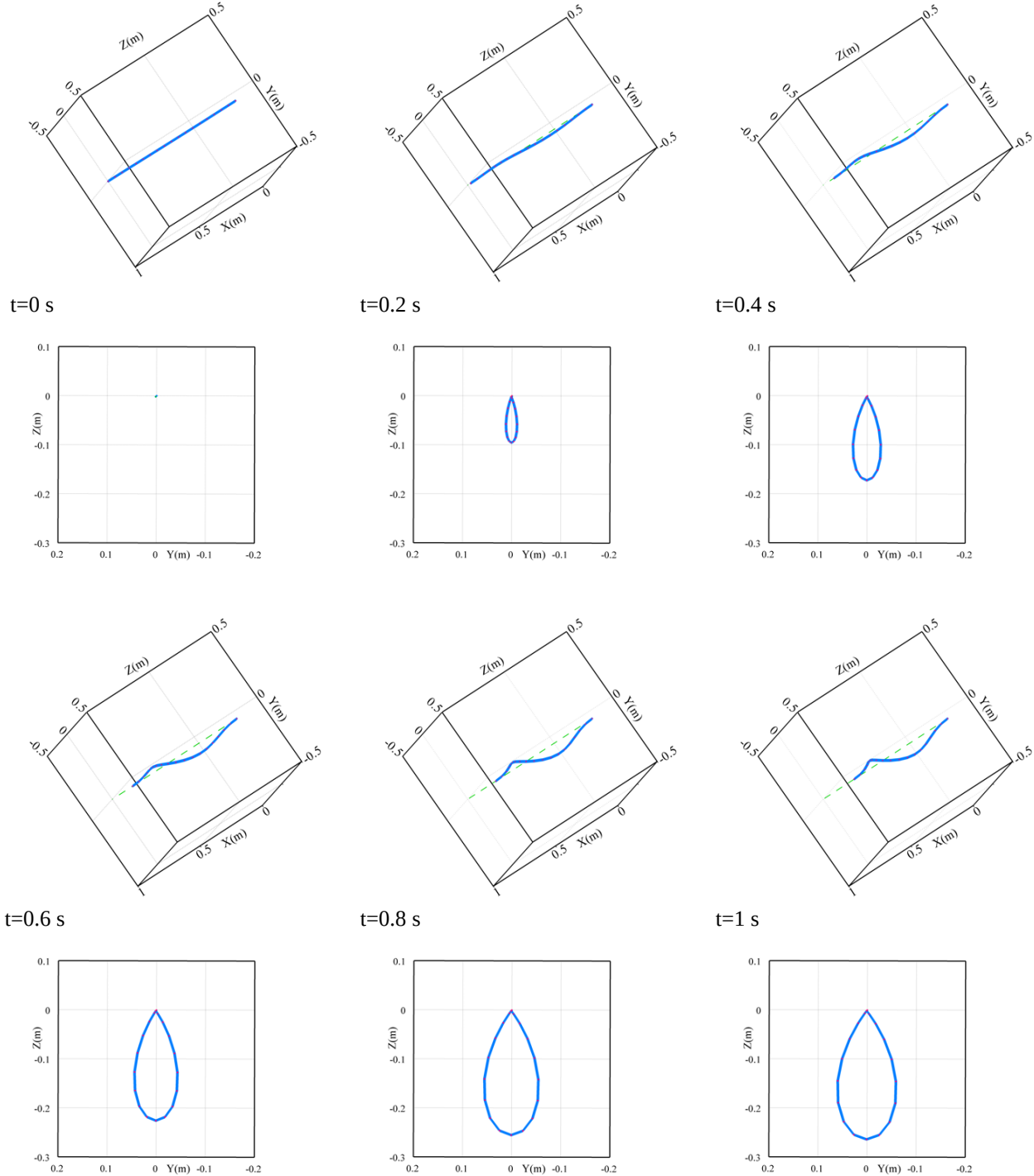


Fig. 12. Deformations of tether cable in twisted-slack tether cable simulation

As can be seen in Fig. 12, the cable subjected to an initial twist angle starts to develop lateral torsional buckling when one of the fixed ends is compressed by a given amount of slack $\Delta L(t)$. It is confirmed that the cable is buckling symmetrically about the x -axis and the deformation increases as the slack increases over time. It has also been observed that the mid node of the tether cable has no lateral displacement in the xy -plane similar to the results discussed in [41, 42], where the slender rods were studied.

The normal stress and torsional shear stress distributions in each layer for three consecutive cross-sections at the mid-span (see Fig. 13) are given in Tables 4, 5 and 6 respectively. As can be seen in Table 4, at the beginning of the simulation, when the cable is subjected to a pure rotation, the shear stresses σ_{yz} distribute symmetrically along the radius in both layers. As expected, the larger stresses occur inside the inner-core layer, which has a higher shear modulus than the outer sheath layer. For a given angle of rotation 180° , the torsional shear stresses across the three consecutive sections are found to be nearly identical.

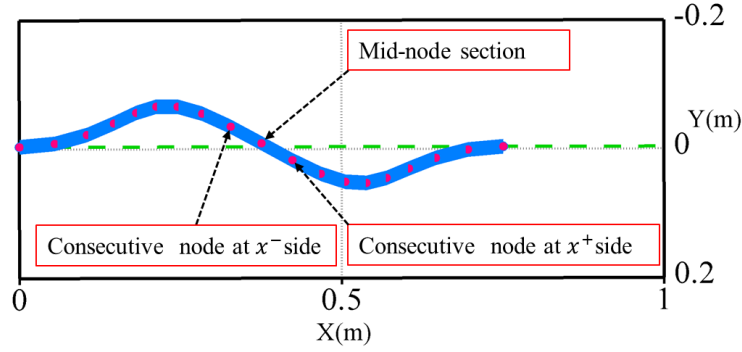
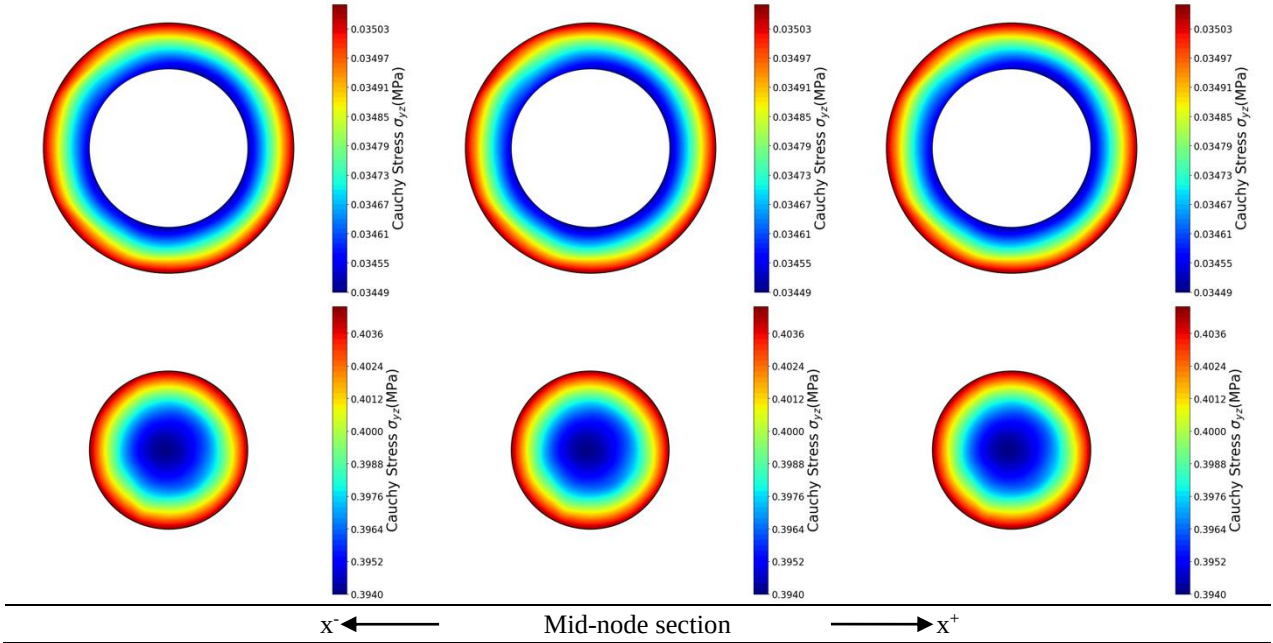


Fig. 13. Deformations of tether cable at $t = 1$ s (xy -plane)

Table 4

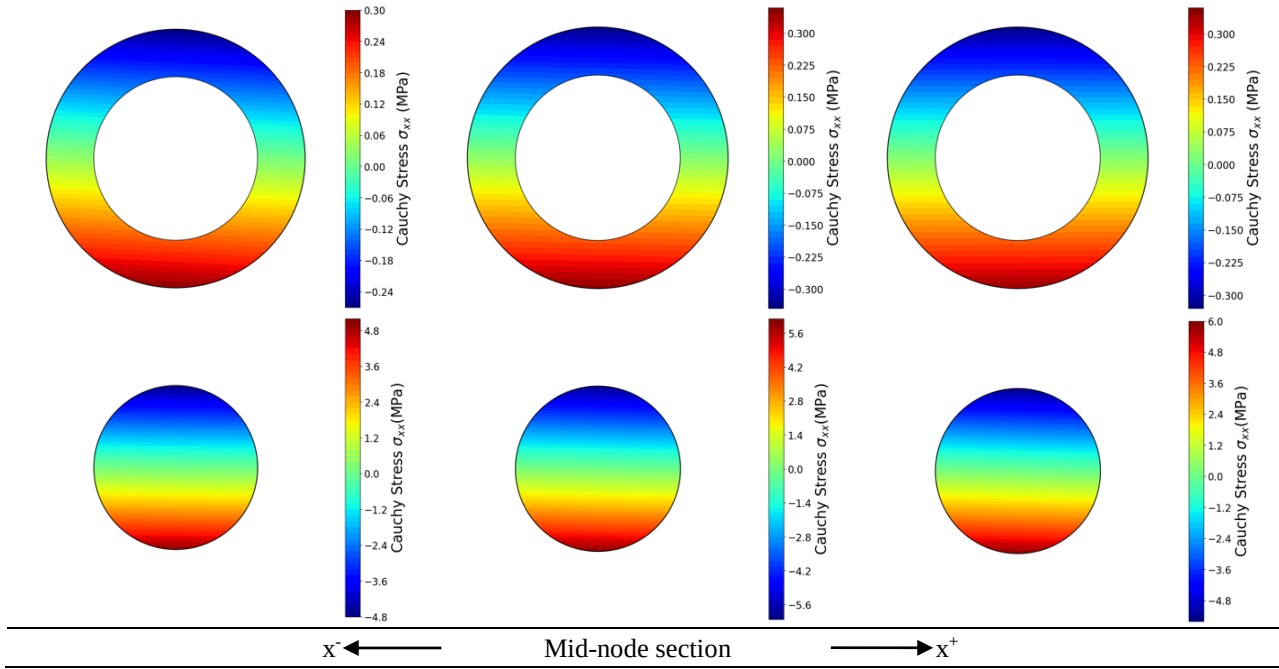
Torsional shear stress (σ_{yz}) in each layer of tether cable when the cable is subjected to rotation angle θ only.



From the normal stresses (σ_{xx}) shown in Table 5, it can be interpreted that the lower half of the tether cable near the mid-span is in tension while the other half in compression following the bending direction due to the gravity. It can also be confirmed that the neutral planes passing through the centers of the respective cross-sections of each layer are not under stress, i.e., the materials on the neutral planes remain unstretched or uncompressed.

Table 5

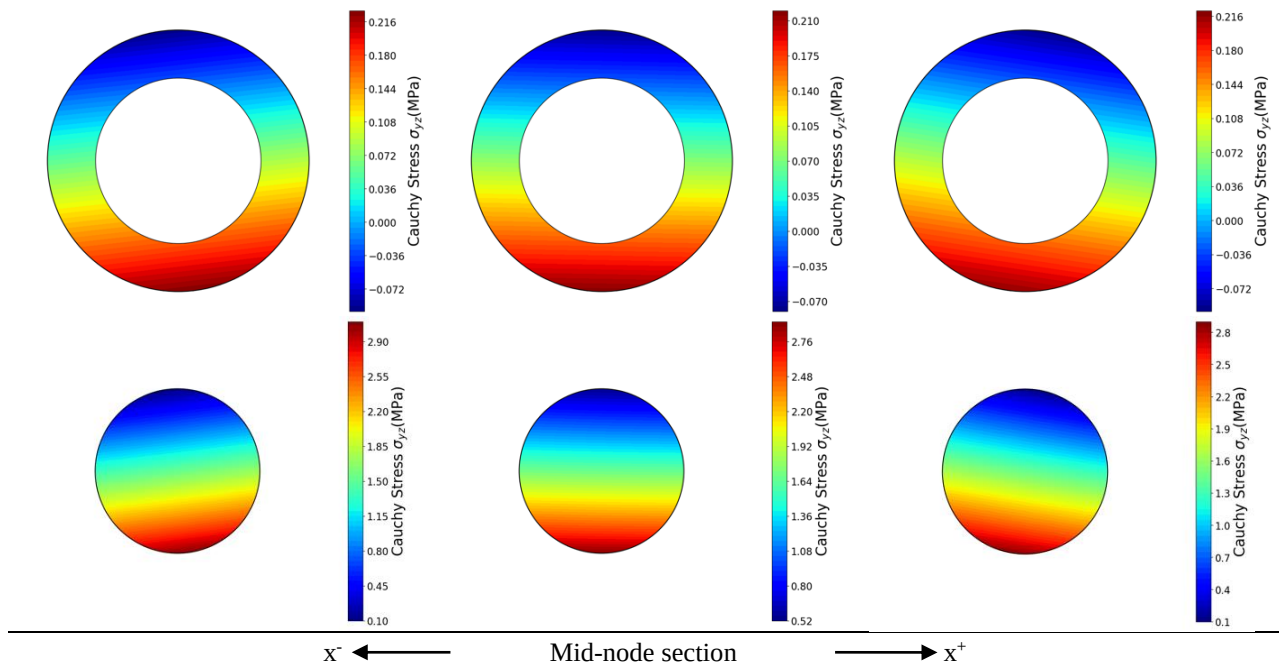
Normal stress (σ_{xx}) in each layer of tether cable at the end of the simulation.



The shear stress (σ_{yz}) profiles shown in Table 6 show that in the case of combined axial, bending and torsional deformations, the shear stresses (σ_{yz}) are distributing on each cross-section of the three consecutive nodes near the mid-span according to the directions of bending and torsion due to the gravity and given initial twist angle respectively. It can also be seen that the maximum values of both the normal stresses σ_{xx} and shear stresses σ_{yz} for all three sections occur at the outermost locations from the respective neutral planes in each layer.

Table 6

Shear stress (σ_{yz}) in each layer of tether cable at the end of the simulation.



7.2.7. Freefall cantilever test of tether cable

To investigate the performance of the developed tether model, a free-falling cantilever experiment was carried out so that the simulation results were compared with experimental results. The tether sample has a free length of 0.9 m and the upper end is clamped as shown in Fig. 14. The beam is initially placed as straight as possible along the horizontal axis.

In the free-falling cantilever beam experiment, it should be noted that the tether sample was not straight and it was cut from the total length of 3.0 m of tether cable connected to CU-ROV, which is shown in Fig. 15. Since the tether was kept winding in a circle shape, the winding induced residual stresses were locked into the tether sample, consequently, the dependency of tether behavior on winding curl was observed, which was also reported in [1, 2]. However, within the scope of this investigation, those dependency and residual stresses were not dealt with. Therefore, in the numerical simulation, the tether model was assumed as straight in the initial configuration and free from any winding induced initial stress. In the experiment, the initial configuration of the tether was placed such that the length from Node 1 to Node 7 was naturally in line with the leveling ruler and the free end's curved direction was pointing upward.

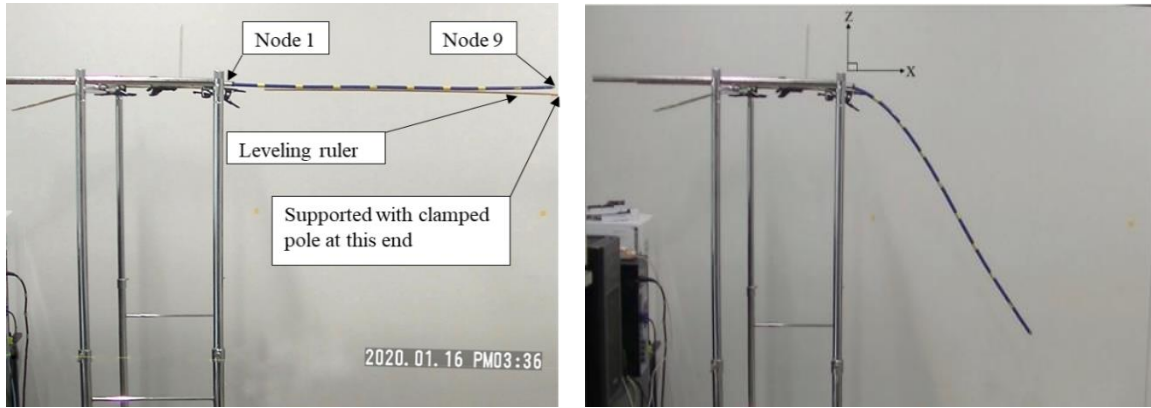


Fig. 14. Free-falling cantilever beam experiment of the tether cable

Three different tether models are used to compare with the experiment results. Model I is the tether model used in the previous work [1], which employed LO-24 ANCF beam elements without applying multilayered modeling and structural weakening parameters. Model II is the tether model, which employs HO-42 ANCF beam elements based on the multilayered modeling without applying structural weakening parameters. Internal damping for both model I and II is incorporated using the approach proposed in [31]. Model III is the proposed tether model in this paper. The dissipation factors are calculated as in the previous cantilever test. The structural weakening parameters are tuned and the results show reasonable agreement with experimental results for $k_{s1} = 0.8$ and $k_{s2} = 0.1$. The calculated damping ratio of 0.2 is used.

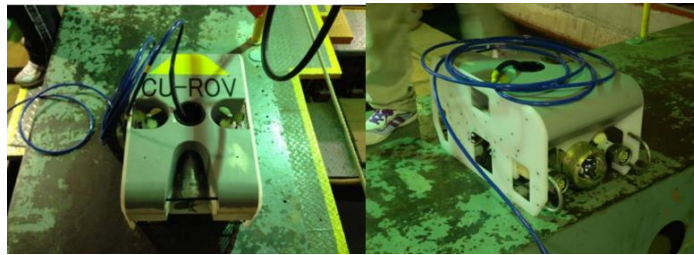


Fig. 15. Tether cable and CU-ROV

The tether continuum is discretized with 20 elements. As can be seen in Fig. 16, the discrepancy between the simulations and experiment becomes apparent when the cable undergoes large displacements and the influence of initial stress becomes predominant. Since the cable's curved direction was placed upward, the cable exhibited stiffer behavior than the simulation results. However, at the beginning of time steps, the proposed tether model utilizing structural weakening parameters and proposed damping model can reproduce the similar behavior of the dynamic response as the experimental results. Besides, the proposed tether model can give the most similar configuration when the cable reaches the final steady configuration.

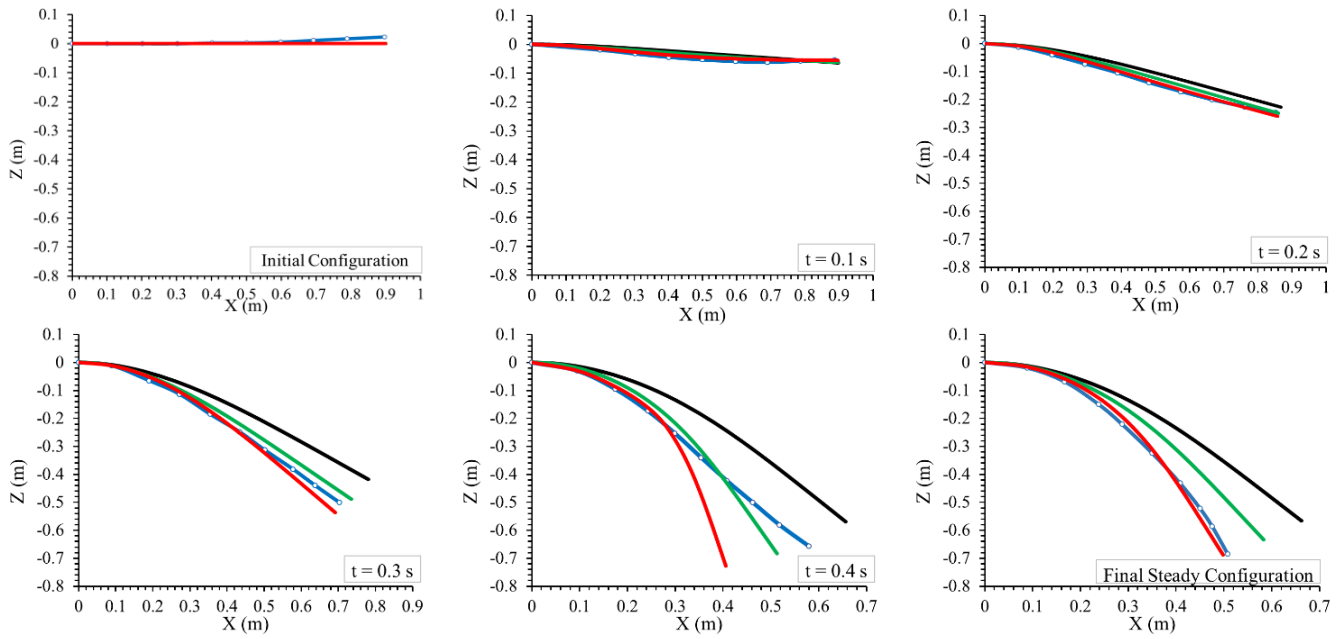


Fig. 16. Comparison of numerical and experimental results in free-falling cantilever test of tether sample (black: model I, green: model II, red: model III, blue: experiment)

8. Summary and conclusions

In this paper, dynamic modeling of a numerical tether model based on the absolute nodal coordinate formulation (ANCF) was proposed. To account for the effects of different material properties across the cross-section, the cross-section is radially divided into sublayers regarding the different material properties in each layer. A Kelvin-Voigt model is employed to account for the viscoelastic behavior of the inner-core matrix, while nearly incompressible Mooney-Rivlin model is applied for the outer sheathing material. Structural weakening parameters are applied to modify the beam's strain energy to be suitable for the composite inner-core layer. An analytical method to determine the dissipation factors for dilatational and deviatoric responses is reviewed, and a new approach is proposed. For the computational efficiency and efficient evaluation of viscoelastic forces, the invariant matrices for the HO42 beam element are presented for the first time. Through validations against the numerical benchmark tests and free-falling cantilever beam experiment, the effectiveness of the proposed beam in modeling radially multi-layered and inhomogeneous beam with a circular cross-section is verified. [A numerical example, which mimics a likely scenario of the tether deformations during ROV operations in real applications is studied, and the cross-sectional stresses in each layer of the tether cable are discussed.](#) In this study, two weakening parameters, k_{s1} and k_{s2} are used to break the isotropic Hookean behavior since they allow one to modulate the values of the axial, torsional, bending and cross-sectional stiffnesses. In practice, subsea cables and ROV's tether cables usually exhibit orthotropic and anisotropic natures due to its complex compositions of materials. Therefore, the present model has the limitation that it could not account for pure orthotropy or complete anisotropy. In future works, anisotropy, as well as the winding induced residual stresses, will be investigated.

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References

- [1] T.Z. Htun, H. Suzuki, A. Kuwano, H. Tomobe, T. Inoue, Numerical motion analysis of ROV coupled with tether applying 24-DOFs absolute nodal coordinate formulation, Proceedings of the 29th International Conference on Offshore and Polar Engineering, ISOPE, 2019,

- pp. 1553-1560.
- [2] H. Suzuki, H. Tomobe, A. Kuwano, T.Z. Htun, T. Inoue, Numerical motion analysis of ROV applying ANCF to tether cable considering its mechanical property, *Proceedings of the 28th International Conference on Offshore and Polar Engineering, ISOPE*, 2018, pp. 365–372.
 - [3] A.A. Shabana, R.Y. Yakoub, Three dimensional absolute nodal coordinate formulation for beam elements: Theory, *ASME J. of Mech. Des.* 123 (4) (2001) 606–613. <https://doi.org/10.1115/1.1410100>.
 - [4] A.A. Shabana, *Dynamics of Multibody Systems*, second ed., Cambridge university press, Cambridge, 1998.
 - [5] M. Berzeri, A.A. Shabana, Development of simple models for the elastic forces in the absolute nodal coordinate formulation, *J. Sound Vib.* 235 (4) (2000) 539–565. <https://doi.org/10.1006/jsvi.1999.2935>.
 - [6] H. Sugiyama, A. Mikkola, A.A. Shabana, A non-incremental nonlinear finite element solution for cable problems, *J. Mech. Des.* 125 (4) (2003) 746–756. <https://doi.org/10.1115/1.1631569>.
 - [7] J. Gerstmayr, A.A. Shabana, Analysis of thin beams and cables using the absolute nodal coordinate formulation, *Non. Dyn.* 45 (1) (2006) 109–130. <https://doi.org/10.1007/s11071-006-1856-1>.
 - [8] Q.T. Wang, Q. Tian, H.Y. Hu, Dynamic simulation of frictional contacts of thin beams during large overall motions via absolute nodal coordinate formulation, *Non. Dyn.* 77 (4) (2014) 1411–1425. <https://doi.org/10.1007/s11071-014-1387-0>.
 - [9] Y. Zhang, C. Wei, Y. Zhao, T. Chunlin, L. Yongjian, Adaptive ANCF method and its application in planar flexible cables, *Acta Mech. Sin.* 34 (1) (2018) 199–213. <https://doi.org/10.1007/s10409-017-0721-4>.
 - [10] C. Liu, Q. Tian, H.Y. Hu, Dynamics of a large scale rigid flexible multibody system composed of composite laminated plates, *Multibody Syst. Dyn.* 26 (3) (2011) 283–305. <https://doi.org/10.1007/s11044-011-9256-9>.
 - [11] M. Patel, G. Orzechowski, Q. Tian, A new multibody system approach for tire modeling using ANCF finite elements, *J. Multibody Dyn.* 230 (1) (2016) 69–84. <https://doi.org/10.1177%2F1464419315574641>.
 - [12] G. Orzechowski, J. Frączek, Volumetric locking suppression method for nearly incompressible nonlinear elastic multi-layer beams using ANCF elements, *J. Theory Appl. Mech.* 55 (3) (2017) 977–990. <https://doi.org/10.15632/jtam-pl.55.3.977>.
 - [13] P. Lan, Q. Tian, Z. Yu, A new absolute nodal coordinate formulation beam element with multilayer circular cross section, *Acta Mech. Sin.* 36 (1) (2020) 82–96. <https://doi.org/10.1007/s10409-019-00897-4>.
 - [14] C. Ronald, J.K. Kyung, A survey of known and new cubature for the unit disk, *Korean J. Comput. Appl. Math.* 7 (3) (2000) 477–485. <https://doi.org/10.1007/BF03012263>.
 - [15] G. Orzechowski, Analysis of beam elements of circular cross section using the absolute nodal coordinate formulation, *Archive of Mech. Eng.* 59 (3) (2012) 283–296. <http://dx.doi.org/10.2478/v10180-012-0014-1>.
 - [16] O.O. Ajayi, O.O. Aribike, Modelling and simulation of subsea umbilical dynamics: A numerical approach, *Int. J. of Comp. Techni.* 2 (1) (2015) 80–91.
 - [17] S. Chucheepeakul, N. Srinil, P. Petchpear, A variational approach for three-dimensional model of extensible marine cables with specified top tension, *Appl. Math. Model.* 27 (10) (2003) 781–803. [https://doi.org/10.1016/S0307-904X\(03\)00089-1](https://doi.org/10.1016/S0307-904X(03)00089-1).
 - [18] J.W. Kamman, R.L. Huston, Multibody dynamics modeling of variable length cable systems, *Multibody Syst. Dyn.* 5 (3) (2001) 211–221. <https://doi.org/10.1023/A:1011489801339>.
 - [19] B. Radanovic, F.R. Driscoll, Development of an efficient general purpose cable model and simulation for marine applications, *Oceans MTS/IEEE, Biloxi, MI, USA*, 71 (4) (2002) 2060–2067. <https://doi.org/10.1109/OCEANS.2002.1191949>.
 - [20] J.L. Tang, G.X. Ren, W.D. Zhu, H. Ren, Dynamics of variable-length tethers with application to tethered satellite deployment, *Commun. Non. Sci. Numer. Simul.* 16 (8) (2011) 3411–3424. <https://doi.org/10.1016/j.cnsns.2010.11.026>.
 - [21] J.L. Escalona, An arbitrary Lagrangian-Eulerian discretization method for modeling and simulation of reeving systems in multibody dynamics, *Mech. Mach. Theory* 112 (2017) 1–21. <https://doi.org/10.1016/j.mechmachtheory.2017.01.014>.
 - [22] E. Passano, S. Abtahi, T. Ottesen, A procedure to include slip damping in a VIV analysis of an umbilical, *Proceedings of the 35th International Conference on Ocean, Offshore and Arctic Engineering (ICOAAE)*, ASME, 2016, pp. 19–24. <https://doi.org/10.1115/OMAE2016-54816>.
 - [23] N. Söðahl, O. Steinkjer, E. Gjørtmesli, K. Hansen-Zahl, Consistent VIV fatigue analysis methodology of umbilicals, *Proceedings of the 30th International Conference on Ocean, Offshore and Arctic Engineering (ICOAAE)*, ASME, 2011, pp. 439–447. <https://doi.org/10.1115/OMAE2011-49459>.
 - [24] Z. Shen, P. Li, C. Liu, G. Hu, Finite element beam model including cross-section distortion in the absolute nodal coordinate formulation, *Non. Dyn.* 77 (3) (2014) 1019–1033. <https://doi.org/10.1007/s11071-014-1360-y>.
 - [25] G. Orzechowski, A.A. Shabana, Analysis of warping deformation modes using higher order ANCF beam element, *J. Sound Vib.* 363 (2016) 428–445. <https://doi.org/10.1016/j.jsv.2015.10.013>.
 - [26] G. Orzechowski, J. Frączek, Nearly incompressible nonlinear material models in the large deformation analysis of beams using ANCF, *Non. Dyn.* 82 (1) (2015) 451–464. <https://doi.org/10.1007/s11071-015-2167-1>.
 - [27] D. García-Vallejo, J. Mayo, J.L. Escalona, J. Domínguez, Efficient evaluation of the elastic forces and the Jacobian in the absolute nodal coordinate formulation, *Non. Dyn.* 35 (4) (2004) 313–329. <https://doi.org/10.1023/B:NODY.0000027747.41604.20>.
 - [28] C. Ma, R. Wang, C. Wei, Y. Zhao, A new absolute nodal coordinate formulation of solid element with continuity condition and viscosity model, *Int. J. of Simul.: Syst. Sci. Techno.* 1 (2016) 101–106. <https://doi.org/10.5013/IJSSST.a.17.21.10>.
 - [29] A.A. Mohamed, A.A. Shabana, A nonlinear viscoelastic constitutive model for large rotation finite element formulations, *Multibody Syst. Dyn.* 26 (1) (2011) 57–79. <https://doi.org/10.1007/s11044-011-9244-0>.
 - [30] Y. Zhang, Q. Tian, L. Chen, J. Yang, Simulation of a viscoelastic flexible multibody system using absolute nodal coordinate and fractional derivative methods, *Multibody Syst. Dyn.* 21 (3) (2009) 281–303. <https://doi.org/10.1007/s11044-008-9142-2>.
 - [31] D. García-Vallejo, J. Valverde, J. Domínguez, An internal damping model for the absolute nodal coordinate formulation, *Non. Dyn.* 42 (4) (2005) 347–369. <https://doi.org/10.1007/s11071-005-6445-1>.
 - [32] K.S. Kerkkänen, D. García-Vallejo, A.M. Mikkola, Modeling of belt-drives using a large deformation finite element formulation, *Non. Dyn.* 43 (3) (2006) 239–256. <https://doi.org/10.1007/s11071-006-7749-5>.
 - [33] J.T. Sopanen, A.M. Mikkola, Description of elastic forces in absolute nodal coordinate formulation, *Non. Dyn.* 34 (1–2) (2003) 53–74. <https://doi.org/10.1023/B:NODY.0000014552.68786.bc>.
 - [34] J. Gerstmayr, A.A. Shabana, Efficient integration of the elastic forces and thin three-dimensional beam elements in the absolute nodal coordinate formulation, *Multibody Dyn. ECCOMAS Thematic Conference*, 2005.
 - [35] L.G. Maqueda, A.A. Shabana, Poisson modes and general nonlinear constitutive models in the large displacement analysis of beams, *Multibody Syst. Dyn.* 18 (3) (2007) 375–396. <https://doi.org/10.1007/s11044-007-9077-z>.
 - [36] A.A. Shabana, *Computational Continuum Mechanics*, first ed., Cambridge University Press, Cambridge, 2008.
 - [37] R. Mooney, A theory of large elastic deformation, *J. Appl. Phys.* 11 (1940) 582–592. <https://doi.org/10.1063/1.1712836>.
 - [38] T. Belytschko, W.K. Liu, B. Moran, *Nonlinear Finite Element for Continua and Structures*, Wiley, New York, 2000.
 - [39] K.J. Bathe, *Finite Element Procedures*, Prentice Hall, New Jersey, 1996.
 - [40] J. Baumgarte, Stabilization of constraints and integrals of motion in dynamical systems, *Comput. Methods Appl. Mech. Eng.* 1 (1) (1972) 1–16. [https://doi.org/10.1016/0045-7825\(72\)90018-7](https://doi.org/10.1016/0045-7825(72)90018-7).

- [41] K.E. Dufva, J.T. Sopanen, A.M. Mikkola, A two-dimensional shear deformable beam element based on the absolute nodal coordinate formulation, *J. Sound Vib.* 280 (3–5) (2005) 719–738. <https://doi.org/10.1016/j.jsv.2003.12.044>.
- [42] K. Kadokura, N. Kobayashi, Reduction of degrees of freedom in three-dimensional ANCF beam of 24-DOF beam element by component mode synthesis, *Trans. of the JSME (in Japanese)*, 80 (814) (2014) DR0172. <https://doi.org/10.1299/transjsme.2014dr0172>.
- [43] S. Goyal, N.C. Perkins, C.L. Lee, Nonlinear dynamics and loop formation in Kirchhoff rods with implications to the mechanics of DNA and cables, *J. of Comput. Phys.* 209 (1) (2005) 371–389. <https://doi.org/10.1016/j.jcp.2005.03.027>.