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by

Nathan Yves Jean-Jacques TOUROUX

**Efficient implicit solver for relativistic hydrodynamics in the
dynamical modeling of heavy-ion collisions**

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Chapter 1

Introduction

Transport coefficients such as the shear viscosity η are useful insight about the nature of the hot plasma produced in heavy-ion collisions (HIC) experiments. In high energy collisions such as at the Large Hadron Collider (LHC), the quarks and gluons which are the degrees of freedom of the Quantum Chromo-Dynamics (QCD) theory deconfine from their hadronic state the Quark Gluon Plasma (QGP). Therefore, properly modeling HIC, and especially the QGP, is pivotal in studying the phase structure of QCD matter.

In this thesis, we focus on the hydrodynamic modeling of the QGP. More specifically, we study the numerical implementation of relativistic hydrodynamic used in the process of simulating HIC. Having an accurate and efficient implementation is crucial in the process of comparing the model results to the experimental data.

This thesis begins in this chapter with the introduction of the basic ideas of QCD and heavy-ion collisions in Secs. 1.1 and 1.2. The history of how hydrodynamics came to be used in HIC modeling will be discussed in Sec. 1.2.2.

Throughout this thesis, the two following conventions will be used. First, we use the natural units where $c = \hbar = k_B = 1$. Second, we use the mostly negative convention for the metric tensor $g_{\mu\nu} = \text{diag}(+, -, -, -)$.

1.1 Quantum Chromo-Dynamics

In this thesis we are interested in a particular state composed of the quarks and gluons called the QGP. The reason the quarks and gluons are so difficult to study, and also the QGP, comes from the laws that rule their behavior called QCD. In this chapter, we will discuss the theory of the QCD and discuss its different states.

1.1.1 Abelian theory

The theory of QCD is described by a Lagrangian including a peculiar type of symmetry. This theory involves many concepts which should be introduced before explaining the theory itself. In this section, we will introduce these concepts by progressively constructing a simpler theory. The discussion in this section and the next one is inspired by chapter 15 of [1].

QCD is a theory constructed on the basis of symmetries. Therefore, we should begin with a simpler theory having a similar symmetry. To do so, we will start with a Dirac field $\psi(x)$. From this field we can consider two types of symmetries.

The first type is global symmetry where the transformation is independent of space. An example of such a transformation is

$$\psi(x) \rightarrow e^{i\alpha} \psi(x), \tag{1.1}$$

which is the global U(1) transformation. From this transformation, we can construct two invariant terms

$$m\bar{\psi}(x)\psi(x) \rightarrow m(\bar{\psi}(x)e^{-i\alpha})(e^{i\alpha}\psi(x)) = m\bar{\psi}(x)\psi(x), \quad (1.2)$$

$$\bar{\psi}(x)\partial_\mu\psi(x) \rightarrow (\bar{\psi}(x)e^{-i\alpha})\partial_\mu(e^{i\alpha}\psi(x)) = \bar{\psi}(x)\partial_\mu\psi(x), \quad (1.3)$$

where the first one is the mass term and the second one the kinetic term. By including the gamma matrices γ^μ from the relativistic Dirac equation, we can construct the Lagrangian describing an interactionless spin-1/2 particle [1]

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x). \quad (1.4)$$

This Lagrangian is invariant under the U(1) transformation.

The second type is local symmetry, or Gauge symmetry. In this case, the transformation depends on the space coordinate x .

The local U(1) transformation is defined as

$$\psi(x) \rightarrow e^{i\theta(x)}\psi(x). \quad (1.5)$$

Under this transformation, the mass term is still invariant, but not the kinetic one

$$m\bar{\psi}(x)\psi(x) \rightarrow m(\bar{\psi}(x)e^{-i\theta(x)})(e^{i\theta(x)}\psi(x)) = m\bar{\psi}(x)\psi(x), \quad (1.6)$$

$$\bar{\psi}(x)\partial_\mu\psi(x) \rightarrow (\bar{\psi}(x)e^{-i\theta(x)})\partial_\mu(e^{i\theta(x)}\psi(x)) = \bar{\psi}(x)\partial_\mu\psi(x) + i\bar{\psi}(x)\psi(x)\partial_\mu\theta(x). \quad (1.7)$$

Because the gauge, here $\theta(x)$, can be chosen freely, every field configuration that can be related through the gauge transformation are actually the same. This means that the only relevant quantities are the gauge invariant one, or said differently, the ones that are invariant after a gauge transformation. Therefore, we want the kinetic term to be invariant in order to be meaningful. To do so, we introduce a covariant derivative D_μ such that

$$D_\mu(e^{i\theta(x)}\psi(x)) = e^{i\theta(x)}D_\mu\psi(x). \quad (1.8)$$

This covariant derivative needs a term additional to ∂_μ in order to subtract the $i\partial_\mu\theta(x)$ contribution in Eq. (1.7) after the transformation is applied. This can be done by introducing a vector boson $A_\mu(x)$ with a constant e which transforms as

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu\theta(x). \quad (1.9)$$

With this vector boson, the covariant derivative read

$$D_\mu = \partial_\mu + ieA_\mu, \quad (1.10)$$

and the kinetic term defined as $\bar{\psi}(x)D_\mu\psi(x)$ is invariant

$$\bar{\psi}(x)D_\mu\psi(x) \rightarrow (\bar{\psi}(x)e^{-i\theta(x)})\left[\partial_\mu + ie\left(A_\mu - \frac{1}{e}\partial_\mu\theta(x)\right)\right](e^{i\theta(x)}\psi(x)) \quad (1.11)$$

$$= \bar{\psi}(x)e^{-i\theta(x)}e^{i\theta(x)}(\partial_\mu + ieA_\mu)\psi(x) \quad (1.12)$$

$$= \bar{\psi}(x)D_\mu\psi(x). \quad (1.13)$$

As we introduce a new vector boson A_μ , we also need the kinetic terms that can be written with it to construct our Lagrangian. More precisely, we want to include all the gauge

invariant terms into the Lagrangian as they are the relevant terms that cannot be changed by changing the gauge.

One way to construct a gauge invariant quantity from A_μ is to realize what happens when the covariant derivative D_μ is applied to the Dirac field $\psi(x)$. The resulting $D_\mu\psi(x)$ transforms in the same way as $\psi(x)$. We can thus apply the covariant derivative again to $\psi(x)$, and it will continue to transform the same. From this reasoning, the commutator of the covariant derivative applied to $\psi(x)$ still transforms the same

$$[D_\mu, D_\nu]\psi(x) \rightarrow e^{i\theta(x)}[D_\mu, D_\nu]\psi(x). \quad (1.14)$$

However, in a flat space-time the partial derivatives ∂_μ commute with each other $[\partial_\mu, \partial_\nu] = 0$. In addition, the components of A_μ are real numbers, so they also commute with each other. Therefore, by developing the commutator

$$[D_\mu, D_\nu]\psi(x) = [\partial_\mu, \partial_\nu]\psi(x) + ie([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu])\psi(x) - e^2[A_\mu, A_\nu]\psi(x) \quad (1.15)$$

$$= ie([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu])\psi(x) \quad (1.16)$$

$$= ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi(x), \quad (1.17)$$

we can see that this commutator is not a derivative. This means that in the right-hand side of Eq. (1.14), the commutator $[D_\mu, D_\nu]$ acts as a simple multiplicative factor. Therefore, the field $\psi(x)$ is the part responsible for the transformation, and the commutator must be invariant. Removing the factor ie , the relevant invariant quantity is the field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (1.18)$$

We can also show that quadratic or higher order terms made of A_μ , such as $A_\mu A^\mu$, are not gauge invariant.

If we impose the parity and time reversal symmetries in addition to the local U(1) gauge symmetry, and we exclude all the non renormalizable terms (see [1]), the Lagrangian is only composed of the invariant terms we introduced before

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (1.19)$$

where the $1/4$ factor is standard. This Lagrangian we constructed is the Quantum Electrodynamics (QED) Lagrangian, where Ψ is the electron field, and A_μ the photon field.

This Lagrangian suggests three behaviors:

1. The kinetic term $\bar{\psi}(\gamma^\mu \partial_\mu - m)\psi$ shows that the field ψ is massive, and therefore particles of the field Ψ will propagate slower than the speed of light.
2. These particles interact with each other through an exchange of a vector boson A_μ with coupling strength e in $\bar{\psi}\gamma^\mu e A_\mu \psi$.
3. The vector bosons A_μ are massless as there is no quadratic term $A_\mu A^\mu$, and they cannot interact with each other as there are no third or higher order terms appearing.

When the coupling constant e is small, such a theory can be solved perturbatively by expansion in e . The story would be completely different if the bosons could interact with each other.

A theory such as the one we constructed where the components of A_μ commute is called an Abelian theory.

1.1.2 QCD, a non-Abelian theory

Compared to the Abelian case discussed in the previous subsection, the theory of QCD is based on the SU(3) local symmetry, whose elements do not commute. More precisely, the SU(3) group act on vectors of three components. These operations can be interpreted as a mixing between three components, which are called colors in QCD. To include these colors, we add an index $i = 1, 2, 3$ to the Dirac field $\psi^i(x)$, which is now the quark field. The SU(3) rotations will then mix the three $\psi^i(x)$.

All the operations permitted by SU(3) can be written as $U = e^{i \sum_a \lambda^a \theta^a(x)}$ where a ranges from 1 to 8. Compared to the previous case where there was a single $\theta(x)$, there is now eight $\theta^a(x)$ associated to eight generators λ^a , which are 3x3 complex matrices. The commutator of these matrices is not null but instead produces a combination of all the matrices

$$[\lambda^b, \lambda^c] = (\lambda^b \lambda^c - \lambda^c \lambda^b) = \sum_a f^{abc} \lambda^a, \quad (1.20)$$

where f_{abc} are the structure constants of the corresponding Lie algebra.

In the process of deriving the covariant derivative for the SU(3) local symmetry, we construct one vector boson $A_\mu^a(x)$ for each $\theta^a(x)$. These bosons are the eight gluons.

The covariant derivative is now a 3x3 matrix which accounts for the three components of the SU(3) vectors

$$(D_\mu)_{ij} = \delta_{ij} \partial_\mu + i \frac{g_s}{2} \sum_a \lambda_{ij}^a A_\mu^a, \quad (1.21)$$

where g_s is the coupling constant.

In the same way, the tensor field became more complex, and there are eight of them

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s \sum_{b,c} f^{abc} A_\mu^b A_\nu^c, \quad a = 1, \dots, 8. \quad (1.22)$$

Compared to the Abelian case, they include a non-zero commutator of the vector bosons.

Once again, by considering the parity and time reversal symmetries, and excluding all the non renormalizable terms (see [1]), the Lagrangian of QCD reads

$$\mathcal{L}_{\text{QCD}} = \sum_{f=1}^6 \bar{\psi}_f^i [i \gamma^\mu (D_\mu)_{ij} - m_f] \psi_f^j - \frac{1}{4} \sum_a F_{\mu\nu}^a F^{a\mu\nu}, \quad (1.23)$$

where the index f in $\psi(x)_f^i$ and m_f account for the six different flavors of quark which have six different masses.

The Lagrangian Eq. (1.23) is very similar to the Abelian one discuss in Sec. 1.1.1. However, there are two main differences. The first one is that the Dirac fields ψ_q^i have three charges which can be exchanged through the interaction with a gluon. The second is the inclusion of the non-zero commutator in $F_{\mu\nu}^a$, which produces third and fourth order terms in A_μ^a , and therefore introduces three and four gluons interactions.

Such a theory where the elements of the symmetry, and therefore the corresponding bosons, do not commute is called a non-Abelian theory.

In summary, QCD is a theory describing the dynamics of quarks and gluons, and its Lagrangian is given by Eq. (1.23). The existence of the gluon self interaction also means that in the process of exchanging colors between quarks, a gluon from one quark could interact with other gluons before finalizing the exchange with the second quark.

1.1.3 Confinement and Asymptotic freedom

The Lagrangian Eq. (1.23) shows the interactions between the quarks and the gluons as a product of them with a coupling constant. However, computing the evolution of the particles from the Lagrangian directly leads to infinities at high momenta called the ultraviolet divergence. For this matter, the renormalization group is a theory that can be applied to cure this divergence in most quantum field theories. Although explaining what is renormalization is outside the scope of this thesis, its main implication is still important.

After renormalization, a theory such as QED or QCD will produce finite results. The way the renormalization cures the divergence is by altering the parameters of the theory. In our case, it makes the coupling constant dependent on the energy scale, so it is not a constant anymore.

In the case of QED, the coupling will increase as the energy scale increases, so the interaction becomes stronger at higher energies. Conversely, it appears that the coupling in QCD has the opposite behavior where it decreases as the energy scale increases.

In the low energy regime, the quarks are always confined into hadrons. This property essentially comes from the large coupling at low energy. More precisely, the color-electric field between the quarks form a flux tube which induces a linear potential [2, 3]. Therefore, when two quarks are far apart and the potential energy becomes greater than the mass of two quarks, it becomes energetically favorable to produce a quark and anti-quark pair resulting in two distinct hadrons and vanishing color-electric potential.

In the asymptotically large energy regime, the coupling will decrease to the point where the strong interaction will vanish. This is called the asymptotic freedom.

The asymptotic freedom was found experimentally before knowing that the local $SU(3)$ symmetry could produce vector bosons and asymptotic freedom. In the 1960s, the SLAC-MIT experiment [4] performed deep inelastic scattering of electrons on protons. At that time the proton structure was not known yet. By performing inelastic scattering, two scenarios could occur: the proton is made of strongly interacting constituents, and thus it would recoil as a whole, or it is composed of weakly interacting constituents which would be deflected independently. The experiment showed that it was the latter. In hard scattering, the electron would be deflected as if it interacted mainly through electrodynamics with an electrically charged elementary particle from inside the proton. A jet of hadrons would be produced in the direction this internal particle should have scattered. What was concluded is that the constituents of the proton would be loosely bound by strong interaction in a short and highly energetic process such as hard electron scattering occurred. This led to the search for an asymptotically free theory with vector bosons, which ended up being the QCD. The fact that the quarks and gluons behave differently at different energy scales motivated the exploration of the different phases of QCD matter.

The previous experiment confirms the asymptotic freedom, but it does not free the quarks of confinement as this freedom is only local in time and space. However, the extreme asymptotic freedom is not necessary to stop confinement. Indeed, at sufficiently large energies such as in HIC, a plasma is produced. In such a state, the amount of color fields increases with the distances and the color-electric potential is no longer linear. It now decreases quickly with the distance due to Debye screening [5, 6]. Because of this, the quarks can move freely in the plasma without being confined into hadrons. This is called the deconfinement.

1.1.4 QGP

The QGP is the state in which the quark have reached deconfinement, and they can individually move almost freely inside the medium.

When produced in a heavy ion collision, the QGP will have a very large pressure in the longitudinal direction, usually called the η direction, where $\eta = \tanh^{-1}(z/t)$ is the position along the longitudinal direction in Milne coordinates. Furthermore, the two nuclei rarely exactly face each other, which produces an elliptic interacting shape in the transverse plane. These two properties lead to anisotropic particle production after the freeze-out.

The anisotropic flow is very useful to study the viscosity of the QGP [7, 8]. However, the hadrons produced after the freeze-out can be either neutral or charged, yet detecting neutral particles is difficult. Therefore, the experiments are usually only focusing on the charged particles as a proxy.

Another useful property of the QGP is its production of jets [9]. The jets, large number of hadrons produced in a similar direction from the scattering of partons, have been well studied especially in the context of proton-proton collisions. These jets have a reduced energy and number of particles in heavy-ion collisions, although they include more participating partons which could scatter than proton-proton collisions. This reduction is a signal of the QGP in which the scattered partons lose energy as they traverse the medium and thus the amount of large energy partons is reduced. This is often called jet quenching.

Similarly to the suppression of jets, the suppression of quarkonia, which are bound state heavy-quark pairs $c\bar{c}$ and $b\bar{b}$, is an interesting probe for the QGP. Due to the color screening in the QGP, the formation of quarkonia should be suppressed [10]. Observations of this suppression such as in [11] is an important probe of the deconfinement of the quarks and gluons in the QGP.

Another way to study the QGP is through its phase transition from the hadronic phase. During phase transitions, the charges tend to fluctuate in very specific ways. Close to the expected critical point of QCD, which has the same universality class as the one of water, the fluctuation of the order parameter should diverge. The order parameter of QCD is the chiral condensate $\sigma = \langle \bar{\psi}\psi \rangle$, which measures the spontaneous breaking of chiral symmetry. This condensate is formed in the vacuum and vanishes at high temperature. Although it is difficult to study it directly, it couples to the net-baryon number, which can be observed by the proxy of charged particles. The fluctuations of the charged particle production can thus be used as a probe for the transition from the QGP to hadronic matter through the critical point [12]. At the boundary of phase transitions, not only a critical point, most conserved charges will have anomalous fluctuation behavior, which means that the study of the fluctuations is a good probe to find the boundary of the QGP state.

One last important probe is the thermal photon and dilepton production to measure the temperature of the QGP. These particles are not made of quarks and gluons and thus are not affected by the strong interaction in the QGP medium, which make them a good probe for the thermal properties of the QGP [13].

1.1.5 Ideal gas limit

Before going further in a theory, it is always convenient to develop intuition with the ideal limit. In our case, there are two interesting limits. The first one is the low temperature limit where the quarks and gluons are confined into colorless hadrons which can be considered to have vanishing strong interaction between them owing to low density. The second one is the infinite temperature limit where the strong coupling vanishes, which leaves the quarks and gluons free.

In this subsection we will go briefly over these two cases by taking inspiration on [14]. In both cases, the general form of an ideal gas partition function is

$$\ln Z_i^1(V, T) = \eta \nu_i V \int \frac{d^3p}{(2\pi)^3} \ln(1 + \eta e^{-\beta(\omega_i - \mu_i)}), \quad (1.24)$$

where $\beta = 1/T$, i indices over the individual particles, $\eta = \{-1, 1\}$ for bosons and fermions, ν_i is the number of degrees of particle i , μ_i is the chemical potential for a conserved charge associated with particle i , and ω_i is the energy of the particle.

The Hadron Resonance Gas

In the low temperature limit, we can consider that the hadrons are approximately free and the partition function of a system of hadrons factorizes in individual partition functions.

The logarithm of the partition function becomes the sum of the individual partition functions $\ln Z \approx \sum_i \ln Z_i^1$, where Z_i^1 is the ideal gas partition function Eq. (1.24). Therefore, the pressure and energy density of such an ideal hadron gas become

$$p = \frac{T}{V} \ln Z = \frac{T}{V} \sum_{i \in \{\text{mesons}\}} \ln Z_i^1(V, T, \mu_i) + \frac{T}{V} \sum_{i \in \{\text{baryons}\}} \ln Z_i^1(V, T, \mu_i) \quad (1.25)$$

$$\epsilon = -\frac{1}{V} \frac{\partial \ln Z}{\partial \beta} = -\frac{1}{V} \sum_{i \in \{\text{mesons}\}} \frac{\partial \ln Z_i^1(V, T, \mu_i)}{\partial \beta} - \frac{1}{V} \sum_{i \in \{\text{baryons}\}} \frac{\partial \ln Z_i^1(V, T, \mu_i)}{\partial \beta} \quad (1.26)$$

where η in Eq. (1.24) is -1 for mesons and 1 for baryons, and i ranges over the different hadrons.

One can use the experimental energies of hadrons and resonances i that were established so far and perform the momentum integrals to obtain the pressure and energy density as a function of the temperature and chemical potential.

Free quarks and gluons, the Stefan-Boltzmann limit

In the high temperature limit, we can regard the system as a gas of free quarks and gluons. In the massless limit for light quarks, the momentum integral in the partition functions can be performed, and it gives the Stefan-Boltzmann pressure

$$p = \alpha_i \nu_i \frac{\pi^2}{90} T^4 \quad (1.27)$$

where $\alpha_i = \{1, 7/8\}$ for bosons and fermions respectively, and ν_i the degeneracy.

There are two polarizations for each of the 8 gluons, and 3 colors and 2 spins for the quarks and anti-quarks. The pressure reads

$$p = (16 + \frac{21}{2} N_f) \frac{\pi^2}{90} T^4 \quad (1.28)$$

where N_f is the number of quark flavors considered.

In this conformal case where quarks are massless, the energy density is

$$\epsilon = 3p. \quad (1.29)$$

1.1.6 Lattice QCD equation of state

The temperature, energy density and the pressure are the most fundamental thermal quantities. Their expression are necessary to model the QGP with relativistic hydrodynamics where the equation of state relating these three quantities is a theoretical input. However, a direct calculation of these thermal quantities from the Lagrangian cannot be done for QCD due to the large coupling constant.

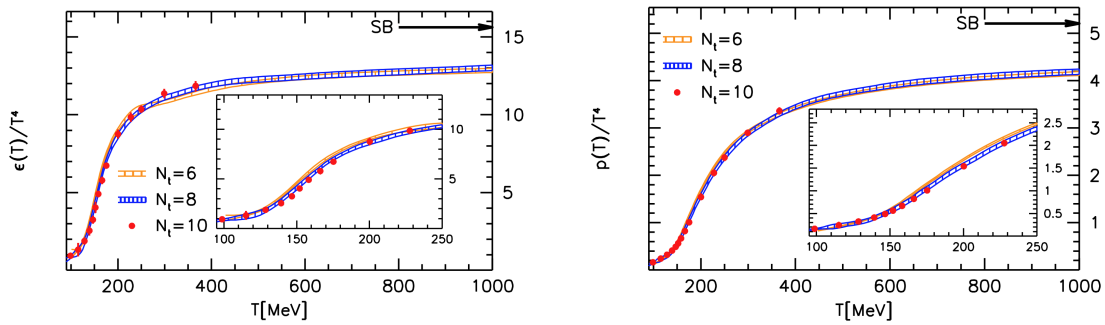


Figure 1.1: Energy density (left) and pressure (right) divided by T^4 from lattice QCD [15] as a function of the temperature T for different time discretizations $N_t = 6, 8, 10$.

To circumvent this, the lattice theory [14] discretizes space and time as a finite lattice where the quarks are on the vertices and the gluons are on the edges. The action of this system is constructed such that the action of the QCD theory is recovered in the continuum limit where the lattice spacing a , the distance between two adjacent vertices, tends to zero. In this prescription, the lattice is expressed in Euclidean space-time so that the action is a real number.

In the Euclidean space-time, the length in the time direction corresponds to the inverse of the temperature $1/T$. This means that the temperature of the system can be controlled by the time discretization by varying the number of sites N_t for fixed a or the opposite. The thermodynamic quantities such as the pressure can then be computed numerically for different space discretization. The continuum limit of these quantities is then obtained by extrapolating to vanishing lattice spacing $a \rightarrow 0$. By varying the time discretization and thus the temperature, the lattice calculations can tabulate the thermodynamic quantities.

In the case of QCD, the $SU(3)$ symmetry is included in the action of the lattice theory. The pressure can then be obtained as a function of the temperature, which is called the equation of state. This is a crucial input necessary to write the equations of relativistic hydrodynamics in the context of heavy-ion collisions as we will do in Cha. 2.

An equation of state for 2 + 1 quarks flavors with physical masses was studied in [15]. The corresponding energy density and pressure divided by T^4 can be seen as a function of the temperature T on the left and right panels of Fig. 1.1. We can see that at high energy they approach the Stefan-Boltzmann limit Eqs. (1.27) and (1.29) we discussed earlier.

1.1.7 QCD phase diagram

The study of quark matter has a similarity to the study of water. It is well known that water undergoes phase transitions depending on the temperature. From liquid, water becomes solid ice when the temperature drops too low, and gas vapor when heated. An additional observation is that in high altitudes, such as in the mountains, the pressure is lower and the water boils at lower temperatures. As water has very different properties in its different states, especially its volume change, it is useful to know in what condition it is in which state. This corresponds to studying its phase diagram as depicted in Fig. 1.2 where we can see the different phases of water as a function of the pressure and temperature. There are two noticeable aspects in this diagram. First, in this diagram, we can see that depending on the temperature and pressure the states can either change spontaneously through a first-order transition depicted by the solid lines, or they can change continuously in a crossover transition where there are no lines in between. Second, when the line of first-order transition reaches the

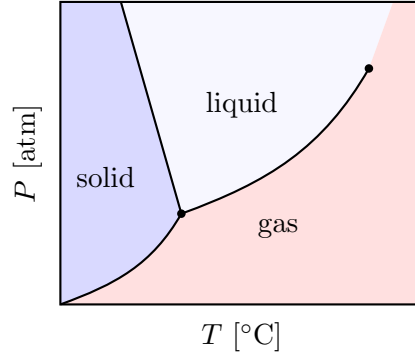


Figure 1.2: Phase diagram of water on the pressure (P) and temperature (T) plane.

crossover region, it just abruptly stops at a point. This specific point happens to be a critical point where a second-order phase transition occurs. On this point, the correlation length tends to infinity and fluctuations grow and diverge. This point is very interesting as it has very visible properties, especially since all frequencies of light can be absorbed and emitted due to the divergence of fluctuations. This makes the water appear completely opaque and white.

A similar story happened for the study of the quark matter, although the first prediction has been made theoretical. In the 1960s, Nambu and Jona-Lasinio (NJL) introduced a superconductive state model (NJL) for the proton-neutron in the vacuum [16]. They will later extend the model to finite μ_B . In the 1970s, Cabibbo and Parisi theorized that the quarks confined in hadrons would reach a deconfined state above a high critical temperature [17]. At that time they thought that the transition would be second order. Even if the actual order of the transition will change as the theories evolve, this started the path of exploring the phases of quark matter. From chiral symmetry restoration calculation on a linear *sigma* model with massless quarks in the 1980s, the transition is expected to be first order [18]. Then finally the transition is found to be a crossover in the 2000s at physical quark masses in the lattice QCD calculations [19].

These predictions at different temperatures and baryon chemical potential motivated the study of the phase diagram of quark matter. Especially, Asakawa and Yazaki predicted a first-order transition at large μ_B in the NJL model [20]. They also predicted a critical point to exist as an end point of the first-order phase transition. Similarly to the water case, a critical point is very interesting due to its peculiar diverging fluctuations. The possibility of experimental search for this critical point was then discussed by many people [21].

These different phases can be summarized in a phase diagram. There are many depictions of this phase diagram, but essentially it has three major phases as depicted in a schematic from [22]* on Fig 1.3. The QGP is located on the top at high temperatures. It is separated from the hadron gas by a crossover transition at low μ_B and a first-order transition depicted by a white line at large μ_B . The expected critical point is depicted in red at the junction between the first-order transition and the crossover. The last color superconductor phase which is still only theoretical is located at very high μ_B and low temperature at the bottom right of the diagram.

All these discussions eventually led to the heavy-ion collisions and the Beam energy scan [24]. The idea consists of colliding nuclei at different energies. At large energy, the temperature greatly increases while the baryon stopping is reduced at mid-rapidity [25],

*Recent overview do not depict a more precise diagram [23]

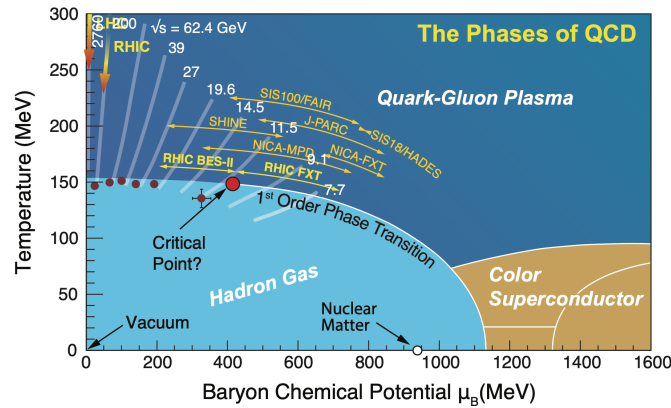


Figure 1.3: Depiction of the QCD phase diagram in the temperature and baryon chemical potential plane from [22]

which reduces the baryon chemical potential. At low energy, the opposite happens where the temperature increases less and the baryon stopping is stronger, leading to a larger μ_B . Thus, the beam energy can scan the QCD phase diagram and search for large fluctuations through higher-order cumulants to find the critical point. The low μ_B region is experimentally obtained by the LHC experiments at $\sqrt{s_{NN}} > 200$ GeV. The lower $3 \text{ GeV} < \sqrt{s_{NN}} < 200$ GeV region is then obtained by RHIC, especially from the beam energy scan program [22]. The High Acceptance DiElectron Spectrometer (HADES) from GSI goes further down to $2 \text{ GeV} < \sqrt{s_{NN}} < 3 \text{ GeV}$ [26].

1.2 Relativistic heavy ion collisions

In a heavy ion collision, the colliding energy can be controlled and the final produced particles can be observed. However, we cannot directly observe what happens from the collision until the produced particles reach the detector. Due to this, the exact evolution cannot be known for certain. The only thing we observe are discrete particles in the final state.

However, much information can be extracted from the distribution of these particles. Especially, the anisotropy of the produced particles tends to reflect the shape of the initial state. The more off center the nuclei collide, the more oval the interacting system will be. In this case, the particles will not be produced isotropically, but instead they will follow this oval shape [27]. Such a behavior strongly suggests that the interacting medium is ruled by fluid dynamics.

Considering the difference between fluid dynamics and freely flying particles, there must be a thermalization from the collision to a hydrodynamic state, and then a chemical freeze-out to some kinetic evolution where particles will fly to the detector. In reality, the stages are not discrete with spontaneous transitions. Instead, the QGP should be composed of simultaneously interacting hadrons, jets, deconfined quarks and gluons, etc. However, it would be too difficult to numerically simulate everything at once. Therefore, the usual way of modeling is to describe the full process as successive independent stages. The specific models used for each of these states will be discussed in the following sections.

1.2.1 Physical evolution of a collision

To numerically solve HIC, we need to have a physical interpretation of the full evolution from the collision to the particles reaching the detector. It is decomposed into 3 major steps. The duration of these steps depends on the collision energy $\sqrt{s_{NN}}$. As this thesis focuses on high energy collisions, the following discussion will consider $\sqrt{s_{NN}} > 200$ GeV.

1. Thermalization ($\tau = 0 \sim 1$ fm)

When the nuclei collide, they are constituted of high-energy hadrons. Then, as they progressively pass through each other, the partons will strongly interact and the majority of the initial kinetic energy will be converted to heat, while a percent of it will be converted into numerous quark and antiquark pairs and gluons. Conversely, the larger the collision energy, the more the baryon chemical potential will decrease at mid-rapidity due to the decrease of the baryon stopping and the increase of matter-antimatter pairs production [25]. Shortly after the nuclei finish passing through each other, less than 1 fm after the collision, the strongly interacting medium had enough time to interact to the point of locally thermalizing [28, 29, 30].

2. The quark gluon plasma ($\tau = 1 \sim 10$ fm)

When the thermalization is established, the temperature is much higher than the critical temperature $T_c \sim 150$ MeV. A such temperature, the quarks and gluons feel a strong Debye screening [5, 6], which leads to the deconfinement and the formation of the QGP. At this early time of $\tau \sim 1$ fm, the average speed of the medium is small in the center of mass frame. However, the pressure is very large, especially in the longitudinal direction. The QGP will then rapidly expand following the pressure gradient. As it expands, the density and temperature will drop progressively. Around a critical temperature of $T_c \sim 150$ MeV reached at $\tau \sim 1 - 10$ fm [31], the deconfinement will cease progressively and the quark and gluons will assemble in color-neutral hadrons.

3. The hadron gas ($\tau = 10 \sim 20$ fm)

After the hadronization at $\tau \sim 10$ fm, the system will only be about 10 fm in radius. The produced hadrons are still close to each other and continue to interact inelastically. After a few more fm of time, the hadrons will have lost energy and the inelastic interaction will stop, leading to the chemical freeze-out. They will continue to interact elastically as they continue to fly away from each other, and around $\tau \sim 20$ fm they will stop interacting altogether [31], reaching the kinetic freeze-out. During all that time, most of the unstable hadrons will also have decayed. What will remain from here will simply fly straight until hitting the detector.

1.2.2 Numerical framework of a collision

The physical evolution described in the previous subsection is usually split into multiple stages. The hadronic transport [32, 33], which we will describe at the end of this section, tried to be used to describe the entire evolution of a collision. However, the results could not reproduce the experimental data so well, especially the flow anisotropy at high energies [34, 35].

By introducing additional models, especially a hydrodynamic stage before the hadronic transport, the agreement with experiments is significantly improved. Nowadays, there is also a thermalization stage between the collision and the hydrodynamics stage [28]. With the particlization stage needed to go from a hydrodynamic representation to a kinetic transport one, this makes five stages in total.

In this section, we will describe the conventional models in the order they appear in a numerical framework.

Collision and initial state

The initial state of a numerical framework for a heavy-ion collision corresponds to a description of the energy and momentum distribution at the moment of impact of the colliding nuclei. The different models that were developed over time tend to either focus on simpler geometrical description or on more complex QCD based descriptions.

One of the earliest and most used initial state model is the Glauber model [36], which is geometrically based. We will go into some details for this model as it was an important foundation in the HIC modeling, and it is the basis of the **T_RENTo** model [37] we use in this thesis. In its first form called the Optical Glauber model, the nucleons from both nuclei fly straight into each other. These nuclei are represented as smooth distribution of nucleons. Inside each nucleus, the nucleon density is parametrized with a Woods-Saxon potential

$$\rho(r) = \rho_0 \frac{1 + w(r/R)^2}{1 + e^{\frac{r-R}{a}}} \quad (1.30)$$

where r is the radius, ρ_0 is the nucleon density in the center of the nucleus, R the nuclear radius, a the skin depth and w the deviation from a spherical shape. These parameters are obtained from experimental data.

The interaction between two nuclei A and B with N_A and N_B nucleons respectively, is modeled through the probability of interaction between each pair of nucleons. This model considers that the nuclei will simply go straight through each other, so the probability of two nucleons to interact in a given transverse area is given by the product of the probability of finding a nucleon from each nucleus in that area. For a unit transverse area s , the probability of finding a nucleon in s corresponds to the integral of the nucleon density over the collision axis z

$$\hat{T}_{A,B}(s) = \int \hat{\rho}_{A,B}(s, z) dz \quad (1.31)$$

where $\hat{\rho}_{A,B}(s, z)$ is the probability to find a nucleon at the location (s, z) . This integral corresponds to the gray tubes in Fig. 1.4 a) from [36], where $\vec{b}$ in both view is the impact parameter and $\vec{s}$ is the location of the unit area from the center of A on the transverse plane.

The joint probability of two nucleons being located in the same area defines the thickness function

$$\hat{T}_{AB}(\vec{b}) = \int \hat{T}_A(\vec{s}) \hat{T}_B(\vec{s} - \vec{b}) d^2 s. \quad (1.32)$$

The probability of having n nucleon-nucleon interactions is given by the binomial distribution

$$P(n, \vec{b}) = \binom{N_A N_B}{n} \left[\hat{T}_{AB}(\vec{b}) \sigma_{\text{inel}}^{NN} \right]^n \left[1 - \hat{T}_{AB}(\vec{b}) \sigma_{\text{inel}}^{NN} \right]^{N_A N_B - n}, \quad (1.33)$$

where the probability of two nucleons interacting correspond to the probability of finding two nucleons at the same location $\hat{T}_{AB}$ multiplied by the nucleon-nucleon inelastic cross-section $\sigma_{\text{inel}}^{NN}$. This cross-section is the second experimental input for this model, which depends on the collision energy.

All the theoretical observables such as the number of collisions

$$N_{\text{coll}}(\vec{b}) = \sum_{n=1}^{N_A N_B} n P(n, \vec{b}) = N_A N_B \hat{T}_{AB}(\vec{b}) \sigma_{\text{inel}}^{NN} \quad (1.34)$$

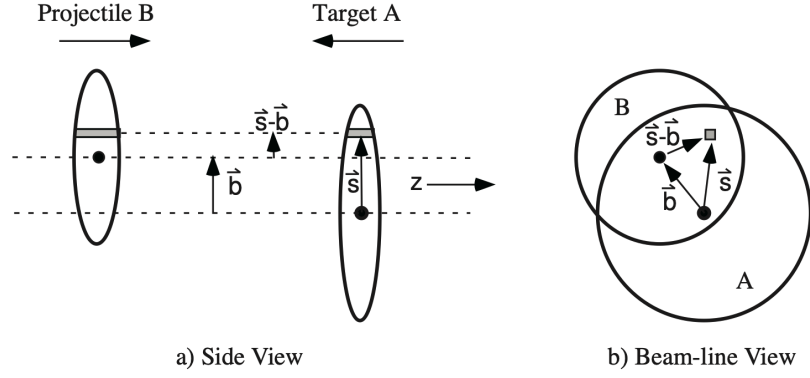


Figure 1.4: Schematic representation of the Optical Glauber model with transverse (a) and longitudinal (b) views from [36]

can be derived from the thickness function and the binomial probability.

The total probability of interaction between A and B , from which the initial energy density can be obtained, is

$$\frac{d^2 \sigma_{\text{inel}}^{AB}}{db^2} = \sum_{n=1}^{N_A N_B} P(n, \vec{b}) = 1 - \left[1 - \hat{T}_{AB}(\vec{b}) \sigma_{\text{inel}}^{NN} \right]^{N_A N_B}. \quad (1.35)$$

Although this optical model is convenient in its simplicity, it suffers from one major drawback, which is its smooth deterministic profile. Realistically, a collision is chaotic and the produced interaction varies event-by-event. This is where the Monte-Carlo Glauber model fix the problem. Instead of considering a smooth homogeneous transverse profile, nucleons are sampled from the transverse probability distribution to form 3-dimensional nuclei. An interaction between nucleons happens when the distance d between them satisfies

$$d \leq \sqrt{\frac{\sigma_{\text{inel}}^{NN}}{\pi}}. \quad (1.36)$$

One interesting aspect of this model, optical or not, is that the number of participating nucleons changes with the impact parameter. By assuming that the number of produced particles in experiments depends on the number of interacting nucleons, we can sort experimental data by number of produced particles in order to also sort by impact parameter. Doing so allows classifying experiments by similar impact parameters, although the exact value is unknown. This way of sorting is called centrality selection and tends to be expressed in percentage ranges instead of number of charged particle, where lower percentage means smaller impact parameter. Numerical results can be matched to experimental data when classified by centrality, in which case the agreement between the results is good [36, 38, 39]. This suggests that this scaling with number of produced particle is correct.

Such a simple geometrical model was a useful start, but it is not sufficient to properly reproduce more complex observables. In particular, fluctuations of densities in the initial state greatly affects the final higher order flow observables, such as higher order anisotropy. Having more physical intrinsic fluctuations would greatly improve the numerical results for such observables. One solution to address this problem is to use the Color Glass Condensate (CGC) effective theory of QCD [40]. This theory describes the nuclei as dense classical gluons fields called Glasma. These color fields are determined by solving the classical Yang-Mills

(CYM) equations $[D_\mu, F^{\mu\nu}] = J^\nu$ where the color current J^μ is generated by the moving nuclei. The solution of the CYM equations, which is obtained numerically using lattice theory, is used to compute the energy density of the initial condition. By solving the CYM equations, QCD fluctuations are inherently included inside the model. When the impact parameter (IP) dependence is also considered, the model is called IP-Glasma. Including this model in a complete heavy-ion collision framework can properly reproduce many non-trivial observables including higher-order anisotropy [41, 42].

Although IP-Glasma is still used nowadays, the convenience and efficiency of simpler geometrical models is still attracting. Therefore, recent geometrical models use more flexible parametrization tuned to reproduce experimental data. They also tend to include additional parametrizable fluctuations for more realistic distributions. For instance, the **T_RENTo** [37] model extends the Glauber model by introducing a generalized mean of parameter p of the transverse probabilities for the calculation of the entropy deposition. This parameter can be adjusted to consider different context. In the case $p = 1$, this corresponds to the classic Monte-Carlo Glauber where nucleons interact over the entire longitudinal direction. However, with $p = 0$, which corresponds to the geometric mean, the entropy production is suppressed in the locations where a small longitudinal thickness is involved, which can be seen as limiting the number of times a nucleon can interact. When comparing with experimental data, the parameter that fit the best is $p \sim 0$ which is also in good agreement with the IP-Glasma model [37]. Compared to the Glauber model which expresses the transverse probability directly from the nucleon distribution, the **T_RENTo** model expresses it as a random-weighted sum of the nucleon distribution where a different weight is sampled for each nucleon. Such extra sources of fluctuations help to accurately reproduce observables that rely on higher order moments, such as the two or four particles anisotropy $v_2\{2\}$ and $v_2\{4\}$ which we will discuss in Sec. 7.2.

Pre-equilibrium dynamics

Landau introduced in the 1950s that the medium produced by colliding nuclei would reach a full stop [43], meaning that the velocity in the center of mass frame would be zero until thermalization. In a complete numerical setup, this corresponds to directly use the densities provided by the initial state models as input for the hydrodynamic stage in Cartesian coordinates. However, this happened to be too much of a simplification.

The system would more likely start to develop velocity and expand as it thermalizes. For that matter, adding a stage with weaker interaction to let the medium evolve before the strongly interacting hydrodynamics is needed. The simplest way to do such thermalization stage is to start with no interaction at all as in a free-stream of particles, followed by instantaneous thermalization [28]. By changing the duration of this free-streaming stage, it is possible to interpolate between short and long thermalization, which corresponds to weakly and strongly coupled early evolution. This additional step produces better results than the purely instantaneous thermalization [28]. This model suggests that a switching time of $\tau \sim 0.1 - 0.6$ fm is the best to reproduce experimental data.

A more complex way to introduce finite time thermalization is from a theoretical point of view. We already discussed one of such theoretical approach for the initial state, which is the CGC. Indeed, the classical Yang-Mills equations describing the glasma can be evolved further than the initial collision. It was shown that during the evolution, the plasma instabilities (Weibel instabilities) progressively drives the system towards isotropy [29]. Essentially, as the glasma field fluctuations grow, the momentum is redistributed progressively which thus lead to thermalization. After about $\sim 0.2 - 0.4$ fm of time, the system is considered to have reached local equilibrium, and the hydrodynamic stage can start.

Another approach is to use perturbative QCD when the collision energy is very large. In this case, the model considers the evolution of high energy hard gluons coming from the nuclei and low energy soft gluons produced from the interaction. On the first hand, the soft gluons are emitted in abundance just after the collision, which make them thermalize very quickly. On the other hand, the hard gluons storing most of the energy will interact with the thermal bath formed by the soft gluon, which will progressively lead to complete thermalization. This model predicts the thermalization time to be $\tau = \alpha_s^{-13/5} Q_s^{-1}$ [44], where at LHC energies $Q_s \sim 2 - 3$ fm and $\alpha_s \sim 0.11$ [45]. These values suggest a thermalization time of $\tau \sim 20$ fm, which would be larger than the duration of the QGP. However, total thermalization is not needed to reach fluid dynamics. Only local thermalization would be needed, which would occur much earlier. One popular code using this approach is called KoMPoST, and it suggests that the switching to hydrodynamics should happen around $\tau \sim 1$ fm [30], which is in agreement with other models.

Essentially, all models predict that the system produced after the collision is very quickly attracted toward hydrodynamics, with a local thermalization time shorter than $\tau \sim 1$ fm.

Local equilibrium dynamics, relativistic hydrodynamics

In recent days, we know from models and experiments that HIC can produce the QGP which is a nearly ideal fluid. Using hydrodynamics to model it is therefore natural. However, hydrodynamics took a long time before being accepted by the community.

It started in the 1950s when Enrico Fermi used a statistical description of multiparticle production in proton collisions [46]. This introduces the idea of considering the system formed in ion collisions as a thermal system. Later in 1953, Landau considered that due to the extreme density, the mean free path would be much smaller than the size of the system, and thus local equilibrium could be reached [47]. He thus modeled the produced system with ideal hydrodynamics.

From then, physicists started to think about producing bulk nuclear matter in heavy ion collisions instead of studying cosmic-rays or proton collisions. A conference in Bear Mountain in 1974 [48] officially started the use of heavy-ion collisions to study the QGP. Following that conference, new experiments started in the 1970s and 1980s at Brookhaven's AGS and CERN's SPS and ISR. The new data acquired started to encourage hydrodynamics description. Especially, Bjorken proposed a 1+1D boost-invariant exact solution for ultra-relativistic collisions [49]. This solution involving a longitudinal boost-invariance (flat rapidity plateau) was in good agreement with the central plateau in rapidity distribution found in JACEE astrophysical HIC [50], and it was surprisingly also in good agreement with the central plateau in rapidity distribution measured at the SPS and ISR, even if these experiments only considered protons and alphas low energy collisions. Although hydrodynamics started to be used, it was puzzling that it could be applicable. Indeed, the system produced in a heavy ion collision happens to be about 10 fm long and only lives for about 10 fm, which makes it surprising that such a system can reach local equilibrium.

This puzzle started another trend to describe QGP with kinetic theory [32, 33]. However, it did not stop hydrodynamics from being used, and the study continued with 2+1D transverse hydrodynamics with boost-invariance in the longitudinal direction in 2000 [51]. It became especially useful to compare to the results from the RHIC experiment first published in 2001 [52], which measured elliptic flow in the transverse direction arising from the initial almond-shape interaction region of two colliding nuclei. Although ideal hydrodynamics already reproduced qualitatively the shape of the elliptic flow as a function of centrality, it could not exactly reproduce it, so the models and simulations were extended to viscous hydrodynamics. The work of Israel and Stewart [53] was especially useful as models started

to use their viscous equations, in (2+1)-dimensions in 2007 [54] and (3+1)-dimensions in 2000 [55]. During that time, the models evolved and in 2014 with [56] started to include fluctuating event-by-event initial conditions before the hydrodynamics and posterior models. The skepticism about the possible local thermalization that led to the development of transport theories was useful as in the end, HIC ends up being composed of both models.

Hadronization and Freeze-out

As the QGP expands during the hydrodynamic stage, it will continuously cool down, which will eventually lead to the freeze-out[†]. This freeze-out correspond to the point where the model switch from hydrodynamics to kinetic theory. Usually, the freeze-out is performed at a constant temperature around the critical temperature $T_c \sim 155$ MeV. This temperature is reached at different times in different location due to the non-homogeneous nature of the very dynamical system.

There are mainly two possibilities to handle the particle production. The first is to produce the particles in the cells as soon as the freeze-out temperature is reached and make them interact with the fluid. The second is to store the information of the cells when they cross the freeze-out temperature, and wait for all of them to cross before completely stopping hydrodynamics and using this space-time hypersurface for only hadronic transport. The latter is much simpler to handle, so it is used the most. In that case, the standard method to produce the particles is the Cooper-Frye prescription [57].

The idea is to integrate the fluid's distribution function over the constant temperature hypersurface σ to obtain the particle distribution

$$E \frac{dN}{d^3p} = \int_{\sigma} f(x, p) p^{\mu} d\sigma_{\mu}, \quad (1.37)$$

where E is the energy, p the momentum, σ_{μ} the vector normal to the hypersurface σ , and $f(x, p)$ is the thermal distribution at hadronization. With this distribution, the particles are usually randomly sampled through Monte-Carlo process.

This method, when followed by hadronic transport which we will describe in the next subsection, can reproduce very well experimental data for flow observables [58, 59, 39], and thus allows to obtain information on the transport coefficients. This suggests that the particlization process indeed happens around T_c . However, this method is only great for the main part which is the thermalized bulk, but it does not handle well high momentum hadrons, which are important in order to properly handle jets for instance. For that matter, other models are used. These models are more complicated, so we will not go into details, but essentially, the jets partons are produced at an earlier stage where they lose energy in the hydrodynamic medium. The partons which have enough energy to leave the bulk will then hadronized. To do so, the main method used is the Lund string fragmentation [60] where color flux tubes (the strings) linking partons are created in high-energy collisions, and as they are stretched apart when the partons leave the bulk, they eventually break into quark-antiquark pairs created from the vacuum. The numerous quarks and antiquarks produced will finally confine into hadrons.

Another important model is the quark coalescence, or recombination model [61]. Indeed, the ratio of produced baryons and mesons is not correct when using Cooper-Frye alone. However, this ratio can be corrected by including recombination, where at higher moment $\sim 2 - 6$ GeV, the quark can recombine by triplets into baryon, increasing the number of

[†]The process of switching from hydrodynamics to hadronic cascade is usually referred as freeze-out, although it is not a thermodynamic freeze-out.

baryons produced. Without such recombination, the amount of baryons randomly produced from the Cooper-Frye formula would be suppressed as they have more constituent than mesons.

Recent studies such as [62] show that the inclusions of all these models together is preferable for a better agreement with experimental data.

Hadronic transport

The hadronic transport nowadays takes charge of the last stage of heavy-ion collisions modeling. However, at first, it was used to simulate entire collisions without hydrodynamics. This was done in the 1990s for low energy proton-proton collisions. A model called UrQMD (Ultra-relativistic Quantum Molecular Dynamics) was introduced to describe the collisions of hadrons as classical particles [63]. More precisely, when two hadrons come close, a collision might happen with a certain probability. In the case of a collision, either new hadrons are produced as resonance or decay, or the momenta of the two colliding hadrons are changed. This continues as a cascade[‡] of interacting hadrons until the mean free path exceeds the system size and interactions cease. In order to be accurate, this model relies heavily on experimental data by including a huge catalog of measured resonances' information such as cross-sections. This cascade representation was sufficient to describe low energy proton-proton and AA collisions, but it could not properly reproduce high energy heavy-ion collisions data coming from RHIC and the LHC [34, 35].

Hydrodynamics could not fully reproduce the data alone either. What ended up reproducing the experimental data the best was to use the hadronic transport after the freeze-out of hydrodynamics as an afterburner. This is how the state-of-the-art models include it.

Although UrQMD, and JAM [64] which was published one year later, are the precursor of hadronic transport, a more recent model called SMASH [65] was created. It aims to consider more physical properties, such as properly reproducing thermodynamic limits from HRG or lattice, or including more recent and precise resonances and cross-sections, while being more modern and maintainable through the use of the C++ programming language. This better control over the included physics, especially the thermodynamics, is in particular useful in the case of Bayesian analysis, whose goal is to fine tune all the model parameters.

1.2.3 Using Bayesian analysis to find the best model parameters

The ever-growing accuracy of the experimental data led to the increase in the complexity of the numerical models. As there is no way to predict what parameters are needed to produce a given value of an observable, one would need to finely sample this more than 10-dimensional parameter space by running the full model every time until the numerical result reproduces the experiment. Such a task is impossible to perform in a reasonable amount of time. To try fixing this issue, one simple clever way could be to first consider a range for each parameter to have a finite volume, and then to sample this volume as uniformly as possible in only a finite amount of points, with Latin hypercube sampling (LHS) [66] for instance. However, even if one of the samples can quite accurately reproduce the data, we would not know the uncertainty associated to the value of the parameters, and therefore it would not be rigorous.

The use of Bayesian analysis was the solution to efficiently find the best fitting set of parameters, while obtaining the associated uncertainties [59, 39]. It consists of finding the posterior distribution of the parameter set $\mathbf{x}$ from Bayes's theorem

$$P(\mathbf{x}|\mathcal{O}) = \frac{P(\mathcal{O}|\mathbf{x})P(\mathbf{x})}{P(\mathcal{O})}, \quad (1.38)$$

[‡]This entire process of hadronic transport is usually referred in short as "cascade".

where $\mathcal{O}$ are the given observations. The observations $\mathcal{O}$ are the values of the experimental observables in our case, which are fixed from the result of the experiments, so $P(\mathcal{O})$ behaves as a constant. If we sample $\mathbf{x}$ uniformly in a volume V with LHS, then the probability distribution $P(x)$ is a constant inside the volume and zero outside. This means that in the sampling volume V , the probability of having the parameter set $\mathbf{x}$ knowing the observations $\mathcal{O}$, so the posterior probability $P(\mathbf{x}|\mathcal{O})$, is proportional to the probability $P(\mathcal{O}|\mathbf{x})$ of having given observations knowing a parameter set $\mathbf{x}$:

$$P(\mathbf{x}|\mathcal{O}) \propto P(\mathcal{O}|\mathbf{x}). \quad (1.39)$$

The posterior probability $P(\mathbf{x}|\mathcal{O})$ is zero outside V by construction, which means that the volume V should be chosen carefully to be large enough to avoid truncating the posterior distribution, while still being small enough to be efficient.

We can compute the value $P(\mathcal{O}|\mathbf{x})$ for a given $\mathbf{x}$, but we do not have its analytical expression for all $\mathbf{x}$, which means we cannot sample directly from the distribution of $P(\mathcal{O}|\mathbf{x})$. However, it is possible to sample from a distribution, in our case the distribution of $P(\mathbf{x}|\mathcal{O})$, by only being able to compute probabilities of a function proportional to it, which in our case is $P(\mathcal{O}|\mathbf{x})$. Indeed, Markov-chain Monte Carlo (MCMC) methods such as the Metropolis algorithm are designed to solve this kind of problem. More specifically, in our case, such an algorithm will uniformly sample candidates $\mathbf{x}$ and keep or reject them based on the ratio of successive probability $P(\mathcal{O}|\mathbf{x})$. After some equilibrating iterations, the remaining samples will follow the desired distribution. After equilibrating, one can obtain the posterior distribution of one parameter by splitting the interval in which this parameter was sampled in a finite number of bins, and count how many samples fall into each bin. Similarly, the relation between two parameters can be obtained by binning in two dimensions to have a bivariate distribution.

To complete the Bayesian process, if we assume the uncertainties to behave in a Gaussian way, we can express the probability $P(\mathcal{O}|\mathbf{x})$ from the experimental observations $\mathcal{O}_{i,exp}$ and the numerical observations $\mathcal{O}_{i,num}(\mathbf{x})$ as

$$P(\mathcal{O}|\mathbf{x}) \approx e^{-\sum_i \frac{(\mathcal{O}_{i,exp} - \mathcal{O}_{i,num}(\mathbf{x}))^2}{2\sigma_i^2}}, \quad (1.40)$$

which is the function proportional to the posterior distribution $P(\mathbf{x}|\mathcal{O})$ needed for the MCMC, where σ_i is the associated uncertainty. Finally, the mean value and uncertainty of each parameter can be obtained by a Gaussian fit of their posterior distribution.

In most cases, this method is not applicable directly. More precisely, performing direct computation such as a full minimum-bias[§] simulations at each step in the MCMC would take excessive time and not be practical. Complete heavy-ion collisions are very computationally demanding. Especially for the hydrodynamical part, where having three physical dimensions makes the system of the order 10^6 cells with at least 10 dynamical variables (4 ideal + 6 viscous) in each. In addition, the random behavior of a collision requires the averaging of observables over many simulations with random initial conditions. Instead of simulating the full model at every MCMC steps, Gaussian Emulators (GE) can be used because Gaussian distribution are expected for the observables [39]. Such a GE only capture the mean and variance of an observable produced by the model as function of the parameters. More precisely, GEs interpolate the means and variances between a finite number of parameter sets. Thereby, using the emulators to produce the observables anywhere in the parameter space is

[§]Minimum-bias simulations correspond to perform many simulations with random impact parameter and then sort by number of charged particles, in order to be as close as the way experimental data are produced and sorted out.

much faster than running the full model. The construction of the GEs still requires running the model at many locations in the parameter space, but it can be done once on a coarse LHS for efficiency. Then, only the GEs will be used to produce interpolated observables during the MCMC process, where much more samples are required to converge to the correct distribution. Thanks to these added GEs, the time required to perform the Bayesian analysis can be greatly reduced.

However, the use of separate GE for each observable would lose the correlation between them, and the computation cost would scale with the amount of observables considered. Instead, a principal component decomposition (PCD) can be performed on the observables produced by the model before constructing the GEs. The principal components (PC) obtained will capture the correlation between the observables, but in addition, most of the time, only a small amount of them will capture most of the relevant information [39]. This means that the other PC can be neglected, and GEs can be constructed for the relevant ones only. The mean and variance of each observable can finally be obtained by inverting the PCD each time the GEs are used. This additional PCD once again shorten the total time needed to perform the Bayesian analysis while taking into account the correlation between the observables.

In the context of HIC, such Bayesian analysis is used to obtain the values and uncertainties of the free-parameters of the models, such as the time at which the hydrodynamics starts, or the value of the shear viscosity η/s [59, 39].

1.3 Purpose and outline of this thesis

The study of the QGP through heavy-ion collisions is a complex and time-consuming process. It requires acquiring a large amount of data through experiments and performing numerous simulations to compare the models results to the experimental data. Therefore, having accurate and efficient numerical implementations of the models is necessary. In particular, the QGP part of heavy-ion collisions is modeled by relativistic hydrodynamics, which is notoriously difficult to implement numerically in an efficient and accurate way.

To this end, the purpose of this thesis is to develop a new implicit integration method to solve the equations of relativistic hydrodynamics used in heavy-ion collision simulations. We especially focus on both accuracy which is important for the quality of the result, and efficiency which is necessary to obtain results in a reasonable time.

In particular, we are interested in the following three subjects:

- First, we study how to use an implicit integrator in order to solve the relativistic hydrodynamic equations. Such an integrator has the great property of being stable, and therefore should not need any manual stabilization which are required when using the usual explicit methods such as in [54, 67, 55, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77].

However, implicit methods need additional prescriptions to be solved. More specifically, they introduce a new set of implicitly defined equations which need to be solved. In the general non-linear case, these equations have to be solved numerically. For this matter, we introduce a precisely optimized fixed-point solver made to outperform explicit methods. We also study the limit of stability of such solver, which will directly limit the stability of the implicit method itself. We already published the application to ideal relativistic hydrodynamics in [78].

- Second, the introduction of viscosity in the hydrodynamic equations introduces complications in the discretization process needed to solve numerically. In the straightforward explicit case, there does not seem to be any easy way to handle these complications. Therefore, approximations are used which greatly decrease the accuracy.

In the process of designing a new implicit method to solve relativistic hydrodynamics, we consider an exact way to discretize the viscous equations which is compatible with the efficient fixed-point solver previously designed. We study its correct accuracy and compare it to the decreased accuracy of explicit methods.

- Third, the purpose of simulating heavy-ion collisions is to be able to compare the results produced by different models with experimental data, and then extract physical quantities from them. Therefore, we include our implicitly solved relativistic hydrodynamics into a complete heavy-ion collision framework. With this complete framework, we compare the results produced with experimental data such as the ones from ALICE [79, 27] to verify the correctness of this newly developed implicit method.

This thesis is organized as follows.

The first two chapters introduce the general concepts needed to solve relativistic hydrodynamics. First, in Chapter 2, we explain in details what is relativistic hydrodynamics. We introduce the equations, derive their viscous extension, and provide exact solutions for specific cases which will be used as reference later in the benchmarks. Then, in Chapter 3, we explain how the hydrodynamic equations are discretized in space and time in order to solve them numerically.

Chapters 4 to 7 are the main parts of this thesis, where we discuss our contribution to numerically solving the hydrodynamic equations in the context of heavy ion collisions.

As an implicit Runge-Kutta method needs an additional implicit solver, we first discuss how we optimized a fixed-point implicit solver for our purpose in Chapter 4.

In Chapter 5, we discuss the specific changes we made to the space and time discretization in the ideal and viscous cases to have more stable and accurate results by considering the implicit nature of our time discretization.

We continue with our benchmark in Chapter 6 where we show that our code properly reproduces the expected space and time discretization accuracies for different exact ideal solutions, and also when using event-by-event heavy-ion initial conditions. We also show that in the viscous case, the traditionally used explicit method is reduced to have the lowest accuracy order of unity[¶] whereas our new implicit method can reach arbitrary order as expected, especially the order two in our benchmark.

Finally, in Chapter 7, we show that we can reproduce with our implicit method non-trivial observables such as the elliptic and high-order flows from ALICE experimental data, and thus we confirm that such implicit methods are suitable for studying heavy-ion collisions.

We conclude with a summary and an outlook in Chapter 8.

[¶]Integration methods usually produce an approximate result with an error proportional to the time step raised to specific power. This power is called the accuracy order.

Chapter 2

Relativistic hydrodynamics

Relativistic hydrodynamics is a field of physics which becomes relevant in extreme cases. The two main cases being when the velocities in a system tend to the speed of light, or when the masses or interactions affect the gravitational field. For instance, such extreme conditions happen in relativistic heavy-ion collisions [54, 67, 55, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77], or in most astrophysical events such as neutron-star mergers [80, 81] or supernovae collapse [82]. All these extreme cases generally involve the evolution of strongly interacting matter, which can be modeled macroscopically by the local conservation of Lorentz covariant quantities. Mainly the energy-momentum tensor and additional relevant charges.

This macroscopic description corresponds to modeling a fluid in relativistic conditions, hence the name relativistic hydrodynamics. Although the equations governing these conservations are conceptually simple, they are mathematically stiff, non-linear, and strongly coupled. They include shocks, rarefaction waves, turbulences and also relaxation dynamics when dissipative corrections are included. Analytic solutions only exist for a few highly symmetric configurations, making numerical simulation an indispensable tool for realistic hydrodynamics.

In this section, we will focus on the general aspects of relativistic hydrodynamics by covering the conservation equations, the second order dissipative extension, the Milne coordinate system usually used in the context of heavy-ion collisions, and two exact solutions which will be used as a reference for the numerical benchmark. The numerical procedures to solve relativistic hydrodynamics in the general case will be discussed later in Cha. 3

2.1 Conservation equations

Hydrodynamics is described by the local conservation of charges. In its relativistic form, we use the conservation of the energy-momentum tensor and conserved charges in the system provided by the Noether theorem.

The conservation of the energy-momentum tensor $T^{\mu\nu}$ reads:

$$\begin{aligned}\partial_{;\mu} T^{\mu\nu} &= \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\alpha\mu} T^{\alpha\nu} + \Gamma^\nu_{\alpha\mu} T^{\mu\alpha} \\ &= 0\end{aligned}\tag{2.1}$$

where $\partial_{;\mu}$ denotes the covariant derivative, and $\Gamma^\nu_{\mu\alpha}$ the associated Christoffel symbols. Covariant derivatives are useful for curved space-time or curvilinear coordinate systems. In this study, we only consider the flat Minkowski space-time. However, we will consider a curvilinear coordinate system called the Milne coordinate system, which is why we use covariant derivatives. In the Cartesian coordinates were $\mu, \nu = t, x, y, z$ and $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$,

the Christoffel symbols vanishes. The rest of this section will use this coordinate system for simplicity.

As $T^{\mu\nu}$ is a Lorentz tensor, one needs to build it by Lorentz covariant quantities. The most basic quantities at our disposal are:

- ϵ : Energy density
- v^i with $i = x, y, z$: Velocity (in the unit of the speed of light c)
- $g_{\mu\nu}$: Metric tensor
- ∂_μ : Four-gradient

where only the velocity is not a Lorentz covariant quantity. However, we can construct a Lorentz vector u^μ from it. First, we compute its norm $\beta^2 = \sum_i (v_i)^2$. Second, we express the Lorentz factor $\gamma = 1/\sqrt{1 - \beta^2}$. Finally, the four-velocity reads:

$$u^\mu = (\gamma, \gamma v^0, \gamma v^1, \gamma v^3). \quad (2.2)$$

This quantity is now a Lorentz vector as any equation including it will remain the same under a Lorentz transformation which would change the referential. For instance, the following equation is true in any referential

$$u^\mu u_\mu = 1. \quad (2.3)$$

From these quantities, we can construct $T^{\mu\nu}$ by order of derivatives $\partial_{;\mu}$. At zeroth order, the only second-rank Lorentz-covariant tensors we can construct are $u^\mu u^\nu$ and $g^{\mu\nu}$. We end up with

$$T^{\mu\nu} = A u^\mu u^\nu - B g^{\mu\nu} \quad (2.4)$$

where A and B are Lorentz scalar.

In the rest frame we know that $T^{\mu\nu} = \text{diag}(\epsilon, p, p, p)$, and the four-velocity is $u^\mu = (1, 0, 0, 0)^T$. This is because in the rest frame, the energy density is fixed in space, and therefore $T^{ti} = 0$. In addition, due to isotropic symmetry, the off-diagonal T^{ij} where $i \neq j$ must all vanish and the diagonal ones should be equal $T^{xx} = T^{yy} = T^{zz}$. Furthermore, in that frame, we can see that the energy density is the eigenvalue of the four velocity

$$u_\mu T^{\mu\nu} = \epsilon u_\nu. \quad (2.5)$$

This equation being made of Lorentz tensors and scalars, it must hold in all referential. Using this property on Eq. 2.4 we have

$$T^{\mu\nu} u_\mu = \epsilon u^\nu \Rightarrow A u^\nu - B u^\nu = \epsilon u^\nu \quad (2.6)$$

$$\Rightarrow A - B = \epsilon. \quad (2.7)$$

We can rewrite $T^{\mu\nu}$ as

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - B \Delta^{\mu\nu} \quad (2.8)$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$, and $u_\mu \Delta^{\mu\nu} = 0$.

By doing so, we see that we have an unknown quantity B orthogonal to the velocity*. This corresponds to the pressure. As we have no more information on it, our only choice is to define it as a function of the other quantities. In the current situation, the only non-constant

* $u_\mu \Delta^{\mu\nu} = 0$

scalar the pressure can depend on is the energy density. Therefore, we define the pressure $P(\epsilon)$, which is then called the equation of state (EOS).

The final energy-momentum tensor zeroth order reads:

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - P \Delta^{\mu\nu}. \quad (2.9)$$

In general, the energy and momentum are not the only conserved charges. Many other charges such as the net baryon number and the electric charge are also conserved. Such charges can be expressed by a conserved current N^μ satisfying

$$\partial_{;\mu} N^\mu = 0, \quad (2.10)$$

which becomes a part of the hydrodynamic equations. Similarly, N^μ should be expressed with its density n and the 4-velocity u^μ :

$$N^\mu = n u^\mu. \quad (2.11)$$

The pressure should also be dependent on the additional charge $P(\epsilon, n)$.

This zeroth order prescription describes what is called the ideal hydrodynamics. This denomination comes from the conservation of the entropy

$$s^\mu = s u^\mu. \quad (2.12)$$

With the relation

$$\epsilon + p = Ts + \mu n, \quad (2.13)$$

where T is the temperature and μ the chemical potential associated to n , and the Gibbs-Duhem relation

$$dp = s dT + n d\mu, \quad (2.14)$$

we can prove from the conservation of the energy momentum $\partial_\mu T^{\mu\nu} = 0$ that the entropy is conserved

$$\partial_\mu s^\mu = 0. \quad (2.15)$$

This makes the zeroth order hydrodynamics ideal.

2.2 Second order IS equations

If we want to stay in the frame that follows the energy, where $u_\mu T^{\mu\nu} = \epsilon u^\nu$, every new term coming from derivatives should be null when projected on u^μ . For simplicity, we will neglect the part involving the baryon number as we will focus on vanishing baryon chemical potential in the rest of this thesis, so we will only consider dissipative contributions to $T^{\mu\nu}$. All such contributions can be summarized by an additional symmetric tensor $D^{\mu\nu}$ with $u_\mu D^{\mu\nu} = 0$.

We can separate this tensor into two contributions, one from its trace and one from the rest. The component from the trace should still be null when projected on u^μ , so it must be proportional to $\Delta^{\mu\nu}$. The dissipation tensor becomes $D^{\mu\nu} = \Pi \Delta^{\mu\nu} + \pi^{\mu\nu}$ where $\pi^{\mu\nu}$ is symmetric, traceless, and $u_\mu \pi^{\mu\nu} = 0$. Π corresponds to an isotropic contribution in the rest frame similar to the pressure, so it affects the bulk of the system, which is why it is called the bulk dissipative pressure. The remaining components in $\pi^{\mu\nu}$ are directional contribution, so it corresponds to directional dissipation usually called shear, which makes it the shear dissipative tensor.

In that case, we can write $T^{\mu\nu}$ in the general form:

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu}. \quad (2.16)$$

By taking into account one derivative $\partial_{;\mu}$, we can express hydrodynamic equation to the first order in derivative expansion. The only non-zero quantities are the scalar $\partial_{;\lambda}u^\lambda$, the operator $u^\lambda\partial_{;\lambda}$, and the two operators $u^\nu\partial^{;\mu}$ and $u^\mu\partial^{;\nu}$. By taking combinations with $(\epsilon, P, u^\mu, g^{\mu\nu})$ we can express Π and $\pi^{\mu\nu}$.

As a scalar, Π can be expressed as:

$$\Pi = A\partial_{;\lambda}u^\lambda + Bu^\lambda\partial_{;\lambda}\epsilon + Cu^\lambda\partial_{;\lambda}P, \quad (2.17)$$

and as a tensor, $\pi^{\mu\nu}$ can be expressed as:

$$\pi^{\mu\nu} = D(u^\nu\partial_{;\lambda}u^\lambda u^\mu + u^\mu\partial_{;\lambda}u^\lambda u^\nu) + E(u^\mu\partial^{;\nu}\epsilon + u^\nu\partial^{;\mu}\epsilon), \quad (2.18)$$

which is expressed in a traceless, orthogonal to u_μ with $u_\mu\pi^{\mu\nu} = 0$, and symmetric form without $g^{\mu\nu}$ and $u^\mu u^\nu$.

These first order expressions are directly proportional to the velocity gradient. Therefore, the viscous reaction to any variation of the velocity is thus instantaneous. This means that a reaction to a variation in the system can propagate at an infinite velocity, which violates causality.

In the non-relativistic limit, the first order hydrodynamic equations correspond to the Navier-Stokes equations, where in that case this non-causality issue is not very relevant as any physical quantities would evolve much slower than the speed of light.

In the relativistic case, higher order terms are needed to avoid such instantaneous response. However, simply going to the second order without further considerations would not necessarily fix this issue.

One solution found by Israel and Stewart [53] is to derive the expression of Π and $\pi^{\mu\nu}$ from an entropy consideration. In their formalism, the entropy current s^μ is defined to contain the dissipative quantities Π and $\pi^{\mu\nu}$ up to second order without derivatives. They also require that the entropy in the rest frame $s^\mu u_\mu$ is maximized in equilibrium where the dissipative currents vanish. This requirement gives the entropy current

$$s^\mu = su^\mu - \frac{u^\mu}{2T}(a_0\Pi^2 + a_1\pi^{\alpha\beta}\pi_{\alpha\beta}) \quad (2.19)$$

with $a_{0,1}$ the positive coupling coefficients. The divergence of Eq. 2.19 is calculated to be

$$\partial_\mu s^\mu = -\frac{\Pi}{T}(\Delta^{\mu\nu}\partial_\mu u_\nu + a_0 u^\mu\partial_\mu\Pi) + \frac{\pi^{\mu\nu}}{T}(\partial_\mu u_\nu - a_1 u^\lambda\partial_\lambda\pi_{\mu\nu}) \quad (2.20)$$

where we neglected the higher-order terms in the dissipative quantities. This expression is composed of two parts with each dissipative quantity Π and $\pi^{\mu\nu}$ in factor.

The second law of thermodynamics requires that $\partial_\mu s^\mu \geq 0$. To guaranty this, the dissipative quantities must be expressed from the expression they are in factors of

$$\Pi = -\zeta(\Delta^{\mu\nu}\partial_\mu u_\nu + a_0 u^\mu\partial_\mu\Pi), \quad (2.21)$$

$$\pi_{\mu\nu} = 2\eta\left\langle\partial_\mu u_\nu - a_1 u^\lambda\partial_\lambda\pi_{\mu\nu}\right\rangle, \quad (2.22)$$

with the symmetric, traceless and orthogonal to u^μ operator $\langle\bullet\rangle$:

$$\langle A^{\mu\nu}\rangle = \left(\frac{1}{2}\Delta_\alpha^\mu\Delta_\beta^\nu + \frac{1}{2}\Delta_\alpha^\nu\Delta_\beta^\mu - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right)A^{\alpha\beta}. \quad (2.23)$$

With the same choice of coefficients as in [55], the IS equations read

$$u^\lambda \partial_{;\lambda} \Pi = -\frac{\Pi - \Pi_{NS}}{\tau_\Pi} - \frac{4}{3} \Pi \partial_{;\lambda} u^\lambda \quad (2.24)$$

$$\langle u^\lambda \partial_{;\lambda} \pi^{\mu\nu} \rangle = -\frac{\pi - \pi_{NS}^{\mu\nu}}{\tau_\pi} - \frac{4}{3} \pi \partial_{;\lambda} u^\lambda \quad (2.25)$$

$$\Pi_{NS} = -\zeta \partial_{;\lambda} u^\lambda, \quad (2.26)$$

$$\pi_{NS}^{\mu\nu} = \eta \left(\Delta^{\mu\lambda} \partial_{;\lambda} u^\nu + \Delta^{\nu\lambda} \partial_{;\lambda} u^\mu \right) - \frac{2}{3} \eta \Delta^{\mu\nu} \partial_{;\lambda} u^\lambda, \quad (2.27)$$

$$\tau_\pi = \tau_\Pi = \frac{5\eta}{\epsilon + P} \quad (2.28)$$

These equations still have a causality constraint which can be approximated from linearization as [83, 84]

$$\frac{\zeta}{\tau_\Pi(\epsilon + P)} + \frac{\eta}{\tau_\pi(\epsilon + P)} \leq (1 - c_s^2), \quad (2.29)$$

with τ_π the shear relaxation time, τ_Π the bulk relaxation time, ϵ the energy density, P the pressure and c_s^2 the speed of sound. With the choice of transport coefficients obtained by the Bayesian analysis [7, 8], ζ is much smaller than η . If we take both relaxation time to be equal and the speed of sound to be ideal $c_s^2 = 1/3$, we have

$$\tau_\pi \geq \frac{3/2\eta}{(\epsilon + P)}. \quad (2.30)$$

The coefficient 5 instead of 3/2 that we took from [85] is in agreement with the above causality constraint.

2.3 Milne coordinates

During heavy-ion collisions, the two colliding nuclei move at a speed close to the speed of light. Applying a Lorentz boost in the collision direction will not change this fact. This symmetry tends to apply to the whole system where applying a Lorentz boost in the collision direction will leave the particle distribution mostly unchanged.

The Lorentz boost affects a slice of space-time which is not Cartesian. Thus, To study a boost invariant system in a fixed size box, it is convenient to choose a coordinate system that expands in the collision axis direction. In contrast, the system will expand much more slowly in the transverse plane. Therefore, we can keep the coordinates in this transverse plane as Cartesian.

Taking the collision axis as the z axis, we can consider the following transformation:

$$t = \tau \cosh \eta, \quad z = \tau \sinh \eta, \quad (2.31)$$

$$\tau = \sqrt{t^2 - z^2}, \quad \eta = \tanh^{-1} \frac{z}{t}, \quad (2.32)$$

$$(2.33)$$

such that z expands proportionally to the proper time τ depending on the rapidity η . A Lorentz boost in the collision axis will only affect η and keep the proper time τ unchanged, which makes this transformation suitable. In addition, the speed of the expansion $z/t = \tanh \eta$ is nearing the speed of light even at relatively small $\eta \sim 5$ with $z/t \sim 0.99991$. This means that taking a fixed box with $-5 < \eta < 5$ should be sufficient to keep the expanding system contained at all time[†].

[†]The box should be taken larger than the initial system in the transverse direction as the system will still expand in these directions even if it is slower.

This coordinate system is often called the Milne coordinate system. The metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -\tau^2)$ is not flat anymore, and non-trivial Christoffel symbols arise $\Gamma_{\tau\eta}^\eta = \Gamma_{\eta\tau}^\eta = 1/\tau$, $\Gamma_{\eta\eta}^\tau = \tau$.

In Milne coordinates, the energy-momentum conservation is represented as

$$\partial_\nu T^{\tau\nu} + \tau T^{\eta\eta} + \frac{1}{\tau} T^{\tau\tau} = 0, \quad (2.34)$$

$$\partial_\nu T^{x\nu} + \frac{1}{\tau} T^{x\tau} = 0, \quad (2.35)$$

$$\partial_\nu T^{y\nu} + \frac{1}{\tau} T^{y\tau} = 0, \quad (2.36)$$

$$\partial_\nu T^{\eta\nu} + \frac{3}{\tau} T^{\eta\tau} = 0. \quad (2.37)$$

One inconvenience is the appearance of the factor $1/\tau$ due to the Christoffel symbols. This factor makes the system change rapidly at small τ , which decreases numerical stability. To get rid of these factors, we will perform a variable change where every time an η appears in an index, a τ factor is multiplied to the variable:

$$\tilde{T}^{\eta\eta} = \tau^2 T^{\eta\eta}, \quad \tilde{\pi}^{\eta\eta} = \tau^2 \pi^{\eta\eta}, \quad (2.38)$$

$$\tilde{T}^{\mu\eta} = \tau T^{\mu\eta}, \quad \tilde{\pi}^{\mu\eta} = \tau \pi^{\mu\eta}, \quad \tilde{N}^\eta = \tau N^\eta, \quad \tilde{u}^\eta = \tau u^\eta, \quad \tilde{\partial}_\eta = \frac{1}{\tau} \partial_\eta, \quad \mu = \tau, x, y, \quad (2.39)$$

$$\tilde{T}^{\mu\nu} = T^{\mu\nu}, \quad \tilde{\pi}^{\mu\nu} = \pi^{\mu\nu}, \quad \tilde{N}^\mu = N^\mu, \quad \tilde{u}^\mu = u^\mu, \quad \tilde{\partial}_\mu = \partial_\mu. \quad (2.40)$$

With this change, Eqs. (2.34)-(2.37) become:

$$\tilde{\partial}_\nu (\tau \tilde{T}^{\tau\nu}) = \tilde{T}^{\eta\eta}, \quad (2.41)$$

$$\tilde{\partial}_\nu (\tau \tilde{T}^{x\nu}) = 0, \quad (2.42)$$

$$\tilde{\partial}_\nu (\tau \tilde{T}^{y\nu}) = 0, \quad (2.43)$$

$$\tilde{\partial}_\nu (\tau \tilde{T}^{\eta\nu}) = \tilde{T}^{\eta\tau}. \quad (2.44)$$

The complete set of second-order equations read

$$\tilde{\partial}_\nu (\tau \tilde{T}^{\mu\nu}) = I^\mu, \quad (2.45)$$

$$\tilde{\partial}_\lambda (u^\lambda \tilde{\pi}^{\mu\nu}) = -\frac{\tilde{\pi}^{\mu\nu} - \tilde{\pi}_{\text{NS}}^{\mu\nu}}{\tau_\pi} - \frac{1}{3} \tilde{\pi}^{\mu\nu} \left(\tilde{\partial}_\lambda \tilde{u}^\lambda - 4 \frac{\tilde{u}^\tau}{\tau} \right) - \left(\tilde{u}^\nu \tilde{\pi}^{\mu\beta} + \tilde{u}^\mu \tilde{\pi}^{\nu\beta} \right) \tilde{u}^\lambda \tilde{\partial}_{;\lambda} \tilde{u}_\beta + \frac{\tilde{u}^\eta}{\tau} I_\pi^{\mu\nu}, \quad (2.46)$$

$$\tilde{\partial}_\lambda (u^\lambda \tilde{\Pi}) = -\frac{\tilde{\Pi} - \tilde{\Pi}_{\text{NS}}}{\tau_\Pi} - \frac{1}{3} \tilde{\Pi} \left(\tilde{\partial}_\lambda \tilde{u}^\lambda - 4 \frac{\tilde{u}^\tau}{\tau} \right). \quad (2.47)$$

where

$$I^\tau = \tilde{T}^{\eta\eta}, \quad I^x = I^y = 0, \quad I^\eta = \tilde{T}^{\eta\tau}, \quad (2.48)$$

and

$$I_\pi^{\eta\eta} = I_\pi^{\tau\tau} = 2\tilde{\pi}^{\tau\eta}, \quad I_\pi^{\tau x} = \tilde{\pi}^{\eta x}, \quad I_\pi^{\eta x} = \tilde{\pi}^{\tau x}, \quad I_\pi^{xx} = I_\pi^{xy} = \tilde{\pi}^{yy} = 0, \quad (2.49)$$

$$I_\pi^{\tau\eta} = \tilde{\pi}^{\tau\tau} + \tilde{\pi}^{\eta\eta}, \quad I_\pi^{\tau y} = \tilde{\pi}^{\eta y}, \quad I_\pi^{\eta y} = \tilde{\pi}^{\tau y}. \quad (2.50)$$

All the covariant derivatives of the 4-velocity are ordinary derivatives except for:

$$\tilde{\partial}_{;\eta} u^\tau = \tilde{\partial}_\eta \tilde{u}^\tau + \frac{\tilde{u}^\eta}{\tau}, \quad \tilde{\partial}_{;\eta} u^\eta = \tilde{\partial}_\eta \tilde{u}^\eta + \frac{\tilde{u}^\eta}{\tau}. \quad (2.51)$$

2.4 Exact solutions

The relativistic hydrodynamic equation does not have an analytic solution in general. However, for some specific initial conditions and choice of the equation of state, exact solutions are known. In this section, we will present two such examples: the Riemann problem [86] in (1+1)-dimensions and the Gubser flow [87] in (2+1)-dimensions.

2.4.1 Riemann problem

Let us consider a system having the translational invariance in the y and z directions in the Cartesian coordinates. We set the x dependency in the initial condition at $t = 0$ as

$$\epsilon(t = 0, x) = \begin{cases} \epsilon_{\max}, & x < 0, \\ \epsilon_{\min}, & x \geq 0, \end{cases} \quad (2.52)$$

$$u^t(t = 0, x) = 1, \quad u^x(t = 0, x) = 0, \quad (2.53)$$

with a discontinuity at $x = 0$. Solving the time evolution of this system is called the Riemann problem [86]. The analytical solution is given for a particular choice of an EOS, $P(\epsilon) = c_s^2 \epsilon$, where c_s is the speed of sound. For $t > 0$, the solution $\epsilon(x, t) = \tilde{\epsilon}(\tilde{x})$ with $\tilde{x} = x/t$ is given piecewise [86]:

$$\tilde{\epsilon}(\tilde{x}) = \begin{cases} \epsilon_{\max}, & \tilde{x} < -c_s, \\ \epsilon_{\max} \left(\frac{(1-c_s)(1-\tilde{x})}{(1+c_s)(1+\tilde{x})} \right)^{\frac{1+c_s^2}{c_s}}, & -c_s \leq \tilde{x} < -v_{\text{r-p}}, \\ \epsilon_{\text{plateau}}, & -v_{\text{r-p}} \leq \tilde{x} < v_{\text{shock}}, \\ \epsilon_{\min}, & v_{\text{shock}} < \tilde{x}, \end{cases} \quad (2.54)$$

$$u^x(\tilde{x}) = \frac{v(\tilde{\epsilon}(\tilde{x}))}{\sqrt{1 - v(\tilde{\epsilon}(\tilde{x}))^2}}, \quad u^t(\tilde{x}) = \sqrt{1 + (u^x(\tilde{x}))^2}, \quad (2.55)$$

where

$$v_{\text{r-p}} = \frac{v(\epsilon_{\text{plateau}}) - c_s}{1 - v(\epsilon_{\text{plateau}})c_s}, \quad (2.56)$$

$$v(\epsilon) = \frac{1 - (\epsilon/\epsilon_{\max})^{2c_s/(1+c_s^2)}}{1 + (\epsilon/\epsilon_{\max})^{2c_s/(1+c_s^2)}}. \quad (2.57)$$

The constants $\epsilon_{\text{plateau}}$ and $v_{\text{shock}} = -v_r$ are obtained by solving the following equations for $(\epsilon_{\text{plateau}}, v_l, v_r)$:

$$\epsilon_{\text{plateau}} \gamma(v_l) v_l = \epsilon_{\min} \gamma(v_r) v_r, \quad (2.58)$$

$$4\epsilon_{\text{plateau}} \gamma(v_l) v_l^2 + \epsilon_{\text{plateau}} = \epsilon_{\min} \gamma(v_r) v_r^2 + \epsilon_{\min}, \quad (2.59)$$

$$v_l = \frac{v(\epsilon_{\text{plateau}}) + v_r}{1 + v(\epsilon_{\text{plateau}})v_r}, \quad (2.60)$$

where $\gamma(v) = 1/\sqrt{1-v^2}$. We note that Eqs. (2.58) and (2.59) represent the Taub equations [88, 89]. The solution $\tilde{\epsilon}(\tilde{x})$ is illustrated in Fig. 2.1. The discontinuity at $\tilde{x} = v_{\text{shock}}$ is called the shock front, and the curve in the range $-c_s \leq \tilde{x} < -v_{\text{r-p}}$ is called the rarefaction wave.

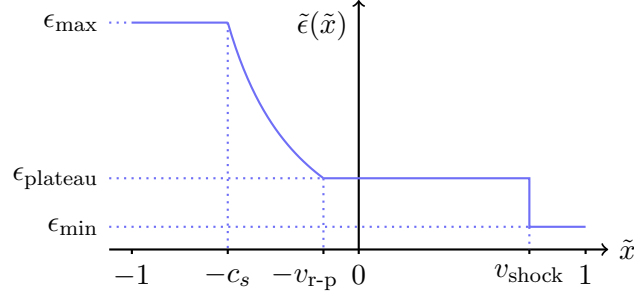


Figure 2.1: Analytical solution to the Riemann problem for $t > 0$ with $\tilde{x} = x/t$.

By setting $\epsilon_{\min} = 0$, Eq. (2.55) becomes

$$\tilde{\epsilon}(\tilde{x}) = \begin{cases} \epsilon_{\max}, & \tilde{x} < -c_s, \\ \epsilon_{\max} \left(\frac{(1-c_s)(1-\tilde{x})}{(1+c_s)(1+\tilde{x})} \right)^{\frac{1+c_s^2}{2c_s}}, & -c_s \leq \tilde{x} < 1, \\ 0, & 1 \leq \tilde{x}. \end{cases} \quad (2.61)$$

This solution mimics the expansion of the matter in the vacuum found in the QGP produced in heavy-ion collisions.

2.4.2 Gubser flow

Another exact solution is the Gubser flow [87], which is a solution for the conformal EOS, $P(\epsilon) = \epsilon/3$, characterized by the boost invariance and the *azimuthal symmetry* (i.e., the rotational symmetry in the x - y plane). The solution is given in the Milne coordinates as

$$\epsilon(\tau, r) = \frac{\epsilon_0 (2q)^{8/3}}{\tau^{4/3} (1 + 2q^2(\tau^2 + r^2) + q^4(\tau^2 - r^2)^2)^{4/3}}, \quad (2.62)$$

$$u^\tau(\tau, r) = \cosh k(\tau, r), \quad (2.63)$$

$$u^x(\tau, r) = \frac{x}{r} \sinh k(\tau, r), \quad (2.64)$$

$$u^y(\tau, r) = \frac{y}{r} \sinh k(\tau, r), \quad (2.65)$$

$$u^\eta(\tau, r) = 0, \quad (2.66)$$

with

$$k(\tau, r) = \tanh^{-1} \frac{2q^2 \tau r}{1 + q^2 \tau^2 + q^2 r^2}, \quad (2.67)$$

$$r = \sqrt{x^2 + y^2}, \quad (2.68)$$

where q and ϵ_0 are free parameters.

Chapter 3

Computational fluid dynamics

To solve the hydrodynamic equations numerically, we must discretize both the spatial and temporal coordinates. In this chapter, we first give a brief review of the Runge-Kutta (RK) integration method with a special focus on the differences between implicit and explicit methods in Sec. 3.1. We then discuss in Sec. 3.2 the Kurganov-Tadmor scheme which is a space discretization scheme often used in hydrodynamics. One of its important features is that it discretizes space independently of time, allowing the use of any time discretization. Therefore, both explicit and implicit time integration schemes can be used in a same formalism. We end this chapter with Sec. 3.3 by discussing the different methods used in the state-of-the-art hydrodynamic codes in the HIC community.

3.1 Implicit and explicit Runge-Kutta methods

The RK methods are common ways to numerically solve a system of ordinary differential equations (ODEs) of the general form,

$$\partial_t \vec{y}(t) = \vec{h}(t, \vec{y}(t)), \quad (3.1)$$

where $\vec{y}(t)$ represents the full set of state variables. The RK method solves the variables $\vec{y}(t_{\text{end}})$ at the end time $t = t_{\text{end}}$ for a given initial condition $\vec{y}(t_{\text{init}})$ at the initial time $t = t_{\text{init}}$ by discretizing the time into steps of length Δt and iteratively updating $\vec{y}(t)$ for every time step. In RK methods, for one time step from t to $t + \Delta t$, the state variable of the next time $\vec{y}(t + \Delta t)$ is constructed from $\vec{y}(t)$ as

$$\vec{y}(t + \Delta t) = \vec{y}(t) + \Delta t \sum_{n=1}^S b_n \vec{k}_{(n)}, \quad (3.2)$$

$$\vec{k}_{(n)} = \vec{h} \left(t + c_n \Delta t, \vec{y}(t) + \Delta t \sum_{m=1}^S a_{nm} \vec{k}_{(m)} \right), \quad (3.3)$$

where S is a parameter called the number of stages which is specific to each RK method, and the coefficients a_{nm} , b_n , c_n for $n, m = 1, \dots, S$ are determined so that the method has desired properties. One specific set of the coefficients defines a particular RK method. The table that summarizes these coefficients is called a Butcher table, whose examples are shown in Table 3.1.

In a RK method, the numerical error of $\vec{y}(t_{\text{end}})$ from the exact solution $\vec{y}^*(t_{\text{end}})$ is known to be suppressed in the limit $\Delta t \rightarrow 0$ as

$$\|\vec{y}(t_{\text{end}}) - \vec{y}^*(t_{\text{end}})\| \leq C(t_{\text{end}}) \Delta t^p, \quad (3.4)$$

where $\|\bullet\|$ is the Euclidean norm, and $C(t)$ is specific to the ODE to be solved and is independent of Δt . The largest integer p satisfying Eq. (3.4) is called the order of accuracy that is specific to each RK method.

The RK methods are classified into two categories: explicit and implicit. A RK method is said to be explicit when $a_{nm} = 0$ for $m \geq n$. In this case, $\vec{k}_{(n)}$ in Eq. (3.3) depend only on $\vec{y}(t)$ and $\vec{k}_{(m)}$ for $m < n$, so $\vec{k}_{(n)}$ are sequentially obtained from $n = 1$ to S by simple substitution of known values. The other RK methods are said to be implicit. For implicit methods, one must solve the set of equations Eq. (3.3) for $\vec{k}_{(n)}(t)$, which are typically nonlinear and practically carried out by a numerical solver. The implicit methods are thus usually considered to be computationally more expensive.

Despite this disadvantage, the implicit methods have a useful property, i.e., they are always linearly stable (the A -stability) as far as the exact solution is not divergent [90]. This means that a larger Δt can be adopted for stiff equations in implicit methods while keeping numerical error finite. This can be illustrated by the test problem

$$\partial_\tau e = -\lambda e \Leftrightarrow \partial_\tau e = -\frac{e-0}{\tau_\lambda}, \quad \tau_\lambda = \frac{1}{\lambda}, \quad (3.5)$$

which is the linearized approximation of any non-linear problem. In that case, it is easy to see that fast dynamics with a small τ_λ corresponds to a steep slope with a large λ . This steep slope is the reason we call a system with fast dynamics a stiff system. This stiffness is directly linked to the stability of RK. Depending on the method and on the time-step, the result will be convergent or not. If we consider the first order Explicit and Implicit Euler method on the left and right respectively

$$e(\tau + \Delta\tau) = e(\tau) + \Delta\tau\left(-\frac{e(\tau)-0}{\tau_\lambda}\right) \quad e(\tau + \Delta\tau) = e(\tau) + \Delta\tau\left(-\frac{e(\tau + \Delta\tau)-0}{\tau_\lambda}\right) \quad (3.6)$$

$$\Leftrightarrow e(\tau + \Delta\tau) = \left(1 - \frac{\Delta\tau}{\tau_\lambda}\right)e(\tau) \quad \Leftrightarrow e(\tau + \Delta\tau) = \frac{1}{1 + \frac{\Delta\tau}{\tau_\lambda}}e(\tau) \quad (3.7)$$

we can see that they will provide finite values depending on the amplification factors $r_{\text{expl}} = 1 - \frac{\Delta\tau}{\tau_\lambda}$ and $r_{\text{impl}} = 1/(1 + \frac{\Delta\tau}{\tau_\lambda})$. If these factors are larger than unity, the solution will diverge, but if they are smaller, the solution will converge. For a positive τ_λ , in the explicit case, $|r_{\text{expl}}| < 1$ when $0 \leq \Delta t < 2\tau_\lambda$. However, in the implicit case, $|r_{\text{impl}}| < 1$ for all Δt . This is why implicit methods are considered stable whereas explicit ones are only conditionally stable. The Israel-Stewart equations Eqs. (2.24, 2.25) contains short relaxation times compared to the typical duration of hydrodynamic evolution, which makes them stiff equations that need a small enough Δt or the use of implicit methods.

Another property of the implicit methods is that they can reach a given order of accuracy in fewer stages than explicit ones. More specifically, explicit methods require as many stages as the order up to $p = 4$ and more stages than the order for $p > 4$. On the other hand, the maximum accuracy for a given S is $p = 2S$ in implicit methods [91].

3.2 Space discretization

In computational hydrodynamics, we discretize the spatial coordinates to represent the state of the fluid using a finite number of degrees. The spatial derivatives are then approximated in terms of discretized fields so that the partial differential equations become a set of ODEs, which can be solved by a RK method. This discretization has to be carefully chosen to suppress numerical diffusion but at the same time to avoid spurious oscillations. Therefore,

		Heun			Midpoint			Gauss-Legendre 1 (GL1)	
Type		Explicit			Explicit			Implicit	
Stage S		2			2			1	
Order p		2			2			2	
c_n	a_{nm}	0	0	0	0	0	0	0.5	0.5
		1	1	0	0.5	0.5	0		
	b_m		0.5	0.5		0	1		1

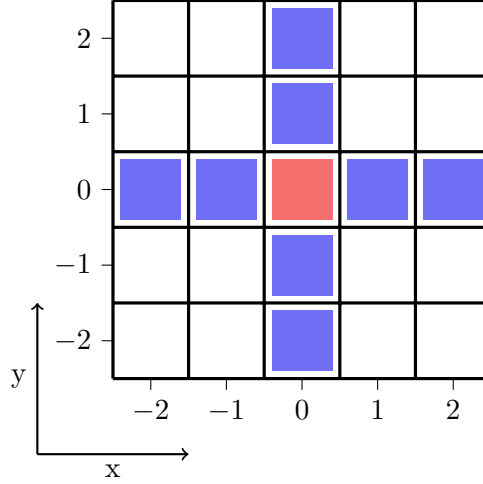


Figure 3.1: Schematic of the stencil used by the KT scheme to compute the space derivatives in a specific cell. Here the cell in which the derivative are evaluated is shown in red and the blue cells are the additional cells required for the computation. The overall shape of the stencil is a cross aligned with the axis. No cells are evaluated in diagonal directions.

In our case, this corresponds to express $T^{x\nu}$ as function of the reconstructed $T^{t\nu}$. We will explain in Sec. 5.6.1 how to extract from $T^{t\nu}$ the quantities such as the energy-density which are needed to express $T^{x\nu}$. The maximum local speed of propagation is given by

$$a_{j+1/2} = \max \left\{ \rho \left(J_{\alpha\beta}([Q^t]_{j+1/2}^+) \right), \rho \left(J_{\alpha\beta}([Q^t]_{j+1/2}^-) \right) \right\}, \quad (3.14)$$

where $J_{\alpha\beta}(Q^t) = \partial T^{x\alpha} / \partial T^{t\beta} |_{T^{t\nu}}$ is the 5×5 Jacobian matrix, and $\rho(M)$ is the spectral radius of a matrix M^* .

In the original study of the KT scheme [92], the quantities on the cell boundary $Q_{j+1/2}^\pm$ are calculated from the cell values Q_j using the MUSCL reconstruction as

$$Q_{j+1/2}^\pm = Q_{j+1/2 \pm 1/2} \mp \frac{\Delta x}{2} [\hat{\partial}_x Q]_{j+1/2 \pm 1/2}, \quad (3.15)$$

$$[\hat{\partial}_x Q]_j = \minmod \left(\minmod \left(\theta \frac{Q_j - Q_{j-1}}{\Delta x}, \frac{Q_{j+1} - Q_{j-1}}{2\Delta x} \right), \theta \frac{Q_{j+1} - Q_j}{\Delta x} \right), \quad (3.16)$$

$$\minmod(a, b) = \frac{\text{sgn}(a) + \text{sgn}(b)}{2} \min(|a|, |b|). \quad (3.17)$$

The flux limiter in Eq. (3.16) guarantees the non-oscillatory property of Q with the parameter θ restricted to $1 \leq \theta \leq 2$. Essentially, it appropriately switches between second order (central fraction) or first order (left and right fractions) slopes depending on their absolute value. Choosing $\theta < 1$ would produce excessive numerical diffusion, whereas choosing $\theta > 2$ would produce unphysical oscillation. We choose $\theta = 1.1$ following Ref. [55], which gives non-oscillatory results with a limited numerical diffusion. We verified that changing the value of θ hardly changes the tendency of our results for accuracy and efficiency in Sec. 6.6 unless θ is too close to two.

*An example of the analytic formula of $\rho(J_{\alpha\beta}(Q^t))$ for relativistic hydrodynamics can be found in Ref. [55].

3.3 Other discretization schemes

In this section we will discuss the general idea behind the mainly used discretization schemes, and we will discuss their advantages and disadvantages. Before discussing them independently, there are a few properties that they all share.

They are all designed to solve partial differential equations of the form

$$\frac{\partial \mathbf{y}}{\partial t} = \mathbf{f}(t, \mathbf{y}) \quad (3.18)$$

where $\mathbf{y}$ is the vector of dynamical quantities and $\mathbf{f}$ is the flux. This type of equations supposes that no time derivatives are involved in the flux, which means that the equations have to be linear in the time derivatives such that all the derivatives can be factored in $\partial \mathbf{y} / \partial t$.

Once the set of equations is in the shape of Eq. 3.18, it is discretized in fixed sized cells by averaging the quantities and considering them constant in these finite volumes. The opposite also exists where the density in each cell is fixed while the cells themselves are deforming and moving. This has the advantage of guarantying that the velocity of information is finite, but it is more complex to implement. To the author's knowledge, such moving cells have not been used in studying the QGP. In addition to this first property, all these schemes contains flux limiters to avoid the growth of non-physical oscillation, while at the same time limiting the numerical diffusion. These two properties are essential to properly propagate shocks, which are bound to happen in very dynamic systems such as the QGP.

The last common property is the guaranty of the positivity of the energy, as it is required physically.

In addition to the common properties, they also all share some common disadvantages. The first comes from the axis aligned space discretization, which introduces preferred directions along the axis. This usually correspond to an increased numerical diffusion along the diagonals. Most schemes tend to discretize time and space simultaneously in a way which reduces this problem. A second disadvantage is the exclusive use of explicit time integration schemes. They have a limited stability which ends up requiring the addition of many prescriptions to stabilize the equations. For viscous hydrodynamics, this usually corresponds to limiting or reducing the value of the viscous components in specific situations, such as when the effective pressure becomes negative. In addition, the trivial way of computing explicit methods incite to solve any additional issues in a similarly trivial manner. We will discuss in more detail in Sec. 3.4 how second order viscous hydrodynamics introduces non-trivial issues which affect the time discretization, effectively reducing every explicit scheme to be first order in accuracy.

In the remaining paragraphs, we will discuss the distinctive properties of each scheme.

KT The combination of KT and RK discretization is quite common [55, 74, 76, 77, 93]. The main advantage of this combination is that KT is independent of the time discretization while RK can discretize time in a very versatile way. More specifically, one can freely choose an RK scheme with any number of stages to reach any accuracy. In addition, one can also adapt the time step in order to reach any accuracy needed and also decrease it as small as needed to remain stable. This stability issue is only present when using an explicit RK scheme. However, an explicit scheme being trivial to compute, all the codes cited previously use them. There are disadvantages with this combination as well. The first one is the shape of the stencil of the KT space discretization. Being a long cross, it creates strongly preferred directions which will deform propagating shapes. For instance, a wave propagating in a circle will end up having a diamond shape. Increasing the number of cells to decrease the lattice spacing will reduce this problem, but at the same time it will increase the computational

cost. The use of a multi-stage RK helps to reduce this issue by smearing the cross shape over the successive stages. The second issue is the use of explicit schemes as explained before, although in this case an implicit integrator could be used.

SHASTA The SHASTA (sharp and smooth transport algorithm) [54, 67, 68, 94, 95] scheme combines both space and time discretization together. It advances from one time interval to the next one in two steps, a propagation of the densities from a geometrical point of view followed by a correction process. In the first step, it considers that the densities are concentrated in a point that will move away from the center at the local velocity. In order for the problem to remain simple, this point should not leave the boundaries of its cell in one time interval, so the time step Δt and the lattice spacing Δx should be chosen such that $v\Delta t/\Delta x \leq 1/2$ with v the velocity of the fluid in that cell. The densities are then interpolated over space between each cell to fill the space. The new density of each cell simply corresponds to the integral of the interpolated density in the cell. This first step actually produced a large numerical diffusion, which is why a correction step follows. This second step essentially corresponds to the subtraction of the numerical diffusion introduced by the first stage. Doing the subtraction of the exact numerical diffusion would lead to numerical oscillations that could change the sign of the density, and in the case of relativistic hydrodynamics produce negative energy. The idea of the SHASTA scheme is to express this numerical diffusion in terms of fluxes such that a limiter can be applied to avoid oscillations. This will leave some numerical diffusion but only in a controlled way of order $O(\Delta x^2)$. This algorithm has the advantage of being simple to implement while keeping all the important properties discussed at the beginning of this section. However, it is explicit by design, so it will end up being effectively first order accuracy in time.

HLLE The HLLE (Harten–Lax–van Leer–Einfeldt) [96, 70, 71, 97, 73] scheme essentially takes the opposite path of SHASTA. Instead of doing a first simple diffusive step and correcting it after, it directly considers how to properly approximate a discontinuity by locally solving the Riemann problem. It does so by first computing the velocities propagating in both directions from the discontinuity and then expressing the flux propagating in both directions with these velocities. To reach second order accuracy with limited oscillations, it then expresses the dynamical quantities at the next time step from the fluxes that were reconstructed at the boundary of each cell while including a slope limiter. The reconstruction at the boundary uses the same MUSCL process as in the KT scheme which we explained in details in Sec. 3.2. In the specific case of viscous hydrodynamic, at each time step this scheme is applied to the ideal part first while the viscous part is computed afterward. It is done that way in order to use the velocity obtained from solving the ideal part to express the time derivatives of the velocity with a finite time difference before actually solving the viscous part. This scheme provides great results for shocks propagation by construction. However, it is explicit by design, and the viscous part is once again first order accuracy in time.

Others There are other types of Godunov-type schemes such as [72, 75] which share similar properties to the previous schemes, especially to HLLE which is itself a Godunov-type scheme.

There is one common issue to all these schemes including the one presented in this thesis. We all suppose that the viscosity stays small, mostly because the IS equations were derived assuming this fact. Because of this, the viscosity will tend to diverge when the variation of the velocity is too large. This essentially only happens near the vacuum where the energy-density is very small, so in locations where hydrodynamics stops to be applicable. In order

to fix this issue, every code will introduce some sort of limits on the norm of the viscosity to keep it small. Such forced limits will alter the results, but as it only happens where the energy-density vanishes, it does not affect the physics at larger energy corresponding to the QGP which we are interested in.

3.4 Time derivatives in constitutive equations

When the equations of interest are non-linear in the time derivatives, such as the Israel-Stewart equations for viscosity, these discretization schemes cannot be applied straightforwardly. By looking at the open-source codes, the main strategy is to approximate the time derivatives that cannot be factored (mainly some $\partial_t u^\nu$) by keeping track of older time steps in order to perform a backward finite time difference. This prescription allows to essentially keep all the properties of the space discretization, but it will degrade the stability and by construction force the time accuracy order to be the one of a finite difference, which is first order. In the case of an implicit time integrator, these remaining problematic time derivatives can be solved exactly which fixes the problem. However, in the case of explicit time integration which is used in all the current state-of-the-art codes, it is unclear if such an exact solving is at all possible for order higher than unity. Nevertheless, all these discretization schemes also aim to limit their complexity in order to simplify the implementation.

Chapter 4

Locally optimized fixed-point implicit solver

The time evolution of the differential equations (3.10), which are obtained by the space discretization of Eq. (3.8), can be solved by RK methods introduced in Sec. 3.1, where $\vec{y}(t)$ and $\vec{h}(\vec{y})$ in Eq. (3.3) correspond to the state variables $\{[T^{t\nu}]_j\}$ and the right-hand side of Eq. (3.10), respectively. For implicit RK methods, we need to numerically solve implicit equations for substantial problems. However, implicit methods usually cannot be efficient without modifications compared to explicit ones.

To overcome this excessive computational cost of implicit methods, we propose an improved solver for implicit RK methods in this study. After a motivation in Sec. 4.1 and a definition of our method in Sec. 4.2, we discuss the convergence condition of our method and its connection to the stiffness and stability in Sec. 4.3. In Sec. 4.4, we finally discuss the accuracy, efficiency, and stability for a specific case of the implicit RK method called the one-stage Gauss-Legendre (GL1) method by relating it to the two-step Adams-Bashforth (AB2) method.

4.1 Motivation

In this research, the goal of using implicit methods is not only to increase the stability but especially to increase the accuracy and efficiency. At first, it would look like using an implicit method would decrease the efficiency. The idea of an implicit method is to define the future state by already knowing its future value. We can see this as guessing the future state all at once. In practice, it takes the shape of a large set of non-linear implicitly defined equations. In general, such a system does not have an analytical expression, so it must be solved numerically. To do so, one has to guess an approximate value of the solution, and iteratively enhance this approximation by applying the equations many times. From the point of view of Runge-Kutta integration, this means that for each time step, we will need to compute the set of equation an amount of time proportional to the number of iterations needed to converge to the solution, where the number of computations per iteration depend on the iterative method chosen. From this alone, it is very clear that the cost of an implicit method will be at least the same as an explicit method which does not need iteration, and most likely it will be much higher.

However, implicit method have the property of requiring fewer stages per time step to reach the same accuracy as an explicit method. This means that in the case where only one iteration is needed, an implicit method will actually need less computation to perform a time step for the same accuracy. Generally, the number of iterations needed to converge to a

solution depends on how close the initial guess is from the solution. Which means that when a system only slightly changes over time, the solution of a previous time step will already be a very close guess. When we look at the about $(20\text{fm})^3$ volume in which the hydrodynamics of the QGP takes place in the simulation, most of it will be a vacuum where barely anything changes. This means that in this large proportion of the volume very few iterations will be needed, where very few could be just one. Only the few locations where the gradients are large will greatly change between time steps, so this is where more iterations will be needed. With this broader picture in mind, it starts to become feasible that an implicit method could yield a higher accuracy for a lower cost.

Another motivation from the implicit framework are the time derivatives of the velocity that appear in the IS equation. They are in general not easily expressible in an explicit space-time discretization. Which means that in this latter case, approximation of these time derivatives must be done independently of the space-time discretization, which would inevitably lead to inconsistencies and hence a decrease in accuracy. In an implicit Runge-Kutta scheme, the time derivatives of the dynamical variables are what is being solved iteratively. We can then assume that we actually know the value of these time derivatives as the approximation we have of them become more and more precise after every iteration. In the same way it is possible to express the velocity from the energy momentum tensor, it is possible to express the time derivative of the velocity from the time derivative of the energy momentum tensor. Due to this, the IS equation can be solved consistently in an implicit framework, which would preserve the expected accuracy. In a Runge-Kutta integration, the accuracy will decrease when the time step increases but so is the total number of steps, which means that the total cost would decrease. This last fact also suggests that for a same accuracy an implicit method could use less time steps and thus require a smaller computational cost.

4.2 Fixed-point solver with local optimization

In implicit RK methods, $\vec{k}_{(n)}$ in Eq. (3.3) are obtained by solving the system of implicit equations. In this study we adopt the fixed-point method as a numerical solver for this procedure. To illustrate this method, we first rewrite Eq. (3.3) in a compact form

$$\vec{K} = \vec{F}(\vec{K}), \quad (4.1)$$

with $\vec{K} = (\vec{k}_{(1)}, \dots, \vec{k}_{(S)})$ and

$$\vec{F}(\vec{K}) = (\vec{f}_{(1)}(\vec{K}), \dots, \vec{f}_{(S)}(\vec{K})), \quad (4.2)$$

$$\vec{f}_{(n)}(\vec{K}) = \vec{h} \left(t + c_n \Delta t, \vec{y}(t) + \Delta t \sum_m a_{nm} \vec{k}_{(m)} \right). \quad (4.3)$$

The fixed-point method solves Eq. (4.1) by starting from an initial guess $\vec{K}^{(0)}$ and iteratively updating the guess as $\vec{K}^{(l+1)} = \vec{F}(\vec{K}^{(l)})$ for $l = 0, 1, 2, \dots$ until the convergence is reached. A schematic illustration of this procedure for a single-variable case is shown in Fig. 4.1.

The convergence in the fixed-point method strongly depends on the choice of the initial guess $\vec{K}^{(0)}$. The initial guess should not be too far from the true solution. In this study, for the initial guess, we use the solution of $\vec{K}$ in the previous RK step. This choice comes from an expectation that $\vec{K}$ only changes at the order of $\mathcal{O}(\Delta t)$, and hence $\vec{K}$ at the previous RK time step is a good guess for the next one. As we will see below, this choice of $\vec{K}^{(0)}$ plays a crucial role in realizing a good efficiency in our method. For the first time step of RK at $t = t_{\text{init}}$, we use $\vec{K}^{(0)} = (0, \dots, 0)$ for the initial guess.

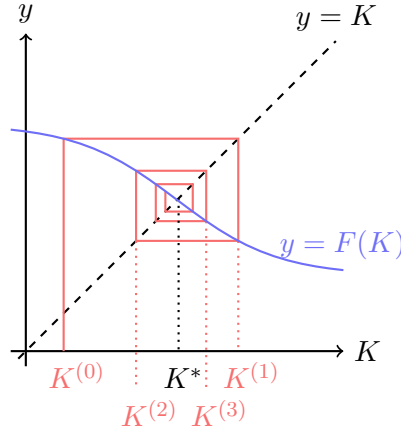


Figure 4.1: Schematic picture of the iterative process $K^{(0)}, K^{(1)}, \dots$ in the fixed-point method toward the solution K^* for the single-variable case.

To improve the efficiency, we utilize the locality of discretized partial differential equations, i.e., a finite number of cells are referenced in an evaluation of the time derivative for one cell. In our case, the right-hand side of Eq. (3.10) depends only on cells up to two neighbors in each direction. By taking this advantage, in our implementation, we check the convergence cell by cell using the criterion

$$\|[\vec{K}^{(l+1)}]_j - [\vec{K}^{(l)}]_j\| = \|[\vec{F}(\vec{K}^{(l)})]_j - [\vec{K}^{(l)}]_j\| < e \frac{\langle T^{tt} \rangle}{\Delta t} \left(\frac{\Delta t}{\Delta x} \right)^p, \quad (4.4)$$

with a pre-determined parameter e , where $[\vec{K}]_j$ represents the set of variables in $\vec{K}$ at cell j , and $\langle T^{tt} \rangle$ is the spatial average of T^{tt} in the entire numerical grid. Note that $[\vec{K}]_j$ includes $4S$ variables corresponding to $T^{t\nu}$ for all stages.

Since $\langle T^{tt} \rangle / \Delta t$ has the same unit as $[\vec{K}^{(l)}]_j$, e is dimensionless. This parameter should be properly tuned for each calculation setup since smaller e leads to better accuracy but degrades the efficiency, and vice versa. In our numerical analysis of the ideal case in Sec. 6.4 we choose $e = 10^{-3}$ and $e = 2 \times 10^{-4}$ for one and two space dimensions, respectively. For the viscous case in Sec. 6.6 we choose $e = 2 \times 10^{-1}$ for both two and three space dimensions.

In Eq. (4.4), we require that the threshold scales as Δt^{p-1} . This Δt dependence is necessary to preserve the accuracy order p of the numerical results. Since the error of $[\vec{K}^{(l+1)}]_j = [\vec{F}(\vec{K}^{(l)})]_j$ is parametrically of one order Δt higher than that of $[\vec{K}^{(l)}]_j$ because $\partial \vec{F}(\vec{K}^{(l)}) / \partial \vec{K}^{(l)} \propto \Delta t$, the error of $[\vec{K}^{(l+1)}]_j$ is of order Δt^p when the condition (4.4) $\propto \Delta t^{p-1}$ is met. This order of $[\vec{K}^{(l+1)}]_j$ is in accordance with the accuracy of RK in Eq. (3.4)*. On the right-hand side of Eq. (4.4), Δx is used to make p dependence dimensionless. In Sec. 6.4 and Sec. 6.6 we show that the expected scaling (3.4) is nicely obtained using the criterion (4.4).

More specifically, we update all the cells for the first iteration. For the succeeding iterations, we update cell j only when any of the following conditions are met:

1. Eq. (4.4) was not satisfied at cell j in the last update.
2. Any of the cells adjacent to cell j was updated in the previous iteration and did not satisfy Eq. (4.4).

*The numerical error of one RK step is of order Δt^{p+1} . Also, notice Δt on the right-hand side of Eq. (3.2).

We update all the cells that need updating at once. We proceed with the iterations until Eq. (4.4) is reached for all the cells. We note that even once the condition (4.4) is satisfied at a cell, the condition can be violated later due to updates of neighboring cells.

As we will show in the next section, this optimization drastically reduces the computational cost. In fact, only a single update is enough for most cells. We refer to this prescription as the *local optimization* in the following.

In the above procedure, we check only the most adjacent cells. Although the right-hand side of Eq. (3.10) also depends on the second neighboring cells, we have checked that their inclusion hardly changes the quality of numerical results despite an extra computational cost[†]. We also notice that an update of a cell does not necessarily lead to the update of its neighboring cells in the next iteration.

Strictly speaking, the local optimization violates the energy-momentum conservation. This is because the flux at a boundary between two cells can be inconsistent when one cell is updated, but the other is not. The numerical error from this violation, however, is controlled within the accuracy as we will show in Sec. 6.4 and Sec. 6.6.

An illustration comparing the explicit and the implicit method including the local optimization is shown of Fig 4.2.

4.3 Convergence condition and stiffness

As is common in iterative methods, the fixed-point method does not necessarily converge to the true solution. A necessary condition for the convergence of the fixed-point method is written in terms of the spectral radius R of the Jacobian matrix of $\vec{F}$ at the solution $\vec{K} = \vec{K}^*$:

$$R = \rho \left(\frac{\partial \vec{F}}{\partial \vec{K}}(\vec{K}^*) \right) < 1, \quad (4.5)$$

Moreover, the convergence is faster for smaller R . From Eqs. (4.2) and (4.3) one finds that the Jacobian matrix $\partial \vec{F} / \partial \vec{K}$, and hence R , is proportional to Δt . Therefore, the convergence is guaranteed for a sufficiently small Δt and is faster for a smaller Δt . However, the convergence is not guaranteed in general. In fact, we will see later that our method can be aborted for $\Delta t / \Delta x \gtrsim 0.4$ due to a failure of the fixed-point solver. The convergence in these cases can be established with additional prescriptions as will be discussed in the forthcoming publications [98]. In the present study, we simply neglect these cases since our method works stably in the range of Δt which is practically important.

The convergence condition (4.5) is related to the stiffness of the system of equations. In fact, Eq. (4.5) is rewritten as

$$\begin{aligned} R &= \rho \left(\Delta t \left[a_{nm} \frac{\partial \vec{h}}{\partial \vec{y}} \right]_{(t+c_n \Delta t, \vec{y} + \Delta t \sum_{l=1}^S a_{nl} \vec{k}_{(l)}^*)} \right)_{nm} \\ &= \Delta t \rho(a_{nm}) \rho \left(\frac{\partial \vec{h}(t, \vec{y})}{\partial \vec{y}} \right) + \mathcal{O}((\Delta t)^2), \end{aligned} \quad (4.6)$$

where $\rho(a_{nm})$ is the spectral radius of the Butcher table of a given RK method. Here, $\rho(\partial \vec{h} / \partial \vec{y})$ characterizes the stiffness of the system since the largest absolute eigenvalue $\max_{\lambda} |\lambda| =$

[†]The success of this prescription may rely on the fact that Eq. (3.10) only depends on at most second neighboring cells. When one employs a space scheme that depends on more distant cells, this procedure may need to be modified to keep the quality of numerical results.

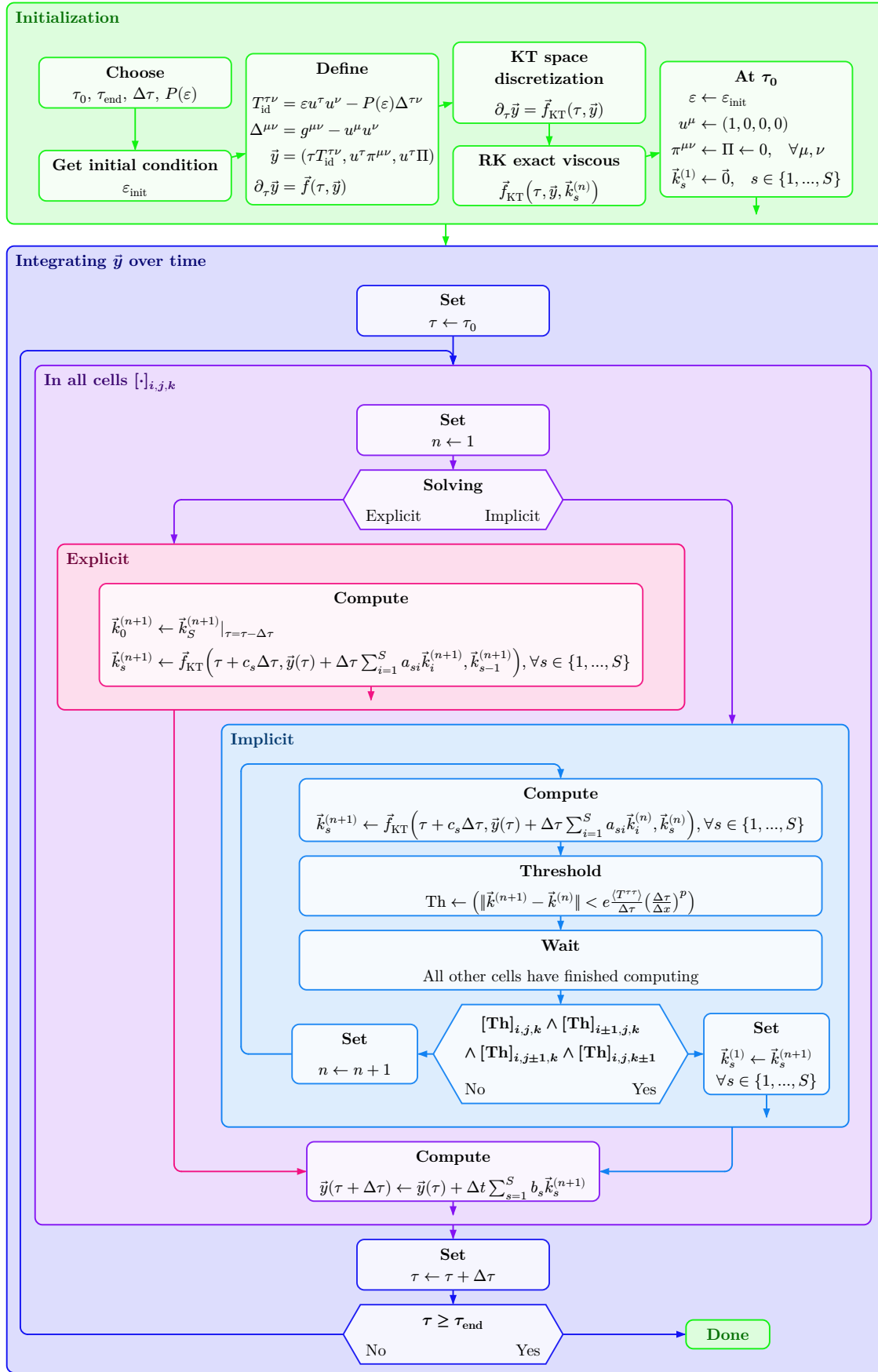


Figure 4.2: Pseudocode comparing the implicit and explicit methods.

$\rho(\partial\vec{h}/\partial\vec{y})$ (i.e., the inverse of the shortest timescale) gives the most severe impact on the stability. Equation (4.6) shows that R is proportional to the stiffness $\rho(\partial\vec{h}/\partial\vec{y})$.

This means that the convergence becomes slow in stiff systems. Since we locally truncate the iteration number of the fixed-point method using Eq. (4.4), the iteration number becomes automatically larger in the stiff spatial region where short-timescale phenomena are happening. In short, our local optimization automatically detects the stiffness of local regions.

4.3.1 Enhancing stability of fixed-point iteration

The locally optimized implicit solver is essentially the same with or without viscosity. The main difference is that viscous equations are stiffer than the ideal ones, which means that the simple fixed-point solver might not converge. The basic fixed-point solver has stricter convergence condition than the stability conditions for the explicit method. This implies that our implicit method would not produce results in some cases where the explicit would produce correct results. To solve this issue, we slightly modify the fixed-point solver to become:

$$\vec{K} = \vec{K} + \alpha (\vec{F}(\vec{K}) - \vec{K}) \quad (4.7)$$

with $0 < \alpha \leq 1$. This is a simple linear interpolation between the initial value $\vec{K}$ and the target value $\vec{F}(\vec{K})$. When $\alpha = 1$, it is the simple fixed-point method $\vec{K} = \vec{F}(\vec{K})$. When $\alpha < 1$, a smaller step is done, which tends to increase the number of iterations needed, but also increase the stability. The convergence condition in this case is:

$$\rho \left(\frac{\partial [\vec{K} + \alpha (\vec{F}(\vec{K}) - \vec{K})]}{\partial \vec{K}} \right) < 1 \Leftrightarrow \rho \left((1 - \alpha)I + \alpha \frac{\partial \vec{F}(\vec{K})}{\partial \vec{K}} \right) < 1 \quad (4.8)$$

where I is the identity matrix. As long as the diagonal of $\partial\vec{F}(\vec{K})/\partial\vec{K}$ is non-positive to suppress $(1 - \alpha)I$, then there must be a value of α such that the spectral radius is smaller than unity. This property is actually ensured when the space discretization complies to the TVD, which is the case for the Kurganov-Tadmor method. Such simple extension should make our implicit method unconditionally stable. It is actually not exactly the case in this paper as we used an approximation to handle the space derivatives in the right-hand side of the evolution equations. As there is no easy way to find the best value for α , we instead default it to $\alpha = 1$, and if the error in any cell increases more than a finite amount of times in a row (i.e. 4 times), we reset $\vec{K} = \vec{0}$ and divide α by 2. We repeat this process until convergence is reached.

Another way to stabilize the convergence which we did not use in this thesis is the Anderson acceleration [99]. This method does not give an exact condition for stability, but at the difference of the relaxation introduced above, it reduces the number of iterations needed instead of increasing it in general. The idea is relatively simple. It wants to reduce the errors

$$\vec{G}^i = \vec{F}(\vec{K}^i) - \vec{K}^i \quad (4.9)$$

over multiple iterations indexed by i . To do so, for each new iteration n , it finds the set of coefficients α_i which simultaneously minimizes the $\vec{G}^i$ over m older iterations,

$$\alpha_i = \operatorname{argmin}_{\alpha_i} \left(\sum_{i=n+1-m}^n \alpha_i \vec{G}^i \right)^2, \quad (4.10)$$

such that $\sum_i \alpha_i = 1$. The next value $\vec{K}^{n+1}$ is computed as

$$\vec{K}^{n+1} = \sum_{i=n+1-m}^n \alpha_i \vec{F}^i. \quad (4.11)$$

This method gives the values of the α_i as a linear system of rank m of scalar products of the $\vec{G}^i$. So the computational cost to find the α_i for small m is small and in general negligible compared to the computation of $\vec{F}(\vec{K})$. In addition, it only requires one computation of $\vec{F}(\vec{K})$ per iteration, so it is as cheap as the simple fixed-point method. Furthermore, it converges in fewer iterations while remaining more stable for stiffer systems.

4.4 Accuracy, efficiency, and stability

In later sections, we particularly consider the one-stage Gauss-Legendre (GL1) method [100, 101, 91] as an implicit method of order $p = 2$, whose Butcher table is given in the right column of Table 3.1. In this subsection, we focus on the locally optimized GL1 and discuss its accuracy, efficiency, and stability.

We first show that the method applied to GL1 keeps the accuracy order $p = 2$ even when the iteration number is fixed to one. The single-iteration GL1 is explicitly written down as

$$\vec{y}(t + \Delta t) = \vec{y}(t) + \Delta t \vec{k}^{(1)}, \quad (4.12)$$

$$\vec{k}^{(1)} = \vec{h} \left(t + \frac{\Delta t}{2}, \vec{y}(t) + \frac{\Delta t}{2} \vec{k}^{(0)} \right), \quad (4.13)$$

where the initial guess $\vec{k}^{(0)}$ of the iteration is chosen to be the solution in the previous time step. By inverting Eq. (4.12) of the previous time step, the initial guess can be expressed as

$$\vec{k}^{(0)} = \frac{\vec{y}(t) - \vec{y}(t - \Delta t)}{\Delta t}. \quad (4.14)$$

Plugging Eqs. (4.13) and (4.14) into Eq. (4.12), we obtain

$$\vec{y}(t + \Delta t) = \vec{y}(t) + \Delta t \vec{h} \left(t + \frac{\Delta t}{2}, \frac{3\vec{y}(t) - \vec{y}(t - \Delta t)}{2} \right). \quad (4.15)$$

The Taylor expansion of this expression matches the analytic expansion up to the second order in Δt :

$$\vec{y}(t + \Delta t) = \vec{y}(t) + \Delta t \vec{h}(t, \vec{y}(t)) + \frac{(\Delta t)^2}{2} [\partial_t + \vec{h}(t, \vec{y}(t)) \cdot \partial_{\vec{y}}] \vec{h}(t, \vec{y}(t)) + \mathcal{O}((\Delta t)^3). \quad (4.16)$$

Thus, the single-iteration fixed-point method for GL1 already has the desired accuracy of order $p = 2$.

Next, we see that the single-iteration GL1 can be related to AB2. By regarding

$$\begin{aligned} \vec{z} \left(t + \frac{\Delta t}{2} \right) &= \frac{3\vec{y}(t) - \vec{y}(t - \Delta t)}{2} \\ &= \vec{y}(t - \Delta t) + \frac{3}{2} \Delta t \vec{h} \left(t - \frac{\Delta t}{2}, \vec{z} \left(t - \frac{\Delta t}{2} \right) \right) \end{aligned} \quad (4.17)$$

as state variables, Eq. (4.15) can be shown to be equivalent to the AB2 method:

$$\vec{z}(t + \Delta t) = \vec{z}(t) + \frac{3}{2}\Delta t \vec{h}(t, \vec{z}(t)) - \frac{1}{2}\Delta t \vec{h}(t - \Delta t, \vec{z}(t - \Delta t)), \quad (4.18)$$

Here, the initial guess $\vec{k}^{(0)}$ plays an important role in the equivalence of the single-iteration GL1 (4.15) to AB2 (4.18). In the single-iteration GL1, the initial guess carries the information of the previous step $\vec{y}(t - \Delta t)$, which is similar to the linear multistep methods such as AB2 using the states of previous steps.

Both the single-iteration GL1 and AB2 are second-order methods with one-stage evaluation of $\vec{h}$, so they are more efficient than the second-order explicit RK with two stages. However, the problem with those one-stage methods is the stability in the stiff equations. Those methods effectively use an extrapolation using the previous step $\vec{y}(t - \Delta t)$ as observed in Eqs. (4.15) and (4.18) and are known to be less stable than the explicit RK. This is a common problem with the linear multistep method such as the Adams-Bashforth method. In our method, this problem is automatically remedied by the iterations. As already discussed, the number of fixed-point iterations increases in the stiff region so that the method approaches the original GL1, which is A -stable. Thus, we can view our method as automatic switching between the two second-order methods, the less stable but efficient AB2 and the stable GL1, based on the local stiffness. In this way, our method benefits from both the efficiency of AB2 and the stability of GL1.

As we will observe in the numerical tests in the later sections, the stiff part appears near the vacuum or the region with large gradients. Such regions typically appear only in limited spacetime volumes in practical applications, so we can expect the computational cost of the locally optimized implicit method would be close to the cost of AB2. Since AB2 with a one-stage evaluation of $\vec{h}(t, \vec{y}(t))$ has half the cost of the second-order explicit RK methods with two stages, we may expect a better performance of the locally optimized implicit method as far as the total size of the stiff regions is not too large.

The application of the idea of the locally optimized implicit method is not limited to relativistic hydrodynamics. For example, the idea can be applied to the general (discretized) field equations with local interactions. In principle, this idea can be applied to any ODEs with the locality, where a degree of freedom directly interacts with a fixed number of other degrees regardless of the size of the entire system. Thus, we expect broader applications of this idea in various fields.

Finally, we comment on the future extension of our analysis to the higher-order implicit RK methods. In this subsection, we considered the specific case of GL1 utilizing its particular form. It should be in principle possible to design the proper choice of the initial guess $\vec{K}^{(0)}$ for the general implicit RK methods by taking the linear combinations of the solutions $\vec{K}$ in the previous time steps so that the accuracy order of the implicit method is preserved even with a single iteration. These analyses are left for future study.

Chapter 5

Numerical scheme

The basic space discretization discussed in Sec. 3.2 is not sufficient to actually solve the hydrodynamic equations. Every set of partial differential equations has its own specific additional requirements. In this chapter, we will discuss how we fulfil the requirements for relativistic hydrodynamics.

5.1 Viscous Numerical Discretization

From this section, every equation in the Milne coordinate system will use the variable change including the factors τ from Sec. 2.3. For convenience reasons, we will drop the $\tilde{\bullet}$ notation. In order to include dissipation through Eqs. (2.45, 2.46, 2.47), we first aim to rewrite them so that they are in a shape that is compatible with both the Kurganov-Tadmor and Runge-Kutta discretizations. One such shape is

$$\partial_\tau (\tau T_{\text{id}}^{\tau\nu}) = -\partial_x (\tau T^{x\nu}) - \partial_y (\tau T^{y\nu}) - \partial_\eta (T^{\eta\nu}) + A(\partial_\tau u^\nu, \partial_\tau \Pi, \partial_\tau \pi^{\tau\nu}) \quad (5.1)$$

$$\partial_\tau (u^\tau \pi^{\mu\nu}) = -\partial_x (u^x \pi^{\mu\nu}) - \partial_y (u^y \pi^{\mu\nu}) - \partial_\eta (u^\eta \pi^{\mu\nu}) + B(\partial_\tau u^\nu, \partial_i u^\nu) \quad (5.2)$$

$$\partial_\tau (u^\tau \Pi) = -\partial_x (u^x \Pi) - \partial_y (u^y \Pi) - \partial_\eta (u^\eta \Pi) + C(\partial_\tau u^\nu, \partial_i u^\nu) \quad (5.3)$$

where

$$A(\partial_\tau u^\nu, \partial_\tau \Pi, \partial_\tau \pi^{\tau\nu}) = I_T^\nu - \partial_\tau (\tau [-\Pi \Delta^{\tau\nu} + \pi^{\tau\nu}]), \quad (5.4)$$

$$B(\partial_\tau u^\nu, \partial_i u^\nu) = -\frac{\tilde{\pi}^{\mu\nu} - \tilde{\pi}_{\text{NS}}^{\mu\nu}}{\tau_\pi} - \frac{1}{3} \tilde{\pi}^{\mu\nu} \left(\tilde{\partial}_\lambda \tilde{u}^\lambda - 4 \frac{\tilde{u}^\tau}{\tau} \right) \quad (5.5)$$

$$- \left(\tilde{u}^\nu \tilde{\pi}^{\mu\beta} + \tilde{u}^\mu \tilde{\pi}^{\nu\beta} \right) \tilde{u}^\lambda \tilde{\partial}_{;\lambda} \tilde{u}_\beta + \frac{\tilde{u}^\eta}{\tau} I_\pi^{\mu\nu}, \quad (5.6)$$

$$C(\partial_\tau u^\nu, \partial_i u^\nu) = -\frac{\tilde{\Pi} - \tilde{\Pi}_{\text{NS}}}{\tau_\Pi} - \frac{1}{3} \tilde{\Pi} \left(\tilde{\partial}_\lambda \tilde{u}^\lambda - 4 \frac{\tilde{u}^\tau}{\tau} \right), \quad (5.7)$$

as long as A , B , C behave as source terms, depending on none of the space and time derivatives. Although they currently do not fulfil this requirement, we will see in the next section how to resolve this issue. Assuming that A , B and C behaves as simple source terms, Eqs. (5.1, 5.2, 5.3) can be discretized exactly as described in the previous section. We solve the ideal part $T_{\text{id}}^{\tau\nu}$ of $T^{\tau\nu}$ as it allows keeping the exact same derivations from the previous section, and also contributes in the procedure to make A , B and C independent of the time derivatives.

5.2 Right hand side time derivatives

Having an analytical expression for the time derivatives $\partial_\tau u^\nu$, $\partial_\tau \Pi$ and $\partial_\tau \pi^{\tau\nu}$ appearing in A , B and C would permit to avoid the need to discretize them. In the case where $\partial_\tau (\tau T_{\text{id}}^{\tau\nu})$, $\partial_\tau (u^\tau \pi^{\mu\nu})$ and $\partial_\tau (u^\tau \Pi)$ are actually known quantities, we can express $\partial_\tau u^\nu$, $\partial_\tau \Pi$ and $\partial_\tau \pi^{\tau\nu}$:

$$\partial_\tau \pi^{\tau\nu} = \frac{\partial_\tau (u^\tau \pi^{\tau\nu}) - \pi^{\tau\nu} \partial_\tau u^\tau}{u^\tau} \quad (5.8)$$

$$\partial_\tau \Pi = \frac{\partial_\tau (u^\tau \Pi) - \Pi \partial_\tau u^\tau}{u^\tau} \quad (5.9)$$

$$\partial_\tau u^\tau = -\frac{u^x \partial_\tau u^x + u^y \partial_\tau u^y + u^\eta \partial_\tau u^\eta}{u^\tau} \quad (5.10)$$

$$(\partial_\tau \epsilon \quad \partial_\tau u^x \quad \partial_\tau u^y \quad \partial_\tau u^\eta)^\top = \mathbb{M}^{-1} \partial_\tau T_{\text{id}}^{\tau\nu} \quad (5.11)$$

$$\partial_\tau T_{\text{id}}^{\tau\nu} = \frac{\partial_\tau (\tau T_{\text{id}}^{\tau\nu}) - T_{\text{id}}^{\tau\nu}}{\tau} \quad (5.12)$$

where the 4 by 4 matrix $\mathbb{M} = \left[\frac{\partial (\partial_\tau T_{\text{id}}^{\tau\nu})}{\partial (\partial_\tau \epsilon \quad \partial_\tau u^x \quad \partial_\tau u^y \quad \partial_\tau u^\eta)} \right]$ can be inverted analytically:

$$\mathbb{M}^{-1} = \begin{bmatrix} \frac{(\epsilon+P)u^\tau(1+\frac{u^2}{(u^\tau)^2})}{\Delta} & \frac{-2(\epsilon+P)\bar{u}^T}{\Delta} \\ \frac{-u^\tau(1+\partial_\epsilon P)\bar{u}}{\Delta} & \frac{(u^\tau)^2(1+\partial_\epsilon P)-\partial_\epsilon P}{\Delta} \frac{\bar{u}\bar{u}^T}{u^2} + \frac{1}{(\epsilon+P)u^\tau} \left(1 - \frac{\bar{u}\bar{u}^T}{u^2}\right) \end{bmatrix} \quad (5.13)$$

where $\bar{u} = (u^x \quad u^y \quad u^\eta)^\top$ and $u^2 = \bar{u}^T \bar{u}$. Detailed explanations are given in Appendix B.

Once these time derivatives are expressed with only the known quantities, the Runge-Kutta procedure can be applied straightforwardly. This corresponds to provide $\vec{k}_{(n)}$, which is a numerical approximation of $\partial_\tau (\tau T_{\text{id}}^{\tau\nu})$, $\partial_\tau (u^\tau \pi^{\mu\nu})$ and $\partial_\tau (u^\tau \Pi)$, as an extra argument in the expression of (4.3)

$$\vec{f}_{(n)}(\vec{K}) = \vec{h} \left(t + c_n \Delta t, \vec{y}(t) + \Delta t \sum_m a_{nm} \vec{k}_{(m)}, \vec{k}_{(n)} \right), \quad (5.14)$$

and replace all the occurrences of the time derivatives in $\vec{f}_{(n)}$ by the solutions found above using the corresponding components of $\vec{k}_{(n)}$.

Such a procedure is usually not used as it makes the equations implicit (defined by themselves). This is not an issue when using an implicit Runge-Kutta method because an implicit solver is already required. However, in an explicit Runge-Kutta method, the quantities $\partial_\tau (\tau T_{\text{id}}^{\tau\nu})$, $\partial_\tau (u^\tau \pi^{\mu\nu})$ and $\partial_\tau (u^\tau \Pi)$ are not known by definition. No known exact solution to this problem seems to have been provided in the literature yet. Backward in time finite differences are used instead as an approximation by keeping track of older time steps. In our case, for consistency between the implicit and explicit cases, we will perform a similar approximation for our explicit case by using the values of $\partial_\tau (\tau T_{\text{id}}^{\tau\nu})$, $\partial_\tau (u^\tau \pi^{\mu\nu})$ and $\partial_\tau (u^\tau \Pi)$ obtained in the previous stage. Our explicit prescription suffers from the same downside as the prescription used in the literature: it involves outdated values which imply the dramatic property of making every explicit Runge-Kutta method first order in accuracy. This will be reflected in the results in Section 6 where the implicit method reaches the expected order in accuracy while the explicit one is stuck at order one.

5.3 Right hand side space derivatives

Approximating the space derivatives $\partial_i u^\mu$ in A , B and C does not have a dramatic implication compared to the approximation of the time derivatives in the explicit case as described in Section 5.2. Therefore, we first reconstruct the values of the fields at the boundaries of cells using the cubic interpolation (here in the x direction by considering that j indexes the x coordinate):

$$[u^\mu]_{j\pm 1/2} \approx -\frac{1}{16}[u^\mu]_{j\pm 1/2-3/2} + \frac{9}{16}[u^\mu]_{j\pm 1/2-1/2} + \frac{9}{16}[u^\mu]_{j\pm 1/2+1/2} - \frac{1}{16}[u^\mu]_{j\pm 1/2+3/2} \quad (5.15)$$

and then perform the finite difference to approximate

$$\partial_x u^\mu \approx \frac{[u^\mu]_{j+1/2} - [u^\mu]_{j-1/2}}{\Delta x}. \quad (5.16)$$

5.4 Spatial reconstruction which guaranty positivity and non-oscillation of the mass m^2

In relativistic hydrodynamics, a naive way of reconstructing the energy-momentum tensor would be to apply spatial reconstruction described in Sec. 3.2 with the simple choice $Q = T_{\text{id}}^{\tau\nu}$, where the boundary values $[T_{\text{id}}^{\tau\nu}]_{j\pm 1/2}^\pm$ are directly obtained from $[T_{\text{id}}^{\tau\nu}]_j$ by Eq. (3.13). However, to enhance the numerical stability of our implementation, we instead choose $Q = (m, T_{\text{id}}^{\tau i})$ for the spatial reconstruction, where m is the mass density of the fluid cell defined by

$$\begin{aligned} m^2 &= g_{\mu\nu} T_{\text{id}}^{\tau\mu} T_{\text{id}}^{\tau\nu} \\ &= g_{\tau\tau} (T_{\text{id}}^{\tau\tau})^2 - K, \end{aligned} \quad (5.17)$$

with $K = \sum_i (T_{\text{id}}^{\tau i})^2$.

More explicitly, we obtain $[T_{\text{id}}^{\tau\nu}]_{j\pm 1/2}^\pm$ as follows.

First, a set of conserved densities $[T_{\text{id}}^{\tau\nu}]_j$ obtained in each intermediate numerical step can fail in reproducing the physical requirement $m^2 \geq 0$. To avoid this situation, whenever a set of $[T_{\text{id}}^{\tau\nu}]_j$ is calculated, we replace $T_{\text{id}}^{\tau\tau}$ with

$$\max(T_{\text{id}}^{\tau\tau}, (1+s)K), \quad (5.18)$$

where a small regulator s is chosen to be 10^{-15} for the present study.

Second, the MUSCL reconstruction would guarantee the non-oscillatory property of the reconstructed variables Q . We confirmed this property when Eqs. (3.15) and (3.16) are applied to $Q = T_{\text{id}}^{\tau\nu}$. However, in this procedure, the non-oscillatory property of m^2 is not guaranteed, and thus the requirement $m^2 > 0$ can be broken. As a practical matter, we are faced with unphysical finite oscillations in ϵ and u^μ obtained from $T_{\text{id}}^{\tau\nu}$ of this reconstruction. To suppress these oscillations, we instead apply the reconstruction by the choice $Q = (m, T_{\text{id}}^{\tau i})$ as in the following procedure:

1. Starting from $[T_{\text{id}}^{\tau\nu}]_j$, construct the local mass density $[m]_j$ using Eq. (5.17).
2. Reconstruct $[m]_{j\pm 1/2}^\pm$ and $[T_{\text{id}}^{\tau i}]_{j\pm 1/2}^\pm$ from $[m]_j$ and $[T_{\text{id}}^{\tau i}]_j$, respectively, using Eq. (3.15).
3. Determine $[T_{\text{id}}^{\tau\tau}]_{j\pm 1/2}^\pm$ from $[m]_{j\pm 1/2}^\pm$ and $[T_{\text{id}}^{\tau i}]_{j\pm 1/2}^\pm$ by inverting Eq. (5.17).

We confirmed that this prescription significantly reduces the oscillating behavior of ϵ and u^μ in our numerical tests.

The KT operators in the other directions, KT_y and KT_η , are defined similarly. The right-hand side of Eq. (3.10) combined with the MUSCL reconstruction Eqs. (3.15)–(3.17) references two cells away from the current cell in every direction. This property plays an important role in optimizing the implicit solver discussed below.

In the Milne coordinates, we solve hydrodynamics equations in the form of Eqs. (2.45), (2.46) and (2.47) with $\tilde{T}_{\text{id}}^{\tau\nu}$ as the state variables. In this case, we use the same prescription as above with the following modifications: To obtain the energy density and velocity, Eqs. (5.19) and (5.22) are applied after transforming $T_{\text{id}}^{\tau\nu}$ to the local orthonormal frame, and the result is transformed back to the Milne basis. In the spatial reconstruction, we use $\tilde{m} = \tau m$ in place of the mass density m . The geometric source term Eq. (2.45) is added to the right-hand side of Eq. (3.10).

5.5 Boundary conditions and vacuum

Since the numerical evolution of hydrodynamics is performed on a finite grid, one has to introduce boundary conditions. In our space scheme the evaluation of $[\text{KT}_i]_j$ references two neighboring cells as previously shown in Fig. 3.1. Therefore, the variables on these cells must be specified when they are outside the numerical grid. In this study, we employ the same boundary condition as that used in Ref. [71]. In this boundary condition, when cells $j \pm 1$ are outside the numerical grid, we use the variables on cell j for those of cell $j \pm 1$. If cells $j \pm 2$ are outside the numerical grid, the variables on cells $j \pm 1$ are substituted there. Since this boundary condition roughly approximates an outflow without reflection, it is sufficient for an application to heavy-ion collisions when the grid is large enough such that the boundary is essentially kept in the vacuum.

In the numerical analysis in Sec. 6.6, we deal with the vacuum region with $\epsilon(\mathbf{x}_j) = 0$, where $\mathbf{x}_j$ represents the spatial coordinates of cell j . However, the treatment of $\epsilon(\mathbf{x}_j) = 0$ is problematic since it leads to an undefined velocity through Eq. (5.22). To avoid this situation, we force the energy density $\epsilon(\mathbf{x}_j)$ to be always at least $\epsilon_{\text{vac}} = 10^{-100} \text{ fm}^{-4}$ in Eq. (5.19).

5.6 Implementation details

5.6.1 Extracting ϵ and u^μ from $T^{\tau\nu}$

At each time step of the Runge-Kutta method, we need to calculate $T^{i\nu}$ as a function of $T^{\tau\nu}$ by using Eq. (2.9). To do so, we first obtain ϵ and u^μ from $T^{\tau\nu}$ and then calculate $T^{i\nu}$. To numerically determine ϵ and u^μ satisfying Eq. (2.9), it is convenient to solve the following equations [71] for the energy density ϵ (> 0) and the velocity v ($\in [0, 1]$) iteratively:

$$\epsilon = T^{\tau\tau} - vK, \quad v = \frac{K}{T^{\tau\tau} + P(\epsilon)}, \quad (5.19)$$

where $K = \sqrt{\sum_{i=1}^3 (T^{\tau i})^2}$. More specifically, we use Newton's method to solve

$$v = \frac{K}{T^{\tau\tau} + P(T^{\tau\tau} - vK)}, \quad (5.20)$$

*We remind the reader that we dropped the $\tilde{\bullet}$ notation, but we still use the variable change from Sec. 2.3, so the metric is flat, and the summation can be done directly

with $v = 0.5$ as an initial guess.

The solution to these equations only exist when the condition $T^{\tau\tau} \geq K$ is satisfied, which is ensured by the prescription described in Sec. 3.2.

We need to force the energy density to be strictly positive to avoid any numerical error such as division by zero. To achieve this, we introduce a lower limit $\epsilon_{\min} = 10^{-100}$ to this quantity. The choice of $\epsilon_{\min}$ is justified by the fact that it lies far bellow the precision of numerical floating point numbers relatively to values of order of 1 in the natural unit system. This leads to the final expression

$$\epsilon = \max\{T^{\tau\tau} - vK, \epsilon_{\min}\}. \quad (5.21)$$

This prescription is applied after the solution for v has been found. It is to be noted that we also force v to be smaller or equal to unity and greater or equal to zero in the Newton's method iterations to stay in the range of valid velocity.

Finally, u^μ is given by

$$u^i = \frac{T^{\tau i}}{\sqrt{(T^{\tau t} + P(\epsilon))(\epsilon + P(\epsilon))}}, \quad u^t = \sqrt{1 + (u^i)^2}. \quad (5.22)$$

5.6.2 Stability constraints on the viscosity

The second order Israel-Stewart equations do not guaranty the positivity of the effective pressure. As the bulk pressure Π and the shear tensor $\pi^{\mu\nu}$ from these equations affect the effective pressure, we need to ensure that when it becomes negative the effective enthalpy does not also become negative. To enforce the positivity of the effective enthalpy manually, we first verify that

$$\epsilon + P(\epsilon) + \Pi + \lambda_{\min} \geq 0, \quad (5.23)$$

where $\lambda_{\min} = \min_\mu \{\lambda^\mu\}$ is the smallest of the eigenvalues λ^μ of the shear tensor $\pi^{\mu\nu}$. If it is not the case, we rescale the bulk pressure and shear tensor as

$$\pi^{\mu\nu} \leftarrow \alpha \pi^{\mu\nu}, \quad (5.24)$$

$$\Pi \leftarrow \alpha \Pi, \quad (5.25)$$

where

$$\alpha = -\frac{\epsilon + P(\epsilon)}{\Pi + \lambda_{\min}}. \quad (5.26)$$

Although this is an ad-hoc prescription, it is only necessary at the vicinity of the vacuum which is at a temperature much lower than the freeze-out of the QGP.

Chapter 6

Numerical Benchmark

In this section we will focus on the correctness of the code and the confirmation of our previous assumptions. First, the correctness will be verified by comparing the numerical results to exact solutions. Second, we will show that the new implicit method is more efficient than its explicit counter-part. Especially, the expected accuracy order of 2 will be confirmed by the cost-error evolution as function of Δt in the ideal case. In the viscous case, we will see that only the implicit method can reach order 2 whereas the explicit method becomes first order.

6.1 Setup

We have tested various implicit and explicit RK methods with different stages and orders. They include the Gauss-Legendre family for the implicit methods, and Heun's and the Euler methods for the explicit. We choose Heun's method as the representative of explicit methods, as it is commonly used in hydrodynamic codes for HIC [55, 76]. This method has two stages ($S = 2$) and second order of accuracy ($p = 2$). Its Butcher table is given in the left column of Table 3.1. For the implicit method, we choose the one-stage Gauss-Legendre method [100, 101, 91], which has the same order of accuracy as Heun's method.

In Sec. 6.4 and Sec. 6.6, we compare the numerical results of these methods.

In the following, we refer to Heun's and one-stage Gauss-Legendre methods as the explicit and implicit methods, respectively.

Another conventional explicit method at the same order of accuracy as Heun's method is the midpoint method, whose Butcher table is given in the middle column of Table 3.1. In Appendix. A, we show that the performance of this method is almost the same as Heun's method in the Riemann problem.

6.2 Dimensions

In this study, we compare the implicit and explicit RK methods in (1+1)-, (2+1)-dimensional and (3+1)-dimensional numerical simulations. For the (1+1)-dimensional case, we deal with the Riemann problem discussed in Sec. 2.4.1 with finite and zero minimum energy density $\epsilon_{\min}$. We also perform (2+1)-dimensional simulations with the boost invariance for the Gubser flow and event-by-event initial conditions for heavy-ion collisions generated by `TRbETo` [37]. For the (3+1)-dimensional case, we only consider event-by-event initial conditions from `TRbETo`. In all cases, we consider a system of length 40 fm in each direction.

To investigate the dependence on the mesh size Δx , we perform numerical simulations with $\Delta x = 0.2$ and 0.4 fm where the spatial extent is divided into 200 and 100 cells, re-

spectively. The mesh size $\Delta x = 0.2$ fm is a typical value used in applications to collisions HIC [55, 96, 71]. The value $\Delta x = 0.4$ fm is employed for comparison.

The time step Δt is fixed per simulation. To study the time step dependence, we perform simulations with different Δt down to $\Delta t = 0.1 \times 2^{-5} \Delta x$, which is determined so that our results in Sec. 6.4 and Sec. 6.6 contain enough points to investigate the accuracy order p through Eq. (3.4). We also checked that our results behave consistently even at $\Delta t = 0.1 \times 2^{-6} \Delta x$ and $0.1 \times 2^{-7} \Delta x$ in several cases.

6.3 Measuring the accuracy and numerical cost

Before showing the results, we first introduce the quantities that we will use to perform the different comparison.

6.3.1 Error estimate

The results obtained by computational hydrodynamics have discretization errors. Numerical simulations are also affected by floating-point errors in general. In the present study, we evaluate these errors using two quantitative measures. First, we compare the energy density obtained by the numerical analysis, $\epsilon_{\text{num}}(\mathbf{x}_j)$, with that of the exact solution $\epsilon_{\text{exact}}(\mathbf{x}_j)$ in the continuum case using the measure

$$(6.1)$$

$$\Delta_{\text{exact}}(\mathbf{x}_j) = D(\epsilon_{\text{num}}(\mathbf{x}_j), \epsilon_{\text{exact}}(\mathbf{x}_j)), \quad (6.2)$$

with

$$D(\epsilon_1, \epsilon_2) = \frac{\epsilon_1 - \epsilon_2}{\max(\epsilon_1, \epsilon_2)}. \quad (6.3)$$

Here, we take the maximum of ϵ_1 and ϵ_2 for the denominator of Eq. (6.3), instead of either one of them, to avoid a situation where Eq. (6.2) becomes extremely large when one of them is vanishingly small. As for the second measure, we choose a numerical result obtained at small Δt , denoted by $\epsilon_{\text{ref}}(\mathbf{x}_j)$, as a reference. We then compare $\epsilon_{\text{num}}(\mathbf{x}_j)$ with the reference as

$$\Delta_{\text{ref}}(\mathbf{x}_j) = D(\epsilon_{\text{num}}(\mathbf{x}_j), \epsilon_{\text{ref}}(\mathbf{x}_j)). \quad (6.4)$$

For the reference $\epsilon_{\text{ref}}(\mathbf{x}_j)$, we use the result by the explicit method with the smallest $\Delta t = 0.1 \times 2^{-5} \Delta x$.

To compare numerical errors of different time discretization methods, $\Delta_{\text{ref}}(\mathbf{x}_j)$ is more convenient than $\Delta_{\text{exact}}(\mathbf{x}_j)$. This is because a finite space-discretization error remains in $\Delta_{\text{exact}}(\mathbf{x}_j)$ even in the $\Delta t \rightarrow 0$ limit, while one can discuss genuinely the time-discretization error using $\Delta_{\text{ref}}(\mathbf{x}_j)$.

In Sec. 6.4 and Sec. 6.6, we inspect the magnitude of numerical error using the maximum and spatial average of $|\Delta_{\text{ref}}(\mathbf{x}_j)|$. However, we found that $|\Delta_{\text{ref}}(\mathbf{x}_j)|$ tends to be close to unity near the vacuum region where $\epsilon_{\text{exact}}(\mathbf{x}_j)$ is vanishingly small. In this region $\epsilon_{\text{num}}(\mathbf{x}_j)$ and $\epsilon_{\text{ref}}(\mathbf{x}_j)$ have non-vanishing values due to the numerical diffusion, i.e., the unphysical propagation from the matter region, which is inevitable in computational hydrodynamics. Nevertheless, $\epsilon_{\text{num}}(\mathbf{x}_j)$ is still negligibly small and the large $|\Delta_{\text{ref}}(\mathbf{x}_j)|$ should not be taken into account as an essential error.

To avoid this apparent error in the vacuum region, in our error analysis we exclude the cells where $\epsilon_{\text{num}}(\mathbf{x}_j)$ is smaller than a threshold value ϵ_{thr} , where ϵ_{thr} is determined so that the sum of the energy in the excluded cells is less than 10^{-6} times the total energy. We also exclude cells within 1 fm from the boundaries to remove possible boundary effects. We then evaluate the maximum and the average of $|\Delta_{\text{ref}}(\mathbf{x}_j)|$ in the remaining cells.

6.3.2 Estimate of computational cost

To compare the performance of different numerical methods, we also need a quantitative measure of the computational cost. However, the actual computational cost depends on the implementation, and its appropriate comparison is difficult in general. In this study, we focus on the fact that the most computationally demanding part of our analysis is the evaluation of the local KT operator $[h_{\text{KT}}]_j$ in Eq. (3.10). This means that the number of evaluations N_{KT} serves as an approximate measure of the computational cost. We thus use the number of evaluations per cell,

$$n_{\text{KT}} = \frac{N_{\text{KT}}}{N_{\text{cell}}}, \quad (6.5)$$

for the measure of the computational cost, where N_{cell} stands for the number of total cells.

For our explicit method with $S = 2$, $[h_{\text{KT}}]_j$ is evaluated twice per cell in a single time step. In the implicit method, the cost of a single time step depends on the number of iterations of the fixed-point solver, where one local iteration needs one evaluation of $[h_{\text{KT}}]_j$ with $S = 1$. Moreover, due to the local optimization, n_{KT} is not necessarily an integer. We have checked that n_{KT} indeed reflects the real computational times well with appropriate optimizations.

6.4 Exact ideal solution Benchmark

6.4.1 1+1d Riemann problem with shock propagation

Now, let us discuss the numerical results. We begin in this subsection to deal with the Riemann problem, whose exact solution is presented in Sec. 2.4.1. This is a (1+1)-dimensional problem in flat spacetime with initial conditions given by Eqs. (2.52) and (2.53). We numerically solve the hydrodynamic evolution from the initial time $t_{\text{init}} = 0$ to $t_{\text{end}} = 15$ fm. We set $c_s^2 = 1/3$ and $\epsilon_{\text{max}} = 10 \text{ fm}^{-4}$ and perform our analysis for the two values of $\epsilon_{\text{min}} = 1$ and 0 fm^{-4} . The latter is important for the application to heavy-ion collisions where the matter expands into the vacuum, which typically causes numerical problems.

Numerical solutions In the upper panel of Fig. 6.1, we first show the numerical result of $\epsilon(\mathbf{x}_j)$ at $t = t_{\text{end}}$ for $\epsilon_{\text{min}} = 1 \text{ fm}^{-4}$ as a function of x/t . The red (blue) curves show the results obtained by the explicit (implicit) method with $\Delta t = 0.1 \times 2^{-5} \Delta x$ while the dotted and dashed lines represent the results for $\Delta x = 0.4$ and 0.2 fm, respectively. The black solid line shows the analytic solution $\epsilon_{\text{exact}}(\mathbf{x}_j)$. In the lower panel, we also show $\Delta_{\text{exact}}(\mathbf{x}_j)$.

One sees that the results of the explicit and implicit methods degenerate well at this Δt . This suggests that the error from the time discretization is well suppressed in both methods. By comparing the results of $\Delta x = 0.4$ and 0.2 fm, one also finds that the numerical result approaches the analytic solution as Δx becomes smaller. Nevertheless, a clear deviation is still observed at the smaller Δx . This is significant around the rarefaction front and shock at $x/t = -c_s$ and v_{shock} , respectively, which are shown in the enlarged subpanels of the upper panel. In particular, a large deviation $\Delta_{\text{exact}} \simeq 0.5$ survives even for the small Δx at $x/t = v_{\text{shock}}$, which is a consequence of the discontinuity of $\epsilon_{\text{exact}}(\mathbf{x}_j)$ at this point.

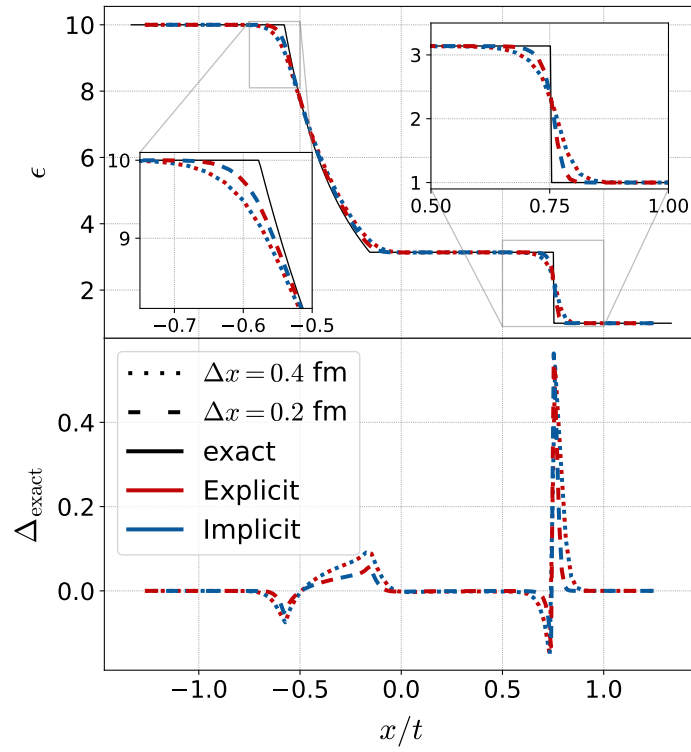


Figure 6.1: Energy density $\epsilon(\mathbf{x}_j)$ (upper panel) and its error compared with the exact solution $\Delta_{\text{exact}}(\mathbf{x}_j)$ (lower panel) for the Riemann problem at $t_{\text{end}} = 15$ fm with $\epsilon_{\text{min}} = 1 \text{ fm}^{-4}$ and $\Delta t = 0.1 \times 2^{-5} \Delta x$.

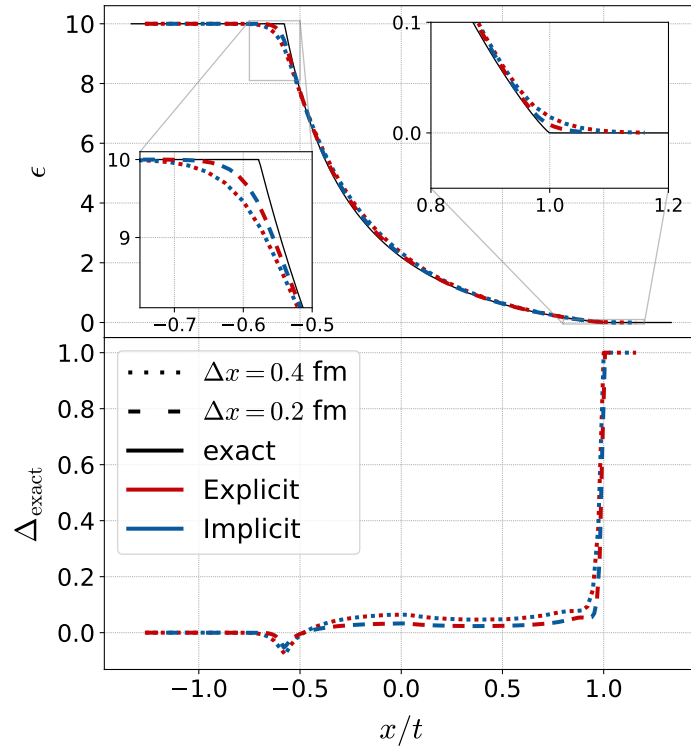


Figure 6.2: Same as Fig. 6.1 but $\epsilon_{\text{min}} = 0$.

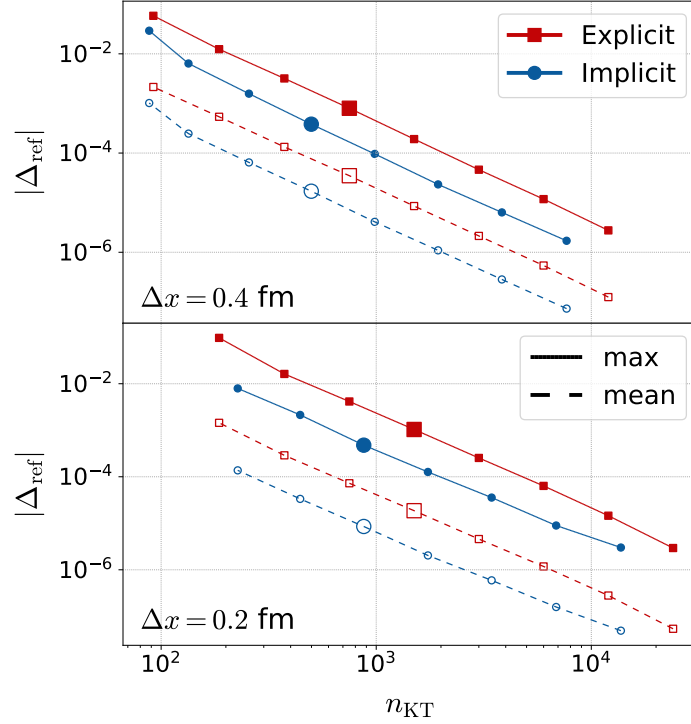


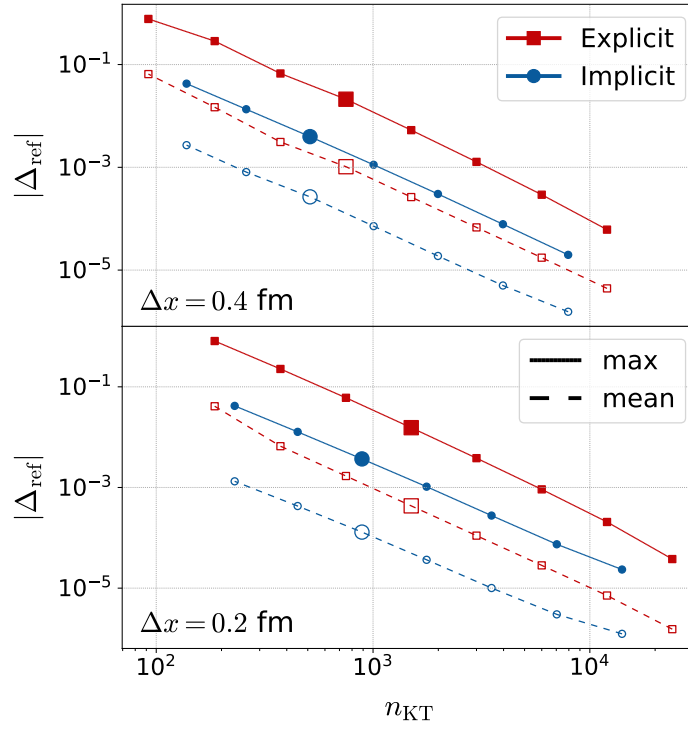
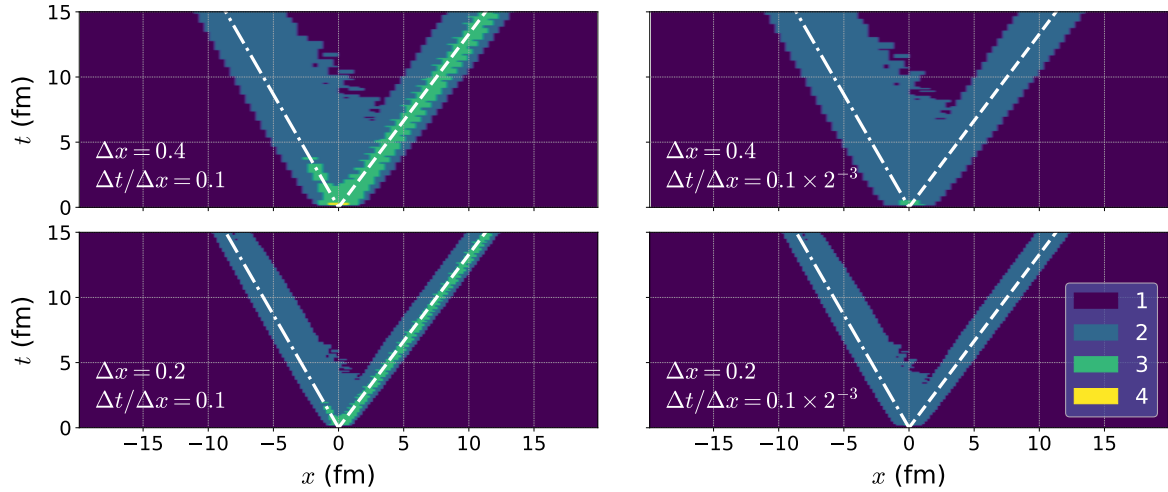
Figure 6.3: Maximum and average of the numerical error $|\Delta_{\text{ref}}|$ defined in Eq. (6.4) as functions of the computational cost n_{KT} for the Riemann problem with $\epsilon_{\text{min}} = 1 \text{ fm}^{-4}$. The maximum and average are shown by the solid and dashed lines, and the explicit and implicit results are shown by red square and blue circle symbols, respectively. The upper and lower panels show the results for $\Delta x = 0.4$ and 0.2 fm , respectively. The large points show the results at $\Delta t/\Delta x = 0.1$, while the other small points are obtained by varying Δt by a factor of 2. Note that the absolute magnitudes of errors with different Δx are not comparable because the reference $\epsilon_{\text{ref}}(\mathbf{x}_j)$ depends on Δx .

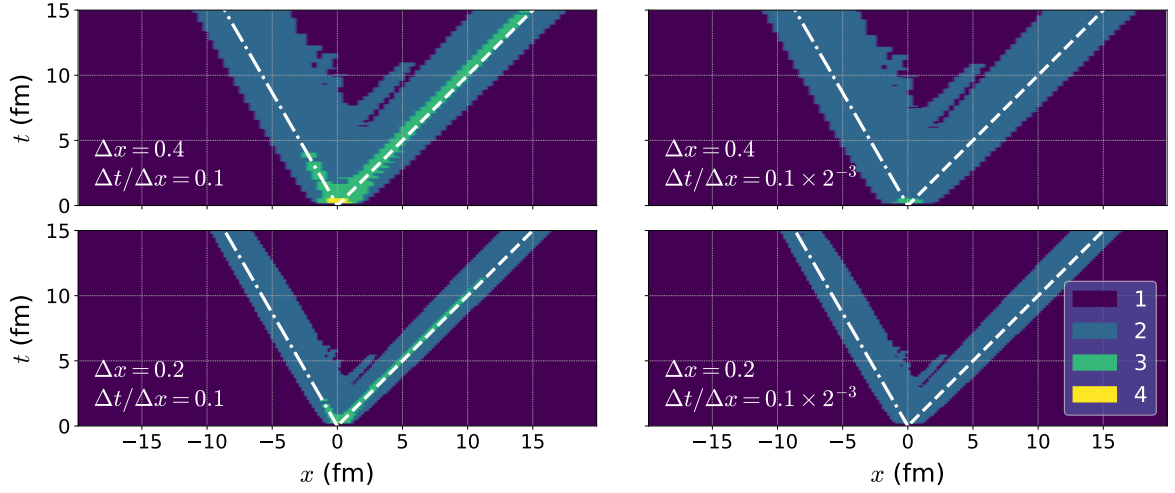
In Fig. 6.2, the numerical result for $\epsilon_{\text{min}} = 0$ is shown in the same manner as Fig. 6.1. The figure shows that $\Delta_{\text{exact}}(\mathbf{x}_j)$ becomes almost unity in the vacuum region $x/t > 1$. As explained in Sec. 6.3.1, this result comes from vanishing $\epsilon_{\text{exact}}(\mathbf{x}_j)$ in the vacuum region.

In the following analysis, we use $\Delta_{\text{ref}}(\mathbf{x}_j)$ defined in Eq. (6.4) in place of $\Delta_{\text{exact}}(\mathbf{x}_j)$ to analyze the time-discretization error exclusively.

Error and computational cost Next, let us compare the numerical error and the cost of the explicit and implicit methods. In Figs. 6.3 and 6.4, we plot the maximum and average of the error defined by Eq. (6.4) at $t = t_{\text{end}}$ for $\epsilon_{\text{min}} = 1$ and 0 fm^{-4} , respectively, for $\Delta x = 0.4 \text{ fm}$ (upper plot) and 0.2 fm (lower plot). The horizontal axis is the computational cost n_{KT} in Eq. (6.5). The red squares and blue circles show the explicit and implicit results, respectively. The large point in each line shows the result for $\Delta t/\Delta x = 0.1$. The distance between the points along one line is obtained from varying Δt by a factor of two. In the explicit method, the cost is exactly $n_{\text{KT}} = S(t_{\text{end}} - t_{\text{init}})/\Delta t = 30 \text{ fm}/\Delta t$.

The numerical simulation fails when the time step Δt is too large, so only successful results are shown in the figures. The failure of the implicit method is not a divergence of the numerical error such as in the explicit case. Instead, it is a limitation of the fixed-point method, which in its simplest form, has very strict convergence conditions. The procedure

Figure 6.4: Same result as Fig. 6.3, but $\epsilon_{\min} = 0$.Figure 6.5: Number of iterations required for the fixed-point solver at each space-time point for the Riemann problem with $\epsilon_{\min} = 1 \text{ fm}^{-4}$. The dashed and dash-dotted lines show the positions of shock and rarefaction front, respectively. Four results for $(\Delta t/\Delta x, \Delta x) = (0.1, 0.4), (0.1 \times 10^{-3}, 0.4), (0.1, 0.2)$ and $(0.1 \times 10^{-3}, 0.2)$ are shown in the top-left, top-right, bottom-left and bottom-right panels.

Figure 6.6: Same result as Fig. 6.5, but $\epsilon_{\min} = 0$.

discussed in Sec. 4.3.1 could greatly enhance the stability, but it was not used for these ideal benchmarks as the plain fixed-point is stable enough for the practical $\Delta t < 0.3\Delta x$.

We observe several notable features in Figs. 6.3 and 6.4. First, all the results are approximately proportional to n_{KT}^{-2} when $\Delta_{\text{ref}} < 10^{-1}$. This behavior is in accordance with the order of accuracy $p = 2$ of our explicit and implicit methods. Second, the computational cost of the implicit method is significantly smaller than the explicit method for a fixed $|\Delta_{\text{ref}}|$ in both maximum and average numerical errors; The former is about 2–3 times as efficient as the latter in both $\epsilon_{\min} = 1 \text{ fm}^{-4}$ and 0 fm^{-4} cases*. This means that with the same accuracy, the implicit method is faster than the explicit one. This is a remarkable outcome of the present study since it is widely considered that implicit methods are slower than explicit ones. The figures also show that this advantage of the implicit method is obtained through the reduction both in the numerical error and computational cost for a given Δt .

To better understand the origin of the efficiency of the implicit method, in Fig. 6.5 we show the number of iterations performed by the fixed-point solver in the implicit method on the t – x plane. The number of iterations is represented by different colors for the numerical simulations with $\Delta t/\Delta x = 0.1$ and $0.1 \times 10^{-3} \text{ fm}$ (left and right), and $\Delta x = 0.4$ and 0.2 fm (top and bottom). The dashed and dash-dotted lines show the locations of the shock and the rarefaction front at which $\epsilon_{\text{exact}}(x)$ and $\partial\epsilon_{\text{exact}}(x)/\partial x$, respectively, have a discontinuity. The figure shows that for almost all cells one iteration is enough to achieve the convergence. This result is reasonable since the hydrodynamic variables would change slowly when they are smooth. We use the solution of the last time step for the initial guess in the fixed-point solver, and it is already a good estimate for the next time step. Moreover, more than one iteration is required only around the shock and rarefaction, where the number of iterations is mostly two except for a few cells with three or four iterations. As a result, the average number of the $[h_{\text{KT}}]_j$ evaluations for an update is smaller than two in the implicit method. This makes the implicit method advantageous compared with the explicit one where two evaluations are required for every update.

In both Figs. 6.5 and 6.6 where the speed of the shock front is different, one finds that the regions of two iterations have similar outlines in the left and right panels where the time

*One might want to compare the values of $|\Delta_{\text{ref}}|$ at the same Δt for different Δx . However, $|\Delta_{\text{ref}}|$ represents the difference from $\epsilon_{\text{ref}}(x)$, not $\epsilon_{\text{exact}}(x)$, so it does not contain the spatial discretization error. Since $\epsilon_{\text{ref}}(x)$ is different for each Δx , their values at different Δx are not comparable.

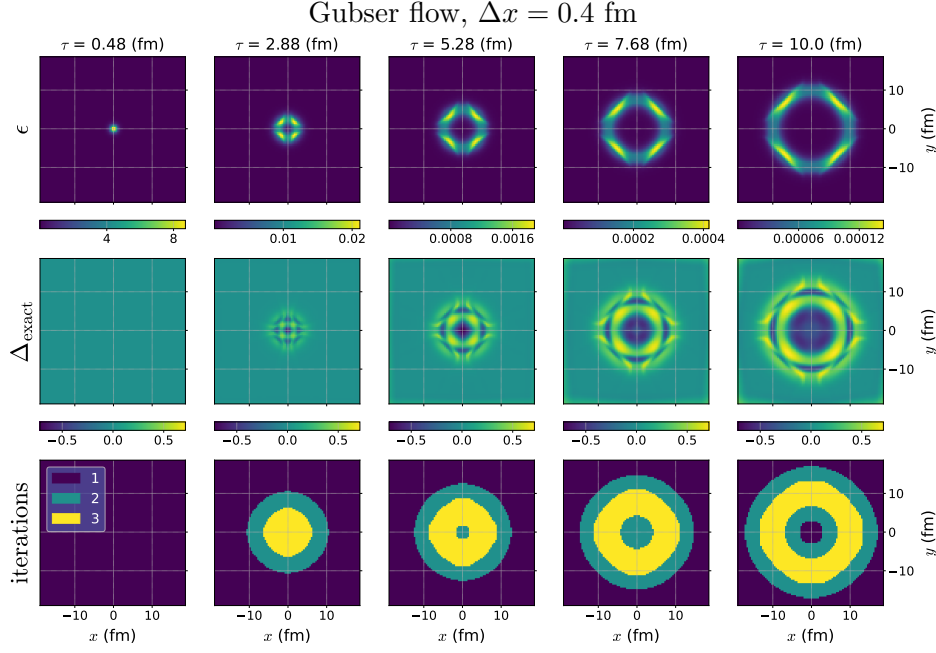


Figure 6.7: Energy density $\epsilon(\mathbf{x}_j)$ (upper panels), error $\Delta_{\text{exact}}(\mathbf{x}_j)$ (middle panels) and the number of iterations of the fixed-point solver in each cell (lower panels) for the Gubser flow at several proper times τ for $\Delta t = 0.1 \times 2^{-5} \Delta x$ and $\Delta x = 0.4$ fm.

step changes. This behavior would be attributed to fine structures that arise due to the space discretization.

As already discussed, in Figs. 6.3 and 6.4 we only show the successful results of the explicit method. We found that the explicit method fails for $\Delta t / \Delta x \geq 0.8$. This result is consistent with the Courant-Friedrichs-Lewy (CFL) condition [102, 103] for stability, $\Delta t / \Delta x < C_{\text{CFL}}$ where $C_{\text{CFL}} = \mathcal{O}(1)$ depends on the detailed scheme. In contrast, the implicit method fails when $\Delta t > 0.3 \Delta x$. However, for completeness, we also tried to use the stabilizing procedure from Sec. 4.3.1. In that case, the implicit method is perfectly stable even when $\Delta t > \Delta x$.

6.5 Gubser flow

Next, we perform the same comparison for the Gubser flow as discussed in Sec. 2.4.2. We solve transverse (2+1)-dimensional hydrodynamics assuming the boost invariance. In this case, the solution contains neither discontinuity nor vacuum region. We employ $\epsilon_0 = 1 \text{ fm}^{-4}$ and $q = 1$ for the parameters in Eq. (2.62) and set $\tau_{\text{init}} = 1$ fm for the initial proper time. We then solve the hydrodynamic equations numerically until the end time $\tau_{\text{end}} = 10$ fm.

In the top panels of Figs. 6.7 and 6.8, we first show the energy density at several values of τ for $\Delta x = 0.4$ and 0.2 fm, respectively, obtained by the implicit method with $\Delta t = 0.1 \times 2^{-5} \Delta x$ fm. Here we do not show the result of the explicit method since its behavior is almost the same as anticipated from the discussion in Sec. 6.4.1. The panels show that the numerical solution has non-circular behaviors, while the exact solution, Eq. (2.62), has azimuthal symmetry. Correspondingly, $\Delta_{\text{exact}}(\mathbf{x}_j)$ shown in the middle panels has oscillating behaviors. These errors are attributed to the space discretization, as we have checked that their structure hardly changes for small Δt . By comparing Figs. 6.7 and 6.8, one also finds that the deviation from the exact solution is suppressed as Δx becomes smaller.

Shown in the lower panels of Figs. 6.7 and 6.8 are the number of iterations in the fixed-

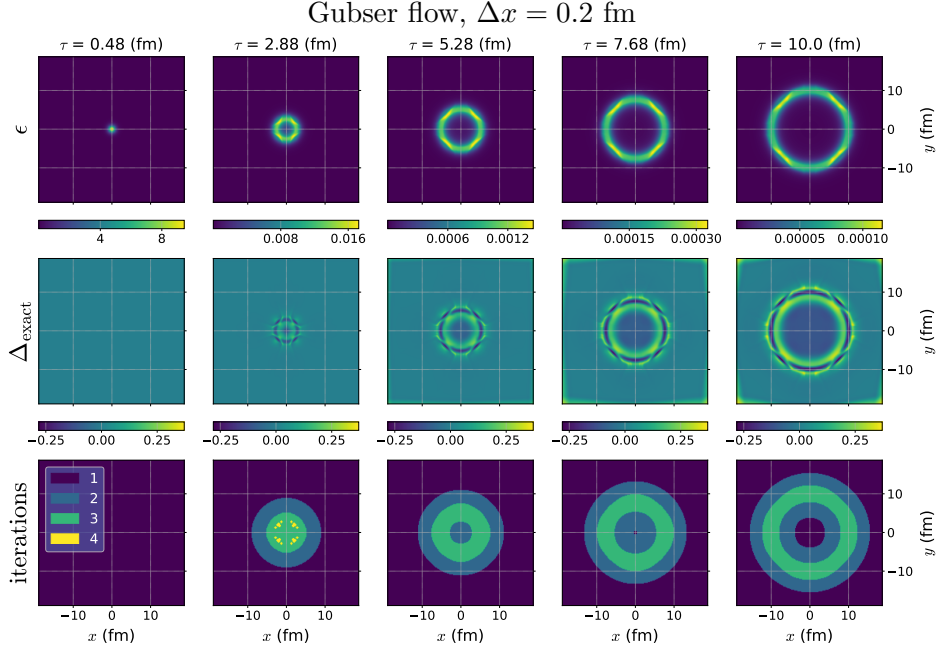


Figure 6.8: Same results as Fig. 6.7, but with $\Delta x = 0.2$ fm.

point solver for each τ . These panels show that more than two iterations are needed around the region where $\epsilon(\mathbf{x}_j)$ changes rapidly, but only one iteration is enough for the other wide area. This result is consistent with the one in Sec. 6.4.1.

In Fig. 6.9, we compare the maximum and the average of $|\Delta_{\text{ref}}(\mathbf{x}_j)|$ in the explicit and implicit methods as functions of n_{KT} . The meanings of the lines and symbols are the same as before. The figure shows that the implicit method is advantageous again in reducing the computational cost when $\Delta t/\Delta x < 0.2$. When Δt becomes larger, the iterative solver fails without the stabilizing procedure from Sec. 4.3.1. Despite A -stability of the implicit method itself, the fixed-point solver is not ensured to converge for large Δt .

6.6 Physical context benchmark

6.6.1 Event-by-event initial condition for heavy-ion collisions

Finally, we apply the same analysis to realistic cases of the relativistic heavy-ion collisions by generating initial conditions using `TRENTo` [37]. For the details about the parameters used for `TRENTo` and the choices of transport coefficients, see the introduction of Cha 7. The only difference is that for this chapter we use a fixed impact parameter of $b = 10.3$ fm for easier comparison of explicit and implicit methods. For the (2+1)-dimensional case, the initial time is chosen to be $\tau_{\text{init}} = 0.48$ fm. In both (2+1)- and (3+1)-dimensional cases, we solve the hydrodynamic evolution until a fixed $\tau_{\text{end}} = 10$ fm, in order to be able to compare the explicit and implicit method at the exact same final time.

In the upper panels of Figs. 6.10 and 6.11, we show the transverse profile of the pressure $p(\epsilon(\mathbf{x}_j))$ for one event in the x - y plane obtained with the finest $\Delta t = 0.1 \times 2^{-5} \Delta x$ in the implicit method for $\Delta x = 0.4$ and 0.2 fm, respectively. The cells satisfying $\epsilon(\mathbf{x}_j) < \epsilon_{\text{thr}}$ are shown by the white color. As discussed in Sec. 6.3.1, these cells are discarded in the error analysis. In the lower panels, the corresponding number of iterations in the fixed-point solver is shown. These results again show that the average number of iterations is smaller than two,

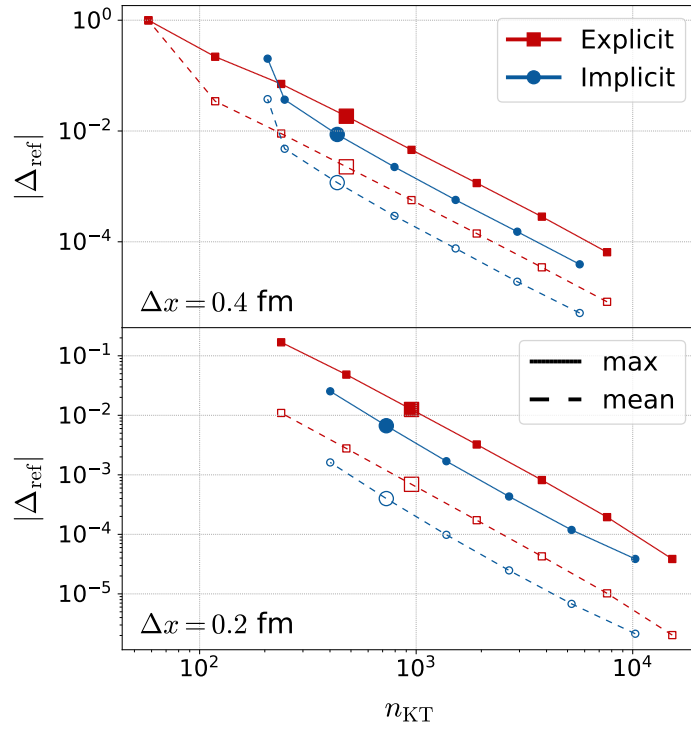


Figure 6.9: Same as Fig. 6.3, but for the analysis of the Gubser flow as in Figs. 6.7 and 6.8.

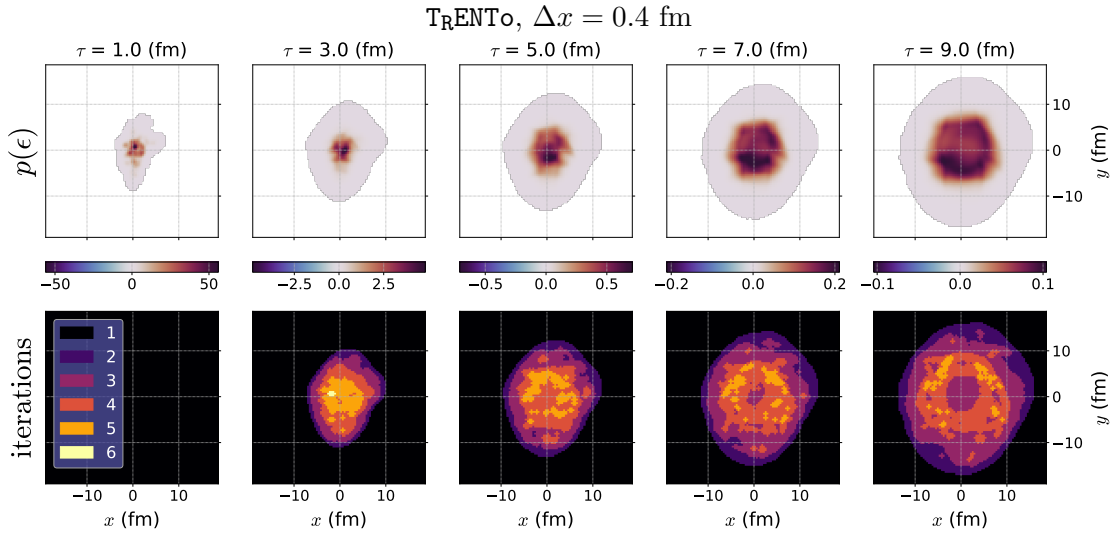
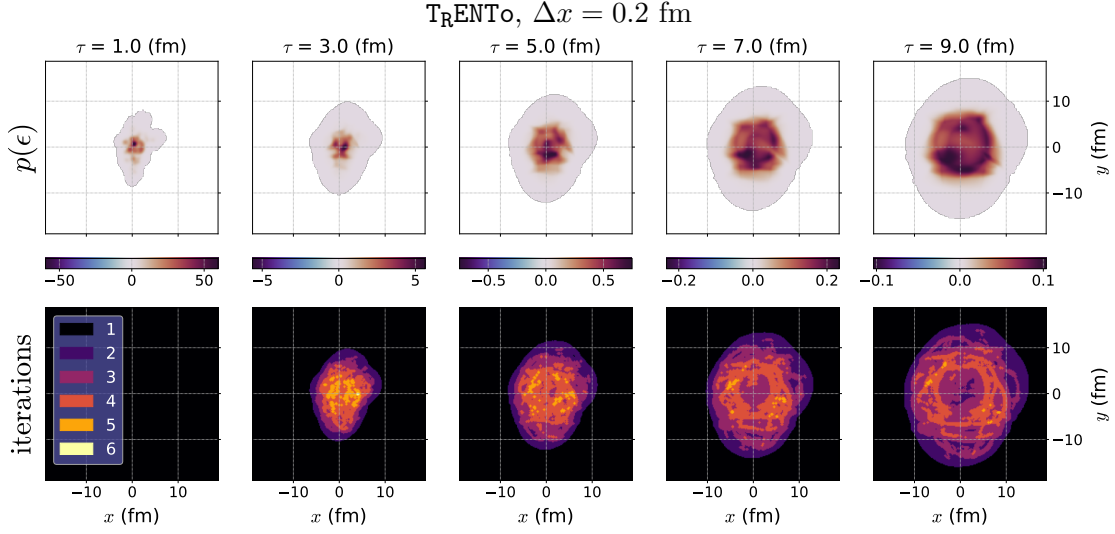


Figure 6.10: Pressure $p(\epsilon(\mathbf{x}_j))$ from the implicit method (upper panels), and the number of iterations of the fixed-point solver in each cell (bottom panels) for a TrENTo initial condition (lead-lead collision with $b = 10.3$ fm and $\sqrt{s_{\text{NN}}} = 5.02$ TeV) at several values of proper time τ for $\Delta t = 0.1 \times 10^{-5} \Delta x$ and $\Delta x = 0.4$ fm.

Figure 6.11: Same as Fig. 6.10, but for $\Delta x = 0.2$ fm.

which is a reasonable result in light of the arguments in the previous subsections. The main difference between the $\Delta x = 0.4$ fm and $\Delta x = 0.2$ fm cases is the sharpness of the result. It is clearer in the $\Delta x = 0.2$ fm case that even at $\tau = 0.9$ fm there are still sharp maxima after the expansion. This is to be expected of an ideal fluid as without dissipation, the energy will entirely flow from maxima to minima which will create an oscillating cycle between maxima and minima. The large number of iterations are also more localized in this case.

By comparing the number of iterations with the pressure profile, it is apparent that the locations where the number of iterations are high mostly correspond to where the pressure is changing the most, or said differently, where the pressure gradient is large. As hydrodynamics reacts to the pressure gradient, this means that the number of iterations follow the largest changes of the system, and thus they are connected to the stiffness of the system.

In Fig. 6.12, we plot the maximum and the average of $|\Delta_{\text{ref}}|$ as functions of n_{KT} for ten initial conditions generated randomly by TrENTO. The figure shows that the implicit method keep the same trend with a smaller computational cost than the explicit method for a given error. Furthermore, all the lines and points are close for the explicit and implicit respectively, which means that even with random initial conditions the cost and accuracy of the hydrodynamics remains the same.

6.6.2 Viscous IS equations Benchmark

Now that we have verified the accuracy and efficiency of our new implicit method in the ideal case where no complications arise, we can verify them in the viscous case where time derivatives appear in the flux. As discussed in Sec. 5.2, it is possible to express the right-hand-side time derivatives with the time derivatives of the dynamical variables. However, this simple solution is only supposed to work for the implicit method where the time derivatives of the dynamical variables are actually solved during the iterative process. Fig. (6.13) shows that the accuracy order $p = 2$ is conserved for the implicit method in blue as expected. The red points of the explicit method only reach an accuracy order of $p = 1$. Contrary to the implicit method, we must use outdated values to express the needed time derivatives which breaks the Runge-Kutta properties and block the order to 1 in the explicit method. We have verified that using the usual backward in time finite difference reproduces the same

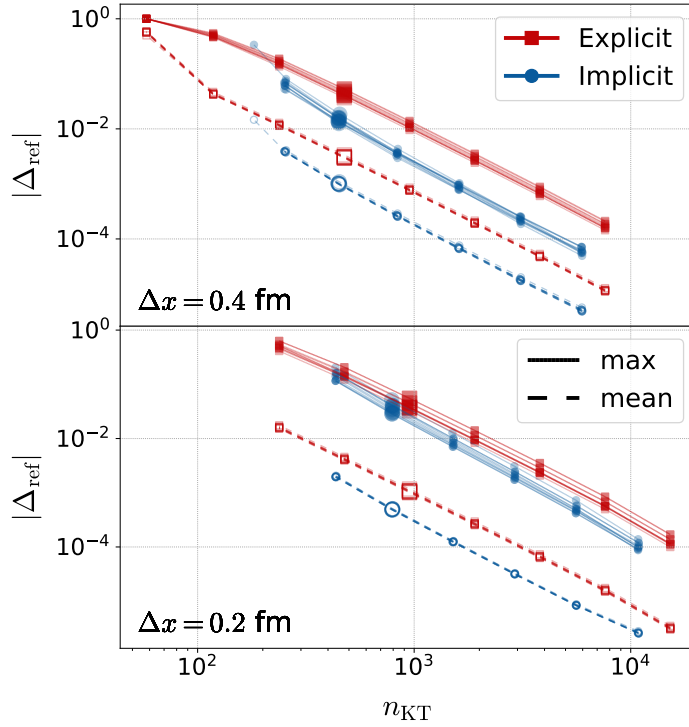


Figure 6.12: Same as Fig. 6.3, but for 10 initial conditions generated by T_RENTO.

slope as our explicit method[†]. This corresponds to first order derivatives which ends up being the bottleneck in accuracy of the explicit method. These behaviors remain the same in (3+1)-dimensions as shown on Fig. (6.14), where the lower panel for $\Delta x = 0.2$ fm stops the simulation early at $\tau_{\text{end}} = 5$ fm with a reference time discretization of only $\Delta t = 0.1 \times 2^{-2} \Delta x$. These limitations originate from the long simulation time needed for such (3+1)-dimensions simulations with fine lattice spacing Δx and time discretization Δt . Reaching later τ_{end} and lower Δt should be possible, although it would require more than a week of simulation on current hardware, which we considered excessive.

On Figs. 6.15, 6.16 are shown from top to bottom the pressure $p(\epsilon)$, the bulk pressure Π , the renormalization factor from 5.6.2 used to force the effective enthalpy to remain positive and the number of iterations as function of the proper time τ . This time, the finer lattice spacing $\Delta x = 0.2$ fm does not provide a particularly finer view of the pressure at late time as the system is mostly smoothed out by the viscosity and no sharp structures remain. However, at early time the finer lattice spacing still permits to better see sharp structures in the pressure

[†]Our first approach was to use the usual backward in time finite difference for both explicit and implicit methods. As theoretically expected, first order results were obtained in both cases. However, we knew that we could express the time derivatives analytically quite easily thanks to the solver involved in implicit Runge-Kutta methods. We thus switched to this new method and confirmed that the correct order was obtained for the implicit case. We noticed that we could use the same analytical solutions from Sec. 5.2 that depend on forward in time values for the explicit method by using backward in time quantities instead. This is only an approximation of the actual implicit analytical solutions as it neglects the changes in time derivatives from one step to the next one. This is by definition a first order approximation. We thus confirmed that it still produces first order results. We then discovered that most of the numerical constraints on the viscosity could be removed thanks to these more stable analytical solutions, so we kept using them even for the explicit method. Furthermore, we also checked that this behavior did not depend on the order of the used Runge-Kutta methods: the explicit ones always produce first order results while the implicit ones produce the expected order even at order 3 and 4.

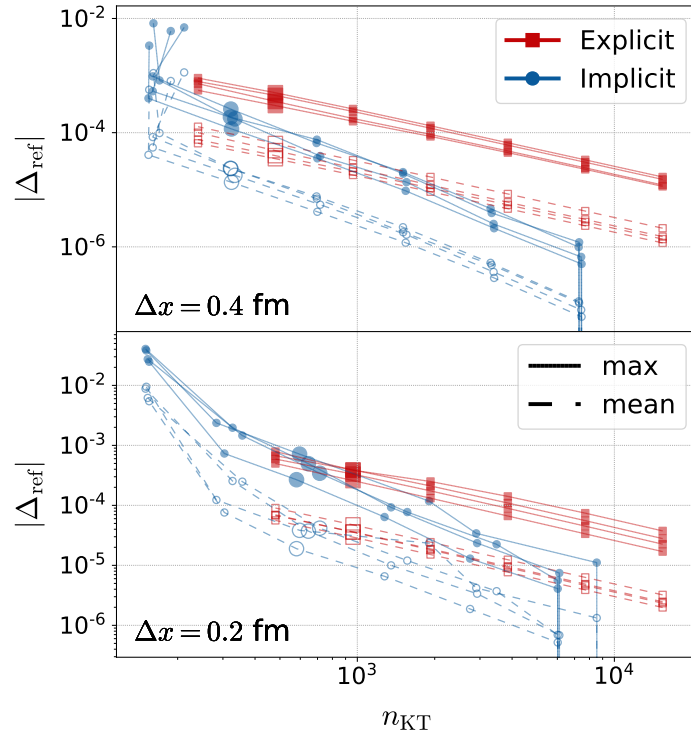


Figure 6.13: Same as Fig. 6.3, but for 4 initial conditions generated by T_{RENTo} , and with viscosity included.

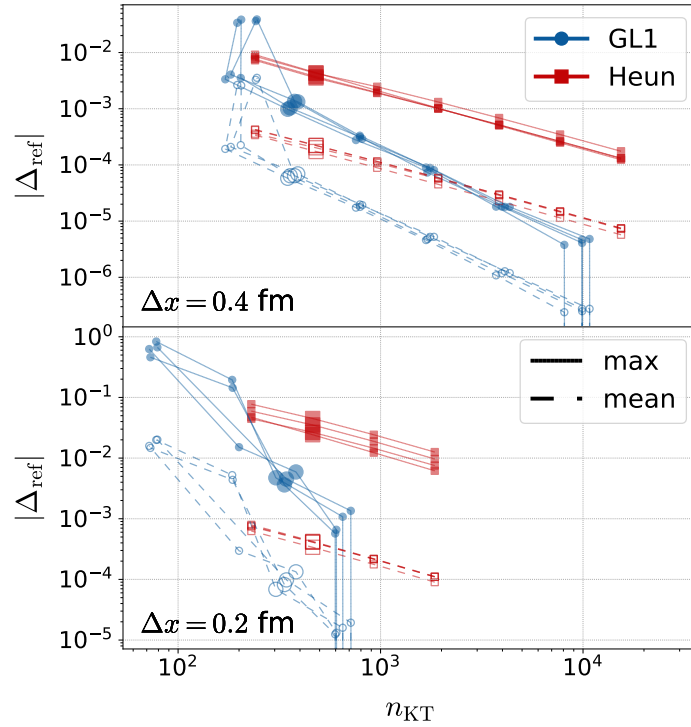


Figure 6.14: Same as Fig. 6.13, but in (3+1)-dimensions. The lower panel for $\Delta x = 0.2$ stops the simulation early at $\tau_{\text{end}} = 5$ fm with a reference time discretization of only $\Delta t = 0.1\Delta x/2^{-2}$.

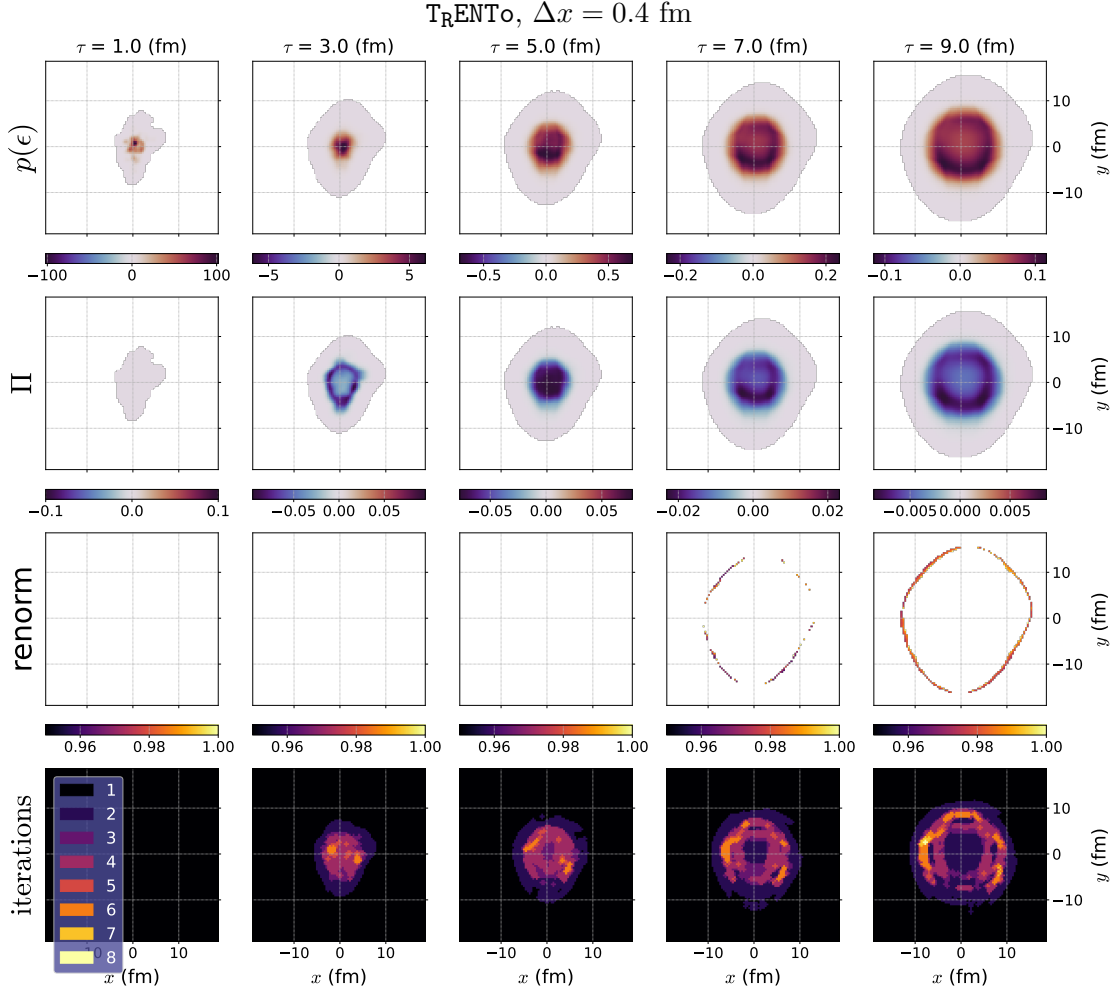


Figure 6.15: Same top and bottom panels as Fig. 6.10, but with viscosity. In addition, the bulk pressure Π (central-up panel) and the renormalization factor to stabilize the viscous equations (central-down panel) are also shown.

and bulk pressure. Such structures can also be seen in the shear stress tensor, although we did not show it as it would involve showing 5 independent quantities. Once again, we can see that the maximum number of iterations are located where the largest changes happen. In particular, at early time $\tau = 3 \text{ fm}$, the large number of iterations spans from the center where the pressure is large up to the further boundary of the bulk pressure.

We can see in the third line of Figs. 6.15, 6.16 that the renormalization factor α takes values different from unity only at late time $\tau \geq 5 \text{ fm}$, and at the vicinity of the vacuum where all the quantities are vanishing. As in this location the temperature is far under the hydrodynamics freeze-out temperature, this forced renormalization of the viscosity should not affect the particle production. It is to be noted that sometimes the number of iteration can greatly increase such as in Fig. 6.16 at $\tau = 9 \text{ fm}$. However, such increase remains finite in space and time, so it does not impact the performance.

On Fig 6.17 is shown a 2-dimensional slice at $\eta = 0$ of a (3+1)-dimensional evolution. The interpretation is essentially the same as the (2+1)-dimensional case, although the patterns in the pressure $p(\epsilon)$ and bulk pressure Π are slightly more complex due to the interactions with the third spatial dimension.

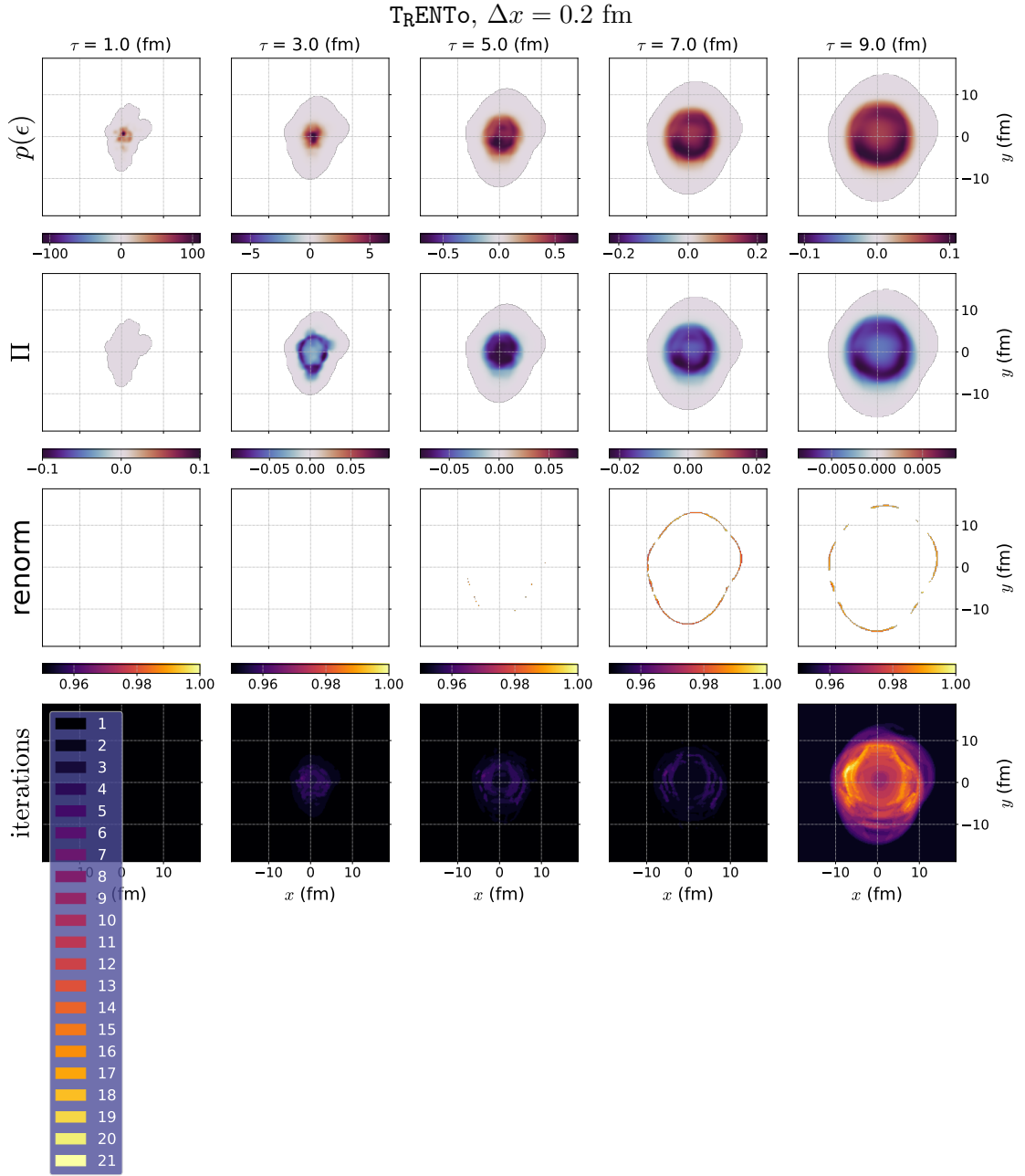


Figure 6.16: Same as Fig. 6.11, but with viscosity.

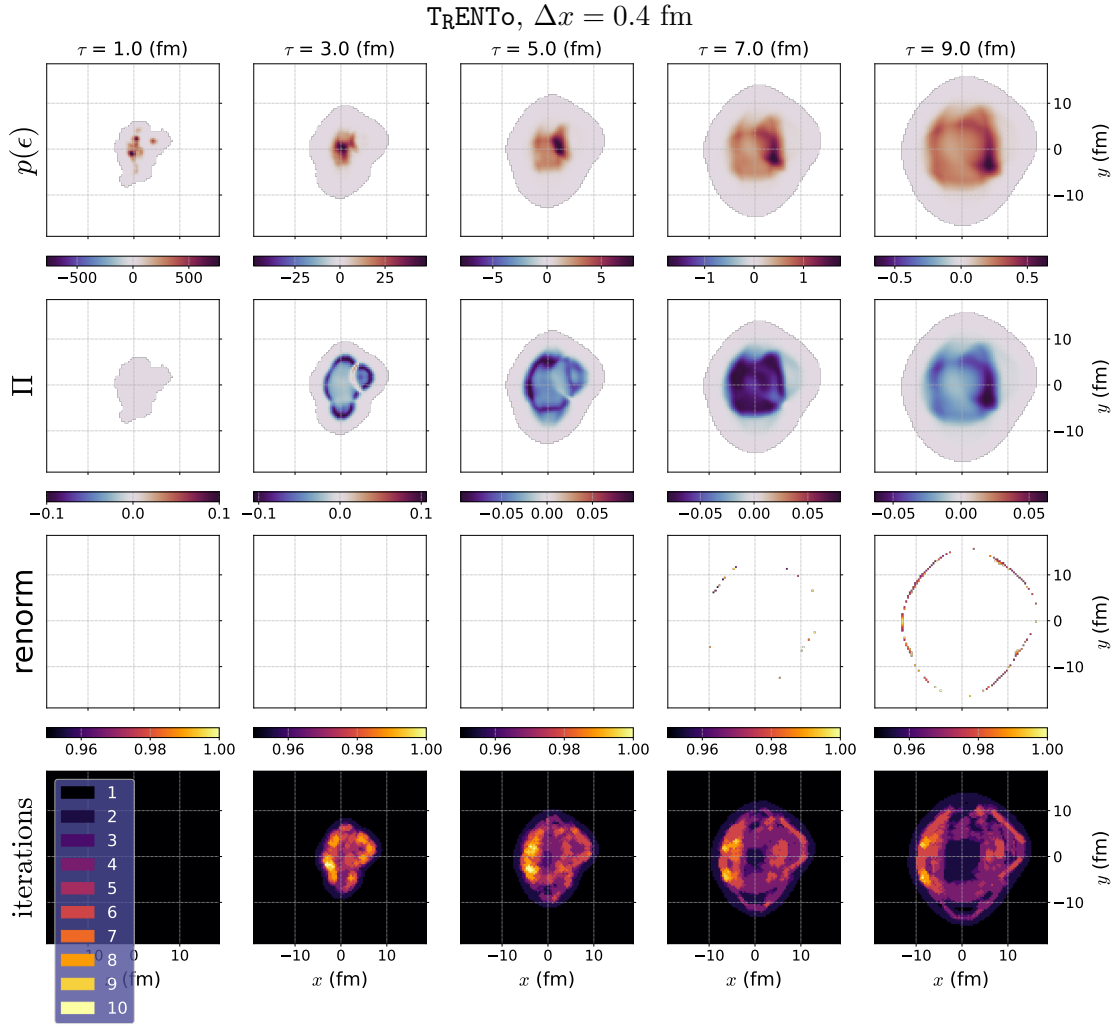


Figure 6.17: Same as Fig. 6.10, but in (3+1)-dimensions.

6.7 Large Δt regime

As shown in the different error-cost figures for the viscous cases using the stabilizing procedure, the implicit method transition to a higher cost at large Δt . This region is shown mostly to attest that in that case the implicit method is still stable. However, our choice of initial guess for the iterative solver assumes that the time step is small. When the time step is large, choosing $\vec{k} = \vec{0}$ would be more suitable.

The fact that even in this unfavorable case the implicit method remains stable, we hope that in an actual stiff case (where for instance there would be very small relaxation times), this new implicit method would allow for the use of much larger time steps than the explicit counterpart while remaining stable and accurate.

Chapter 7

Heavy-ion collision modeling

In this chapter we want to investigate if our new implicit method can reproduce non-trivial experimental data, such as the elliptic flow v_2 , as correctly as the existing explicit methods can do. To do so, we perform (3+1)-dimensional event-by-event simulations for lead-lead collisions with random impact parameters and collision energy $\sqrt{s_{\text{NN}}} = 5.02$ TeV. The initial entropy density is provided by the 3-dimensional version of the `TRENTo` model [37]. Instead of using a free-streaming model during the thermalization stage such as in [7], we simply hold the initial entropy density provided by `TRENTo` until $t_0 = 0.6$ fm and multiply it by a normalization factor $N = 11.3$. A normalization factor is needed because `TRENTo` does not provide the entropy itself but instead a quantity proportional to it. All the other coefficients of the `TRENTo` model are taken from the Bayesian analysis [7, 8], where they calculate the probability distribution that the coefficients of the model, including the initial condition parameters, reproduce the experimental data. The equation of state used in the hydrodynamics is the parametrization form [74, 104] of the Wuppertal-Budapest lattice EOS [15]. We also use the bulk ζ/s and shear η/s temperature dependent parametrization from [7, 8], which we will discuss in more details in Sec. 7.1.3.

In order to produce the particles after the freeze-out, we output the zigzag hyper-surface (see Sec. 7.1.4) at constant temperature $T = 151$ MeV, whose details will be discussed in Sec. 7.1.4. We then produce 20 random particlizations for each event by using the Cooper-Frye formulas with the `frzout` [105] code. For each of these particlizations we then compute the cascade with the UrQMD afterburner [106]. The observables are first averaged over the 20 particlizations for each event, and then the statistical error is computed over the events. A total of 1000 initial conditions with random impact parameter are performed.

In order to verify the effectiveness of our implicit method in the case of heavy-ion collisions, we consider particle yield $dN_{\text{ch}}/d\eta$ as function of centrality and η , and we also consider the anisotropy v_n as function of centrality determined by the number of charged particles produced. We use the experimental data from the ALICE experiment [79, 27] as a reference.

A more detailed explanation of the models used in this thesis is presented in the next section Sec. 7.1.

7.1 Numerical framework of a collision

The physical process described previously is generally split into 5 models as described in Sec. 1.2. In the following subsection, we will describe the 5 models that will be used in this thesis. Although the initial thermalization has an impact on the final results, for efficiency and energy consumption reasons, it was not used for the results presented in this thesis, which means only 4 of the following models were used. The missing model is still

presented in Sec. 7.1.2 as it would be important to include it for more accurate results. This framework comes from the heavy-ion collisions event generator from Duke-QCD collaboration [107] which was used for the Bayesian analysis in [7, 8]. The differences in this thesis is that the hydrodynamic part is replaced by our new implicit method, and the free-streaming thermalization part has been skipped.

7.1.1 Collision: Trento initial conditions

The most computationally efficient initial conditions are the geometrically inspired ones such as **T_RENTO** [37] which directly produce the initial energy density without any evolution. Such geometrical model has the advantage to be computationally efficient and also the model in itself tends to be simpler at the cost of not actually describing the physics of the collision.

The **T_RENTO** model [37] is essentially a 2-dimensional transverse entropy density modelization at mid-rapidity which is then extended to finite rapidity to reach a full 3-dimensional description at a starting time τ_0 . The entropy density produced by the collision is proportional to the product of a transverse function $f(x)$ at mid-rapidity and a rapidity-dependent function $g(x, \eta_s)$:

$$s(x, \eta_s)|_{\tau=\tau_0} \propto f(x)g(x, \eta_s) \quad (7.1)$$

with $x = (x, y)$ the position in the transverse plane and η_s the spacetime rapidity.

The transverse function $f(x)$ is the general mean of the thickness function T_A and T_B of the two participating nuclei

$$f(x) \propto \left(\frac{T_A^p + T_B^p}{2} \right)^{1/p}. \quad (7.2)$$

In a multiple constituent nucleus, the thickness function is a random-weighted superposition of nucleons

$$T_{A,B} = \sum_{i=1}^{N_{\text{part}}} \omega_i T_p(x - x_i) \quad (7.3)$$

with ω_i a random weight sampled from the gamma distribution of unit mean and variance $1/k$, where k is a free parameter, x_i is the position of the i_{th} nucleon, and T_p is the proton thickness function

$$T_p(x) = \frac{1}{2\pi w^2} e^{-\frac{x^2}{2w^2}}. \quad (7.4)$$

The nucleon positions are sampled from a Woods-Saxon potential and projected to the transverse plane. For both nucleus, the positions are shifted in the opposite direction by half the impact parameter b .

Not all the nucleons participate in the collision, so they should not all be summed in Eq. (7.3). Participating nucleons are sampled for the pairwise inelastic probability

$$\frac{d\sigma_{NN}^{\text{inel}}}{2\pi b db} = 1 - e^{-\sigma_{gg} T_{pp}(b)} \quad (7.5)$$

where σ_{gg} is an effective partonic cross-section to be fitted from experimental data, and T_{pp} is the proton-proton overlap function

$$T_{pp}(b) = \int d^2x T_p(x) T_p(x - b). \quad (7.6)$$

The extension to finite rapidity is performed through an inverse Fourier transform

$$g(x, y) = F^{-1} \{g(x, k)\} \quad (7.7)$$

where y here is the rapidity such that $g(x, \eta) = g(x, y) \frac{dy}{d\eta}$. The goal of using an inverse Fourier transform is to describe the rapidity extension by its cumulants

$$\log(g) = i\mu k - \frac{1}{2}\sigma^2 k^2 - \frac{1}{6}i\gamma\sigma^3 k^3 + \dots \quad (7.8)$$

with μ the mean and σ^2 the variance.

7.1.2 Thermalization: free-streaming

Although we could directly use the initial conditions as input for the hydrodynamic equations used to evolve the QGP, it would suppose that the system reach local equilibrium instantly. This can be a simple approximation in the same way that assuming boost-invariance greatly simplifies the problem, but it is not realistic. One extra step should be happening between the collision and the QGP part.

One simple model would be to consider that the particles produced during the collision will simply freely fly without interactions for a short time before the system starts to strongly interact and become a fluid [28]. This simple prescription allows interpolating between a non-interacting and strongly interacting medium and thus gives time for the strong interaction to take place and for the system to reach local equilibrium. This addition allows to better reproduce the experimental results [28], which suggest that the system needs some finite time to reach local equilibrium.

In more details, the specific procedure from [28] consists of the free propagation of massless partons through the collisionless Boltzmann equation

$$p^\mu \partial_\mu f(\mathbf{x}, \tau; \mathbf{p}) = 0 \quad (7.9)$$

with $f(x, p)$ the distribution function. In this case, the solution is simply

$$f(\mathbf{x}, \tau_s; \mathbf{p}) = f(\mathbf{x} - (\tau_s - \tau_0)\hat{\mathbf{p}}, \tau_0; \mathbf{p}) \quad (7.10)$$

where t_0 is the starting time, and $\hat{\mathbf{p}} = \mathbf{p}/p$.

After the thermalization is reached at τ_s , the energy momentum tensor can be obtained with

$$T^{\mu\nu}(\mathbf{x}, \tau_s; \mathbf{p}) = \frac{g}{(2\pi)^3} \int \frac{d^3p}{E} p^\mu p^\nu f(\mathbf{x}, \tau_s; \mathbf{p}) \quad (7.11)$$

where g is a degeneracy factor. In the case of boost-invariance, it can be shown that $T^{\mu\nu}(\mathbf{x}, \tau_s; \mathbf{p})$ can be directly expressed as function of $T^{\mu\nu}(\mathbf{x}, \tau_0; \mathbf{p})$, which means that there is no need to sample partons to perform free-streaming from T_{RENT}o initial conditions.

In the local fluid rest frame, the viscous hydrodynamic form reads

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (p + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \quad (7.12)$$

where ϵ and p are the energy density and pressure, Π is the bulk viscous pressure, $\pi^{\mu\nu}$ is the shear pressure tensor, u^μ is the fluid velocity and $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ the projector orthogonal to the velocity.

The Landau matching consists of first extracting the velocity u^μ from $T^{\mu\nu}$ as its time-like eigenvector, with the energy density ϵ as its eigenvalue

$$T^{\mu\nu} u_\nu = \epsilon u^\mu \quad (7.13)$$

with $u^\mu u_\mu = 1$. As $T^{\mu\nu}$ has at most one time-like eigenvector, this solution is unique when it exists.

The remaining components can be extracted as follows when an equation of state $p(\epsilon)$ has been chosen.

$$\Pi = -\frac{1}{3}(\Delta_{\mu\nu}T^{\mu\nu}) - p(\epsilon) \quad (7.14)$$

and

$$\pi^{\mu\nu} = T^{\mu\nu} - \epsilon u^\mu u^\nu + (p(\epsilon) + \Pi)\Delta^{\mu\nu}. \quad (7.15)$$

Although hydrodynamics is known to greatly increase flow and especially elliptic flow, this additional thermalization step actually enhance even more the final flow [28].

This step affects the final results but not in a dramatic way. For time and energy consumption reasons, we do not include it in our framework.

7.1.3 QGP: viscous hydrodynamics

The QGP formed in HIC behaves like a fluid, especially when observing anisotropy in particle production, even in context such as proton-lead collision or even proton-proton collisions where the typical scales suggest it should not. For this reason, relativistic hydrodynamics is used to model the QGP. The detailed equations and derivations are discussed in Chapter 2.

The evolution of the energy-density, the velocities, the bulk pressure and the shear tensor are described by the hydrodynamic equations. However, the EOS describing the thermodynamic quantities as function of the conserved charges needs to be parametrized, and the viscous transport coefficients as well. For the EOS, it is possible to parametrize it from theory at vanishing baryon chemical potential μ_B where lattice QCD provides thermodynamic solutions at high temperature [19]. The low temperature corresponds to the Hadron resonance gas (HRG) [108] which has analytical solutions. Furthermore, as the experimental results suggest, the phase transition between QGP and HRG at vanishing μ_B is a crossover [19]. Therefore, a complete EOS spanning from low to high temperature can be made by connecting the HRG and the lattice QCD EOS with a smooth transition, as done in the Wuppertal-Budapest EOS [15]. Such EOS do not have an analytical expression, and they are thus provided either as table data or interpolation polynomial.

The transport coefficient for the bulk viscosity ζ and the shear viscosity η had been chosen to be constant for many years. However, they have been recently parametrized as function of the temperature in [7, 8] to provide better quality results

$$(\eta/s)(T) = (\eta/s)_{\min} + (\eta/s)_{\text{slope}} \cdot (T - T_c) \left(\frac{T}{T_c} \right)^{(\eta/s)_{\text{crv}}}, \quad (7.16)$$

$$(\zeta/s)(T) = \frac{(\zeta/s)_{\max}}{1 + \left(\frac{T - (\zeta/s)_{\text{peak}}}{(\zeta/s)_{\text{width}}} \right)^2}, \quad (7.17)$$

where the values of the parameters are provided in [7, 8].

The relaxation time for the bulk pressure Π and shear tensor $\pi^{\mu\nu}$ is usually chosen from a causality consideration as discussed in Sec. 2.2 Eq. (2.28).

7.1.4 Particlization: frzout

While expanding, the QGP cools down and the deconfinement progressively stop being effective. The quarks will therefor confine again into particles. However, this process happens sporadically in the system, which means that particles that are usually modeled by kinetic theory cohabit with the fluid modeled by hydrodynamics. Simulating these two models simultaneously is very challenging. As a simplification, they are used one after the other.

The hydrodynamics is first pursued until all cells pass below the hydrodynamical freeze-out temperature. Only then all the particles are produced, and the model is switched to kinetic theory. However, The particles are supposed to be produced at the freeze-out temperature, which means that in these two stages modeling it is important to keep track of when and where the fluid cross the freeze-out temperature. More precisely, we need to track the full space-time hypersurface corresponding to the constant freeze-out temperature.

Hypersurface extraction

The hydrodynamic equations are usually solved numerically on a discretized lattice made of fixed sized cubes containing the averaged values of the fields. One very simple way to find the hypersurface corresponding to the freeze-out temperature is to search for its normal. When two contiguous cells either in space or time have a temperature on each side of the freeze-out, we know that the direction from the high temperature cell to the low temperature is orthogonal to the hypersurface of constant freeze-out temperature. This direction aligned to one of the discretization axis is the normal. By taking one unit of length or time in the three remaining axis, the corresponding volume of the hypersurface can be obtained. It then suffices to store the values of the fields in all these hypersurface volumes and their location to have the complete hypersurface needed to produce the particles. This prescription is often called zigzag.

The zigzag prescription is very simple and thus not particularly accurate. It is possible to obtain a smoother hypersurface with algorithms such as Marching cubes which do not simply consider the main direction but instead look at the value of the quantity of interest (here the temperature) at each of the corners of a cell and then split each edges at the distance between the corners where the quantity would cross the desired threshold (here the freeze-out temperature). The mainly used variation of such algorithm is CORNELIUS [109]. For simplicity, we use the zigzag prescription.

Cooper-Frye formula

In order to switch from hydrodynamics described by the energy-momentum tensor to a particle description with mass and momentum, we need a way to sample particles. To do so, we consider that the system changes from fluid to gas at a fixed temperature T_c . We can then use the Cooper-Frye formula [57] to sample the particles on this constant temperature hypersurface.

The idea behind the Cooper-Frye formula is to count the number of particles that cross a boundary around the system such that the total energy of these particles is in agreement with the total energy obtained from the energy-momentum tensor on the surface. Given a Fermi-Dirac or Bose-Einstein statistics $f(x, p)$, and a hypersurface σ with normal vector σ^μ corresponding to the boundary of the system, we can count the number of particles that cross σ as

$$N(\sigma) = \int Dp \int_{\sigma} f(x, p) p^\mu d\sigma_\mu, \quad (7.18)$$

with $Dp = 2\delta^+(p^2 - m^2)d^4p$ the causal differential tangent to particle crossing outward the boundary. From here, we can write the invariant single-particle distribution in momentum space [57] of the particles on σ

$$E \frac{dN}{d^3p} = \int_{\sigma} f(x, p) p^\mu d\sigma_\mu. \quad (7.19)$$

This last formula Eq. (7.19) can be used as a distribution to sample particles on the hypersurface produced by any of the algorithms described in Sec. 7.1.4. This sampling is performed in the python code called frzout [105], which we use in this thesis.

7.1.5 Hadronic transport: UrQMD

The Ultra-relativistic Quantum Molecular Dynamics (UrQMD) model [106] is used to describe low energy (below 100 GeV) ion collisions. It is based on hadron propagation through the Boltzmann relativistic equation by including more than 100 hadron species. It especially includes rescattering of hadrons and decay.

This model can thus be used to propagate the particles produced by the Cooper-Frye formula on the freeze-out temperature hypersurface until they would reach the detector. This stage is also important as the particle distribution changes before reaching the detector due to the rescattering and decay.

7.2 QGP observables

In order to understand the results that will be showed in the next section, an introduction to the observables of interest is necessary.

The simplest observable is the number of charged particles N_{ch} .

To study other observables, it would be convenient to know the impact parameter which is the distance between the center of the nuclei during the collision. This value cannot be directly observed as only the final produced particles can be seen. However, the usual proxy is to use the number of charged particle to approximate how central a collision is. When the impact parameter is small and the collision is very central, a larger number of particles will be produced. Therefore, instead of the impact parameter, the centrality is obtained by sorting numerous events by their number of charged particles, and then splitting this list in small bins of centrality percentages. The first bin near 0% centrality correspond to the highest number of charged particles, so most likely the collisions with the lowest impact parameter.

If we look at the number of charged particle for central η as function of the centrality from ALICE data [79] on the left of Fig 7.1, we can see that the number of charged particles decreases as the centrality increases, which is expected by construction.

To go a little further, we can look at the longitudinal direction dependency. Due to the extreme longitudinal pressure after the collision, the system will expand at nearly the speed of light at high rapidity. However, if we represent our system in the Milne expanding coordinates, it will not expand but mainly stay still as the coordinate system expand as quickly as the system. In this case, the system will have the general shape of a ball where the density will be smaller on the outside. Therefore, the density is maximum at vanishing eta, and the amount of produced charged particles should also be maximum. If we look at the number of charged particles as function of η on the right of Fig. 7.1, we can see that it decreases as η increases as expected.

A less trivial and more interesting observable is the particle anisotropy which can be obtained by a Fourier decomposition of the azimuthal distribution of particle production in the transverse plane [27]

$$\frac{dN}{d\phi} \propto 1 + 2 \sum_{n=1}^{\infty} v_n \cos [n(\phi - \psi_n)], \quad (7.20)$$

where v_n is the amplitude of the n directional particle emission, and ψ_n is the symmetry plane. The most prominent anisotropy will come from the initial elliptic shape for non-central collisions. Because the pressure is larger inside the medium and lower at its boundary,

hydrodynamics which follows the gradient of pressure will make the medium expands toward the normal of its surface. In the case of an elliptic shape with two preferred directions, there will be an enhancement of the anisotropy in two directions corresponding to v_2 .

If we also consider that the content of the nuclei are not perfectly homogeneous, we can expect that the elliptic shape will be slightly deformed toward a higher number of directions, producing high order anisotropy.

The experimental data of v_n as function of centrality from the ALICE experiment [27] shown in black dots on Fig. 7.2 confirm that at vanishing centrality all anisotropy vanish, and as the centrality increases, the anisotropy increases.

7.2.1 Computing v_n

There are many ways to compute the anisotropy v_n . Some are more suited for experimental data and other for simulation results. The two most straightforward ones would be the event plane and the two particle correlation methods.

In the event plane one, we first find the angle ψ_n corresponding to the event plane in which an anisotropy with n harmonics takes place. It is computed as

$$\psi_n = \frac{1}{n} \arctan \frac{\langle \sin(n\phi) \rangle}{\langle \cos(n\phi) \rangle}, \quad (7.21)$$

where ϕ is the azimuthal angle of each particle and $\langle \cdot \rangle$ is the average over all the particles. The anisotropy is then computed as

$$v_n = \langle \cos[n(\phi - \psi_n)] \rangle. \quad (7.22)$$

This computation is linear in the number of particles, but it is sensitive to non-flow effects such as jet or resonance decays.

The two-particle correlation is more straightforward as it computes the anisotropy in one stage by averaging over all pairs ϕ_1 and ϕ_2 of particle angles

$$v_n\{2\} = \sqrt{\langle \cos[n(\phi_1 - \phi_2)] \rangle}. \quad (7.23)$$

This computation is also subject to non-flow events, but in addition it is quadratic in the number of particles. However, by only including the pairs of particles with a large enough difference in η (the longitudinal axis of the collision), it is possible to suppress the non-flow effects. This is usually called an η -gap and is used in experimental data analysis [27].

In the various other methods, there is one which can be used to analyze data in linear time while reducing non-flow effects described in [110]. It consists of computing the flow vector for the harmonic n for the M particles of an event

$$Q_n = \sum_{i=1}^M e^{in\phi_i}, \quad (7.24)$$

and the second order cumulant

$$\langle 2 \rangle = \frac{|Q_n|^2 - M}{W_{\langle 2 \rangle}}, \quad (7.25)$$

with associated weight

$$W_{\langle 2 \rangle} = M(M - 1). \quad (7.26)$$

The subtraction of the number of particles in $\langle 2 \rangle$ corresponds to removing the contribution of particle with itself from the square $|Q_n|^2$, and the weight $W_{\langle 2 \rangle}$ are chosen to minimize multiplicity variations. The second order cumulant flow is constructed as

$$v_n\{2\} = \sqrt{\frac{\sum_{i \in \text{events}} (W_{\langle 2 \rangle})_i \langle 2 \rangle_i}{\sum_{i \in \text{events}} (W_{\langle 2 \rangle})_i}}. \quad (7.27)$$

This is the method we used to analyze the numerical results shown in the next section Sec. 7.3.

7.3 Results

In this section we want to show the results from the heavy-ion collision framework described in the previous sections including our implicit method. In Fig. 7.1 and Fig. 7.2 we show the reference ALICE experimental data [79, 27] in black dots, the numerical results of our implicit method in continuous lines, and the numerical results for the VISH2+1 explicit code in dashed lines. There are two important differences between our hydrodynamics and the one of VISH2+1. The first one appears in the name where "2+1" means (2+1)-dimensional with boost-invariance, contrary to our (3+1)-dimensional implicit hydrodynamics. The boost-invariance is only an approximation, which is why most hydrodynamics codes are becoming (3+1)-dimensional, including our implicit hydrodynamics. The second difference is in the framework in which the hydrodynamics is embedded. In the case of VISH2+1 results, the complete framework including the thermalization stage is used, compared to our simplified framework where we skip this thermalization part which corresponds to an instantaneous thermalization approximation.

In Fig. 7.1 we show the particle yield $dN_{\text{ch}}/d\eta$ as function of centrality for mid-rapidity on the left and as function of η for different centrality class on the right. We can see that the particle yield slightly deviates from the experimental data around 30% centrality, although the results for VISH2+1 from [7] do not have such deviation. The difference is most likely caused by our simplified framework which skips the thermalization stage, although the transport coefficients we took from [7] were tuned with that thermalization stage included. An additional possible cause is the boost-invariance assumption used in VISH2+1, which is not used in our (3+1)-dimensions implicit hydrodynamics. Consequently, the parameters obtained in [7] might not be directly usable without the boost-invariance approximation. We still decided to perform the analysis in (3+1)-dimensions to verify that our new method can work properly in the dimensions of real physics, and also be sure that it would be suitable for the upcoming Bayesian analysis in (3+1)-dimensions.

In Fig. 7.2 we show the particle anisotropy $v_n\{2\}$ at mid-rapidity as function of centrality. We can see that v_2 , v_3 and v_4 are well reproduced within the statistical error until 55% centrality, similarly to the results for VISH2+1 from [7]. From 60% centrality and above v_2 is underestimated, contrary to VISH2+1 results. However, older studies using VISH2+1 which did not include a thermalization stage before the hydrodynamics had the same underestimation issue as shown in Fig. 4 of [111].

Considering three spatial dimensions is important to properly reproduce the longitudinal η dependence. However, the results from Fig. 7.1 and Fig. 7.2 also show how important the initial thermalization stage is. Especially, the (2+1)-dimensional boost-invariant results which include the thermalization stage reproduce better the experimental data at mid-rapidity than our (3+1)-dimensional hydrodynamics which skips this thermalization. Overall, these results suggest that the new implicit method should properly reproduce experimental data at both finite and mid-rapidity in a complete framework including the free-streaming stage.

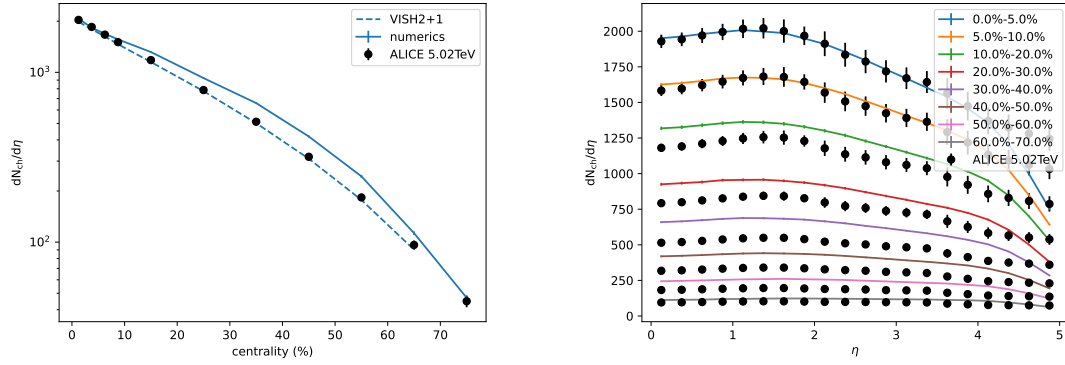


Figure 7.1: Comparison between the numerical results in continuous lines and the ALICE [79] experimental results in black points for the particle yield $dN_{ch}/d\eta$ as function of centrality for mid-rapidity on the left and as function of η for different centrality class on the right.

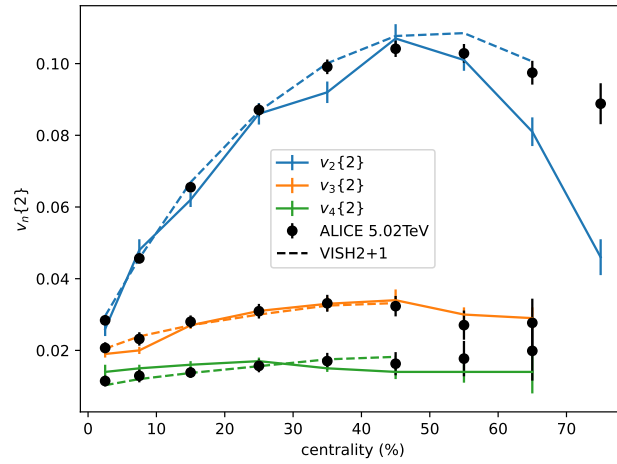


Figure 7.2: Comparison between the numerical results in continuous lines and the ALICE [27] experimental results in black points for the particle anisotropy $v_n\{2\}$ as function of centrality.

Chapter 8

Conclusion

8.1 Summary

Over the past few decades, the phase diagram of Quantum Chromodynamics (QCD) has been probed intensively from both theoretical and experimental angles. At sufficiently high temperatures the theory predicts, and experiments confirm, the emergence of a quark–gluon plasma (QGP), which is now routinely created in ultra-relativistic heavy-ion collisions at facilities such as the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC). A striking outcome of these studies is that the space-time evolution of the QGP is remarkably well described by relativistic hydrodynamics with low viscosity. As a result, cutting-edge computational fluid-dynamical simulations constitute the cornerstone of modern dynamical models of heavy-ion collisions.

In this thesis, we studied how to numerically solve relativistic hydrodynamics used to model the QGP in HIC. We showed that the use of implicit methods with a locally optimized fixed-point solver can outperform explicit methods in both efficiency and accuracy, especially when viscosity is considered. These two properties are essential to explore the phase diagram of QCD matter and extract physical quantities such as the viscosity of the QGP through Bayesian analysis.

In Chapter 3, we investigated how relativistic hydrodynamics is solved numerically in the context of HIC. We discussed existing models which all focus on careful space discretization with straightforward explicit time discretization. We emphasized that viscous equations require additional approximations to be usable with explicit time discretization, while they can be solved exactly in an implicit context.

In Chapter 4, we provided a motivation for implicit methods and investigated how to solve them numerically. We studied the general principle of numerically solving implicitly defined equations, and focused on the particular case of the fixed-point solver. We discussed its stability, efficiency and optimization, and showed that when applied to the one stage Gauss-Legendre (GL1) implicit method, it automatically reduces to the efficient and accurate Adams-Bashforth method in stable regions and keeps using the more stable GL1 in stiffer regions.

In Chapter 5, we investigated how to solve the viscous equations exactly in conjunction with an implicit solver. We also studied how to reduce oscillations of the energy-density and velocity in the space discretization process.

In Chapter 6, we investigated the correctness, efficiency and accuracy of both explicit and implicit methods. We first compared the numerical result to exact solutions of hydrodynamics in the specific cases of the Riemann problem and the Gubser flow. We then studied the numerical error as function of the computational cost for both explicit and implicit methods.

Finally, we showed that in the ideal case, they both reproduce the expected accuracy, with the implicit method being about twice as efficient. In the viscous case, the implicit method could reproduce the expected accuracy while the explicit one had reduced accuracy.

In Chapter 7, we concluded the study with an application of our implicit method in a complete framework for HIC. We showed that a non-trivial experimental observable such as the flow anisotropy v_n is well reproduced by the numerical results including the implicit method, which suggests that implicit methods are suitable for the study of the QCD phase diagram.

8.2 Outlook

There are multiple interesting extensions to this study.

First, we should use our new method to study more extensively heavy-ion collisions. One way to extract meaningful physical quantities from HIC modeling is to use Bayesian analysis, which requires performing numerous simulations. Therefore, our new efficient implicit method will be advantageous to perform the analysis in (3+1)-dimension, which is necessary to accurately study the η dependence of physical observables.

Second, it is highly likely that including a two stage Anderson acceleration in the fixed-point method would keep it numerically cheap while making it more efficient and more stable. Studying its cost-error benchmark in stiff contexts, such as magneto-hydrodynamics, might show that such a simple implicit solver can be efficient, accurate and stable at the same time. If efficiency is indeed improved, it might be preferable to include Anderson acceleration before performing a Bayesian analysis.

Third, our method was designed with simplicity in mind, so including additional charges should be straightforward. It would be interesting to include charges such as the baryon number and the electric charge to reach the finite μ_B region of the phase diagram.

Fourth, including fluctuations in the dissipative equations would be useful to study the phase transition between the QGP and hadronic matter. Including fluctuations in numerical hydrodynamics is known to be challenging and especially leads to instabilities. It would be interesting to see how the fixed-point integrator can handle the discontinuities produced by the fluctuations. It would also be interesting to see if stabilization such as the Anderson acceleration can reduce the numerical instabilities introduced by these fluctuations, and therefore help to search for the critical point.

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Appendices

Appendix A

Comparison with the midpoint method

In Sec. 6.4 and Sec. 6.6, we compare the computational efficiencies of Heun's and GL1 methods as representatives of explicit and implicit methods, respectively. In this appendix, we employ the midpoint method as another explicit method and compare its efficiency with Heun's and GL1 methods. The Butcher table of the midpoint method is shown in the middle column of Table 3.1. As shown there, this method has the same order of accuracy $p = 2$ as Heun's and GL1 methods.

In Figs. A.1 and A.2, we plot the maximum and the average of the numerical error $|\Delta_{\text{ref}}|$ as functions of n_{KT} for the (1+1)-dimensional Riemann problem with $\epsilon_{\text{min}} = 1$ and 0 fm^{-4} , respectively. The results show that the performance of the midpoint method is almost the same as Heun's method.

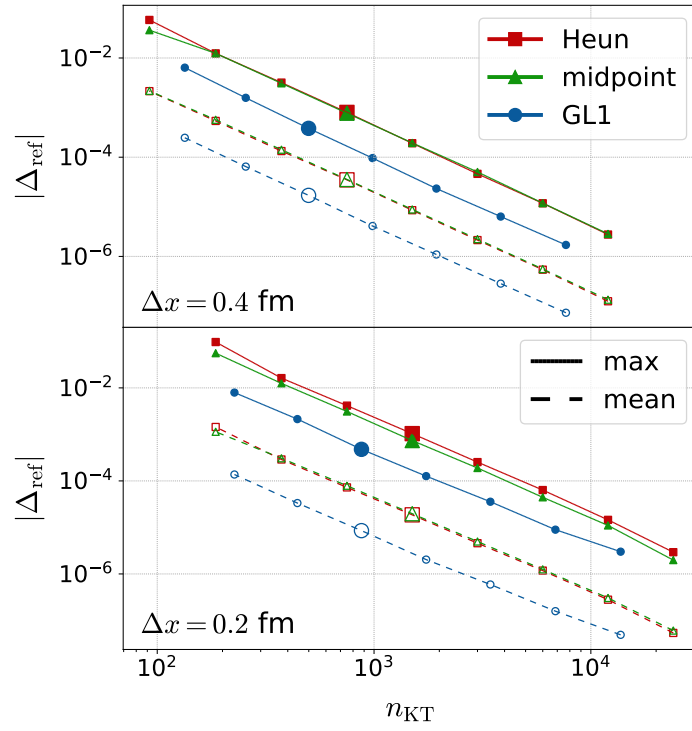


Figure A.1: Maximum and average of the numerical error $|\Delta_{\text{ref}}|$ as functions of n_{KT} for the Riemann problem with $\epsilon_{\min} = 1 \text{ fm}^{-4}$. The result of the midpoint method is shown by the green triangle symbols. The results for Heun's and GL1 methods are the same as Fig. 6.3.

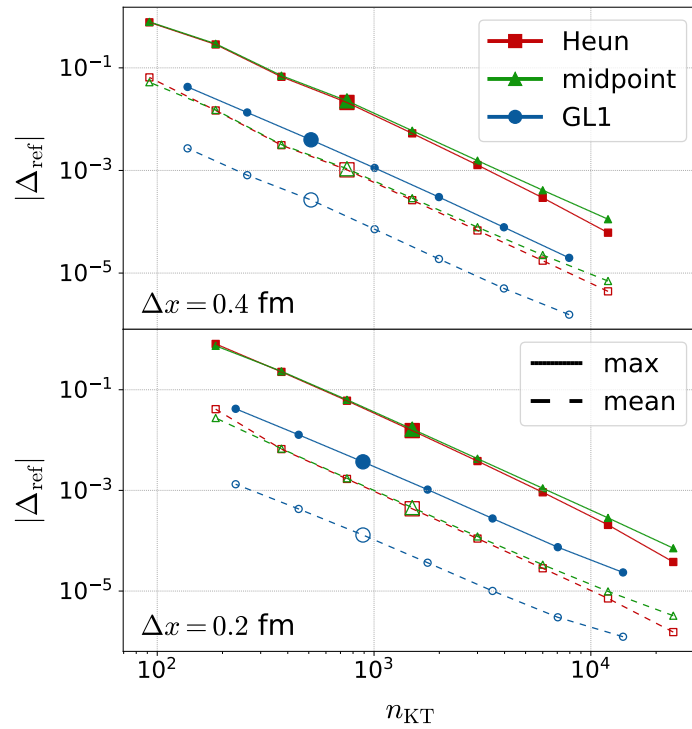


Figure A.2: Same result as Fig. A.1, but $\epsilon_{\min} = 0 \text{ fm}^{-4}$.

Appendix B

Obtaining $\{\partial_\tau \epsilon, \partial_\tau u^x, \partial_\tau u^y, \partial_\tau u^\eta\}$

In order to obtain the expression of the time derivatives of the components of the ideal energy-momentum tensor Eq. (5.11),

$$V = (\partial_\tau \epsilon, \partial_\tau u^x, \partial_\tau u^y, \partial_\tau u^\eta), \quad (\text{B.1})$$

we first have to express $\partial_\tau T_{\text{id}}^{\tau\nu}$ as function of the components of V in the Milne coordinates with the change of variable introduced in Sec. 2.3:

$$\partial_\tau T_{\text{id}}^{\tau\nu} = \partial_\tau (\epsilon u^\tau u^\nu - P \Delta^{\tau\nu}) \quad (\text{B.2})$$

$$= u^\tau u^\nu \partial_\tau (\epsilon + P) + (\epsilon + P) u^\tau \partial_\tau u^\nu + (\epsilon + P) u^\nu \partial_\tau u^\tau - g^{\tau\nu} \partial_\tau P \quad (\text{B.3})$$

$$= (u^\tau u^\nu (1 + \partial_\epsilon P) - g^{\tau\nu} \partial_\epsilon P) \partial_\tau \epsilon + (\epsilon + P) u^\tau (\partial_\tau u^\nu + u^\nu \frac{\partial_\tau u^\tau}{(u^\tau)^2}). \quad (\text{B.4})$$

In this specific coordinate system, $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$, which is why all $\partial_\tau g^{\mu\nu} = 0$ were discarded.

By considering that

$$u^\tau \partial_\tau u^\tau = u^x \partial_\tau u^x + u^y \partial_\tau u^y + u^\eta \partial_\tau u^\eta, \quad (\text{B.5})$$

we can see that $\partial_\tau T_{\text{id}}^{\tau\nu}$ is linear in $\{\partial_\tau \epsilon, \partial_\tau u^x, \partial_\tau u^y, \partial_\tau u^\eta\}$. We can thus write

$$\partial_\tau T_{\text{id}}^{\tau\nu} = \mathbb{M} V = \begin{bmatrix} (u^\tau)^2 (1 + \partial_\epsilon P) - \partial_\epsilon P & 2(\epsilon + P) \bar{u}^T \\ u^\tau (1 + \partial_\epsilon P) \bar{u} & (\epsilon + P) u^\tau (1 + \frac{\bar{u} \bar{u}^T}{(u^\tau)^2}) \end{bmatrix} \begin{pmatrix} \partial_\tau \epsilon \\ \partial_\tau \bar{u} \end{pmatrix} \quad (\text{B.6})$$

where $\bar{u} = (u^x \ u^y \ u^\eta)^T$. By inverting it we obtain

$$V = \mathbb{M}^{-1} \partial_\tau T_{\text{id}}^{\tau\nu}. \quad (\text{B.7})$$

At this point we can consider several ways to explicit $\mathbb{M}^{-1}$. One basic but tedious approach consists in inverting it analytically by hand with Gauss pivot, or use a symbolic algebra software. However, in this case, a more clever approach is to consider rotation symmetry of the velocity u^i . Indeed, in the base $\mathcal{B}'$ where $\bar{u} = (u \ 0 \ 0)^T$ with $u = \sqrt{\bar{u} \cdot \bar{u}}$, the problem reduces to a block diagonal matrix:

$$\mathbb{M}_{\mathcal{B}'} = \begin{bmatrix} (u^\tau)^2 (1 + \partial_\epsilon P) - \partial_\epsilon P & 2(\epsilon + P) u & 0 & 0 \\ u^\tau (1 + \partial_\epsilon P) u & (\epsilon + P) u^\tau (1 + \frac{u^2}{(u^\tau)^2}) & 0 & 0 \\ 0 & 0 & (\epsilon + P) u^\tau & 0 \\ 0 & 0 & 0 & (\epsilon + P) u^\tau \end{bmatrix} \quad (\text{B.8})$$

which can easily be inverted as:

$$\mathbb{M}_{\mathcal{B}'}^{-1} = \begin{bmatrix} \frac{(\epsilon+P)u^\tau(1+\frac{u^2}{(u^\tau)^2})}{\Delta} & \frac{-2(\epsilon+P)u}{\Delta} & 0 & 0 \\ \frac{-u^\tau(1+\partial_\epsilon P)u}{\Delta} & \frac{(u^\tau)^2(1+\partial_\epsilon P)-\partial_\epsilon P}{\Delta} & 0 & 0 \\ 0 & 0 & \frac{1}{(\epsilon+P)u^\tau} & 0 \\ 0 & 0 & 0 & \frac{1}{(\epsilon+P)u^\tau} \end{bmatrix} \quad (\text{B.9})$$

with $\Delta = ((u^\tau)^2(1 + \partial_\epsilon P) - \partial_\epsilon P)(\epsilon + P)u^\tau(1 + \frac{u^2}{(u^\tau)^2}) - 2u^\tau(1 + \partial_\epsilon P)(\epsilon + P)u^2$.

By noticing that $\bar{u}\bar{u}^T/u^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ when $\bar{u} = (u \ 0 \ 0)^T$, we can factor out the 3 by 3 bottom-right block:

$$\mathbb{M}^{-1} = \begin{bmatrix} \frac{(\epsilon+P)u^\tau(1+\frac{u^2}{(u^\tau)^2})}{\Delta} & \frac{-2(\epsilon+P)\bar{u}^T}{\Delta} \\ \frac{-u^\tau(1+\partial_\epsilon P)\bar{u}}{\Delta} & \frac{(u^\tau)^2(1+\partial_\epsilon P)-\partial_\epsilon P}{\Delta} \frac{\bar{u}\bar{u}^T}{u^2} + \frac{1}{(\epsilon+P)u^\tau} \left(1 - \frac{\bar{u}\bar{u}^T}{u^2}\right) \end{bmatrix} \quad (\text{B.10})$$

and one can easily check that this form of $\mathbb{M}^{-1}$ is the actual inverse matrix of $\mathbb{M}$ in the general case.

Bibliography

- [1] M. E. Peskin, D. V. Schroeder, An Introduction to quantum field theory, Addison-Wesley, Reading, USA, 1995. doi:10.1201/9780429503559.
- [2] Z. Dehghan, M. Faber, What do we know about the confinement mechanism? (12 2024). arXiv:2412.10767.
- [3] R. Alkofer, J. Greensite, Quark Confinement: The Hard Problem of Hadron Physics, J. Phys. G 34 (2007) S3. arXiv:hep-ph/0610365, doi:10.1088/0954-3899/34/7/S02.
- [4] R. E. Taylor, Deep inelastic scattering: The early years, Rev. Mod. Phys. 63 (1991) 573–595. doi:10.1103/RevModPhys.63.573.
URL <https://link.aps.org/doi/10.1103/RevModPhys.63.573>
- [5] M. Laine, P. Schicho, Y. Schröder, A QCD Debye mass in a broad temperature range, Phys. Rev. D 101 (2) (2020) 023532. arXiv:1911.09123, doi:10.1103/PhysRevD.101.023532.
- [6] J. Jankowski, D. Blaschke, O. Kaczmarek, Debye mass at the QCD transition in the PNJL model (8 2015). arXiv:1508.05322.
- [7] J. S. Moreland, J. E. Bernhard, S. A. Bass, Bayesian calibration of a hybrid nuclear collision model using p-pb and pb-pb data at energies available at the cern large hadron collider, Phys. Rev. C 101 (2020) 024911. doi:10.1103/PhysRevC.101.024911.
URL <https://link.aps.org/doi/10.1103/PhysRevC.101.024911>
- [8] J. E. Bernhard, J. S. Moreland, S. A. Bass, Bayesian estimation of the specific shear and bulk viscosity of quark–gluon plasma, Nature Physics 15 (11) (2019) 1113–1117. doi:10.1038/s41567-019-0611-8.
URL <https://doi.org/10.1038/s41567-019-0611-8>
- [9] L. Cunqueiro, A. M. Sickles, Studying the QGP with Jets at the LHC and RHIC, Prog. Part. Nucl. Phys. 124 (2022) 103940. arXiv:2110.14490, doi:10.1016/j.pnpnp.2022.103940.
- [10] T. Matsui, H. Satz, J/ψ Suppression by Quark-Gluon Plasma Formation, Phys. Lett. B 178 (1986) 416–422. doi:10.1016/0370-2693(86)91404-8.
- [11] B. Alessandro, et al., A New measurement of J/ψ suppression in Pb-Pb collisions at 158-GeV per nucleon, Eur. Phys. J. C 39 (2005) 335–345. arXiv:hep-ex/0412036, doi:10.1140/epjc/s2004-02107-9.
- [12] M. Asakawa, M. Kitazawa, Fluctuations of conserved charges in relativistic heavy ion collisions: An introduction, Prog. Part. Nucl. Phys. 90 (2016) 299–342. arXiv:1512.05038, doi:10.1016/j.pnpnp.2016.04.002.

- [13] O. Massen, G. Nijs, M. Sas, W. van der Schee, R. Snellings, Effective temperatures of the QGP from thermal photon and dilepton production (12 2024). [arXiv:2412.09671](#).
- [14] O. Philipsen, The QCD equation of state from the lattice, *Prog. Part. Nucl. Phys.* 70 (2013) 55–107. [arXiv:1207.5999](#), [doi:10.1016/j.ppnp.2012.09.003](#).
- [15] S. Borsanyi, G. Endrodi, Z. Fodor, A. Jakovac, S. D. Katz, S. Krieg, C. Ratti, K. K. Szabo, The QCD equation of state with dynamical quarks, *JHEP* 11 (2010) 077. [arXiv:1007.2580](#), [doi:10.1007/JHEP11\(2010\)077](#).
- [16] Y. Nambu, G. Jona-Lasinio, Dynamical model of elementary particles based on an analogy with superconductivity. II., *Phys. Rev.* 124 (1961) 246–254. [doi:10.1103/PhysRev.124.246](#).
- [17] N. Cabibbo, G. Parisi, Exponential hadronic spectrum and quark liberation, *Physics Letters B* 59 (1) (1975) 67–69. [doi:https://doi.org/10.1016/0370-2693\(75\)90158-6](#).
URL <https://www.sciencedirect.com/science/article/pii/0370269375901586>
- [18] R. D. Pisarski, F. Wilczek, Remarks on the chiral phase transition in chromodynamics, *Phys. Rev. D* 29 (1984) 338–341. [doi:10.1103/PhysRevD.29.338](#).
URL <https://link.aps.org/doi/10.1103/PhysRevD.29.338>
- [19] Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz, K. K. Szabo, The Order of the quantum chromodynamics transition predicted by the standard model of particle physics, *Nature* 443 (2006) 675–678. [arXiv:hep-lat/0611014](#), [doi:10.1038/nature05120](#).
- [20] M. Asakawa, K. Yazaki, Chiral restoration at finite density and temperature (Nov 1989).
- [21] M. Stephanov, K. Rajagopal, E. Shuryak, Signatures of the tricritical point in qcd, *Phys. Rev. Lett.* 81 (1998) 4816–4819. [doi:10.1103/PhysRevLett.81.4816](#).
URL <https://link.aps.org/doi/10.1103/PhysRevLett.81.4816>
- [22] D. Tlusty, The RHIC Beam Energy Scan Phase II: Physics and Upgrades, in: 13th Conference on the Intersections of Particle and Nuclear Physics, 2018. [arXiv:1810.04767](#).
- [23] J. N. Guenther, Overview of the QCD phase diagram: Recent progress from the lattice, *Eur. Phys. J. A* 57 (4) (2021) 136. [arXiv:2010.15503](#), [doi:10.1140/epja/s10050-021-00354-6](#).
- [24] N. R. Sahoo, Recent results on event-by-event fluctuations from the RHIC Beam Energy Scan program in the STAR experiment, *J. Phys. Conf. Ser.* 535 (2014) 012007. [arXiv:1407.1554](#), [doi:10.1088/1742-6596/535/1/012007](#).
- [25] S. Acharya, et al., Measurements of Chemical Potentials in Pb-Pb Collisions at $\sqrt{s_{NN}}=5.02$ TeV, *Phys. Rev. Lett.* 133 (9) (2024) 092301. [arXiv:2311.13332](#), [doi:10.1103/PhysRevLett.133.092301](#).
- [26] G. Kornakov, the HADES Collaboration, Exploring hot and dense qcd matter with hades, *Journal of Physics: Conference Series* 1643 (1) (2020) 012012. [doi:10.1088/1742-6596/1643/1/012012](#).
URL <https://dx.doi.org/10.1088/1742-6596/1643/1/012012>

- [27] J. Adam, et al., Anisotropic flow of charged particles in Pb-Pb collisions at $\sqrt{s_{NN}} = 5.02$ TeV, Phys. Rev. Lett. 116 (13) (2016) 132302. [arXiv:1602.01119](#), [doi:10.1103/PhysRevLett.116.132302](#).
- [28] J. Liu, C. Shen, U. Heinz, Pre-equilibrium evolution effects on heavy-ion collision observables, Phys. Rev. C 91 (2015) 064906. [doi:10.1103/PhysRevC.91.064906](#).
URL <https://link.aps.org/doi/10.1103/PhysRevC.91.064906>
- [29] P. Romatschke, R. Venugopalan, The Unstable Glasma, Phys. Rev. D 74 (2006) 045011. [arXiv:hep-ph/0605045](#), [doi:10.1103/PhysRevD.74.045011](#).
- [30] A. Kurkela, A. Mazeliauskas, J.-F. Paquet, S. Schlichting, D. Teaney, Effective kinetic description of event-by-event pre-equilibrium dynamics in high-energy heavy-ion collisions, Phys. Rev. C 99 (3) (2019) 034910. [arXiv:1805.00961](#), [doi:10.1103/PhysRevC.99.034910](#).
- [31] I. Karpenko, M. Bleicher, P. Huovinen, H. Petersen, Effects of EoS in viscous hydro + cascade model for the RHIC Beam Energy Scan, Nucl. Phys. A 956 (2016) 834–837. [arXiv:1601.00800](#), [doi:10.1016/j.nuclphysa.2016.02.030](#).
- [32] U. Heinz, Kinetic theory for plasmas with non-abelian interactions, Phys. Rev. Lett. 51 (1983) 351–354. [doi:10.1103/PhysRevLett.51.351](#).
URL <https://link.aps.org/doi/10.1103/PhysRevLett.51.351>
- [33] H.-T. Elze, U. W. Heinz, Quark - Gluon Transport Theory, Phys. Rept. 183 (1989) 81–135. [doi:10.1016/0370-1573\(89\)90059-8](#).
- [34] H. Petersen, Q. Li, X. Zhu, M. Bleicher, Directed and elliptic flow in heavy ion collisions at GSI-FAIR and CERN-SPS, Phys. Rev. C 74 (2006) 064908. [arXiv:hep-ph/0608189](#), [doi:10.1103/PhysRevC.74.064908](#).
- [35] H. Petersen, J. Steinheimer, G. Burau, M. Bleicher, A Transport calculation with an embedded (3+1)d hydrodynamic evolution: Elliptic flow as a function of transverse momentum at SPS energies, Nucl. Phys. A 830 (2009) 283C–286C. [arXiv:0907.2169](#), [doi:10.1016/j.nuclphysa.2009.10.024](#).
- [36] M. L. Miller, K. Reygers, S. J. Sanders, P. Steinberg, Glauber modeling in high energy nuclear collisions, Ann. Rev. Nucl. Part. Sci. 57 (2007) 205–243. [arXiv:nucl-ex/0701025](#), [doi:10.1146/annurev.nucl.57.090506.123020](#).
- [37] J. S. Moreland, J. E. Bernhard, S. A. Bass, Alternative ansatz to wounded nucleon and binary collision scaling in high-energy nuclear collisions, Phys. Rev. C 92 (2015) 011901. [doi:10.1103/PhysRevC.92.011901](#).
URL <https://link.aps.org/doi/10.1103/PhysRevC.92.011901>
- [38] B. Muller, J. Schukraft, B. Wyslouch, First Results from Pb+Pb collisions at the LHC, Ann. Rev. Nucl. Part. Sci. 62 (2012) 361–386. [arXiv:1202.3233](#), [doi:10.1146/annurev-nucl-102711-094910](#).
- [39] J. E. Bernhard, P. W. Marcy, C. E. Coleman-Smith, S. Huzurbazar, R. L. Wolpert, S. A. Bass, Quantifying properties of hot and dense qcd matter through systematic model-to-data comparison, Phys. Rev. C 91 (2015) 054910. [doi:10.1103/PhysRevC.91.054910](#).
URL <https://link.aps.org/doi/10.1103/PhysRevC.91.054910>

- [40] B. Schenke, P. Tribedy, R. Venugopalan, Fluctuating glasma initial conditions and flow in heavy ion collisions, *Phys. Rev. Lett.* 108 (2012) 252301. doi:10.1103/PhysRevLett.108.252301.
URL <https://link.aps.org/doi/10.1103/PhysRevLett.108.252301>
- [41] S. McDonald, S. Jeon, C. Gale, 3 + 1D initialization and evolution of the glasma, *Phys. Rev. C* 108 (2023) 064910. doi:10.1103/PhysRevC.108.064910.
URL <https://link.aps.org/doi/10.1103/PhysRevC.108.064910>
- [42] S. Acharya, et al., Characterizing the initial conditions of heavy-ion collisions at the LHC with mean transverse momentum and anisotropic flow correlations, *Phys. Lett. B* 834 (2022) 137393. arXiv:2111.06106, doi:10.1016/j.physletb.2022.137393.
- [43] L. D. Landau, On the multiparticle production in high-energy collisions, *Izv. Akad. Nauk Ser. Fiz.* 17 (1953) 51–64.
- [44] R. Baier, A. H. Mueller, D. Schiff, D. T. Son, 'Bottom up' thermalization in heavy ion collisions, *Phys. Lett. B* 502 (2001) 51–58. arXiv:hep-ph/0009237, doi:10.1016/S0370-2693(01)00191-5.
- [45] V. Chekhovsky, et al., Determination of the strong coupling and its running from measurements of inclusive jet production (12 2024). arXiv:2412.16665.
- [46] E. Fermi, High energy nuclear events, *Progress of Theoretical Physics* 5 (4) (1950) 570–583. arXiv:<https://academic.oup.com/ptp/article-pdf/5/4/570/5430247/5-4-570.pdf>, doi:10.1143/ptp/5.4.570.
URL <https://doi.org/10.1143/ptp/5.4.570>
- [47] L. D. Landau, On Multiple Production of Particles during Collisions of Fast Particles, *Izv. Akad. Nauk Ser. Fiz.* 15 (1953). doi:10.1016/b978-0-08-010586-4.50079-1.
- [48] B. N. Lab., Report of the workshop on gev/nucleon collisions of heavy ions: how and why, november 29–december 1, 1974, bear mountain, new york, Brookhaven National Lab., Upton, N.Y. (USA), 1974.
URL <https://www.osti.gov/biblio/4061527>
- [49] J. D. Bjorken, Highly Relativistic Nucleus-Nucleus Collisions: The Central Rapidity Region, *Phys. Rev. D* 27 (1983) 140–151. doi:10.1103/PhysRevD.27.140.
- [50] T. H. Burnett, S. Dake, M. Fuki, J. C. Gregory, T. Hayashi, R. Holynski, J. Iwai, W. V. Jones, A. Jurak, J. J. Lord, O. Miyamura, H. Oda, T. Ogata, T. A. Parnell, T. Saito, T. Tabuki, Y. Takahashi, T. Tominaga, B. Wilczynska, R. J. Wilkes, W. Wolter, B. Wosiek, Extremely high multiplicities in high-energy nucleus-nucleus collisions, *Phys. Rev. Lett.* 50 (1983) 2062–2065. doi:10.1103/PhysRevLett.50.2062.
URL <https://link.aps.org/doi/10.1103/PhysRevLett.50.2062>
- [51] P. F. Kolb, J. Sollfrank, U. W. Heinz, Anisotropic transverse flow and the quark hadron phase transition, *Phys. Rev. C* 62 (2000) 054909. arXiv:hep-ph/0006129, doi:10.1103/PhysRevC.62.054909.
- [52] K. H. Ackermann, N. Adams, C. Adler, Z. Ahammed, S. Ahmad, C. Allgower, J. Amsbaugh, M. Anderson, E. Anderssen, H. Arnesen, L. Arnold, G. S. Averichev, A. Baldwin, J. Balewski, O. Barannikova, L. S. Barnby, J. Baudot, M. Beddo, S. Bekele, V. V. Belaga, R. Bellwied, S. Bennett, J. Bercovitz, J. Berger, W. Betts, H. Bichsel,

- F. Bieser, L. C. Bland, M. Bloomer, C. O. Blyth, J. Boehm, B. E. Bonner, D. Bonnet, R. Bossingham, M. Botlo, A. Boucham, N. Bouillo, S. Bouvier, K. Bradley, F. P. Brady, E. S. Braithwaite, W. Braithwaite, A. Brandin, R. L. Brown, G. Brugalette, C. Byrd, H. Caines, M. Calderón de la Barca Sánchez, A. Cardenas, L. Carr, J. Carroll, J. Castillo, B. Caylor, D. Cebra, S. Chatopadhyay, M. L. Chen, W. Chen, Y. Chen, S. P. Chernenko, M. Cherney, A. Chikanian, B. Choi, J. Chrin, W. Christie, J. P. Coffin, L. Conin, C. Consiglio, T. M. Cormier, J. G. Cramer, H. J. Crawford, V. I. Danilov, D. Dayton, M. DeMello, W. S. Deng, A. A. Derevschikov, M. Dialinas, H. Diaz, P. A. DeYoung, L. Didenko, D. Dimassimo, J. Dioguardi, W. Dominik, C. Drancourt, J. E. Draper, V. B. Dunin, J. C. Dunlop, V. Eckardt, W. R. Edwards, L. G. Efimov, T. Eggert, V. Emelianov, J. Engelage, G. Eppley, B. Erazmus, A. Etkin, P. Fachini, C. Feliciano, D. Ferenc, M. I. Ferguson, H. Fessler, E. Finch, V. Fine, Y. Fisysak, D. Flierl, I. Flores, K. J. Foley, D. Fritz, N. Gagunashvili, J. Gans, M. Gazdzicki, M. Germain, F. Geurts, V. Ghazikhanian, C. Gojak, J. Grabski, O. Grachov, M. Grau, D. Greiner, L. Greiner, V. Grigoriev, D. Grosnick, J. Gross, G. Guilloux, E. Gushin, J. Hall, T. J. Hallman, D. Hardtke, G. Harper, J. W. Harris, P. He, M. Heffner, S. Heppelmann, T. Herston, D. Hill, B. Hippolyte, A. Hirsch, E. Hjort, G. W. Hoffmann, M. Horsley, M. Howe, H. Z. Huang, T. J. Humanic, H. Hümmler, W. Hunt, J. Hunter, G. J. Igo, A. Ishihara, Y. I. Ivanshin, P. Jacobs, W. W. Jacobs, S. Jacobson, R. Jared, P. Jensen, I. Johnson, P. G. Jones, E. Judd, M. Kaneta, M. Kaplan, D. Keane, V. P. Kenney, A. Khodinov, J. Klay, S. R. Klein, A. Klyachko, G. Koehler, A. S. Konstantinov, V. Kormilitsyne, L. Kotchenda, I. Kotov, A. D. Kovalenko, M. Kramer, P. Kravtsov, K. Krueger, T. Krupien, P. Kuczewski, C. Kuhn, G. J. Kunde, C. L. Kunz, R. K. Kutuev, A. A. Kuznetsov, L. Lakehal-Ayat, J. Lamas-Valverde, M. A. C. Lamont, J. M. Landgraf, S. Lange, C. P. Lansdell, B. Lasiuk, F. Laue, A. Lebedev, T. LeCompte, W. J. Leonhardt, V. M. Leontiev, P. Leszczynski, M. J. LeVine, Q. Li, Q. Li, Z. Li, C.-J. Liaw, J. Lin, S. J. Lindenbaum, V. Lindenstruth, P. J. Lindstrom, M. A. Lisa, H. Liu, T. Ljubicic, W. J. Llope, G. LoCurto, H. Long, R. S. Longacre, M. Lopez-Noriega, D. Lopiano, W. A. Love, J. R. Lutz, D. Lynn, L. Madansky, R. Maier, R. Majka, A. Maliszewski, S. Margetis, K. Marks, R. Marstaller, L. Martin, J. Marx, H. S. Matis, Y. A. Matulenko, E. A. Matyushevski, C. McParland, T. S. McShane, J. Meier, Y. Melnick, A. Meschanin, P. Middlekamp, N. Mikhalin, B. Miller, Z. Milosevich, N. G. Minaev, B. Minor, J. Mitchell, E. Mogavero, V. A. Moiseenko, D. Moltz, C. F. Moore, V. Morozov, R. Morse, M. M. de Moura, M. G. Munhoz, G. S. Mutchler, J. M. Nelson, P. Nevski, T. Ngo, M. Nguyen, T. Nguyen, V. A. Nikitin, L. V. Nogach, T. Noggle, B. Norman, S. B. Nurushev, T. Nussbaum, J. Nystrand, G. Odyniec, A. Ogawa, C. A. Ogilvie, K. Olchanski, M. Oldenburg, D. Olson, G. A. Ososkov, G. Ott, D. Padrazo, G. Paic, S. U. Pandey, Y. Panebratsev, S. Y. Panitkin, A. I. Pavlinov, T. Pawlak, M. Pentia, V. Perevotchikov, W. Peryt, V. A. Petrov, W. Pinganaud, S. Pirogov, E. Platner, J. Pluta, I. Polk, N. Porile, J. Porter, A. M. Poskanzer, E. Potrebenikova, D. Prindle, C. Pruneau, J. Puskar-Pasewicz, G. Rai, J. Rasson, O. Ravel, R. L. Ray, S. V. Razin, D. Reichhold, J. Reid, R. E. Renfordt, F. Retiere, A. Ridiger, J. Riso, H. G. Ritter, J. B. Roberts, D. Roehrich, O. V. Rogachevski, J. L. Romero, C. Roy, D. Russ, V. Rykov, I. Sakrejda, R. Sanchez, Z. Sandler, J. Sandweiss, P. Sappenfield, A. C. Saulys, I. Savin, J. Schambach, R. P. Scharenberg, J. Scheblien, R. Scheetz, R. Schlueter, N. Schmitz, L. S. Schroeder, M. Schulz, A. Schüttauf, J. Sedlmeir, J. Seger, D. Seliverstov, J. Seyboth, P. Seyboth, R. Seymour, E. I. Shakaliev, K. E. Shestermanov, Y. Shi, S. S. Shimanskii, D. Shuman, V. S. Shvetcov, G. Skoro, N. Smirnov, L. P. Smykov, R. Snellings, K. Solberg, J. Sowinski, H. M. Spinka, B. Srivastava, E. J.

- Stephenson, R. Stock, A. Stolpovsky, N. Stone, R. Stone, M. Strikhanov, B. Stringfellow, H. Stroebele, C. Struck, A. A. P. Suaide, E. Sugarbaker, C. Suire, T. J. M. Symons, J. Takahashi, A. H. Tang, A. Tarchini, J. Tarzian, J. H. Thomas, V. Tikhomirov, A. Szanto de Toledo, S. Tonse, T. Trainor, S. Trentalange, M. Tokarev, M. B. Tonjes, V. Trofimov, O. Tsai, K. Turner, T. Ullrich, D. G. Underwood, I. Vakula, G. Van Buren, A. M. VanderMolen, A. Vanyashin, I. M. Vasilevski, A. N. Vasiliev, S. E. Vigdor, G. Visser, S. A. Voloshin, C. Vu, F. Wang, H. Ward, D. Weerasundara, R. Weidenbach, R. Wells, R. Wells, T. Wenaus, G. D. Westfall, J. P. Whitfield, C. Whitten, H. Wiedenmann, R. Willson, K. Wilson, J. Wirth, J. Wisdom, S. W. Wissink, R. Witt, J. Wolf, L. Wood, N. Xu, Z. Xu, A. E. Yakutin, E. Yamamoto, J. Yang, P. Yepes, A. Yokosawa, V. I. Yurevich, Y. V. Zanevski, J. Zhang, W. M. Zhang, J. Zhu, D. Zimmerman, R. Zoukharneev, A. N. Zubarev, Elliptic flow in $au + au$ collisions at $\sqrt{s_{NN}} = 130\text{GeV}$, Phys. Rev. Lett. 86 (2001) 402–407. doi:10.1103/PhysRevLett.86.402.
URL <https://link.aps.org/doi/10.1103/PhysRevLett.86.402>
- [53] W. Israel, J. Stewart, Transient relativistic thermodynamics and kinetic theory, Annals of Physics 118 (2) (1979) 341–372. doi:[https://doi.org/10.1016/0003-4916\(79\)90130-1](https://doi.org/10.1016/0003-4916(79)90130-1).
URL <https://www.sciencedirect.com/science/article/pii/0003491679901301>
- [54] H. Song, U. W. Heinz, Causal viscous hydrodynamics in 2+1 dimensions for relativistic heavy-ion collisions, Phys. Rev. C 77 (2008) 064901. arXiv:0712.3715, doi:10.1103/PhysRevC.77.064901.
- [55] B. Schenke, S. Jeon, C. Gale, (3+1)D hydrodynamic simulation of relativistic heavy-ion collisions, Phys. Rev. C 82 (2010) 014903. arXiv:1004.1408, doi:10.1103/PhysRevC.82.014903.
- [56] C. Shen, Z. Qiu, H. Song, J. Bernhard, S. Bass, U. Heinz, The iEBE-VISHNU code package for relativistic heavy-ion collisions, Comput. Phys. Commun. 199 (2016) 61–85. arXiv:1409.8164, doi:10.1016/j.cpc.2015.08.039.
- [57] F. Cooper, G. Frye, Comment on the Single Particle Distribution in the Hydrodynamic and Statistical Thermodynamic Models of Multiparticle Production, Phys. Rev. D 10 (1974) 186. doi:10.1103/PhysRevD.10.186.
- [58] J. Novak, K. Novak, S. Pratt, J. Vredevoogd, C. Coleman-Smith, R. Wolpert, Determining Fundamental Properties of Matter Created in Ultrarelativistic Heavy-Ion Collisions, Phys. Rev. C 89 (3) (2014) 034917. arXiv:1303.5769, doi:10.1103/PhysRevC.89.034917.
- [59] S. Pratt, E. Sangaline, P. Sorensen, H. Wang, Constraining the equation of state of superhadronic matter from heavy-ion collisions, Phys. Rev. Lett. 114 (2015) 202301. doi:10.1103/PhysRevLett.114.202301.
URL <https://link.aps.org/doi/10.1103/PhysRevLett.114.202301>
- [60] B. Andersson, G. Gustafson, G. Ingelman, T. Sjostrand, Parton Fragmentation and String Dynamics, Phys. Rept. 97 (1983) 31–145. doi:10.1016/0370-1573(83)90080-7.
- [61] R. J. Fries, B. Muller, C. Nonaka, S. A. Bass, Hadronization in heavy ion collisions: Recombination and fragmentation of partons, Phys. Rev. Lett. 90 (2003) 202303. arXiv:nucl-th/0301087, doi:10.1103/PhysRevLett.90.202303.

- [62] W. Zhao, W. Ke, W. Chen, T. Luo, X.-N. Wang, From Hydrodynamics to Jet Quenching, Coalescence, and Hadron Cascade: A Coupled Approach to Solving the RAA $\otimes$ v2 Puzzle, *Phys. Rev. Lett.* 128 (2) (2022) 022302. [arXiv:2103.14657](#), [doi:10.1103/PhysRevLett.128.022302](#).
- [63] M. Bleicher, et al., Relativistic hadron hadron collisions in the ultrarelativistic quantum molecular dynamics model, *J. Phys. G* 25 (1999) 1859–1896. [arXiv:hep-ph/9909407](#), [doi:10.1088/0954-3899/25/9/308](#).
- [64] Y. Nara, N. Otuka, A. Ohnishi, K. Niita, S. Chiba, Relativistic nuclear collisions at 10a gev energies from $p + \text{Be}$ to $\text{au}+\text{au}$ with the hadronic cascade model, *Phys. Rev. C* 61 (1999) 024901. [doi:10.1103/PhysRevC.61.024901](#).
URL <https://link.aps.org/doi/10.1103/PhysRevC.61.024901>
- [65] J. Weil, et al., Particle production and equilibrium properties within a new hadron transport approach for heavy-ion collisions, *Phys. Rev. C* 94 (5) (2016) 054905. [arXiv:1606.06642](#), [doi:10.1103/PhysRevC.94.054905](#).
- [66] M. D. McKay, R. J. Beckman, W. J. Conover, A comparison of three methods for selecting values of input variables in the analysis of output from a computer code, *Technometrics* 21 (2) (1979) 239–245.
URL <http://www.jstor.org/stable/1268522>
- [67] A. K. Chaudhuri, Viscous fluid dynamics in Au+Au collisions at RHIC (1 2008). [arXiv:0801.3180](#).
- [68] V. Roy, A. K. Chaudhuri, 2+1 dimensional hydrodynamics including bulk viscosity: A Systematics study, *Phys. Rev. C* 85 (2012) 024909, [Erratum: *Phys.Rev.C* 85, 049902 (2012)]. [arXiv:1109.1630](#), [doi:10.1103/PhysRevC.85.024909](#).
- [69] P. Bozek, Flow and interferometry in 3+1 dimensional viscous hydrodynamics, *Phys. Rev. C* 85 (2012) 034901. [arXiv:1110.6742](#), [doi:10.1103/PhysRevC.85.034901](#).
- [70] L. Del Zanna, V. Chandra, G. Inghirami, V. Rolando, A. Beraudo, A. De Pace, G. Pagliara, A. Drago, F. Becattini, Relativistic viscous hydrodynamics for heavy-ion collisions with ECHO-QGP, *Eur. Phys. J. C* 73 (2013) 2524. [arXiv:1305.7052](#), [doi:10.1140/epjc/s10052-013-2524-5](#).
- [71] I. Karpenko, P. Huovinen, M. Bleicher, A 3+1 dimensional viscous hydrodynamic code for relativistic heavy ion collisions, *Comput. Phys. Commun.* 185 (2014) 3016–3027. [arXiv:1312.4160](#), [doi:10.1016/j.cpc.2014.07.010](#).
- [72] Y. Akamatsu, S.-i. Inutsuka, C. Nonaka, M. Takamoto, A new scheme of causal viscous hydrodynamics for relativistic heavy-ion collisions: A Riemann solver for quark-gluon plasma, *J. Comput. Phys.* 256 (2014) 34–54. [arXiv:1302.1665](#), [doi:10.1016/j.jcp.2013.08.047](#).
- [73] K. Murase, Causal hydrodynamic fluctuations and their effects on high-energy nuclear collisions, Ph.D. thesis, Tokyo U. (7 2015). [doi:10.15083/00072981](#).
- [74] D. Bazow, U. W. Heinz, M. Strickland, Massively parallel simulations of relativistic fluid dynamics on graphics processing units with CUDA, *Comput. Phys. Commun.* 225 (2018) 92–113. [arXiv:1608.06577](#), [doi:10.1016/j.cpc.2017.01.015](#).

- [75] K. Okamoto, Y. Akamatsu, C. Nonaka, A new relativistic hydrodynamics code for high-energy heavy-ion collisions, *Eur. Phys. J. C* 76 (10) (2016) 579. [arXiv:1607.03630](#), [doi:10.1140/epjc/s10052-016-4433-x](#).
- [76] L.-G. Pang, H. Petersen, X.-N. Wang, Pseudorapidity distribution and decorrelation of anisotropic flow within the open-computing-language implementation CLVisc hydrodynamics, *Phys. Rev. C* 97 (6) (2018) 064918. [arXiv:1802.04449](#), [doi:10.1103/PhysRevC.97.064918](#).
- [77] L. Du, U. Heinz, (3+1)-dimensional dissipative relativistic fluid dynamics at non-zero net baryon density, *Comput. Phys. Commun.* 251 (2020) 107090. [arXiv:1906.11181](#), [doi:10.1016/j.cpc.2019.107090](#).
- [78] N. Touroux, M. Kitazawa, K. Murase, M. Nahrgang, Efficient Solver of Relativistic Hydrodynamics with an Implicit Runge–Kutta Method, *PTEP* 2024 (6) (2024) 063D01. [arXiv:2306.12696](#), [doi:10.1093/ptep/ptae058](#).
- [79] J. Adam, et al., Centrality Dependence of the Charged-Particle Multiplicity Density at Midrapidity in Pb-Pb Collisions at $\sqrt{s_{NN}} = 5.02$ TeV , *Phys. Rev. Lett.* 116 (22) (2016) 222302. [arXiv:1512.06104](#), [doi:10.1103/PhysRevLett.116.222302](#).
- [80] S. Fujibayashi, K. Kiuchi, N. Nishimura, Y. Sekiguchi, M. Shibata, Mass Ejection from the Remnant of a Binary Neutron Star Merger: Viscous-Radiation Hydrodynamics Study, *Astrophys. J.* 860 (1) (2018) 64. [arXiv:1711.02093](#), [doi:10.3847/1538-4357/aabafd](#).
- [81] Y. Gao, K. Hayashi, K. Kiuchi, A. T.-L. Lam, H.-J. Kuan, M. Shibata, Convective stability analysis of massive neutron stars formed in binary mergers (1 2025). [arXiv:2501.19053](#).
- [82] T. Kuroda, K. Kotake, T. Takiwaki, Fully General Relativistic Simulations of Core-Collapse Supernovae with An Approximate Neutrino Transport, *Astrophys. J.* 755 (2012) 11. [arXiv:1202.2487](#), [doi:10.1088/0004-637X/755/1/11](#).
- [83] G. S. Denicol, T. Kodama, T. Koide, P. Mota, Stability and Causality in relativistic dissipative hydrodynamics, *J. Phys. G* 35 (2008) 115102. [arXiv:0807.3120](#), [doi:10.1088/0954-3899/35/11/115102](#).
- [84] S. Pu, T. Koide, D. H. Rischke, Does stability of relativistic dissipative fluid dynamics imply causality?, *Phys. Rev. D* 81 (2010) 114039. [arXiv:0907.3906](#), [doi:10.1103/PhysRevD.81.114039](#).
- [85] G. S. Denicol, S. Jeon, C. Gale, Transport coefficients of bulk viscous pressure in the 14-moment approximation, *Phys. Rev. C* 90 (2014) 024912. [doi:10.1103/PhysRevC.90.024912](#).
URL <https://link.aps.org/doi/10.1103/PhysRevC.90.024912>
- [86] F. D. Lora-Clavijo, J. P. Cruz-Perez, F. S. Guzman, J. A. Gonzalez, Exact solution of the 1d riemann problem in newtonian and relativistic hydrodynamics (2013). [arXiv:1303.3999](#).
- [87] S. S. Gubser, Symmetry constraints on generalizations of bjorken flow, *Physical Review D* 82 (8) (oct 2010). [doi:10.1103/physrevd.82.085027](#).
URL <https://doi.org/10.1103/physrevd.82.085027>

- [88] A. H. Taub, Relativistic Rankine-Hugoniot Equations, *Phys. Rev.* 74 (1948) 328–334. doi:[10.1103/PhysRev.74.328](https://doi.org/10.1103/PhysRev.74.328).
- [89] L. D. Landau, E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, 1987.
- [90] E. Hairer, G. Wanner, Stiff differential equations solved by radau methods, *Journal of Computational and Applied Mathematics* 111 (1) (1999) 93–111. doi:[https://doi.org/10.1016/S0377-0427\(99\)00134-X](https://doi.org/10.1016/S0377-0427(99)00134-X).
URL <https://www.sciencedirect.com/science/article/pii/S037704279900134X>
- [91] I. Arie, *A First Course in the Numerical Analysis of Differential Equations*, Cambridge University Press, 1996.
- [92] A. Kurganov, E. Tadmor, New high-resolution central schemes for nonlinear conservation laws and convection–diffusion equations, *Journal of Computational Physics* 160 (1) (2000) 241–282. doi:<https://doi.org/10.1006/jcph.2000.6459>.
URL <https://www.sciencedirect.com/science/article/pii/S0021999100964593>
- [93] M. McNelis, D. Bazow, U. Heinz, Anisotropic fluid dynamical simulations of heavy-ion collisions, *Comput. Phys. Commun.* 267 (2021) 108077. arXiv:2101.02827, doi:10.1016/j.cpc.2021.108077.
- [94] H. Petersen, J. Steinheimer, G. Burau, M. Bleicher, H. Stöcker, A Fully Integrated Transport Approach to Heavy Ion Reactions with an Intermediate Hydrodynamic Stage, *Phys. Rev. C* 78 (2008) 044901. arXiv:0806.1695, doi:10.1103/PhysRevC.78.044901.
- [95] H. Holopainen, H. Niemi, K. J. Eskola, Event-by-event hydrodynamics and elliptic flow from fluctuating initial state, *Phys. Rev. C* 83 (2011) 034901. arXiv:1007.0368, doi:10.1103/PhysRevC.83.034901.
- [96] T. Hirano, K. Tsuda, K. Kajimoto, Hydrodynamic analysis of noncentral Pb + Pb collisions at 158-A-GeV (11 2000). arXiv:nucl-th/0011087.
- [97] Y. Tachibana, Hydrodynamic response to jet propagation in quark-gluon plasma, Ph.D. thesis, Tokyo U. (2014). doi:10.15083/00008070.
- [98] N. Touroux, M. Kitazawa, K. Murase, M. Nahrgang (in preparation).
- [99] D. G. Anderson, Iterative procedures for nonlinear integral equations, *J. ACM* 12 (4) (1965) 547–560. doi:10.1145/321296.321305.
URL <https://doi.org/10.1145/321296.321305>
- [100] J. Kuntzmann, Neuere entwicklungen der methode von runge und kutta, *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* 41 (S1) (1961) T28–T31. doi:<https://doi.org/10.1002/zamm.19610411317>.
URL <https://onlinelibrary.wiley.com/doi/abs/10.1002/zamm.19610411317>
- [101] J. C. Butcher, Implicit runge-kutta processes, *Mathematics of computation* 18 (85) (1964) 50–64.
- [102] R. Courant, K. Friedrichs, H. Lewy, Über die partiellen differenzengleichungen der mathematischen physik, *Mathematische Annalen* 100 (1) (1928) 32–74. doi:10.1007/BF01448839.
URL <https://doi.org/10.1007/BF01448839>

- [103] R. Courant, K. Friedrichs, H. Lewy, On the partial difference equations of mathematical physics, IBM Journal of Research and Development 11 (2) (1967) 215–234. doi:10.1147/rd.112.0215.
- [104] D. Bazow, (3+1)-dimensional viscous hydrodynamic code on GPUs using CUDA (2017).
URL <https://github.com/bazow/gpu-vh>
- [105] J. Bernhard, Particlization model (cooper-frye sampler) for relativistic heavy-ion collisions (2018).
URL <https://github.com/Duke-QCD/frzout>
- [106] S. A. Bass, et al., Microscopic models for ultrarelativistic heavy ion collisions, Prog. Part. Nucl. Phys. 41 (1998) 255–369. arXiv:nuc1-th/9803035, doi:10.1016/S0146-6410(98)00058-1.
- [107] J. Bernhard, Duke-qcd heavy-ion collisions event generator (2015).
URL <https://github.com/Duke-QCD/hic-eventgen>
- [108] L. M. Satarov, M. N. Dmitriev, I. N. Mishustin, Equation of state of hadron resonance gas and the phase diagram of strongly interacting matter, Phys. Atom. Nucl. 72 (2009) 1390–1415. arXiv:0901.1430, doi:10.1134/S1063778809080146.
- [109] P. Huovinen, H. Petersen, Particlization in hybrid models, Eur. Phys. J. A 48 (2012) 171. arXiv:1206.3371, doi:10.1140/epja/i2012-12171-9.
- [110] A. Bilandzic, R. Snellings, S. Voloshin, Flow analysis with cumulants: Direct calculations, Phys. Rev. C 83 (2011) 044913. arXiv:1010.0233, doi:10.1103/PhysRevC.83.044913.
- [111] H. Song, S. A. Bass, U. Heinz, Elliptic flow in 200 A GeV Au+Au collisions and 2.76 A TeV Pb+Pb collisions: insights from viscous hydrodynamics + hadron cascade hybrid model, Phys. Rev. C 83 (2011) 054912, [Erratum: Phys.Rev.C 87, 019902 (2013)]. arXiv:1103.2380, doi:10.1103/PhysRevC.83.054912.

Titre: Solveur implicite efficient pour l'hydrodynamique relativiste dans la modélisation dynamique des collisions d'ions lourds

Mots clés: Collisions d'ions lourds, Hydrodynamique, Runge-Kutta implicite

Résumé: La riche structure de phase de la chromodynamique quantique (QCD) fait l'objet d'études approfondies depuis des décennies, tant sur le plan théorique qu'expérimental. En particulier, les propriétés du plasma de quark et gluon (QGP), qui se forme dans un milieu à haute température, ont été activement étudiées dans le cadre de collisions relativistes d'ions lourds dans de grandes installations expérimentales telles que le Relativistic Heavy Ion Collider (RHIC) et le Large Hadron Collider (LHC). L'une des conclusions remarquables de ces expériences est que l'évolution temporelle du QGP dans le milieu chaud créé par les collisions est bien décrite par l'hydrodynamique relativiste à faible viscosité. Dans la modélisation dynamique des collisions d'ions lourds, l'hydrodynamique nu-

merique joue donc un rôle central.

Dans cette thèse, nous proposons une nouvelle méthode pour résoudre les équations d'hydrodynamique relativiste dans la modélisation dynamique des collisions d'ions lourds. Cette méthode utilise une méthode implicite de Runge-Kutta pour l'intégration temporelle avec un solveur itératif à point fixe optimisé localement. Nous démontrons que la précision et le coût de calcul de la nouvelle méthode sont meilleurs que ceux des méthodes explicites utilisées habituellement. L'avantage est plus prononcé lorsque la méthode est appliquée à des équations hydrodynamiques visqueuses. Nous implémentons le solveur dans un modèle dynamique de collisions d'ions lourds et effectuons des simulations pour reproduire les données expérimentales.

Title: Efficient implicit solver for relativistic hydrodynamics in the dynamical modeling of heavy-ion collisions

Keywords: Heavy-ion collisions, Hydrodynamics, Implicit Runge-Kutta

Abstract: The rich phase structure of Quantum Chromodynamics (QCD) has been extensively studied for decades, both theoretically and experimentally. In particular, properties of the quark-gluon plasma (QGP), which is realized in the high-temperature medium, have been actively investigated in relativistic heavy-ion collisions at large experimental facilities such as the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC). One of the remarkable findings of these experiments is that the time evolution of the QGP in the hot medium created by the collisions is well described by relativistic hydrodynamics with low viscosity. In the dynamical modeling of heavy-ion collisions, therefore, computational hydrodynam-

ics plays a central role.

In this thesis, we propose a new method to solve the relativistic hydrodynamic equations in the dynamical modeling of heavy-ion collisions. This method uses an implicit Runge-Kutta method for time integration with a locally optimized fixed-point iterative solver. We demonstrate that the accuracy and computational cost of the new method are better than those of conventionally used explicit ones. The advantage is more pronounced when the method is applied to viscous hydrodynamic equations. We implement the solver into a dynamical model of heavy-ion collisions and perform simulations to reproduce experimental data.