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THE CUT LOCUS AND THE DIASTASIS OF A HERMITIAN SYMMETRIC SPACE OF COMPACT TYPE

Hiroyuki TASAKI

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1. Introduction

For a complete Riemannian manifold M and a point p in M, we denote by $C_p(M)$ the cut locus of M with respect to p. As a property of the cut locus of a simply connected compact symmetric space M, it is known in [2] that the cut locus $C_p(M)$ coincides with the first conjugate locus of M with respect to p. Sakai [5] proved that in general the cut locus of a compact symmetric space is determined by that of its maximal totally geodesic flat submanifold (see Section 4 for details). Using this, Sakai [6] and Takeuchi [8], [9] gave stratifications of the cut loci of compact symmetric spaces.

Calabi [1] introduced the notion "diastasis" to study Kähler imbeddings. The diastasis of a Kähler manifold M is a real analytic function defined on its domain of real analyticity in $M\times M$ containing the diagonal set and behaves like as the square of the geodesic distance in the small (see Section 2 for the definition). The most characteristic property of the diastasis proved by Calabi will be that the diastasis of a Kähler submanifold N of a Kähler manifold M coincides with the restriction of the diastasis of M to N. Making use of these properties, Calabi obtained various fundamental results of Kähler imbeddings. In particular, he proved the rigidity of a Kähler submanifold of a space of constant holomorphic sectional curvature.

It seems to be interesting to study relations between the geodesic distance and the diastasis in the large. In this note we shall show a relation between the cut locus and the diastasis of a Hermitian symmetric space of compact type. More precisely, the main result of this note is the following:

Theorem. *Let M be a Hermitian symmetric space of compact type and D be the diastasis of M. Then, for each point p in M, the cut locus $C_p(M)$ is equal to the set of points q at which $D(p, q)$ cannot be defined.*

In other words, $M-C_p(M)$ is the domain of real analyticity of the real analytic function $q\mapsto D(p, q)$. This result gives a relation between the cut locus of a Hermitian symmetric space of compact type and that of its symmetric

Kähler submanifold (Corollary 8). In particular, the cut locus of a symmetric Kähler submanifold of a complex projective space is a hyperplane section of the submanifold.

2. The diastasis

In this section we shall give the definition of the diastasis of a Kähler manifold due to Calabi [1] and state some basic properties of the diastasis. At the end of this section we shall show that, for $P=\boldsymbol{P}^1(\boldsymbol{C})\times\cdots\times\boldsymbol{P}^1(\boldsymbol{C})$ with the product metric of Hermitian symmetric metrics on $\boldsymbol{P}^1(\boldsymbol{C})$ and a point p in P, the cut locus $C_p(P)$ is the set of points q at which the diastasis $D(p, q)$ cannot be defined.

Let M be a k-dimensional complex manifold with an analytic Kähler metric and $\bar{M}$ be its conjugate manifold. For each point p in M, the point corresponding to p in $\bar{M}$ is denoted by $\bar{p}$. For each complex coordinate system $(z^1, z^2, \cdots, z^k)$ in M, put

$$z^{\alpha*}(\bar{q}) = \overline{z^{\alpha}(q)}\,.$$

Then $(z^{1*}, \cdots, z^{k*})$ is a complex coordinate system in $\bar{M}$ and $(z^1, \cdots, z^k, z^{1*}, \cdots, z^{k*})$ is a complex coordinate system in the product complex manifold $M\times\bar{M}$. Imbedding M into $M\times\bar{M}$ as the diagonal set $\{(p, \bar{p}); p\in M\}$, we can uniquely extend a real analytic functional element in M to a complex analytic functional element in $M\times\bar{M}$.

Let

$$ds^2 = g_{\alpha\beta^*}(z, \bar{z})dz^{\alpha}dz^{\beta*} \tag{1}$$

be the Kähler metric of M, then there exists a real analytic function $\Phi(z, \bar{z})$ such that

$$g_{\alpha\beta^*}(z, \bar{z}) = \frac{\partial^2\Phi(z, \bar{z})}{\partial z^{\alpha}\partial z^{\beta*}}\,. \tag{2}$$

We can extend $\Phi(z, \bar{z})$ to a complex analytic function defined on an open subset of $M\times\bar{M}$. For each points p and q in the open set on which the complex coordinate system $(z^1, \cdots, z^k)$ is defined, we define the functional element

$$\begin{aligned} D(p, q) = {} & \Phi(z(p), \overline{z(p)})+\Phi(z(q), \overline{z(q)}) \\ & -\Phi(z(p), \overline{z(q)})-\Phi(z(q), \overline{z(p)})\,. \end{aligned} \tag{3}$$

These functional elements generate a real analytic function, which is called the *diastasis of M* and denoted by D.

The following proposition is due to Calabi [1].

Proposition 1. *Let M be a Kähler submanifold of a Kähler manifold N* *with an analytic Kähler metric. Then the diastasis of M is the restriction of the diastasis of N to M.*

The following lemma follows from (1), (2), and (3).

Lemma 2. *Let $M_1, \cdots, M_n$ be Kähler manifolds with analytic Kähler metrics ahd D_i be the diastasis of M_i for $i=1, \cdots, n$, then the diastasis of $M_1\times \cdots \times M_n$ is equal to $D_1+ \cdots +D_n$.*

At the end of this section we consider the diastasis of $\boldsymbol{P}^1(\boldsymbol{C})\times\cdots\times\boldsymbol{P}^1(\boldsymbol{C})$.

Proposition 3. *For $P=\boldsymbol{P}^1(\boldsymbol{C})\times \cdots \times\boldsymbol{P}^1(\boldsymbol{C})$ with the product metric of Hermitian symmetric metrics on $\boldsymbol{P}^1(\boldsymbol{C})$ and a point p in P, the cut locus $C_p(P)$ is equal to the set of points q at which the diastasis $D(p, q)$ cannot be defined.*

Proof. We denote by $[z^0, z^1]$ the homogeneous coordinate of $\boldsymbol{P}^1(\boldsymbol{C})$. Without loss of generality it may be assumed that the homogeneous coordinate of p is $[1, 0]$. Let

$$O= \{q\in\boldsymbol{P}^1(\boldsymbol{C});\ z^0(q)\neq 0\}\ .$$

O is an open subset containing p and z^1/z^0 is a complex coordinate on O. The diastasis D of $\boldsymbol{P}^1(\boldsymbol{C})$ is given by

$$D(p, q) = \alpha\log\left[1+\left|\frac{z^1(q)}{z^0(q)}\right|^2\right]$$

for some $\alpha>0$. Let p' be the point whose homogeneous coordinate is $[0, 1]$, then the set of points q at which $D(p, q)$ cannot be defined is $\{p'\}$, which is equal to the cut locus $C_p(\boldsymbol{P}^1(\boldsymbol{C}))$. This proves Proposition 3 for $P=\boldsymbol{P}^1(\boldsymbol{C})$. Proposition 3 follows from Lemma 2 and the fact that, for complete Riemannian manifolds $M_1, \cdots, M_n$ and points p_i in M_i $(1\leqq i\leqq n)$,

$$\begin{aligned} &C_{(p_1,\cdots,p_n)}(M_1\times \cdots \times M_n)\\ &= C_{p_1}(M_1)\times M_2\times \cdots \times M_n\cup M_1\times C_{p_2}(M_2)\times M_3\times \cdots \times M_n\\ &\quad\cup \cdots \cup M_1\times M_2\times \cdots \times M_{n-1}\times C_{p_n}(M_n). \end{aligned}$$

3. Certain submanifolds of a Hermitian symmetric space of compact type

Let M be a Hermitian symmetric space of compact type. In this section we shall construct a maximal totally geodesic flat submanifold A of M and a totally geodesic Kähler submanifold P of M which includes A. For details about the results without proofs, see Helgason [3].

Let $(\mathfrak{u}, \theta)$ be the orthogonal symmetric Lie algebra associated with M. We have the canonical direct sum decomposition of $\mathfrak{u}$:

$$\mathfrak{u} = \mathfrak{k} + \mathfrak{p} ,$$

where

$$\mathfrak{k} = \{X \in \mathfrak{u};\, \theta(X) = X\} \quad \text{and} \quad \mathfrak{p} = \{X \in \mathfrak{u};\, \theta(X) = -X\} .$$

Take a maximal Abelian subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. We denote the complexifications of $\mathfrak{u}$, $\mathfrak{k}$, $\mathfrak{p}$, and $\mathfrak{h}$ by $\mathfrak{g}$, $\tilde{\mathfrak{k}}$, $\tilde{\mathfrak{p}}$, and $\tilde{\mathfrak{h}}$ respectively. $\tilde{\mathfrak{h}}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the set of nonzero roots of $\mathfrak{g}$ with respect to $\tilde{\mathfrak{h}}$. For each root α in Δ, put

$$\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g};\, [H, X] = \alpha(H)X \qquad \text{for each } H \in \tilde{\mathfrak{h}}\} .$$

Since $\tilde{\mathfrak{h}} \subset \tilde{\mathfrak{k}}$, for each α in Δ, $\mathfrak{g}^{\alpha} \subset \tilde{\mathfrak{k}}$ or $\mathfrak{g}^{\alpha} \subset \tilde{\mathfrak{p}}$. A root α is called *compact* [resp. *noncompact*], if $\mathfrak{g}^{\alpha} \subset \tilde{\mathfrak{k}}$ [resp. $\mathfrak{g}^{\alpha} \subset \tilde{\mathfrak{p}}$]. By the root space decomposition of $\mathfrak{g}$, we obtain

$$\tilde{\mathfrak{k}} = \tilde{\mathfrak{h}} + \sum_{\alpha\,:\,\text{compact}} \mathfrak{g}^{\alpha}, \quad \tilde{\mathfrak{p}} = \sum_{\beta\,:\,\text{non-compact}} \mathfrak{g}^{\beta} .$$

We introduce a lexicographic order in the dual of the real vector space $\sqrt{-1}\,\mathfrak{h}$. Note that each root is real valued on $\sqrt{-1}\,\mathfrak{h}$.

Let Q be the set of positive noncompact roots in Δ and r be the rank of M. Then there is a strongly orthogonal root system $\{\gamma_1, \cdots, \gamma_r\}$ in Q, that is, $\gamma_i \pm \gamma_j \notin \Delta$ for $1 \leqq i, j \leqq r$. We can choose nonzero vectors $X_{\alpha} \in \mathfrak{g}^{\alpha}$ for each roots α in Δ such that

$$X_{\alpha} - X_{-\alpha},\ \sqrt{-1}(X_{\alpha} + X_{-\alpha}) \in \mathfrak{u} , \tag{4}$$

$$[X_{\alpha}, X_{-\alpha}] = \frac{2}{\alpha(H_{\alpha})} H_{\alpha} , \tag{5}$$

where H_{α} is the dual vector of α with respect to the Killing form of $\mathfrak{g}$. Since $\gamma_i \pm \gamma_j \notin \Delta$,

$$[X_{\pm\gamma_i}, X_{\pm\gamma_j}] = [H_{\pm\gamma_i}, X_{\pm\gamma_j}] = 0, \quad \text{if } i \neq i . \tag{6}$$

By this property

$$\mathfrak{a}_{\mathfrak{p}} = \sum_{i=1}^{r} \boldsymbol{R} \sqrt{-1}\,(X_{\gamma_i} + X_{-\gamma_i}) \tag{7}$$

is a maximal Abelian subspace in $\mathfrak{p}$.

Let U be a simply connected Lie group with Lie algebra $\mathfrak{u}$ and K be the analytic subgroup of U with Lie algebra $\mathfrak{k}$. Since M is simply connected, $M = U/K$. The action of U on M is isometric and holomorphic.

Put

$$A = \exp(\mathfrak{a}_{\mathfrak{p}})o ,$$

where o is the origin of $M = U/K$. The submanifold A is a maximal totally geodesic flat submanifold of M, because $\mathfrak{a}_\mathfrak{p}$ is a maximal Abelian subspace in $\mathfrak{p}$.

We define a linear map ϕ_i: $\mathfrak{su}(2) \to \mathfrak{u}$ by

$$\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \mapsto X_{\gamma_i} - X_{-\gamma_i},$$
$$\begin{bmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{bmatrix} \mapsto \sqrt{-1}\,(X_{\gamma_i} + X_{-\gamma_i}),$$
$$\begin{bmatrix} \sqrt{-1} & \\ & \sqrt{-1} \end{bmatrix} \mapsto \frac{2\sqrt{-1}}{\gamma_i(H_{\gamma_i})} H_{\gamma_i}.$$

By (4) and (5) ϕ_i is a well-defined injective Lie algebra homomorphism of $\mathfrak{su}(2)$ to $\mathfrak{u}$. Since

$$[\phi_i(\mathfrak{su}(2)), \phi_j(\mathfrak{su}(2))] = \{0\}, \quad 1 \leqq i \neq j \leqq r,$$

by (6), we can define an injective Lie algebra homomorphism ϕ from the r-fold direct sum $\mathfrak{su}(2)^r$ of $\mathfrak{su}(2)$ into $\mathfrak{u}$ by

$$\phi(X_1, \cdots, X_r) = \sum_{i=1}^{r} \phi_i(X_i) \qquad \text{for } X_i \in \mathfrak{su}(2).$$

ϕ also denotes the homomorphism from the r-fold direct product $SU(2)^r$ of $SU(2)$ into U induced by the Lie algebra homomorphism ϕ. Then ϕ induces an equivariant holomorphic imbedding

$$\rho\colon SU(2)^r/S(U(1) \times U(1))^r \to M$$
$$xS(U(1) \times U(1))^r \mapsto \phi(x)o \qquad \text{for } x \in SU(2)^r.$$

Note that the r-fold direct product $\boldsymbol{P}^1(\boldsymbol{C})^r$ of $\boldsymbol{P}^1(\boldsymbol{C})$ is canonically identified with $SU(2)^r/S(U(1) \times U(1))^r$.

We denote by P the image of the imbedding ρ. By the definition of ϕ and (7), $\mathfrak{a}_\mathfrak{p} \subset \phi(\mathfrak{su}(2)^r)$, so $A \subset P$. Since

$$\phi(\mathfrak{su}(2)^r) = \mathfrak{k} \cap \phi(\mathfrak{su}(2)^r) + \mathfrak{p} \cap \phi(\mathfrak{su}(2)^r),$$

P is a totally geodesic submanifold of M. The induced metric on $\boldsymbol{P}^1(\boldsymbol{C})^r$ is the product metric of Hermitian symmetric metrics on $\boldsymbol{P}^1(\boldsymbol{C})$, because the imbedding ρ is equivariant.

By a theorem of Cartan to the effect that M is given by

$$M = \bigcup_{k \in K} kA,$$

we obtain

$$M = \bigcup_{k \in K} kP.$$

The following proposition summarizes this section.

Proposition 4. a) *A is a maximal totally geodesic flat submanifold of M through o.*
b) *$P \cong \boldsymbol{P}^1(\boldsymbol{C})^r$ is a totally geodesic Kähler submanifold of M which includes A and its metric is the product metric of Hermitian symmetric metrics on $\boldsymbol{P}^1(\boldsymbol{C})$.*
c)
$$M = \bigcup_{k \in K} kP .$$

REMARK. The imbedding ρ was used by Takagi and Takeuchi [6] in order to determine the degree of symmetric Kähler submanifolds of a complex projective space.

4. The cut locus and the diastasis of a Hermitian symmetric space of compact type

In this section we shall prove the following main theorem stated in Introduction.

Theorem 5. *Let M be a Hermitian symmetric space of compact type and D be the diastasis of M. Then, for each point p in M, the cut locus $C_p(M)$ is equal to the set of points q at which $D(p, q)$ cannot be defined.*

We retain the notations in Section 3.

Lemma 6.
$$C_o(M) \cap A = C_o(A) \text{ and } C_o(M) = \bigcup_{k \in K} kC_o(A) .$$
This lemma is due to Sakai [5].

Lemma 7.
$$C_o(M) = \bigcup_{k \in K} kC_o(P) .$$

Proof. Put
$$U_1 = \phi(SU(2)^r) \text{ and } K_1 = U_1 \cap K .$$
Then (U_1, K_1) is a Riemannian symmetric pair and $P = U_1/K_1$.

Since P is a totally geodesic submanifold of M, A is also a maximal totally geodesic flat submanifold of P. Applying Lemma 6 to $P = U_1/K_1$ and A, we have
$$\begin{aligned} C_o(P) &= \bigcup_{k_1 \in K_1} k_1 C_o(A) \\ &= \bigcup_{k_1 \in K_1} k_1(A \cap C_o(M)) \\ &= P \cap C_o(M) , \end{aligned}$$

hence from c) of Proposition 4

$$\bigcup_{k\in K} kC_o(P) = \bigcup_{k\in K} k(P\cap C_o(M)) = M\cap C_o(M) = C_o(M).$$

Now we shall prove Theorem 5. Without loss of generality we may assume that p is the origin o of $M=U/K$. Since P is a Kähler submanifold of M, the restriction of D to P is the diastasis of P by Proposition 1. The action of K on M is isometric and holomorphic, hence

$$\begin{aligned} &\{q\in M;\ D(p,q) \text{ cannot be defined}\} \\ &= \bigcup_{k\in K} k\{q\in P;\ D(p,q) \text{ cannot be defined}\} \\ &= \bigcup_{k\in K} kC_o(P) \\ &= C_o(M). \end{aligned}$$

This completes the proof of Theorem 5.

Corollary 8. *Let M_1 and M_2 be Hermitian symmetric spaces of compact type. If M_1 is a Kähler submanifold of M_2, then*

$$C_p(M_1) = M_1\cap C_p(M_2)$$

for each point p in M_1.

REMARK. In case of $M_2=\boldsymbol{P}^n(\boldsymbol{C})$, Theorem 4.3 in Nakagawa and Takagi [4] implies that the imbedding of M_1 into $\boldsymbol{P}^n(\boldsymbol{C})$ is equivariant. So we can describe the behavior of a geodesic of M_1 in $\boldsymbol{P}^n(\boldsymbol{C})$ and directly show the assertion of Corollary 8 in this case.

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Department of Mathematics
Tokyo Gakugei University
Koganei, Tokyo 184
Japan