OUKA
The University of Osaka
Institutional Knowledge Archive

Title	Leaf as a Poincaré convex domain associated with an endomorphism on a real inner product space
Author(s)	小川, 裕之
Citation	大阪大学, 2025, 博士論文
Version Type	VoR
URL	https://doi.org/10.18910/105278
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

The University of Osaka

Leaf as a Poincaré convex domain associated with an endomorphism on a real inner product space

リーフ：実内積空間上の自己準同型写像に伴うポアンカレ凸領域

Hiroyuki Ogawa

July, 2025

Department of Mathematics, Graduate School of Science,
The University of Osaka

Dedicated to late Professor Yoshihiko Yamamoto

Contents

Abstract **4**

Acknowledgments **5**

1 Introduction **6**

2 Preliminaries **9**
2.1 Normal endomorphisms 9
2.2 The Poincaré metric on $\mathfrak{H}$ 10

3 Leaf and real eigenvalues **12**

4 Numerical range **14**
4.1 Numerical range and the fundamental theorems 14
4.2 Numerical range vs. leaf 15
4.3 Proof of the fundamental theorems of numerical range 16

5 Leaf of an endomorphism of dimension at most two **19**

6 Leaf in $\overline{\mathfrak{H}}$ and eigenvalues with nonnegative imaginary part **23**
6.1 Subleaf and eigenvalue inclusion of leaf 23
6.2 Closed curve in the leaf corresponding to a two-dimensional subspace . . . 24
6.3 Proofs of the Poincaré convexity 27

7 Leaf of a normal endomorphism **29**
7.1 Subleaf corresponding to the real part of an eigenspace 29
7.2 The geodesic bridge 34
7.3 Leaf of a normal endomorphism 35

Appendix : Note on the contents of this thesis **37**

Bibliography **38**

Abstract

The leaf is defined as a subset of the closed upper half-plane associated with an endomorphism on a real inner product space. For spaces of dimension three or higher, the leaf is convex with respect to the Poincaré metric and contains all eigenvalues whose imaginary parts are nonnegative. Moreover, the leaf of a normal endomorphism is the minimal Poincaré convex domain containing all eigenvalues whose imaginary parts are nonnegative. The most commonly studied convex domain containing eigenvalues is numerical range. The numerical range is convex with respect to the Euclidean metric on $\mathbb{C}$, and thus it contains less information about real eigenvalues than the leaf. We propose a new visual approach to the study of endomorphisms.

Keywords : Numerical range, Poincalré metric, Real inner product, Leaf

MSC 2020 : 15A60, 47A12

e-mail : ogawa.hiroyuki.sci@osaka-u.ac.jp

Acknowledgments

The author would like to thank Hiroaki Nakamura, Yu Yasufuku, Michihisa Wakui, and Kanji Namba for their invaluable remarks and for providing valuable information on related research. Special thanks are due to Hiroaki Nakamura for his continued support and insightful suggestions throughout the course of the author's research.

Chapter 1

Introduction

In this paper we define a subset of the closure of the complex upper half plane associated with an endomorphism on a real inner product space, which is called the leaf, and show the structure theorems (Theorem A, B and C). The structure theorems on leaf correspond to the fundamental theorems on numerical range (Theorem 4.1, 4.2 and 4.3), which is the well-known convex domain containing all eigenvalues, as below. Both leaf and numerical range are simply connected domains containing eigenvalues, but leaf is convex with respect to the Poincaré metric on the closure of the complex upper half plane, and numerical range is convex with respect to the Euclidean metric on $\mathbb{C}$. The structure theorems on leaf and the fundamental theorems on numerical range correspond to each other, but they are proved in the different ways because of the difference in the shape of leaf and numerical range. The most significant difference lies in their intersections with the real axis: the intersection of the leaf and the real axis is exactly the set of all real eigenvalues, whereas that of the numerical range is a real interval containing all real eigenvalues. Since the leaf captures information about individual real eigenvalues, it is possible to study the behavior of an endomorphism near the eigenspace corresponding to a real eigenvalue.

Let V be a finite dimensional inner product space over $\mathbb{R}$. Since the dimension of the space V is important in this paper, an endomorphism on a d-dimensional space V is sometimes simply referred to as an endomorphism of dimension d. The angle $\theta = \theta(\mathsf{v}_1, \mathsf{v}_2) \in [0, \pi] \subset \mathbb{R}$ of two nonzero vectors $\mathsf{v}_1, \mathsf{v}_2 \in V^\times$ $(= V \setminus \{\mathsf{0}\})$ is defined by $\mathsf{v}_1 \cdot \mathsf{v}_2 = \|\mathsf{v}_1\| \, \|\mathsf{v}_2\| \cos\theta$. Let φ be an endomorphism on V. We consider the continuous map $\mathbb{L}_\varphi : V^\times \to \mathbb{C}$ is defined by

$$\mathbb{L}_\varphi(\mathsf{v}) = \frac{\|\varphi(\mathsf{v})\|}{\|\mathsf{v}\|} \, e^{\sqrt{-1}\,\theta(\mathsf{v},\, \varphi(\mathsf{v}))} \qquad \text{for } \mathsf{v} \in V^\times.$$

When $\varphi(\mathsf{v}) = \mathsf{0}$, we don't define the angle $\theta(\mathsf{v}, \varphi(\mathsf{v}))$, but can put $\mathbb{L}_\varphi(\mathsf{v}) = 0$ because of

$\|\varphi(\mathsf{v})\| = 0$. Since $0 \leq \theta(\mathsf{v}, \varphi(\mathsf{v})) \leq \pi$, the imaginary part of $\mathbb{L}_\varphi(\mathsf{v})$ is nonnegative. $\mathbb{L}_\varphi(\mathsf{v})$ belongs to the closure of the upper half plane $\overline{\mathfrak{H}} = \mathfrak{H} \cup \mathbb{R} \cup \{\infty\}$, where $\mathfrak{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.

Definition 1.1. The image of $\mathbb{L}_\varphi$

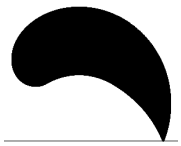
$$\Psi(\varphi) = \{\mathbb{L}_\varphi(\mathsf{v}) \mid \mathsf{v} \in V^\times\} \subset \overline{\mathfrak{H}}$$

is called the *leaf* of the endomorphism φ.

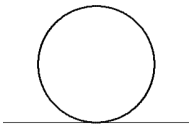
Example 1.2. Figure 1 contains leaves of five endomorphisms on an real Euclid spaces $\mathbb{R}^n$ with standard inner product. Such endomorphisms are given by some square matrices and direct sums of square matrices. Put $\mathsf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathsf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathsf{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $J(\lambda;\, d)$ be the Jordan cell with eigenvalue $\lambda \in \mathbb{R}$ and of degree d. The straight line at the bottom on each figure is the real axis.



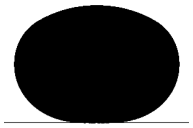
(a)
$(1) \oplus (3\,\mathsf{J}{+}2\,\mathsf{K})$
$\oplus(-4\,\mathsf{E}{+}10\,\mathsf{J})$



(b) $\begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$



(c) $J(0;2)$



(d) $J(0;3)$



(e) $(-2) \oplus (6\,\mathsf{J})$
$\oplus(2\,\mathsf{E} + \mathsf{J})$

Figure 1.1: Examples of leaves

Leaf is a bounded closed set in the Euclidean metric on $\mathbb{C}$. Numerical range is convex with respect to the Euclidean metric on $\mathbb{C}$, but, in generally, leaf is not convex on the Euclidean metric on $\mathbb{C}$. In Chapter 4, we will show that leaf is path-connected (see Proposition 3.2), and that leaf and the real axis share only all real eigenvalues (see Theorem 3.3). In Chapter 5, we will see that leaf of endomorphism of dimension at most two is simple (see Theorem 5.1). The purpose of this paper is to show the following structure theorems of leaf of endomorphism of dimension three or higher.

Theorem A (Poincaré Convexity)**.** *Leaf is convex with respect to the Poincaré metric.*

Theorem B (Eigenvalue inclusion)**.** *Leaf contains all eigenvalues whose imaginary parts are nonnegative.*

The smallest convex domain containing all eigenvalues with nonnegative imaginary part is the convex filled geodesic polygon whose vertices are eigenvalues, that is called the *eigenvalue geodesic polygon*. The structure theorems A and B lead that the leaf of an endomorphism of dimension three or higher includes the eigenvalue geodesic polygon.

Theorem C (Normal case)**.** *Leaf of a normal endomorphism coincides with the eigenvalue geodesic polygon.*

Remark 1.3. While this paper focuses on finite-dimensional cases, the concept of the leaf can be extended to any bounded linear operator on an infinite-dimensional inner product space by taking the closure, and the structure theorems still hold in that setting.

In Chapter 2, we review some basic facts on a normal endomorphism and on the Poincaré metric. Chapter 3 introduces the notion of a leaf of an endomorphism and analyzes its intersection with the real axis. In Chapter 4, we describe the difference between the numerical range and the leaf. In Chapter 5, we determine the leaf of an endomorphism of dimension at most two. In Chapters 6 and 7, we prove the structure theorems for the leaf of any endomorphism on a real inner product space of dimension three or higher.

Chapter 2

Preliminaries

2.1 Normal endomorphisms

Let V be a real inner product space of finite dimension, and φ a normal endomorphism on V. Then V is the orthogonal direct sum of φ-stable subspaces of dimension at most two (see Theorem 10.10 in [5]). In this section, we provide a brief introduction to this fact.

Put $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \sqrt{-1}V$ the coefficient extension of V to $\mathbb{C}$. There exists a Hermitian inner product on $V_{\mathbb{C}}$ which extends the real inner product on V. The same dot symbol is used for both inner products. Each $\mathsf{v} \in V_{\mathbb{C}} = V \oplus \sqrt{-1}V$ is expressed by $\mathsf{v}_r + \sqrt{-1}\mathsf{v}_i$ for some v_r, $\mathsf{v}_i \in V$. v_r is called the real part of v, denoted by $\operatorname{Re}\mathsf{v}$, and v_i is called the imaginary part of v, denoted by $\operatorname{Im}\mathsf{v}$. $\overline{\mathsf{v}} = \mathsf{v}_r - \sqrt{-1}\mathsf{v}_i$ is called the complex conjugate of v. Each endomorphism φ on V can be extended to the endomorphism on $V_{\mathbb{C}}$ over $\mathbb{C}$. The same symbol φ is used for both endomorphisms.

Let $\mathsf{v} \in V_{\mathbb{C}}$ be an eigenvector of φ corresponding to an eigenvalue $\lambda \in \mathbb{C}$.

$$\begin{aligned}
&\varphi(\mathsf{v}) = \varphi(\operatorname{Re}\mathsf{v} + \sqrt{-1}\operatorname{Im}\mathsf{v}) = \varphi(\operatorname{Re}\mathsf{v}) + \sqrt{-1}\,\varphi(\operatorname{Im}\mathsf{v}) \\
&\varphi(\mathsf{v}) = \lambda\mathsf{v} = (\operatorname{Re}\lambda\operatorname{Re}\mathsf{v} - \operatorname{Im}\lambda\operatorname{Im}\mathsf{v}) + \sqrt{-1}\,(\operatorname{Im}\lambda\operatorname{Re}\mathsf{v} + \operatorname{Re}\lambda\operatorname{Im}\mathsf{v}) \\
&\quad\therefore\quad (\varphi(\operatorname{Re}\mathsf{v}),\ \varphi(\operatorname{Im}\mathsf{v})) = (\operatorname{Re}\mathsf{v},\ \operatorname{Im}\mathsf{v})\,(\operatorname{Re}\lambda\,\mathsf{E} + \operatorname{Im}\lambda\,\mathsf{J})
\end{aligned}$$

where E and J are the quadratic matrices defined in Example 1.2. The complex conjugate $\overline{\mathsf{v}}$ is an eigenvector of φ corresponding to the eigenvalue $\overline{\lambda}$. If $\lambda \in \mathbb{R}$, then $\operatorname{Re}\mathsf{v}$, $\operatorname{Im}\mathsf{v}$ and $\overline{\mathsf{v}}$ are eigenvectors of φ corresponding to λ. If $\lambda \notin \mathbb{R}$, since v and $\overline{\mathsf{v}}$ are linearly independent on $\mathbb{C}$, $\{\operatorname{Re}\mathsf{v},\ \operatorname{Im}\mathsf{v}\}$ is a basis of the φ-stable subspace $\langle \operatorname{Re}\mathsf{v},\ \operatorname{Im}\mathsf{v}\rangle = (\mathbb{C}\,\mathsf{v} + \mathbb{C}\,\overline{\mathsf{v}}) \cap V$. Especially, if v and $\overline{\mathsf{v}}$ are orthogonal in $V_{\mathbb{C}}$, then $\operatorname{Re}\mathsf{v}$ and $\operatorname{Im}\mathsf{v}$ are orthogonal in V.

Because

$$\mathrm{Re}\,\mathsf{v}\cdot\mathrm{Im}\,\mathsf{v}=\frac{\mathsf{v}+\overline{\mathsf{v}}}{2}\cdot\frac{\mathsf{v}-\overline{\mathsf{v}}}{2\sqrt{-1}}=\frac{-1}{4}(\|\mathsf{v}\|^2-\mathsf{v}\cdot\overline{\mathsf{v}}+\overline{\mathsf{v}}\cdot\mathsf{v}-\|\overline{\mathsf{v}}\|^2)=\frac{1}{2}\mathrm{Re}\,(\mathsf{v}\cdot\overline{\mathsf{v}})=0.$$

$\{\mathrm{Re}\,\mathsf{v}/\|\mathrm{Re}\,\mathsf{v}\|,\,\mathrm{Im}\,\mathsf{v}/\|\mathrm{Im}\,\mathsf{v}\|\}$ is an orthonormal basis of $\langle\mathrm{Re}\,\mathsf{v},\,\mathrm{Im}\,\mathsf{v}\rangle$. Moreover, since $\|\mathrm{Re}\,\mathsf{v}\|=\|\mathrm{Im}\,\mathsf{v}\|=\|\mathsf{v}\|/\sqrt{2}$, the matrix $\mathrm{Re}\,\lambda\,\mathsf{E}+\mathrm{Im}\,\lambda\,\mathsf{J}$ is the matrix representation of the restriction of φ to the subspace $\langle\mathrm{Re}\,\mathsf{v},\,\mathrm{Im}\,\mathsf{v}\rangle$ with respect to the orthonormal basis $\{\mathrm{Re}\,\mathsf{v}/\|\mathrm{Re}\,\mathsf{v}\|,\,\mathrm{Im}\,\mathsf{v}/\|\mathrm{Im}\,\mathsf{v}\|\}$.

The adjoint endomorphism φ^* of φ is defined by the endomorphism on V satisfying that $\varphi(\mathsf{v}_1)\cdot\mathsf{v}_2=\mathsf{v}_1\cdot\varphi^*(\mathsf{v}_2)$ for any $\mathsf{v}_1,\mathsf{v}_2\in V$. The adjoint endomorphism of φ as an endomorphism on $V_{\mathbb{C}}$ is equal to the extension to $V_{\mathbb{C}}$ of φ^* on V. An endomorphism that commutative with its adjoint is said to be normal. Each normal endomorphism on $V_{\mathbb{C}}$ is diagonalizable by an orthonormal basis consisting eigenvectors on $V_{\mathbb{C}}$. Every normal endomorphism on V has as the matrix representation the direct sum of the real eigenvalues and of the $a\,\mathsf{E}+b\,\mathsf{J}$-type quadratic matrices ($a,b\in\mathbb{R}$, $b\neq 0$) corresponding to the non-real eigenvalues ($a+b\sqrt{-1}$), given an appropriate orthonormal basis.

2.2 The Poincaré metric on $\mathfrak{H}$

The Poincaré metric on the complex upper half plane $\mathfrak{H}$ is defined by $ds=|dz|/\mathrm{Im}\,z$ for $z\in\mathfrak{H}$. The distance $\rho(z_1,\,z_2)$ between two points $z_1,\,z_2\in\overline{\mathfrak{H}}$ is given by

$$\rho(z_1,\,z_2)=\int_{z_1}^{z_2}|ds|=\log\frac{1+\delta(z_1,\,z_2)}{1-\delta(z_1,\,z_2)}\qquad\text{where}\quad\delta(z_1,\,z_2)=\frac{|z_1-z_2|}{|z_1-\overline{z_2}|}.$$

We see that $0\leq\delta(z_1,\,z_2)\leq 1$, and the function $\log\frac{1+k}{1-k}$ is monotonic increasing on k. $\mathbb{R}\cup\{\infty\}$ is called the line at infinity, because $\delta(z_1,\,z_2)=1$ and $\rho(z_1,\,z_2)=\infty$, if and only if at least one of z_1 and z_2 lies on $\mathbb{R}\cup\{\infty\}$. A geodesic in $\mathfrak{H}$ with respect to the Poincaré metric is either a vertical line segment parallel to the imaginary axis or a semicircular arc with its center on the real axis. A geodesic circle with respect to the Poincaré metric whose center is $z_0\in\mathfrak{H}$ and radius is $r=\log\frac{1+k}{1-k}$ for $0\leq k\leq 1$ is defined by

$$C_{\mathrm{P}}(z_0;\,k)=\{z\in\mathfrak{H}\mid\delta(z,\,z_0)=k\}=\{z\in\mathfrak{H}\mid\rho(z,\,z_0)=r\}.$$

By putting $z=x+y\sqrt{-1}$, $z_0=p+q\sqrt{-1}\in\mathfrak{H}\cup\mathbb{R}$ ($x,y,p,q\in\mathbb{R}$, $y\geq 0$, $q\geq 0$), the equation $\delta(z,\,z_0)=k$ or $\rho(z,\,z_0)=r$ is equivalent to

$$(x-p)^2+(y-\frac{1+k^2}{1-k^2}q)^2=((\frac{1+k^2}{1-k^2})^2-1)\,q^2=(\frac{2\,k}{1-k^2}\,q)^2.$$

Therefore, the geodesic circle $C_{\mathrm{P}}(z_0;\, k)$ is an ordinary circle in $\mathbb{C}$ with center $p+\frac{1+k^2}{1-k^2}q\sqrt{-1}$ and radius $\frac{2\,k}{1-k^2}q$. A geodesic circle with respect to the Poincaré metric is simply referred to as a geodesic circle, while an ordinary circle in the complex plane is referred to simply as a circle.

Chapter 3

Leaf and real eigenvalues

Let φ be an endomorphism on a real inner product space V of finite dimension. The continuous map $\mathbb{L}_\varphi$ on $V^\times$ $(= V \setminus \{\mathsf{0}\})$ to $\mathbb{C}$ defining by

$$\mathbb{L}_\varphi(\mathsf{v}) = \frac{\|\varphi(\mathsf{v})\|}{\|\mathsf{v}\|}\, e^{\sqrt{-1}\,\theta(\mathsf{v},\,\varphi(\mathsf{v}))} \qquad (\mathsf{v} \in V^\times),$$

where the angle $\theta = \theta(\mathsf{v},\, \varphi(\mathsf{v})) \in [0,\, \pi] \subset \mathbb{R}$ between v and $\varphi(\mathsf{v})$ is defined by $\mathsf{v} \cdot \varphi(\mathsf{v}) = \|\mathsf{v}\|\, \|\varphi(\mathsf{v})\| \cos\theta$. When $\varphi(\mathsf{v}) = \mathsf{0}$, we don't define the angle θ,but can put $\mathbb{L}_\varphi(\mathsf{v}) = 0$ because of $\|\varphi(\mathsf{v})\| = 0$. $\mathbb{L}_\varphi(k\mathsf{v}) = \mathbb{L}_\varphi(\mathsf{v})$ holds for any $k \in \mathbb{R}^\times$, because $\|\varphi(k\mathsf{v})\|/\|k\mathsf{v}\| = \|\varphi(\mathsf{v})\|/\|\mathsf{v}\|$ and $\theta(k\mathsf{v},\, \varphi(k\mathsf{v})) = \theta(\mathsf{v},\, \varphi(\mathsf{v}))$. So the continuous map $\mathbb{L}_\varphi$ is considered as the continuous map on the projective space $\mathbb{P}(V) = V^\times/\mathbb{R}^\times$.

For any $\mathsf{v}_1,\, \mathsf{v}_2 \in V^\times$, put $\sigma(\mathsf{v}_1,\, \mathsf{v}_2) = \sqrt{\|\mathsf{v}_1\|^2\, \|\mathsf{v}_2\|^2 - (\mathsf{v}_1 \cdot \mathsf{v}_2)^2}$. Then we have

$$\mathbb{L}_\varphi(\mathsf{v}) = \frac{\mathsf{v} \cdot \varphi(\mathsf{v}) + \sigma(\mathsf{v},\, \varphi(\mathsf{v}))\, \sqrt{-1}}{\|\mathsf{v}\|^2}.$$

The real part of $\mathbb{L}_\varphi(\mathsf{v})$ is equal to the Rayleigh quotient $\frac{\mathsf{v}\cdot\varphi(\mathsf{v})}{\|\mathsf{v}\|^2}$ of φ, and the square of absolute value of $\mathbb{L}_\varphi(\mathsf{v})$ to that of $\varphi^*\varphi$. The operator norm $\|\varphi\|$ of φ is the supremum of the absolute value of $\mathbb{L}_\varphi(\mathsf{v})$.

Definition 3.1 (leaf)**.** The image of $\mathbb{L}_\varphi$

$$\Psi(\varphi) = \mathbb{L}_\varphi(V^\times) = \mathbb{L}_\varphi(\mathbb{P}(V))$$

is called the *leaf* of the endomorphism φ.

Proposition 3.2. *$\Psi(\varphi)$ is path connected.*

Proof. Let $\lambda_1,\, \lambda_2 \in \Psi(\varphi)$, and take $\mathsf{v}_1,\, \mathsf{v}_2 \in V^\times$ satisfying that $\lambda_1 = \mathbb{L}_\varphi(\mathsf{v}_1)$ and $\lambda_2 = \mathbb{L}_\varphi(\mathsf{v}_2)$. The path $[0,\, 1] \ni t \mapsto \mathbb{L}_\varphi((1-t)\,\mathsf{v}_1 + t\,\mathsf{v}_2) \in \Psi(\varphi)$ in $\Psi(\varphi)$ connects λ_1 and λ_2. Hence, the leaf $\Psi(\varphi)$ is path connected. □

The leaf $\Psi(\varphi)$ is a bounded closed set in $\mathbb{C}$ because the projective space $\mathbb{P}(V)$ is compact. Since an argument θ belongs to the closed interval $[0, \pi]$, the value of $\mathbb{L}_\varphi$ belongs to the closure of the upper half plane $\overline{\mathfrak{H}} = \mathfrak{H} \cup \mathbb{R} \cup \{\infty\}$. The leaf is included in $\overline{\mathfrak{H}}$. We will see that the Poincaré metric on $\mathfrak{H}$ plays an important role in the study of the leaf in later chapters.

In closing this chapter, we present the following theorem concerning the real eigenvalues.

Theorem 3.3. *The intersection of the leaf and the real axis is the set of real eigenvalues.*

Proof. Let φ an endomorphism on a real inner product space of finite dimension V. Let $\lambda \in \mathbb{R}$ a real eigenvalue of φ, and $\mathsf{v} \in V$ an eigenvector of φ corresponding to λ. If $\lambda = 0$, then $\varphi(\mathsf{v}) = \mathsf{0}$, so $\mathbb{L}_\varphi(\mathsf{v}) = 0 = \lambda$. Assume that $\lambda \neq 0$. Since $\varphi(\mathsf{v}) = \lambda\,\mathsf{v}$ is parallel to v, the angle $\theta = \theta(\mathsf{v}, \varphi(\mathsf{v}))$ is equal to 0 if $\lambda > 0$, and to π if $\lambda < 0$. Since $\|\varphi(\mathsf{v})\|/\|\mathsf{v}\| = \|\lambda\,\mathsf{v}\|/\|\mathsf{v}\| = |\lambda|$, and $e^{\theta\sqrt{-1}} = \mathrm{sign}\lambda = \lambda/|\lambda|$, we have that $\mathbb{L}_\varphi(\mathsf{v}) = \lambda$. Thus $\lambda = \mathbb{L}_\varphi(\mathsf{v}) \in \Psi(\varphi) \cap \mathbb{R}$.

Conversely, we will show that each element $\lambda \in \Psi(\varphi) \cap \mathbb{R}$ is an eigenvalue of φ. Take $\mathsf{v} \in V^\times$ satisfying that $\lambda = \mathbb{L}_\varphi(\mathsf{v})$. In the case that $\lambda = 0$, it follows from $\mathbb{L}_\varphi(\mathsf{v}) = \lambda = 0$ that $\|\varphi(\mathsf{v})\| = 0$, i.e. $\varphi(\mathsf{v}) = \mathsf{0} = 0\,\mathsf{v}$, so 0 is an eigenvalue of φ. Assume that $\lambda \neq 0$. Since $\|\varphi(\mathsf{v})\| = |\mathbb{L}_\varphi(\mathsf{v})|\,\|\mathsf{v}\| = |\lambda|\,\|\mathsf{v}\| \neq 0$, the angle $\theta = \theta(\mathsf{v}, \varphi(\mathsf{v}))$ is defined. And θ is also the angle of the real number $\lambda = \mathbb{L}_\varphi(\varphi)$, so $\theta = 0$ or π. $\varphi(\mathsf{v})$ is parallel to v, namely v is an eigenvector of φ. Put λ' be the eigenvalue of φ satisfying that $\varphi(\mathsf{v}) = \lambda'\mathsf{v}$, then it follows from the first step in this proof that $\mathbb{L}_\varphi(\mathsf{v}) = \lambda'$. Hence $\lambda = \mathbb{L}_\varphi(\mathsf{v}) = \lambda'$ is an eigenvalue of φ. □

Chapter 4

Numerical range

4.1 Numerical range and the fundamental theorems

A well-known convex domain that contains all eigenvalues is the numerical range (cf. [2], [3]). Here, we introduce the fundamental theorems that served as the starting point for the study of the numerical range, as presented in Chapter 1 of Gustafson and Rao's textbook ([3]).

Let φ be a bounded linear operator on a Hermite inner product space V. The numerical range of φ is the subset in $\mathbb{C}$ defined by

$$W(\varphi) = \{\frac{\varphi(\mathsf{v}) \cdot \mathsf{v}}{\|\mathsf{v}\|^2} \mid \mathsf{v} \in V^\times\} = \{\varphi(\mathsf{v}) \cdot \mathsf{v} \mid \|\mathsf{v}\| = 1\},$$

that is the range of the Rayleigh quotients $\varphi(\mathsf{v}) \cdot \mathsf{v}/\|\mathsf{v}\|^2$, $\mathsf{v} \in V^\times$. The fundamental theorems of numerical range are following.

Theorem 4.1 (Convexity, Toeplitz-Hausdorff, [4], [7])**.** *The numerical range of an operator is convex with respect to the Euclidean metric on* $\mathbb{C}$*.*

Theorem 4.2 (Spectral inclusion, [8])**.** *The spectrum of a bounded linear operator is contained in the closure of its numerical range.*

Theorem 4.3 ([1], [6])**.** *The closure of numerical range of a normal bounded linear operator is the convex full of its spectrum.*

Theorem 4.4 (Equivalent norm)**.** *Put* $w(\varphi) = \sup\{|\lambda| \mid \lambda \in W(\varphi)\}$ *(the numerical radius), and* $\|\varphi\| = \sup\{\|\varphi(\mathsf{v})\|/\|\mathsf{v}\| \mid \mathsf{v} \in V^\times\}$ *(the operator norm). Then* $w(\varphi) \leq \|\varphi\| \leq 2w(\varphi)$*.*

4.2 Numerical range vs. leaf

The structure theorems of the leaf (Theorem A, B, and C) correspond to the first three fundamental theorems presented above. The result corresponding to the last fundamental theorem, in the context of the leaf, is that

$$\|\varphi\| = \sup\{|\lambda| \mid \lambda \in \Psi(\varphi)\}$$

which follows directly from the definition of the leaf.

For an endomorphism φ on a real inner product space, the value $\varphi(v) \cdot \mathsf{v}/\|\mathsf{v}\|^2$, which appears in the definition of the numerical range, corresponds to the real part of $\mathbb{L}_\varphi(\mathsf{v}) = (\|\varphi(\mathsf{v})\|/\|\mathsf{v}\|)\, e^{\theta\sqrt{-1}}$, which appears in the definition of the leaf. Therefore, the numerical range is the projection of the leaf onto the real axis. Compared to the numerical range, the leaf is expected to naturally retain richer structural information about the endomorphism.

There is an essential difference between Theorem A and Theorem 4.1: the leaf is convex with respect to the Poincaré metric, whereas the numerical range is convex with respect to the Euclidean metric. Different metrics result in different representations of the real eigenvalues in the figures. The intersection of the leaf with the real axis is precisely the set of all real eigenvalues (Theorem 3.3), whereas the intersection of the numerical range with the real axis is an interval containing all real eigenvalues (Theorems 4.1, 4.2). Information about each individual real eigenvalue is preserved in the leaf but lost in the numerical range.

Theorem B and Theorem 4.2 have different ways of proving. In the case of numerical range, the Rayleigh quotient of an eigenvector is the eigenvalue. But, since any endomorphism on a real vector space has no real eigenvector with respect to the non-real eigenvalue which is a non-real root of the characteristic polynomial, the same method cannot be use for leaf as for numerical range. We need to use Theorem A, to find a real vector for which the value of the function $\mathbb{L}_\varphi$ is the non-real eigenvalue (Proposition 6.2).

The proof of Theorem A is slightly more complex than that of Theorem 4.1. The image of a two-dimensional subspace under the Rayleigh quotient forms an ellipse (ellipse lemma, Proposition 4.5). For any two points in the numerical range, the ellipse obtained as the image, under the Rayleigh quotient, of the subspace spanned by the two corresponding vectors is contained in the numerical range. Hence, the line segment connecting the two points is also contained in the numerical range. On the other hand, in the case of the leaf, considering the image of a two-dimensional subspace in the same way yields a closed curve that contains a geodesic, since the coefficients are taken over the real numbers (Proposition

6.3). To fill the interior of this closed curve, a third vector is required (Proposition 6.5). This is why the dimension of the real inner product space must be three or higher.

In the case of a normal endomorphism, both the numerical range and the leaf are convex sets that contain all eigenvalues. However, since the underlying metrics are different, the corresponding geodesics have different shapes. In particular, the edges connecting eigenvalues along geodesics differ in shape between the numerical range and the leaf. For a normal endomorphism, the numerical range can be explicitly described using an orthonormal basis of eigenvectors, and it forms a polygon whose vertices are the eigenvalues. In contrast, the leaf does not admit such a simple representation. To address this, we use the fact (see Proposition 7.6) that the image of the orthogonal direct sum of invariant subspaces is formed by connecting the images of each subspace via geodesic segments—referred to as geodesic bridges—to show that the leaf forms a polygon whose edges are geodesics.

4.3 Proof of the fundamental theorems of numerical range

In this section, we provide proofs of the fundamental theorems (4.1, 4.2, 4.3, 4.4) of numerical range, according to chapter 1 of Guatafson-Rao's text book ([3]). The fundamental theorems still hold when V is infinite-dimensional. For simplicity, in this section we assume that V is finite-dimensional.

Let φ be an endomorphism on a finite-dimensional Hermite inner product space V.

$$W(\varphi) = \{\frac{\varphi(\mathsf{v}) \cdot \mathsf{v}}{\|\mathsf{v}\|^2} \mid \mathsf{v} \in V^\times\} = \{\varphi(\mathsf{v}) \cdot \mathsf{v} \mid \|\mathsf{v}\| = 1\}.$$

First, we show the Theorem 4.2. Let $\lambda \in \mathbb{C}$ be an eigenvalue of φ, and $\mathsf{v} \in V$ an eigenvector with respect to λ. Then $\varphi(\mathsf{v}) \cdot \mathsf{v} = \lambda \mathsf{v} \cdot \mathsf{v} = \lambda \|\mathsf{v}\|$, so $\lambda \in W(\varphi)$. When V is finite dimensional, the spectrum is the set on eigenvalues. As an aside, it is shown as follows, in the case of infinite-dimensional. For any approximate point spectrum λ of φ, we consider the series of unit vectors in V satisfying that $\|(\varphi - \lambda)\mathsf{v}_n\| \to 0$ as $n \to \infty$. Since $\varphi(\mathsf{v}_n) \cdot \mathsf{v}_n \to \lambda$ as $n \to \infty$, λ lies on $W(\varphi)$. The boundary of the spectrum is included in the approximate point spectrum, so the convexity of $W(\varphi)$ (Theorem 4.1) leads the Theorem 4.2.

Secons, we show the Theorem 4.3. When φ is normal, V has an orthonormal basis $\{\mathsf{v}_j\}_j$ consisting eigenvectors of φ, namely $\varphi(\mathsf{v}_j) = \lambda_j \mathsf{v}_j$ and $\mathsf{v}_i \cdot \mathsf{v}_j = \delta_{ij}$. By easy calculation,

we have that

$$W(\varphi) = \{x_1\lambda_1 + \cdots + x_n\lambda_n \mid x_1, \cdots, x_n > 0,\ |x_1|^2 + \cdots + |x_n|^2 = 1\},$$

where $n = \dim V$. The right hand side is the convex full of the eigenvalues of φ.

Third, we compare the operator norm with the numerical radius. Let $\lambda = \varphi(\mathsf{v}) \cdot \mathsf{v} \in W(\varphi)$, where $\mathsf{v} \in V$ with $\|\mathsf{v}\| = 1$. By the Schwarz inequality, we have that $|\lambda| = |\varphi(\mathsf{v}) \cdot \mathsf{v}| \leq \|\varphi(\mathsf{v})\|\,\|\mathsf{v}\| = \|\varphi(\mathsf{v})\|$, so $w(\varphi) \leq \|\varphi\|$. To show that $\|\varphi\| \leq 2w(\varphi)$, we use the following identity:

$$\begin{aligned}\varphi(\mathsf{u}) \cdot \mathsf{v} = {} & \varphi(\mathsf{u}+\mathsf{v}) \cdot (\mathsf{u}+\mathsf{v}) - \varphi(\mathsf{u}-\mathsf{v}) \cdot (\mathsf{u}-\mathsf{v}) \\ & + \varphi(\mathsf{u}+\sqrt{-1}\mathsf{v}) \cdot (\mathsf{u}+\sqrt{-1}\mathsf{v}) - \varphi(\mathsf{u}-\sqrt{-1}\mathsf{v}) \cdot (\mathsf{u}-\sqrt{-1}\mathsf{v}).\end{aligned}$$

By the Schwarz inequality, we have that

$$\begin{aligned}\varphi(\mathsf{u}) \cdot \mathsf{v} \ &\leq w(\varphi)(\|\mathsf{u}+\mathsf{v}\| + \|\mathsf{u}-\mathsf{v}\| + \|\mathsf{u}+\sqrt{-1}\mathsf{v}\| + \|\mathsf{u}-\sqrt{-1}\mathsf{v}\| \\ &\leq 4w(\varphi)(\|\mathsf{u}\| + \|\mathsf{v}\|)\end{aligned}$$

Take $\mathsf{u} = \mathsf{v}$ and $\|\mathsf{v}\| = 1$, then $\varphi(\mathsf{v}) \cdot \mathsf{v} \leq 2w(\varphi)$. So, we obtain that $\|\varphi\| \leq 2w(\varphi)$.

Finally, we prove the Theorem 4.1. Let $\alpha = \varphi(\mathsf{u}) \cdot \mathsf{u}$, $\beta = \varphi(\mathsf{v}) \cdot \mathsf{v} \in W(\varphi)$, where u, $\mathsf{v} \in V$ with $\|\mathsf{u}\| = \|\mathsf{v}\| = 1$. Put $V_0 = \langle \mathsf{u}, \mathsf{v} \rangle$. Let $\pi : V \to V_0$ be the orthogonal projection of V on V_0, and $\iota : V_0 \to V$ the inclusion map of V_0 to V. $\pi\varphi\iota$ the endomorphism on V_0. We will show that $W(\pi\varphi\iota) \subset W(\varphi)$. Let $\mathsf{w} \in V_0$. It is easy to show that $\pi = \iota^*$, so we have that

$$\pi\varphi\iota(\mathsf{w}) \cdot \mathsf{w} = \iota^*\varphi\iota(\mathsf{w}) \cdot \mathsf{w} = \varphi(\iota(\mathsf{w})) \cdot \iota(\mathsf{w}) \in W(\varphi).$$

Hence, $W(\pi\varphi\iota) \subset W(\varphi)$. We refer to the following lemma on the numerical range of an endomorphism on the two-dimensional vector space.

Lemma 4.5 (Ellipse Lamma, [3] Lemma 1.1-1)**.** *The numerical range of a linear operator φ on a two-dimensional Hilbert space is an ellipse whose foci are the eigenvalues of φ.*

This lemma is demonstrated by detailed calculations involving the values of bivariate quadratic forms. Here, we will limit ourselves to citing the assertion. The last lemma leads us that the numerical range $W(\pi\varphi\iota)$ is an ellipse, especially $W(\pi\varphi\iota)$ is convex. The line segment containing α and β is contained in $W(\pi\varphi\iota) \subset W(\varphi)$. Hence the numerical range $W(\varphi)$ is convex.

The ellipse lemma plays a central role in proving the convexity of the numerical range. Since the image of a 2-dimensional subspace corresponding to two points in the numerical

range is an ellipse, the line segment connecting those two points lies within the numerical range. This implies that the numerical range is convex. In contrast, the convexity of the leaf relies on Propositions 6.3 and 6.5. In this case, convexity is established by first constructing a closed curve that encloses the geodesic segment connecting the two points (Proposition 6.3), and then filling its interior (Proposition 6.5).

Chapter 5

Leaf of an endomorphism of dimension at most two

In this chapter, we will show the following theorem.

Theorem 5.1. *Leaf of an endomorphism of dimension at most two is either a single point set, a circle or a bended circle.*

A *bended circle* is defined as the figure obtained by bending a circle that intersects the real axis upward along the real axis. The center and radius of a bended circle are defined to be the same as those of the original (unbent) circle. Let C be a circle in $\mathbb{C}$ crossing the real axis, and put C' be the complex conjugate of C. Then $(C \cup C') \cap \overline{\mathfrak{H}}$ is a bended circle.

Clearly, the leaf of any endomorphism on a 1-dimensional space is a single point set. So, we consider the leaf of any endomorphism on a two-dimensional space.

Proposition 5.2. *Let V be a real inner product space of dimension two, and φ an endomorphism on V.*

(a) *If φ has $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a matrix representation with respect to an orthonormal basis of V, then the leaf $\Psi(\varphi)$ contains $a+|b|\sqrt{-1}$, $a+|c|\sqrt{-1}$, $d+|b|\sqrt{-1}$ and $d+|c|\sqrt{-1}$.*

(b) *Put $\alpha = \frac{a+d}{2} + \frac{|b-c|}{2}\sqrt{-1}$ and $r = \frac{1}{2}\sqrt{(a-d)^2+(b+c)^2}$. Both α and r are independent of choice of orthonormal basis of V.*

(c) *Let $C \subset \mathbb{C}$ be the circle with center α and radius r. If φ does not have any real eigenvalues, then $\Psi(\varphi) = C$. Moreover, if $r > 0$, then the eigenvalue with positive imaginary part is located on the circumference or on the interior of C.*

(d) *If all eigenvalues of φ are real, then C crosses the real axis at the eigenvalues, and $\Psi(\varphi)$ is a bended circle $(C \cup C') \cap \overline{\mathfrak{H}}$, where C' is the complex conjugate of C.*

Proof. (a) Let $\{\mathsf{u}_1, \mathsf{u}_2\}$ be an orthonormal basis of V satisfying that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix representation of φ with respect to this basis. Then $\varphi(\mathsf{u}_1) = a\,\mathsf{u}_1 + c\,\mathsf{u}_2$. Since $\|\mathsf{u}_1\| = 1$, $\|\varphi(\mathsf{u}_1)\|^2 = a^2 + c^2$, $\mathsf{u}_1 \cdot \varphi(\mathsf{u}_1) = a$ and $\sigma(\mathsf{u}_1, \varphi(\mathsf{u}_1)) = |c|$, we have $\mathbb{L}_\varphi(\mathsf{u}_1) = a + |c|\sqrt{-1}$. Similarly, we have $\mathbb{L}_\varphi(\mathsf{u}_2) = d + |b|\sqrt{-1}$, $\mathbb{L}_\varphi((a-d)\mathsf{u}_1 + (b+c)\mathsf{u}_2) = a + |b|\sqrt{-1}$ and $\mathbb{L}_\varphi((b+c)\mathsf{u}_1 - (a-d)\mathsf{u}_2) = d + |c|\sqrt{-1}$. Their four points belong to $\Psi(\varphi)$.

(b) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ be the matrix representations of φ with respect to the two orthonormal bases of V. Put P be the translation matrix between their orthonormal bases, then $A' = P^{-1}AP = {}^tPAP$, because P is an orthogonal matrix. The column vectors p and q where $P = (\mathsf{p}\ \ \mathsf{q})$ are an orthonormal basis of $\mathbb{R}^2$. Comparing the $(1,2)$ and $(2,1)$ components of $A' = {}^tPAP$ yields $b' = {}^t\mathsf{p}A\mathsf{q}$ and $c' = {}^t\mathsf{q}A\mathsf{p} = {}^t\mathsf{p}{}^tA\mathsf{q}$. And then $b' - c' = {}^t\mathsf{p}(A - {}^tA)\mathsf{q} = {}^t\mathsf{p}(b-c)\mathsf{J}\mathsf{q} = (b-c){}^t\mathsf{p}\mathsf{J}\mathsf{q}$, where J is defined in Example 1.2. J is the $\pi/2$-rotation on $\mathbb{R}^2$, so $\mathsf{J}\mathsf{q} = \pm\mathsf{p}$. Hence, $|b' - c'| = |(b-c)(\pm\|\mathsf{p}\|^2)| = |b-c|$.

The trace and the determinant of a matrix representation are independent of choice of basis. That is $a + d = a' + d'$ and $ad - bc = a'd' - b'c'$, then

$$\alpha = \frac{a+d}{2} + \frac{|b-c|}{2}\sqrt{-1} = \frac{a'+d'}{2} + \frac{|b'-c'|}{2}\sqrt{-1}$$

and

$$\begin{aligned}(2r)^2 &= (a-d)^2 + (b+c)^2 = (a+d)^2 + (b-c)^2 - 4(ad-bc) \\ &= (a'+d')^2 + (b'-c')^2 - 4(a'd'-b'c') = (a'-d')^2 + (b'+c')^2.\end{aligned}$$

We have that α and r are independent of choice of orthonormal basis of V.

(c) Let D be the discriminant of the characteristic polynomial of φ. φ does not have any real eigenvalues, so D is negative. Since $D = (a+d)^2 - 4(ad-bc) = (a-d)^2 + 4bc$, we have $bc < 0$. Four points of (a) form the rectangle with center α and diagonal length $2r$. Therefore, four points in (a) lie on C.

Let $\lambda' = \mathbb{L}_\varphi(\mathsf{v}) \in \Psi(\varphi)$ where $\mathsf{v} \in V^\times$, and put $\mathsf{u} = \mathsf{v}/\|\mathsf{v}\|$. There exists an orthonormal basis of V containing u. Since C is independent of choice of orthonormal basis of V, $\lambda' = \mathbb{L}_\varphi(\mathsf{v}) = \mathbb{L}_\varphi(\mathsf{u})$ lies on C. $\Psi(\varphi)$ is included in C.

We will show that $C \subset \Psi(\varphi)$. The leaf is path connected (Proposition 3.2), so the complement $C \smallsetminus \Psi(\varphi)$ is at most one connected component. For a matrix representation $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ of φ with respect to an orthonormal basis of V, two points $a' + |c'|\sqrt{-1}$ and

$d'+|b'|\sqrt{-1} \in \Psi(\varphi)$ are on the diagonal on C. $\Psi(\varphi)$ is point symmetric with respect to the center α of C. The complement $C \setminus \Psi(\varphi)$ is also point symmetric. Thus, the complement $C \setminus \Psi(\varphi)$ is the empty set or whole of C. Since $\Psi(\varphi)$ is not empty, $\mathbb{C} \subset \Psi(\varphi)$. So $\Psi(\varphi) = C$.

Let λ be an eigenvalue of φ with positive imaginary part. Then we have $\lambda = \frac{a+d}{2} + \frac{\sqrt{D}}{2}$. Since $D+(b-c)^2 = (a-d)^2+(b+c)^2 = (2r)^2 \geq 0$ and $D < 0$, we have that $\sqrt{|D|} \leq |b-c|$. The distance between λ and α, the center of C, is at most the radius of C, as below.

$$|\lambda-\alpha|^2 = \frac{(\sqrt{|D|} - |b-c|)^2}{4} = \frac{|D| + (b-c)^2 - 2\sqrt{|D|}\,|b-c|}{4} \leq \frac{|D| + (b-c)^2 - 2|D|}{4} = r^2.$$

Namely, λ is located on the circumference or on the interior of C.

(d) Let λ be a real eigenvalue of φ, which is denoted by $\frac{a+d\pm\sqrt{D}}{2}$.

$$|\lambda - \alpha|^2 = (\lambda - \frac{a+d}{2})^2 + (\frac{|b-c|}{2})^2 = \frac{D + (b-c)^2}{4} = r^2.$$

Hence, each real eigenvalue lies on the circumference of the circle C. Let C' be the circle with center $\overline{\alpha} = \frac{a+d}{2} - \frac{b-c}{2}\sqrt{-1}$ and radius r. C' is symmetric with C about the real axis. Hence, each real eigenvalue also lies on the circumference of the circle C'. We consider eight points $a \pm b\sqrt{-1}$, $a \pm c\sqrt{-1}$, $d \pm b\sqrt{-1}$ and $d \pm c\sqrt{-1}$. Four points $a + b\sqrt{-1}$, $a - c\sqrt{-1}$, $d + b\sqrt{-1}$ and $d - c\sqrt{-1}$ are the vertices of the rectangle with center α and diagonal length $2r$, and are on C. Another four points $a - b\sqrt{-1}$, $a + c\sqrt{-1}$, $d - b\sqrt{-1}$ and $d + c\sqrt{-1}$ are the vertices of the rectangle with center $\overline{\alpha}$ and diagonal length $2r$, and are on C'. Therefore these eight points are on the union of the circles C and C'. Four of eight points, $a + |b|\sqrt{-1}$, $a + |c|\sqrt{-1}$, $d + |b|\sqrt{-1}$ and $d + |c|\sqrt{-1}$ belong to $\overline{\mathfrak{H}}$, and then are on the bended circle $(C \cup C') \cap \overline{\mathfrak{H}}$ which is the shape made by bending the circle C upward along the real axis. We have that $\Psi(\varphi)$ is included in $(C \cup C') \cap \overline{\mathfrak{H}}$. The reverse inclusive relation $(C \cup C') \cap \overline{\mathfrak{H}} \subset \Psi(\varphi)$ is valid in the same way as the proof of (c). □

Proposition 5.3. *Let φ be an endomorphism on a real inner product space V of dimension two or higher. Assume that φ has two different real eigenvalues λ_1 and λ_2. Let v_1 and $\mathsf{v}_2 \in V$ be unit eigenvectors of φ corresponding to λ_1 and λ_2, respectively, and put $\beta = \mathsf{v}_1 \cdot \mathsf{v}_2$. Then $\mathbb{L}_\varphi(\langle \mathsf{v}_1, \mathsf{v}_2 \rangle^\times)$ is the bended circle with center α and radius r, where $\alpha = \frac{\lambda_1+\lambda_2}{2} + t\sqrt{-1}$, and t and r are nonnegative real numbers satisfying that*

$$r^2 = (\frac{\lambda_1 - \lambda_2}{2})^2 + t^2, \qquad t = \frac{|\lambda_1 - \lambda_2|}{2}\frac{|\beta|}{\sqrt{1-\beta^2}}, \quad \textit{and} \quad r = \frac{|\lambda_1 - \lambda_2|}{2}\frac{1}{\sqrt{1-\beta^2}}.$$

Proof. Since the last proposition leads that λ_1 and λ_2 lie on the circle with center α and radius r, we have that $|\lambda_1 - \alpha| = |\lambda_2 - \alpha| = r$, namely

$$(\frac{\lambda_1 - \lambda_2}{2})^2 + t^2 = r^2$$

Put $\mathsf{w} = \mathsf{v}_1 + \mathsf{v}_2 \in \langle \mathsf{v}_1, \mathsf{v}_2 \rangle^\times$, then

$$\|\mathsf{w}\|^2 = 2(1+\beta), \quad \|\varphi(\mathsf{w})\|^2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\beta, \quad \text{and} \quad \mathsf{w} \cdot \varphi(\mathsf{w}) = (\lambda_1 + \lambda_2)(1+\beta)$$
$$\therefore\ |\mathbb{L}_\varphi(\mathsf{w})|^2 = \frac{\|\varphi(\mathsf{w})\|^2}{\|\mathsf{w}\|^2} = \frac{\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\beta}{2(1+\beta)}, \quad \operatorname{Re}\mathbb{L}_\varphi(\mathsf{w}) = \frac{\mathsf{w} \cdot \varphi(\mathsf{w})}{\|\mathsf{w}\|^2} = \frac{\lambda_1 + \lambda_2}{2}$$

Hence

$$(\operatorname{Im}\mathbb{L}_\varphi(\mathsf{w}))^2 = |\mathbb{L}_\varphi(\mathsf{w})|^2 - (\operatorname{Re}\mathbb{L}_\varphi(\mathsf{w}))^2 = (\frac{\lambda_1 - \lambda_2}{2})^2 \frac{1-\beta}{1+\beta}.$$

Since the real part of $\mathbb{L}_\varphi(\mathsf{w})$ coincides with that of the center α of the bended circle $\mathbb{L}_\varphi(\langle \mathsf{v}_1, \mathsf{v}_2 \rangle^\times)$.

$$\operatorname{Im}\mathbb{L}_\varphi(\mathsf{w}) = (\text{the radius of bended circle}) \pm \operatorname{Im}\alpha = r \pm t.$$

We obtain that

$$r^2 - t^2 = (\frac{\lambda_1 - \lambda_2}{2})^2, \quad \text{and} \quad r \pm t = \frac{\lambda_1 - \lambda_2}{2}\sqrt{\frac{1-\beta}{1+\beta}}$$

Therefore

$$r = \frac{\lambda_1 - \lambda_2}{2}\,\frac{1}{2}\left(\sqrt{\frac{1-\beta}{1+\beta}} + \sqrt{\frac{1+\beta}{1-\beta}}\right) = \frac{\lambda_1 - \lambda_2}{2}\frac{1}{\sqrt{1-\beta^2}},$$
$$t = \frac{\lambda_1 - \lambda_2}{2}\,\frac{1}{2}\left|\sqrt{\frac{1-\beta}{1+\beta}} - \sqrt{\frac{1+\beta}{1-\beta}}\,\right| = \frac{\lambda_1 - \lambda_2}{2}\frac{|\beta|}{\sqrt{1-\beta^2}}.$$

□

Chapter 6

Leaf in $\overline{\mathfrak{H}}$ and eigenvalues with nonnegative imaginary part

6.1 Subleaf and eigenvalue inclusion of leaf

There are various shapes that can appear within the leaves of endomorphisms on spaces of dimension three or higher. We will present the structure theorems on the leaf. The first theorem concerns the geometric property that each leaf is convex with respect to the Poincaré metric and is simply connected. The second theorem concerns the algebraic property that each leaf contains all eigenvalues with nonnegative imaginary part. The third theorem concerning the leaf of a normal endomorphism will be presented in the next chapter.

Let V be a real inner product space of finite dimension, and φ an endomorphism on V. Let W be a φ-stable subspace in V. Since the restriction $\varphi|_W$ of φ on W is the endomorphism on W, the subset $\mathbb{L}_\varphi(W^\times)$ of $\Psi(\varphi)$ is equal to the leaf $\Psi(\varphi|_W)$ of $\varphi|_W$. $\mathbb{L}_\varphi(W^\times) = \Psi(\varphi|_W)$ is called the subleaf corresponding to W and denoted by $\Psi(\varphi; W)$.

Proposition 6.1. *A subleaf corresponding to a two-dimensional stable subspace is one of three kinds of figures on Theorem 5.1.*

Proposition 6.2. *If the leaf is simply connected, then it contains all eigenvalues with nonnegative imaginary parts.*

Proof. It is already shown that the leaf contains all real eigenvalues, by Theorem 3.3. Let $\lambda \in \mathfrak{H}$ be an eigenvalue of φ with positive imaginary part, and $\mathsf{v} \in V_{\mathbb{C}}$ an eigenvector of φ corresponding to λ. $W = \langle \mathrm{Re}\,\mathsf{v}, \mathrm{Im}\,\mathsf{v} \rangle$ is a φ-stable subspace of dimension two. By Propositions 6.1 and 5.2, λ is located on the circumference or on the inside of the circle

$\Psi(\varphi; W) = \mathbb{L}_\varphi(W^\times)$. By the assumption that $\Psi(\varphi)$ is simply connected, $\Psi(\varphi)$ includes the circumference or on the interior of the circle $\Psi(\varphi; W)$. Hence, λ belongs to $\Psi(\varphi)$. □

The second structure theorem (Theorem B, eigenvalue inclusion) follows from the first structure theorem (Theorem A, Poincaré convexity). In this chapter, we focus on proving the Poincaré convexity of the leaf.

6.2 Closed curve in the leaf corresponding to a two-dimensional subspace

Put $\ell(W) = \mathbb{L}_\varphi(W^\times) = \mathbb{L}_\varphi(\mathbb{P}(W))$ for any two-dimensional subspace W in V. $\ell(W)$ is a closed curve, because the projective space $\mathbb{P}(W) = W^\times/\mathbb{R}^\times$ is homeomorphic to the circle. Let $\mathcal{D}(W)$ be the simple connected domain bounded by the closed curve $\ell(W)$, and put $\mathcal{D}(W)^\circ = \mathcal{D}(W) \setminus \ell(W)$ the interior of $\mathcal{D}(W)$.

Proposition 6.3. *Let λ_1, $\lambda_2 \in \Psi(\varphi) \subset \overline{\mathfrak{H}}$ with $\lambda_1 \neq \lambda_2$. Take v_1, $\mathsf{v}_2 \in V^\times$ with $\|\mathsf{v}_1\| = \|\mathsf{v}_2\| = 1$, $\lambda_1 = \mathbb{L}_\varphi(\mathsf{v}_1)$ and $\lambda_2 = \mathbb{L}_\varphi(\mathsf{v}_2)$, and put $W = \langle \mathsf{v}_1, \mathsf{v}_2 \rangle$ which is a two-dimensional subspace of V. Let $s(\lambda_1, \lambda_2)$ be the geodesic connecting λ_1 to λ_2 on $\overline{\mathfrak{H}}$.*

(a) *The geodesic $s(\lambda_1, \lambda_2)$ is included in the domain $\mathcal{D}(W)$.*

(b) *If $s(\lambda_1, \lambda_2)$ does not pass through the interior $\mathcal{D}(W)^\circ (= \mathcal{D}(W) \setminus \ell(W))$, then the interior $\mathcal{D}(W)^\circ$ is empty, and $\ell(W) (= \partial\mathcal{D}(W))$ is a geodesic including $s(\lambda_1, \lambda_2)$.*

(c) *If $s(\lambda_1, \lambda_2)$ passes through the interior $\mathcal{D}(W)^\circ$, then $s(\lambda_1, \lambda_2) \cap \ell(W) = \{\lambda_1, \lambda_2\}$.*

(d) *If the interior $\mathcal{D}(W)^\circ$ is not empty, then $\ell(W)$ is a closed simple curve.*

Proof. We observe the positional relationship between the closed curve $\ell(W)$ and the geodesic $s(\lambda_1, \lambda_2)$. Express a typical $\mathsf{w} \in \mathbb{P}(W) = W^\times/\mathbb{R}^\times$ is the form by $x\mathsf{v}_1 + \mathsf{v}_2$ for some $x \in \mathbb{P}^2(\mathbb{R}) = (\mathbb{R}^2)^\times/\mathbb{R}^\times = \mathbb{R} \cup \{\infty\}$. Note that $\mathsf{v}_1 = \infty\mathsf{v}_1 + \mathsf{v}_2$ in this notation. We have that

$$\begin{aligned}
\|\mathsf{w}\|^2 &= x^2 + 2x(\mathsf{v}_1 \cdot \mathsf{v}_2) + 1, \\
\|\varphi(\mathsf{w})\|^2 &= x^2\|\varphi(\mathsf{v}_1)\|^2 + 2x(\varphi(\mathsf{v}_1) \cdot \varphi(\mathsf{v}_2)) + \|\varphi(\mathsf{v}_2)\|^2, \\
\mathsf{w} \cdot \varphi(\mathsf{w}) &= x^2(\mathsf{v}_1 \cdot \varphi(\mathsf{v}_1)) + x(\mathsf{v}_1 \cdot \varphi(\mathsf{v}_2) + \mathsf{v}_2 \cdot \varphi(\mathsf{v}_1)) + \mathsf{v}_2 \cdot \varphi(\mathsf{v}_2).
\end{aligned}$$

A geodesic on the Poincaré metric is a part of a line parallel to the imaginary axis and semicircle with center on the real axis. $s(\lambda_1,\, \lambda_2)$ is a part of a line, if $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2$, or a part of a semicircle, otherwise.

(i) In the case that $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2$. Put $a = \operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2$ $(= \mathsf{v}_1 \cdot \varphi(\mathsf{v}_1) = \mathsf{v}_2 \cdot \varphi(\mathsf{v}_2))$. $s(\lambda_1,\, \lambda_2)$ is a part of the line whose real part is a.

$$\operatorname{Re}\mathbb{L}_\varphi(\mathsf{w}) - a = \frac{\mathsf{w}\cdot\varphi(\mathsf{w})}{\|\mathsf{w}\|^2} - \mathsf{v}_1\cdot\varphi(\mathsf{v}_1) = R\,\frac{x}{\|\mathsf{w}\|^2},$$

where

$$R = \mathsf{v}_1\cdot\varphi(\mathsf{v}_2) + \mathsf{v}_2\cdot\varphi(\mathsf{v}_1) - a\,(\mathsf{v}_1\cdot\mathsf{v}_2).$$

Since $\|\mathsf{w}\|^2 = O(x^2)$ as $x\to\infty$, we remark that $\operatorname{Re}\mathbb{L}_\varphi(x\,\mathsf{v}_1+\mathsf{v}_2) - a = O(x^{-1})$ as $x\to\infty$.

(ii) In the case that $\operatorname{Re}\lambda_1 \neq \operatorname{Re}\lambda_2$. $s(\lambda_1, \lambda_2)$ is a part of semicircle. Let $c\in\mathbb{R}$ be the center of the semicircle and d the radius. Since

$$d^2 = |\lambda_i - c|^2 = (\operatorname{Re}\lambda_i - c)^2 + (\operatorname{Im}\lambda_i)^2 = |\lambda_i|^2 - 2c\operatorname{Re}\lambda_i + c^2 \quad (i=1,\,2),$$

we have that

$$d^2 - c^2 = |\lambda_i|^2 - 2c\operatorname{Re}\lambda_i, \qquad c = \frac{|\lambda_1|^2 - |\lambda_2|^2}{2(\operatorname{Re}\lambda_1 - \operatorname{Re}\lambda_2)}.$$

Thus,

$$|\mathbb{L}_\varphi(\mathsf{w}) - c|^2 - d^2 = |\mathbb{L}_\varphi(\mathsf{w})|^2 - 2c\operatorname{Re}\mathbb{L}_\varphi(\mathsf{w}) = R\,\frac{x}{\|\mathsf{w}\|^2},$$

where

$$R = 2(\varphi(\mathsf{v}_1)\cdot\varphi(\mathsf{v}_2) + c(\mathsf{v}_1\cdot\varphi(\mathsf{v}_2) + \mathsf{v}_2\cdot\varphi(\mathsf{v}_1)) + (c^2-d^2)(\mathsf{v}_1\cdot\mathsf{v}_2)).$$

In this case, we also remark that $|\mathbb{L}_\varphi(x\,\mathsf{v}_1+\mathsf{v}_2) - c|^2 - d^2 = O(x^{-1})$ as $x\to\infty$.

In both cases (i) and (ii), the sign of $R\,t/\|\mathsf{w}\|^2$ shows the position on $\mathbb{L}_\varphi(\mathsf{w})$ relative to $s(\lambda_1,\, \lambda_2)$. Let K be the geodesic extending from the both endpoints of $s(\lambda_1,\, \lambda_2)$ to $\mathbb{R}\cup\{\infty\}$. K divides the upper half plane $\overline{\mathfrak{H}}\setminus K$ into two connected areas K_1 and K_2. In the case that K is a line, put K_1 be the right area of K, and K_2 the left area. In the case that K is a semicircle, put K_1 be the outer area of K, and K_2 the inner area. Then $\mathbb{L}_\varphi(\mathsf{w})$ belongs to K if $R\,x/\|\mathsf{w}\|^2 = 0$, to K_1 if $R\,x/\|\mathsf{w}\|^2 > 0$, and to K_2 if $R\,x/\|\mathsf{w}\|^2 < 0$. Since R is independent of t, and $\|\mathsf{w}\|$ is always positive, the sign change of $R\,t/\|\mathsf{w}\|^2$ coincides with the sign change of t. We separate two cases $R=0$ and $R\neq 0$.

(A) In the case that $R=0$, $\mathbb{L}_\varphi(\mathsf{w})$ is always on K, so $\ell(W)$ is a part of K. $\ell(W)$ contains λ_1 and λ_2. Hence $s(\lambda_1,\, \lambda_2)\subset\ell(W)$.

(B) In the case that $R\neq 0$. If $x\neq 0,\,\infty$, then $R\,x/\|\mathsf{w}\|^2\neq 0$, so $\mathbb{L}_\varphi(\mathsf{w})$ does not belong to K. $\mathbb{L}_\varphi(\mathsf{w})$ belongs to one of K_1 or K_2 if $x>0$, and to the other if $x<0$. $\ell(W)$

goes around $s(\lambda_1, \lambda_2)$ as below: starts at λ_1 which is one of endpoints of the geodesic $s(\lambda_1, \lambda_2)$, goes through K_1, crosses K at λ_2 which is the other endpoint of $s(\lambda_1, \lambda_2)$, goes through K_2, and returns to λ_1. Hence, $s(\lambda_1, \lambda_2)$ is included in $\mathcal{D}(W)$.

In both cases (A) and (B), $s(\lambda_1, \lambda_2)$ is included in $\mathcal{D}(W)$, so (a) holds. (b) follows from (A), and (c) from (B).

Finally, we will show that (d) holds. Let λ_3 be a self-intersection point of $\ell(W)$. If $\ell(W)$ has a normal crossing at λ_3, then there exist two geodesics that tangent to $\ell(W)$ at λ_3. Each geodesic between these geodesic intersects $\ell(W)$ in three or higher points. This contradicts (c), so there exists only one tangent geodesic at λ_3. There exists a geodesic that displaces the tangent geodesic just a little and crosses $\ell(W)$ by more than two points. This also contradicts (c). Therefore, $\ell(W)$ does not intersect itself. □

Proposition 6.4. *Let* $\mathsf{v}_1, \mathsf{v}_2 \in V^\times$. *Assume that* v_1 *and* v_2 *satisfy the orthogonal condition:*

$$\langle \mathsf{v}_1, \varphi(\mathsf{v}_1)\rangle \perp \langle \mathsf{v}_2, \varphi(\mathsf{v}_2)\rangle.$$

Then the curve $\ell(\langle \mathsf{v}_1, \mathsf{v}_2\rangle)$ *is the geodesic connecting* $\mathbb{L}_\varphi(\mathsf{v}_1)$ *to* $\mathbb{L}_\varphi(\mathsf{v}_2)$.

Proof. Put $\lambda_1 = \mathbb{L}_\varphi(\mathsf{v}_1)$, $\lambda_2 = \mathbb{L}_\varphi(\mathsf{v}_2)$ and $W = \langle \mathsf{v}_1, \mathsf{v}_2\rangle$. The orthogonal condition leads that $R = 0$, where R is defined in the proof of the last proposition. Hence, $\ell(W)$ is a geodesic connecting λ_1 to λ_2.

Express a typical $\mathsf{w} \in V^\times$ is the form by $x\,\mathsf{v}_1 + y\,\mathsf{v}_2$ for some $x, y \in \mathbb{R}$ with $(x, y) \neq (0, 0)$. It follows from the orthogonal condition that

$$\begin{aligned}
\|\mathsf{w}\|^2 &= x^2\|\mathsf{v}_1\|^2 + y^2\|\mathsf{v}_2\|^2,\\
\|\varphi(\mathsf{w})\|^2 &= x^2\|\varphi(\mathsf{v}_1)\|^2 + y^2\|\varphi(\mathsf{v}_2)\|^2,\\
\mathsf{w}\cdot\varphi(\mathsf{w}) &= x^2\mathsf{v}_1\cdot\varphi(\mathsf{v}_1) + y^2\mathsf{v}_2\cdot\varphi(\mathsf{v}_2).
\end{aligned}$$

Hence,

$$\begin{aligned}
\operatorname{Re}\mathbb{L}_\varphi(\mathsf{w}) &= \frac{\mathsf{w}\cdot\varphi(\mathsf{w})}{\|\mathsf{w}\|^2} = \frac{(x\,\|\mathsf{v}_1\|)^2\operatorname{Re}\lambda_1 + (y\,\|\mathsf{v}_2\|)^2\operatorname{Re}\lambda_2}{(x\,\|\mathsf{v}_1\|)^2 + (y\,\|\mathsf{v}_2|)^2},\\
|\mathbb{L}_\varphi(\mathsf{w})|^2 &= \frac{\|\varphi(\mathsf{w})\|^2}{\|\mathsf{w}\|^2} = \frac{(x^2\,\|\mathsf{v}_1\|)^2|\lambda_1|^2 + (y\,\|\mathsf{v}_2\|)^2|\lambda_2|^2}{(x\,\|\mathsf{v}_1\|)^2 + (y\,\|\mathsf{v}_2|)^2}.
\end{aligned}$$

$\operatorname{Re}\mathbb{L}_\varphi(\mathsf{w})$ divides $\operatorname{Re}\lambda_1$ and $\operatorname{Re}\lambda_2$ into the ratio $(y\|\mathsf{v}_2\|)^2 : (x\|\mathsf{v}_1\|)^2$. And $|\mathbb{L}_\varphi(\mathsf{w})|^2$ divides $|\lambda_1|^2$ and $|\lambda_2|^2$ into the same ratio. Since each point on $\ell(W)$ lies between λ_1 and λ_2, $\ell(W)$ is the geodesic with the endpoints λ_1 and λ_2. □

6.3 Proofs of the Poincaré convexity

Proposition 6.5. *If* $\dim V \geq 3$*, then* $\mathcal{D}(W) \subset \Psi(\varphi)$ *for any two-dimensional subspace* $W \subset V$*.*

Proof. It is enough to show the assertion in the case that $\mathcal{D}(W)$ has an interior point. In this case, $\ell(W)$ is a closed simple curve. Let U be a three-dimensional subspace of V including W. Put $\mathbb{P}^+(U) = U^\times/\mathbb{R}_{>0}$. $\mathbb{P}^+(U)$ is the quotient set of $U^\times$ identified by multiplying by positive real numbers, namely the set of half-lines starting at the origin. $\mathbb{P}^+(U)$ is homeomorphic to a spherical surface, and double covering of the projective plane $\mathbb{P}(U)$. The map $\mathbb{L}_\varphi$ is also regarded as a map on $\mathbb{P}^+(U)$. We have that $\mathbb{L}_\varphi(U^\times) = \mathbb{L}_\varphi(\mathbb{P}(U)) = \mathbb{L}_\varphi(\mathbb{P}^+(U))$. For any two-dimensional subspace $W_1 \subset U$, the quotient space $\mathbb{P}^+(W_1) = W_1^\times/\mathbb{R}_{>0}$ is a great circle on a spherical surface $\mathbb{P}^+(U)$.

Let $\{\mathsf{v}_0, \mathsf{v}_1, \mathsf{v}_2\}$ be a basis of U satisfying that $\{\mathsf{v}_0, \mathsf{v}_1\}$ is a basis of W. Put $\lambda_0 = \mathbb{L}_\varphi(\mathsf{v}_0)$, $\lambda_1 = \mathbb{L}_\varphi(\mathsf{v}_1)$, $\lambda_2 = \mathbb{L}_\varphi(\mathsf{v}_2) \in \overline{\mathfrak{H}}$, and $T = \langle \mathsf{v}_1, \mathsf{v}_2 \rangle$. For any $\mathsf{t} \in \mathbb{P}^+(T)$, we define the closed oriented curve $c(\mathsf{t})$ by

$$c(\mathsf{t}) : [0,\, 1] \ni \theta \mapsto \mathbb{L}_\varphi(\cos(\pi\,\theta)\mathsf{v}_0 + \sin(\pi\,\theta)\mathsf{t})) \in \ell(\langle \mathsf{v}_0, \mathsf{t} \rangle) \subset \mathbb{L}_\varphi(U^\times) \subset \Psi(\varphi).$$

$c(\mathsf{t})$ is a closed curve passing through λ_0, because $c(\mathsf{t})(0) = c(\mathsf{t})(1) = \mathbb{L}_\varphi(\pm\mathsf{v}_0) = \lambda_0$. $\{c(\mathsf{t})\}_\mathsf{t}$ is the continuous deformation of a closed curve on the complex plane. $c(\mathsf{v}_1)$ and $c(-\mathsf{v}_1)$ are the same curve $\ell(W)$, but have different orientations. Through the continuous deformation $\{c(\mathsf{t})\}_\mathsf{t}$, the closed curve $\ell(W)$ is superimposed on itself in the opposite orientation. Therefore, the continuous deformation passes through all interior points of $\mathcal{D}(W)$. Any interior point of $\mathcal{D}(W)$ lies on a curve $c(\mathsf{t})$ for some $\mathsf{t} \in \mathbb{P}^+(T)$. So, $\mathcal{D}(W) \subset \Psi(\varphi)$ holds. □

Theorem A. *Leaf is convex on the Poincaré metric, for any endomorphism of dimension three or higher.*

Proof. Let φ be an endomorphism on a real inner product space V of dimension three or higher. Let $\lambda_1,\, \lambda_2 \in \Psi(\varphi)$, $\mathsf{v}_1,\, \mathsf{v}_2 \in V$ with $\lambda_1 = \mathbb{L}_\varphi(\mathsf{v}_1)$ and $\lambda_2 = \mathbb{L}_\varphi(\mathsf{v}_2)$, and $s(\lambda_1,\, \lambda_2)$ be the geodesic connecting λ_1 to λ_2. By Proposition 6.3, $s(\lambda_1,\, \lambda_2)$ is included in $\mathcal{D}(\langle \mathsf{v}_1,\, \mathsf{v}_2 \rangle)$. By Proposition 6.5, $\mathcal{D}(\langle \mathsf{v}_1,\, \mathsf{v}_2 \rangle)$ is included in $\Psi(\varphi)$. Hence, the leaf is convex on the Poincaré metric. □

The last theorem leads that the leaf of an endomorphism on a space of dimension three or higher is simply connected. Therefore, the following theorem holds by Proposition 6.2. It is interesting to note that the following algebraic property is derived from the geometric property.

Theorem B. *Leaf contains all eigenvalues whose imaginary parts are nonnegative, for any endomorphism of dimension three or higher.*

Chapter 7

Leaf of a normal endomorphism

7.1 Subleaf corresponding to the real part of an eigenspace

For any endomorphism on a finite-dimensional real inner product space, the smallest convex domain in $\overline{\mathfrak{H}}$ containing all eigenvalues with nonnegative imaginary part is a filled geodesic polygon whose vertexes are those eigenvalues with nonnegative imaginary part. This domain is called the eigenvalue geodesic polygon. By the structure theorems (Theorems A and B), the leaf contains the eigenvalue geodesic polygon, for any endomorphism on a real inner product space of dimension three or higher. In general, the leaf is larger than the eigenvalue geodesic polygon. We show that they coincide for every normal endomorphism (Theorem C) in this section.

Let φ be an endomorphism on a real inner product space V of finite dimension.

Proposition 7.1. *Let $\lambda \in \mathfrak{H}$ be an eigenvalue of φ with positive imaginary part, and $\mathsf{v} \in V_{\mathbb{C}}$ an eigenvector of φ corresponding to λ. Put $W = \langle \operatorname{Re}\mathsf{v}, \operatorname{Im}\mathsf{v}\rangle \subset V$ and $\ell(W) = \mathbb{L}_{\varphi}(W^{\times})$.*

(a) *W is a two-dimensional φ-stable subspace of V. Hence $\ell(W)$ is the subleaf.*

(b) *Put $\lambda^{+} = \mathbb{L}_{\varphi}(\operatorname{Re}\mathsf{v})$, $\lambda^{-} = \mathbb{L}_{\varphi}(\operatorname{Im}\mathsf{v})$ and $\beta = \mathsf{v}\cdot\overline{\mathsf{v}}/\|\mathsf{v}\|^2$. Then*

$$|\lambda^{\pm}|^2 = \frac{|\lambda|^2 \pm \operatorname{Re}(\lambda^2\beta)}{1 \pm \operatorname{Re}\beta}, \quad \operatorname{Re}\lambda^{\pm} = \frac{\operatorname{Re}\lambda \pm \operatorname{Re}(\lambda\beta)}{1 \pm \operatorname{Re}\beta}, \quad \operatorname{Im}\lambda^{\pm} = \operatorname{Im}\lambda\,\frac{\sqrt{1-|\beta|^2}}{1 \pm \operatorname{Re}\beta}.$$

(c) *$\lambda^{\pm} = \lambda$ holds, if and only if $\mathsf{v} \perp \overline{\mathsf{v}}$.*

(d) *If $\mathsf{v} \perp \overline{\mathsf{v}}$, then $\ell(W) = \{\lambda\}$.*

(e) *Three points* λ, λ^+ *and* $\lambda^- \in \Psi(\varphi)$ *lie on the same geodesic.*

(f) $\ell(W)$ *is a geodesic circle* $C_{\mathrm{P}}(\lambda; \frac{1-\sqrt{1-|\beta|^2}}{|\beta|})$. *(rem.* $\frac{1-\sqrt{1-|\beta|^2}}{|\beta|} \to 0$ *as* $\beta \to 0$*)*

Proof. (a) The assertion is already shown in §2.

(b) Since $\mathrm{Re}\,\mathsf{v} = (\mathsf{v} + \overline{\mathsf{v}})/2$, $\mathrm{Im}\,\mathsf{v} = (\mathsf{v} - \overline{\mathsf{v}})/(2\sqrt{-1})$, and $\overline{\mathsf{v}}$ is an eigenvector of φ corresponding to $\overline{\lambda}$, we have that

$$\|\mathrm{Re}\,\mathsf{v}\|^2 = \frac{\|\mathsf{v}\|^2}{2}(1 + \mathrm{Re}\,\beta), \qquad \|\mathrm{Im}\,\mathsf{v}\|^2 = \frac{\|\mathsf{v}\|^2}{2}(1 - \mathrm{Re}\,\beta),$$
$$\|\varphi(\mathrm{Re}\,\mathsf{v})\|^2 = \frac{\|\mathsf{v}\|^2}{2}(|\lambda|^2 + \mathrm{Re}\,\lambda^2\beta), \qquad \|\varphi(\mathrm{Im}\,\mathsf{v})\|^2 = \frac{\|\mathsf{v}\|^2}{2}(|\lambda|^2 - \mathrm{Re}\,\lambda^2\beta),$$
$$\mathrm{Re}\,\mathsf{v} \cdot \varphi(\mathrm{Re}\,\mathsf{v}) = \frac{\|\mathsf{v}\|^2}{2}(\mathrm{Re}\,\lambda + \mathrm{Re}\,\lambda\beta), \qquad \mathrm{Im}\,\mathsf{v} \cdot \varphi(\mathrm{Im}\,\mathsf{v}) = \frac{\|\mathsf{v}\|^2}{2}(\mathrm{Re}\,\lambda - \mathrm{Re}\,\lambda\beta),$$

and then

$$|\lambda^+|^2 = \|\mathbb{L}_\varphi(\mathrm{Re}\,\mathsf{v})\|^2 = \frac{|\lambda|^2 + \mathrm{Re}\,\lambda^2\beta}{1 + \mathrm{Re}\,\beta}, \qquad \mathrm{Re}\,\lambda^+ = \mathrm{Re}\,\mathbb{L}_\varphi(\mathrm{Re}\,\mathsf{v}) = \frac{\mathrm{Re}\,\lambda + \mathrm{Re}\,\lambda\beta}{1 + \mathrm{Re}\,\beta},$$
$$|\lambda^-|^2 = \|\mathbb{L}_\varphi(\mathrm{Im}\,\mathsf{v})\|^2 = \frac{|\lambda|^2 - \mathrm{Re}\,\lambda^2\beta}{1 - \mathrm{Re}\,\beta}, \qquad \mathrm{Re}\,\lambda^- = \mathrm{Re}\,\mathbb{L}_\varphi(\mathrm{Im}\,\mathsf{v}) = \frac{\mathrm{Re}\,\lambda - \mathrm{Re}\,\lambda\beta}{1 - \mathrm{Re}\,\beta}.$$

The imaginary parts of $\lambda^\pm$ (> 0) are given by $\sqrt{|\lambda^\pm|^2 - (\mathrm{Re}\,\lambda^\pm)^2}$.

$$(\mathrm{Im}\,\lambda^\pm)^2 = |\lambda^\pm|^2 - (\mathrm{Re}\,\lambda^\pm)^2 = \frac{|\lambda|^2 \pm \mathrm{Re}\,\lambda^2\beta}{1 \pm \mathrm{Re}\,\beta} - (\frac{\mathrm{Re}\,\lambda \pm \mathrm{Re}\,\lambda\beta}{1 \pm \mathrm{Re}\,\beta})^2 = \frac{(\mathrm{Im}\,\lambda)^2(1 - |\beta|^2)}{(1 \pm \mathrm{Re}\,\beta)^2}.$$

Since Cauchy-Schwarz inequality leads $|\beta| \leq 1$ and $|\mathrm{Re}\,\beta| \leq 1$, we have that

$$\mathrm{Im}\,\lambda^\pm = \mathrm{Im}\,\lambda\,\frac{\sqrt{1 - |\beta|^2}}{1 \pm \mathrm{Re}\,\beta}.$$

(c) By the above formulae, $|\lambda^\pm| = |\lambda|$ if and only if $\lambda\beta \in \mathbb{R}$ and $\lambda\beta \leq 0$. $\mathrm{Re}\,\lambda^\pm = \mathrm{Re}\,\lambda$ if and only if $\beta \in \mathbb{R}$. Hence, $\lambda^\pm = \lambda$ if and only if $\beta = 0$, i.e. $\mathsf{v} \perp \overline{\mathsf{v}}$.

(d) λ and $\overline{\lambda}$ are the eigenvalues of the restriction $\varphi|_W$ of φ on the two-dimensional φ-stable subspace W. By Propositions 6.1 and 5.2, the closed curve $\ell(W)$ is a circle (or a single point set), and λ lies on the circumference or the inside of $\ell(W)$. By §2, $\mathrm{Re}\,\lambda\,\mathsf{E} + \mathrm{Im}\,\lambda\,\mathsf{J}$ is the matrix representation of $\varphi|_W$ with respect to the basis $\mathrm{Re}\,\mathsf{v}$, $\mathrm{Im}\,\mathsf{v}$ of W. Since v and $\overline{\mathsf{v}}$ are orthogonal, $\mathrm{Re}\,\mathsf{v}$ and $\mathrm{Im}\,\mathsf{v}$ are also orthogonal and equal in length. $\ell(W)$ is the circle with radius 0 by Proposition 5.2, so the single point set $\{\lambda\}$.

(e) Each geodesic of $\mathfrak{H}$ is a part of line parallel to the imaginary axis or a semicircle whose center is on the real axis. The geodesic connecting through λ^+ and λ^- is a part of line if $\mathrm{Re}\,\lambda^+ = \mathrm{Re}\,\lambda^-$, or a part of semicircle if $\mathrm{Re}\,\lambda^+ \neq \mathrm{Re}\,\lambda^-$.

• In the case that $\operatorname{Re}\lambda^+ = \operatorname{Re}\lambda^-$. It follows from (b) that

$$(\operatorname{Re}\lambda^+ =)\ \frac{\operatorname{Re}\lambda + \operatorname{Re}\lambda\beta}{1+\operatorname{Re}\beta} = \frac{\operatorname{Re}\lambda - \operatorname{Re}\lambda\beta}{1-\operatorname{Re}\beta}\ (= \operatorname{Re}\lambda^-)$$
$$(\operatorname{Re}\lambda + \operatorname{Re}\lambda\beta)(1-\operatorname{Re}\beta) = (\operatorname{Re}\lambda - \operatorname{Re}\lambda\beta)(1+\operatorname{Re}\beta)$$
$$\therefore\ \operatorname{Re}\lambda \operatorname{Re}\beta = \operatorname{Re}\lambda\beta$$

$$\therefore\ \operatorname{Re}\lambda^\pm = \frac{\operatorname{Re}\lambda \pm \operatorname{Re}\lambda\beta}{1\pm\operatorname{Re}\beta} = \frac{\operatorname{Re}\lambda \pm \operatorname{Re}\lambda\operatorname{Re}\beta}{1\pm\operatorname{Re}\beta} = \operatorname{Re}\lambda.$$

λ lies on the line which is the geodesic connecting through λ^+ and λ^-.

• In the case that $\operatorname{Re}\lambda^+ \neq \operatorname{Re}\lambda^-$. The geodesic connecting through λ^+ and λ^- is a semicircle. Put $c \in \mathbb{R}$ be the center of the semicircle, and d the radius. The calculation in the proof of Proposition 5.2 leads that

$$d^2 - c^2 = |\lambda^\pm|^2 - 2c\operatorname{Re}\lambda^\pm \qquad c = \frac{|\lambda^+|^2 - |\lambda^-|^2}{2(\operatorname{Re}\lambda^+ - \operatorname{Re}\lambda^-)}$$

To show that λ is on the semicircle, we need only show that $|\lambda - c| = d$ holds. Since the equality $|\lambda - c| = d$ is equivalent to

$$d^2 = |\lambda - c|^2 = |\lambda|^2 - 2c\operatorname{Re}\lambda + c^2,$$

we show that $d^2 - c^2 + 2c\operatorname{Re}\lambda - |\lambda|^2 = 0$.

$$\begin{aligned} d^2 - |\lambda - c|^2 &= d^2 - c^2 + 2c\operatorname{Re}\lambda - |\lambda|^2 = |\lambda^\pm|^2 - 2c\,\lambda^\pm + 2c\operatorname{Re}\lambda - |\lambda|^2 \\ &= |\lambda^\pm|^2 - \frac{|\lambda^+|^2 - |\lambda^-|^2}{\operatorname{Re}\lambda^+ - \operatorname{Re}\lambda^-}\operatorname{Re}\lambda^\pm + \frac{|\lambda^+|^2 - |\lambda^-|^2}{\operatorname{Re}\lambda^+ - \operatorname{Re}\lambda^-}\operatorname{Re}\lambda - |\lambda|^2 \\ &= \frac{|\lambda^-|^2\operatorname{Re}\lambda^+ - |\lambda^+|^2\operatorname{Re}\lambda^-}{\operatorname{Re}\lambda^+ - \operatorname{Re}\lambda^-} + \frac{|\lambda^+|^2 - |\lambda^-|^2}{\operatorname{Re}\lambda^+ - \operatorname{Re}\lambda^-}\operatorname{Re}\lambda - |\lambda|^2 \\ &= \frac{(|\lambda^-|^2 - |\lambda|^2)(\operatorname{Re}\lambda^+ - \operatorname{Re}\lambda) - (|\lambda^+|^2 - |\lambda|^2)(\operatorname{Re}\lambda^- - \operatorname{Re}\lambda)}{\operatorname{Re}\lambda^+ - \operatorname{Re}\lambda^-} \end{aligned}$$

Substituting (b) into the right-hand side of the last equation, we have that

$$|\lambda^-|^2 - |\lambda|^2 = \frac{|\lambda|^2\operatorname{Re}\beta - \operatorname{Re}\lambda^2\beta}{1-\operatorname{Re}\beta}, \qquad \operatorname{Re}\lambda^+ - \operatorname{Re}\lambda = \frac{\operatorname{Re}\lambda\beta - \operatorname{Re}\lambda\operatorname{Re}\beta}{1+\operatorname{Re}\beta},$$
$$|\lambda^+|^2 - |\lambda|^2 = \frac{|\lambda|^2\operatorname{Re}\beta - \operatorname{Re}\lambda^2\beta}{1+\operatorname{Re}\beta}, \qquad \operatorname{Re}\lambda^- - \operatorname{Re}\lambda = \frac{\operatorname{Re}\lambda\beta - \operatorname{Re}\lambda\operatorname{Re}\beta}{1-\operatorname{Re}\beta}.$$

So

$$(|\lambda^-|^2 - |\lambda|^2)(\operatorname{Re}\lambda^+ - \operatorname{Re}\lambda) = (|\lambda^+|^2 - |\lambda|^2)(\operatorname{Re}\lambda^- - \operatorname{Re}\lambda).$$

Hence $|\lambda - c| = d$. Namely, λ lies on the geodesic connecting through λ^+ and λ^-.

(f) In §3, we already see that the geodesic circle $C_{\mathrm{P}}(\lambda;\, k)$ with center $\lambda = p + q\sqrt{-1}$ for some $0 \leq k \leq 1$ is the circle with center $p + \frac{1+k^2}{1-k^2} q\sqrt{-1}$ and radius $\frac{2\,k}{1-k^2} q$. We give $k \in \mathbb{R}$ such that $\ell(W) = C_{\mathrm{P}}(\lambda;\, k)$. In the case $\mathsf{v} \perp \overline{\mathsf{v}}$, we already get $\ell(W) = \{\lambda\}$ in (c). Since $\frac{1-\sqrt{1-\beta^2}}{|\beta|} \to 0$ as $\beta \to 0$, we consider that $C_{\mathrm{P}}(\lambda;\, \frac{1-\sqrt{1-\beta^2}}{|\beta|}) = C_{\mathrm{P}}(\lambda;\, 0) = \{\lambda\}$ for $\beta = 0$. $\mathsf{v} \perp \overline{\mathsf{v}}$ means $\beta = 0$, so the assertion holds in this case.

Assume that $\mathsf{v} \not\perp \overline{\mathsf{v}}$, i.e. $\beta \neq 0$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix representation of $\varphi|_W$ with respect to an orthonormal basis of W. Then $\lambda = \frac{(a+d)+\sqrt{D}}{2}$, where $D = (a+d)^2 - 4(ad-bc)$. By Proposition 5.2, the subleaf $\ell(W) = \Psi(\varphi;\, W)$ corresponding to W is the circle with center $\frac{(a+d)+|b-c|\sqrt{-1}}{2}$ and radius $\frac{\sqrt{(a-d)^2+(b+c)^2}}{2}$. For the centers of circles $C_{\mathrm{P}}(\lambda;\, k)$ and $\ell(W)$ to coincide, p, q and k satisfy that

$$p = \frac{a+d}{2}, \qquad q = \frac{\sqrt{-D}}{2}, \qquad \frac{1+k^2}{1-k^2}\, q = \frac{|b-c|}{2}.$$

And then the radiuses of the circles coincide as the following.

$$(\frac{2\,k}{1-k^2}\, q)^2 = ((\frac{1+k^2}{1-k^2})^2 - 1)\, q^2 = (\frac{|b-c|}{2})^2 - (\frac{\sqrt{-D}}{2})^2 = (\frac{\sqrt{(a-d)^2+(b+c)^2}}{2})^2.$$

Hence $\ell(W) = C_{\mathrm{P}}(\lambda;\, k)$. λ^+ lies on the geodesic circle $C_{\mathrm{P}}(\lambda;\, k)$, so we have that

$$k^2 = \delta(\lambda,\, \lambda^+)^2 = \frac{|\lambda - \lambda^+|^2}{|\lambda - \overline{\lambda^+}|^2} = \frac{(\operatorname{Re}\lambda - \operatorname{Re}\lambda^+)^2 + (\operatorname{Im}\lambda - \operatorname{Im}\lambda^+)^2}{(\operatorname{Re}\lambda - \operatorname{Re}\lambda^+)^2 + (\operatorname{Im}\lambda + \operatorname{Im}\lambda^+)^2} = \frac{1-\sqrt{1-|\beta|^2}}{1+\sqrt{1-|\beta|^2}}.$$

Hence,

$$\sqrt{1-|\beta|^2} = \frac{1-k^2}{1+k^2} \qquad \therefore\ |\beta| = \frac{2k}{1+k^2}.$$

Since k is the root of the quadratic equation $|\beta|\, k^2 - 2\, k + |\beta| = 0$ lying on the interval $[0,\, 1]$, we obtain that

$$k = \frac{1-\sqrt{1-|\beta|^2}}{|\beta|}.$$

□

Corollary 7.2. *Let λ be an eigenvalue of φ with positive imaginary part. If there exist two $\mathbb{C}$-linearly independent eigenvectors corresponding to λ, then λ belongs to $\Psi(\varphi)$.*

Proof. Let $\mathsf{v}_1,\ \mathsf{v}_2 \in V_{\mathbb{C}}$ be $\mathbb{C}$-linearly independent eigenvectors corresponding to λ. If $\mathsf{v}_1 \perp \overline{\mathsf{v}_1}$, then the last proposition (c) leads that $\mathbb{L}_\varphi(\operatorname{Re}\mathsf{v}_1) = \lambda$, i.e. $\lambda \in \Psi(\varphi)$. Assume that $\mathsf{v}_1 \not\perp \overline{\mathsf{v}_1}$. Put $\mathsf{v} = t\,\mathsf{v}_1 + \mathsf{v}_2 \in V_\lambda$ $(t \in \mathbb{C})$. The quadratic equation $\mathsf{v} \cdot \overline{\mathsf{v}} = (\mathsf{v}_1 \cdot \overline{\mathsf{v}_1})\, t^2 +$

$2\operatorname{Re}(\mathsf{v}_1 \cdot \overline{\mathsf{v}_2})\, t + (\mathsf{v}_2 \cdot \overline{\mathsf{v}_2}) = 0$ has a root $t_0 \in \mathbb{C}$, because the coefficient of the leading term is nonzero. Put $\mathsf{v}_0 = t_0\mathsf{v}_1 + \mathsf{v}_2$, and then $\mathsf{v}_0 \perp \overline{\mathsf{v}_0}$. The last proposition (c) leads that $\mathbb{L}_\varphi(\operatorname{Re}\mathsf{v}_0) = \lambda$, i.e. $\lambda \in \Psi(\varphi)$. □

The last corollary is included in Theorem B. By Theorem B, there exists a vector for which the value of the function $\mathbb{L}_\varphi$ is the given eigenvalue. However, the last corollary provides an explicit such vector in terms of eigenvectors, provided that the eigenspace has dimension two or higher.

Let $\lambda \in \overline{\mathfrak{H}}$ be an eigenvalue of φ with nonnegative imaginary part, and $V_\lambda \subset V_{\mathbb{C}}$ the eigenspace corresponding to λ. The real part $\operatorname{Re} V_\lambda = \{\operatorname{Re}\mathsf{v} \mid \mathsf{v} \in V_\lambda\}$ of V_λ is a subspace of V. It is easy to show that $\operatorname{Re} V_\lambda = (V_\lambda + \overline{V_\lambda}) \cap V = \operatorname{Im} V_\lambda$ and $\overline{V_\lambda} = V_{\overline{\lambda}}$. Moreover, since $\operatorname{Re} V_\lambda$ is φ-stable, $\mathbb{L}_\varphi((\operatorname{Re} V_\lambda)^\times)$ is the subleaf $\Psi(\varphi;\, \operatorname{Re} V_\lambda)$ corresponding to $\operatorname{Re} V_\lambda$. In Proposition 7.1, we use the closed curve $\ell(\langle \operatorname{Re}\mathsf{v},\, \operatorname{Im}\mathsf{v}\rangle)$ for some $\mathsf{v} \in V_\lambda$, to see $\Psi(\varphi)$ around λ. We need the subleaf $\Psi(\varphi;\, \operatorname{Re} V_\lambda)$ to look at the relationship between λ and the other point on $\Psi(\varphi)$.

Proposition 7.3. *Let $\lambda \in \overline{\mathfrak{H}}$ be an eigenvalue with non-negative imaginary part.*

(a) *If $\lambda \in \mathbb{R}$, then $\Psi(\varphi;\, \operatorname{Re} V_\lambda) = \{\lambda\}$.*

(b) *Assume that $\lambda \notin \mathbb{R}$. $\Psi(\varphi;\, \operatorname{Re} V_\lambda) = \{\lambda\}$ holds, if and only if $V_\lambda \perp \overline{V_\lambda}$.*

(c) *Assume that $\lambda \notin \mathbb{R}$ and $V_\lambda \not\perp \overline{V_\lambda}$. Then $\Psi(\varphi;\, \operatorname{Re} V_\lambda)$ is a geodesic circle with center λ when $\dim \operatorname{Re} V_\lambda = 2$, and a geodesic disc (a filled geodesic circle) with center λ when $\dim \operatorname{Re} V_\lambda > 2$.*

Proof. (a) Since $\lambda \in \mathbb{R}$, $\operatorname{Re} V_\lambda$ is the eigenspace in V corresponding to λ. It follows from the proof of Proposition 3.3 that $\Psi(\varphi;\, \operatorname{Re} V_\lambda) = \mathbb{L}_\varphi((\operatorname{Re} V_\lambda)^\times) = \{\mathbb{L}_\varphi(\mathsf{v}) \mid \mathsf{v} \in (\operatorname{Re} V_\lambda)^\times\} = \{\lambda\}$.

(b) Assume that $V_\lambda \perp \overline{V_\lambda}$. Each $\mathsf{v}_1 \in V_\lambda$ is orthogonal to its conjugate. Proposition 7.1 (c) leads that $\mathbb{L}_\varphi(\operatorname{Re}\mathsf{v}_1) = \lambda$. Then, $\Psi(\varphi;\, \operatorname{Re} V_\lambda) = \mathbb{L}_\varphi((\operatorname{Re} V_\lambda)^\times) = \{\lambda\}$.

Conversely, we assume that $V_\lambda \not\perp \overline{V_\lambda}$. Take $\mathsf{v}_1, \mathsf{v}_2 \in V_\lambda$ satisfying that $\mathsf{v}_1 \not\perp \overline{\mathsf{v}_2}$. At least one of v_1, v_2 and $\mathsf{v}_1 + \mathsf{v}_2$ is not orthogonal to the conjugate of itself. Because, if $\mathsf{v}_1 \perp \overline{\mathsf{v}_1}$ and $\mathsf{v}_2 \perp \overline{\mathsf{v}_2}$, then $(\mathsf{v}_1 + \mathsf{v}_2) \not\perp \overline{\mathsf{v}_1 + \mathsf{v}_2}$ as the following:

$$(\mathsf{v}_1 + \mathsf{v}_2) \cdot (\overline{\mathsf{v}_1 + \mathsf{v}_2}) = \mathsf{v}_1 \cdot \overline{\mathsf{v}_1} + \mathsf{v}_1 \cdot \overline{\mathsf{v}_2} + \mathsf{v}_2 \cdot \overline{\mathsf{v}_1} + \mathsf{v}_2 \cdot \overline{\mathsf{v}_2} = \mathsf{v}_1 \cdot \overline{\mathsf{v}_2} + \mathsf{v}_2 \cdot \overline{\mathsf{v}_1} = 2\,\mathsf{v}_1 \cdot \overline{\mathsf{v}_2} \neq 0.$$

Put v_0 one of v_1, v_2, $\mathsf{v}_1 + \mathsf{v}_2$ which is not orthogonal to the conjugate of itself. Proposition 7.1 (c) leads that $\mathbb{L}_\varphi(\operatorname{Re}\mathsf{v}_0) \neq \lambda$. Hence, $\Psi(\varphi;\, \operatorname{Re} V_\lambda) = \mathbb{L}_\varphi((\operatorname{Re} V_\lambda)^\times) \neq \{\lambda\}$.

(c) If $\dim \operatorname{Re} V_\lambda = 2$, then the subleaf $\Psi(\varphi;\, \operatorname{Re} V_\lambda)$ is a geodesic circle with center

λ. Assume that $\dim V_\lambda \geq 3$. Then the subleaf $\Psi(\varphi;\, \mathrm{Re}\, V_\lambda)$ is simply connected. By Proposition 7.1 (f), $\ell(\langle \mathrm{Re}\, \mathsf{v},\, \mathrm{Im}\, \mathsf{v}\rangle)$ is a geodesic circle with center λ for any $\mathsf{v} \in V_\lambda$. Since $\mathrm{Re}\, V_\lambda = \bigcup_{\mathsf{v}\in V_\lambda} \langle \mathrm{Re}\, \mathsf{v},\, \mathrm{Im}\, \mathsf{v}\rangle$, we see that $\Psi(\varphi;\, \mathrm{Re}\, V_\lambda) = \bigcup_{\mathsf{v}\in V_\lambda} \ell(\langle \mathrm{Re}\, \mathsf{v},\, \mathrm{Im}\, \mathsf{v}\rangle)$, the union of geodesic circle with same center. The simply connected domain $\Psi(\varphi;\, \mathrm{Re}\, V_\lambda)$ is a geodesic disc with center λ. □

Remark 7.4. In the proof of part (c) of the last proposition, the geometric property of the structure theorem for the leaf was used. However, the proof can be carried out without relying on this property. The subleaf $\Psi(\varphi;\, \mathrm{Re}\, V_\lambda)$ is the union of concentric geometric circles $\ell(\langle \mathrm{Re}\, \mathsf{v},\, \mathrm{Im}\, \mathsf{v}\rangle)$ ($\mathsf{v} \in \mathrm{Re}\, V_\lambda$) with center λ, by Proposition 7.1 (f). Moreover, Proposition 7.1 (f) and the argument of the proof of Corollary 7.2 leads that the set of radiuses of the geometric circles $\ell(\langle \mathrm{Re}\, \mathsf{v},\, \mathrm{Im}\, \mathsf{v}\rangle)$ is a closed interval from 0 to some nonnegative real number. So, the leaf $\Psi(\varphi;\, \mathrm{Re}\, V_\lambda)$ is a geodesic disc with center λ.

7.2 The geodesic bridge

Definition 7.5. For any subset A, $B \subset \overline{\mathfrak{H}}$, the union of all geodesics connecting a point on A to a point on B is called the *geodesic bridge* connecting A to B, denoted by $A \asymp B$.

The geodesic bridge $\asymp$ is a commutative semigroup on the power set of $\overline{\mathfrak{H}}$, that is, it satisfies the associativity: $(A \asymp B) \asymp C = A \asymp (B \asymp C)$, and the commutativity: $A \asymp B = B \asymp A$. The geodesic bridge connecting finite convex subsets is the smallest convex set including all those subsets. Hence, the eigenvalue geodesic polygon of an endomorphism φ is represented by $\{\lambda_1\} \asymp \cdots \asymp \{\lambda_r\}$ where $\lambda_1, \ldots, \lambda_r$ are the eigenvalue of φ with nonnegative imaginary part.

Proposition 7.6. *Let φ be an endomorphism on a real inner product space V, and W_1 and W_2 two nonzero φ-stable subspaces of V satisfying that $W_1 \perp W_2$. Then*

$$\Psi(\varphi;\, W_1 \oplus W_2) = \Psi(\varphi;\, W_1) \asymp \Psi(\varphi;\, W_2).$$

Proof. $\dim(W_1 + W_2) \leq 2$ leads $\dim W_1 = \dim W_2 = 1$, so $\Psi(\varphi;\, W_1)$ and $\Psi(\varphi;\, W_2)$ are single point sets. The elements λ_1 and $\lambda_2 \in \mathbb{R}$ of these sets are eigenvalues of $\varphi|_{W_1}$ and $\varphi|_{W_2}$, respectively. Since $W_1 \perp W_2$, the representation matrix of $\varphi|_{W_1\oplus W_2}$ of an orthonormal bases is the diagonal matrix whose components of diagonal are λ_1 and λ_2. By Proposition 5.2, the subleaf $\Psi(\varphi;\, W_1 \oplus W_2)$ is a geodesic connecting through λ_1 and λ_2,

that is the geodesic bridge connecting $\{\lambda_1\}$ to $\{\lambda_2\}$. Hence, $\Psi(\varphi;\, W_1 \oplus W_2) = \Psi(\varphi;\, W_1) \asymp \Psi(\varphi;\, W_2)$.

Assume that $\dim(W_1 \oplus W_2) \geq 3$. The subleaf $\Psi(\varphi;\, W_1 \oplus W_2)$ is convex, so $\Psi(\varphi;\, W_1 \oplus W_2)$ includes $\Psi(\varphi;\, W_1) \asymp \Psi(\varphi;\, W_2))$. We will show the inverse inclusion relation. By looking at the end points of the geodesics, $\Psi(\varphi;\, W_1)$ and $\Psi(\varphi;\, W_2)$ are included in $\Psi(\varphi;\, W_1 \oplus W_2)$. Let $\lambda \in \Psi(\varphi\, W_1 \oplus W_2) \setminus (\Psi(\varphi;\, W_1) \cup \Psi(\varphi;\, W_2))$. Take $\mathsf{v} = \mathsf{w}_1 + \mathsf{w}_2 \in W_1 \oplus W_2$ such that $\lambda = \mathbb{L}_\varphi(\mathsf{v})$, $\mathsf{w}_1 \in W_1$ and $\mathsf{w}_2 \in W_2$. Since w_1 and w_2 satisfy the orthogonal condition in Proposition 6.4, the curve $\ell(\langle \mathsf{w}_1,\, \mathsf{w}_2 \rangle)$ is the geodesic connecting $\mathbb{L}_\varphi(\mathsf{w}_1)$ to $\mathbb{L}_\varphi(\mathsf{w}_2)$. Since $\lambda = \mathbb{L}_\varphi(\mathsf{w}_1 + \mathsf{w}_2) \in \ell(\langle \mathsf{w}_1,\, \mathsf{w}_2 \rangle)$, $\mathbb{L}_\varphi(\mathsf{w}_1) \in \Psi(\varphi;\, W_1)$ and $\mathbb{L}_\varphi(\mathsf{w}_2) \in \Psi(\varphi;\, W_2)$, λ lies on the geodesic bridge connecting $\Psi(\varphi;\, W_1)$ to $\Psi(\varphi;\, W_2)$. Hence,

$$\Psi(\varphi;\, W_1 \oplus W_2) = \Psi(\varphi;\, W_1) \asymp \Psi(\varphi;\, W_2).$$

□

7.3 Leaf of a normal endomorphism

Theorem C. *Leaf of a normal endomorphism coincides with the eigenvalue geodesic polygon.*

Proof. Let φ be a normal endomorphism on a real inner product space V of finite dimension. Then, eigenvectors of φ corresponding to different eigenvalues are orthogonal. V is the orthogonal sum of real parts of eigenspaces of all eigenvalues with nonnegative imaginary part. Let $\Lambda = \{\lambda_1,\, \ldots,\, \lambda_r\}$ be the set of all eigenvalues with nonnegative imaginary part, $\overline{\Lambda}$ the set of all eigenvalues, and V_λ the eigenspace corresponding to $\lambda \in \overline{\Lambda}$. Since φ is normal, $V_\lambda \perp V_{\lambda'}$ for any $\lambda,\, \lambda' \in \overline{\Lambda}$ with $\lambda \neq \lambda'$. And then $\Psi(\varphi;\, \mathrm{Re}\, V_\lambda) = \{\lambda\}$ for any $\lambda \in \Lambda$ by Proposition 7.3, and V is the φ-stable orthogonal sum of $\mathrm{Re}\, V_{\lambda_1},\, \cdots,\, \mathrm{Re}\, V_{\lambda_r}$. Namely, $V = \mathrm{Re}\, V_{\lambda_1} \oplus \cdots \oplus \mathrm{Re}\, V_{\lambda_r}$, $\mathrm{Re}\, V_{\lambda_i} \perp \mathrm{Re}\, V_{\lambda_j}$ $(i \neq j)$, and each $\mathrm{Re}\, V_{\lambda_i}$ is φ-stable. By Proposition 7.6,

$$\Psi(\varphi) = \Psi(\varphi;\, \mathrm{Re}\, V_{\lambda_1}) \asymp \cdots \asymp \Psi(\varphi;\, \mathrm{Re}\, V_{\lambda_r}) = \{\lambda_1\} \asymp \{\lambda_2\} \asymp \cdots \asymp \{\lambda_r\}$$

□

Remark 7.7. In general, the converse of the last theorem does not hold. For example, take the endomorphism $\varphi = (-1) \oplus (0) \oplus (1) \oplus ((3\,\mathsf{J} + \mathsf{K})/4)$ on $\mathbb{R}^5$, where J and K are quadratic square matrices defined in Example 1.2. $(3\,\mathsf{J} + \mathsf{K})/4$ is not normal, so φ is also not. The eigenvalues with nonnegative imaginary part are -1, 0, 1 and $\sqrt{-1/2}$. The

filled geodesic triangle Δ with vertexes -1, 0 and 1 is surrounded by three semicircles with center $1/2$ and radius $1/2$, with center $-1/2$ and radius $1/2$, and with center 0 and radius 1. The eigenvalue $\lambda = \sqrt{-1/2}$ lies on the interior of Δ, so Δ is also the eigenvalue geodesic polygon of φ. Put $W = \operatorname{Re} V_\lambda$, then we have that $\varphi|_W = (3\,\mathsf{J} + \mathsf{K})/4$ and $\varphi|_{W^\perp} = (-1) \oplus (0) \oplus (1)$. The subleaf $\Psi(\varphi;\, W)$ is a circle with center $\alpha = 3\sqrt{-1}/4$ and radius $1/4$ by Proposition 5.2, and the subleaf $\Psi(\varphi;\, W^\perp) = \Delta$ by the last theorem. We see that the circle $\Psi(\varphi;\, W)$ is included in Δ, by calculating the distance between the center α of the circle and the edges of Δ. Therefore, the leaf $\Psi(\varphi)$ is the eigenvalue geodesic polygon. φ and $\varphi|_{W^\perp}$ have the same leaf. By replacing or increasing the direct sum factor appropriately, it is possible to obtain infinitely many endomorphisms with the same leaf.

Appendix

Note on the contents of this thesis

The body of this thesis mainly consists of the author's article published in *Linear Algebra and its Applications* Volume 724 (2025), 62–82. The inclusion of the original texts of the aforementioned article from *Linear Algebra and its Applications* into the author's thesis and its subsequent compilation at the author's discretion are permitted in the "Publishing Agreement" between Elsevier Inc. and the author.

The revisions made in this thesis are as follows:

- While the published article was structured only by sections, in this thesis, related sections have been reorganized into chapters, with the original sequence preserved.
- §4.3 was newly added to Chapter 4 to include the proofs of the fundamental theorems of numerical range for the reader's convenience.
- In the proof of Proposition 7.1 (e), the derivation of $|\lambda - c| = d$ has been revised to provide a clearer and more concise explanation.
- Several minor corrections have been made, including the revision of typographical errors, grammatical mistakes, and unclear expressions identified after publication.

Bibliography

[1] *S. K. Berberian*, The Numerical Range of a Normal Operator, Duke Math. J. **31** (1964), 479–483.

[2] *M. Elin*, *S. Reich* and *D. Shoikhet*, Numerical Range of Holomorphic Mappings and Applications, Birkhäuser, 2019

[3] *Karl E. Gustafson* and *Duggirala K. M. Rao*, Numerical Range, Springer-Verlag, 1995

[4] *F. Hausdorff*, Der Wertvorrat einer Bilinearform, Math. Z. **3** (1919), 314–316

[5] *S. Roman*, Advanced Linear algebra (3rd ed), Springer GTM, 2007

[6] *M. H. Stone*, Linear Transformations in Hilbert Space, American Mathematical Society, R.I., 1932

[7] *O. Toeplitz*, Das algebraische Analogon zu einem Satz von Fejer, Math. Z. **2** (1918), 187–197.

[8] *A. Wintner*, Zur Theorie der beschrankten Bilinearformen, Math. Z. **30** (1929), 228–282.