Osaka University Knowledge Archive

| Title | Affine rulings of normal rational surfaces |
| :---: | :--- |
| Author(s) | Daigle, Daniel; Russell, Peter |
| Citation | Osaka Journal of Mathematics. 2001, 38(1), p. <br> $37-100$ |
| Version Type | VoR |
| URL | https://doi.org/10.18910/10530 |
| rights |  |
| Note |  |

Osaka University Knowledge Archive : OUKA
https://ir. library.osaka-u.ac.jp/

# AFFINE RULINGS OF NORMAL RATIONAL SURFACES 

Daniel DAIGLE and Peter RUSSELL

(Received December 21, 1998)

Given an algebraic surface $X$ satisfying:
$X$ is a complete normal rational surface, $X$ is affine ruled and $\operatorname{rank}\left(\operatorname{Pic} X_{s}\right)=1$,
where $X_{s}$ denotes the smooth locus of $X$, consider:
Problem 1. Find all affine rulings of $X$.
Problem 2. Find all pairs of curves $C_{1}, C_{2}$ on $X$ such that $X \backslash\left(C_{1} \cup C_{2}\right)$ is isomorphic to $\mathbb{P}^{2}$ minus two lines.

Problem 3. Find all curves $C$ in $X$ such that $\bar{\kappa}\left(X_{s} \backslash C\right)=-\infty$.
This paper investigates Problem 1 for an arbitrary $X$ satisfying ( $\dagger$ ). We define (Definition 1.14) the notion of a "basic" affine ruling of $X$ and our main results describe how to construct all affine rulings of $X$, assuming that the basic ones are known. In the case where $X$ is a weighted projective plane, the basic affine rulings of $X$ are given in [6]; the present paper and [6] therefore constitute a solution to Problem 1 in that case.

Problem 3 (with $X=\mathbb{P}^{2}$ ) has been considered by several authors ([8], [9], [18], [19], [14]). In his review of [14] (see MR 82k:14013), M. H. Gizatullin mentions some unpublished examples found by V. I. Danilov and himself, and which seem to correspond to the list of basic affine rulings of $\mathbb{P}^{2}$. The case $X=\mathbb{P}^{2}$ was finally solved in [10]. Our generalization to weighted projective planes seems to be new, as well as our method-valid for any $X$ satisfying ( $\dagger$ )-which reduces the general problem to the determination of the basic affine rulings.

Let us briefly indicate how problems 1-3 are related to each other. Consider the stronger condition ( $\ddagger$ ) on a surface $X$ :

$$
X \text { satisfies }(\dagger) \text { and every singular point of } X \text { is a cyclic quotient sin- }
$$ gularity.

As an example, note that the weighted projective planes satisfy ( $\ddagger$ ) (they even satisfy $\operatorname{Pic}\left(X_{s}\right)=\mathbb{Z}$; see [6] for these claims). Also note the following by-product of section 1 : A surface satisfying ( $\ddagger$ ) cannot have more than 3 singular points (see Corollary 1.16).

It is clear that any solution $\left(C_{1}, C_{2}\right)$ to Problem 2 gives rise to an affine ruling of $X$; by Theorem 1.15 , the converse holds if $X$ satisfies ( $\ddagger$ ), so:

For any surface $X$ satisfying ( $\ddagger$ ), Problems 1 and 2 are equivalent.
The exact relation between Problem 3 and the other two is given by the following statement, which will be proved in 1.17, below: Given X satisfying ( $\ddagger$ ) and a curve $C$ on $X$, the following are equivalent:
(i) $\bar{\kappa}\left(X_{s} \backslash C\right)=-\infty$;
(ii) there exists at least one affine ruling ${ }^{1} \Lambda$ of $X$ such that $n C \in \Lambda$ for some $n>0$.

For instance, if $C \subset \mathbb{P}^{2}$ is Yoshihara's rational quintic ([19], Proposition 3, case $N=1$ ), then infinitely many affine rulings $\Lambda$ of $\mathbb{P}^{2}$ contain multiples of $C$.

By way of motivation, we now explain the connection between problems $1-3$ and locally nilpotent derivations. Consider the polynomial ring $B=\mathbf{k}\left[X_{1}, X_{2}, X_{3}\right]$, where $\mathbf{k}$ is an algebraically closed field of characteristic zero. It is known ([13], [3]) that describing the locally nilpotent derivations $D: B \rightarrow B$ is equivalent to answering: Which pairs of polynomials $f, g \in B$ have the property that $\mathbf{k}[f, g]$ is the kernel of a locally nilpotent derivation of $B$ ? If we restrict ourselves to the case where $D$ is (or equivalently $f$ and $g$ are) homogeneous with respect to weights $w\left(X_{i}\right)=a_{i}$, where $a_{1}, a_{2}, a_{3}$ are relatively prime positive integers, then we can think of $f$ and $g$ as defining curves in the weighted projective plane $\mathbb{P}\left(a_{1}, a_{2}, a_{3}\right)=\operatorname{Proj} B$; then [4] gives the following result:

Theorem. For $w$-homogeneous elements $f, g \in B$ satisfying $\operatorname{gcd}(w(f), w(g))=1$, the following are equivalent:
(1) There exists a w-homogeneous locally nilpotent derivation $D$ of $B$ such that ker $D=\mathbf{k}[f, g]$;
(2) $f$ and $g$ are irreducible elements of $B$ and the algebraic surface $\operatorname{Proj} B \backslash V(f g)$ is isomorphic to $\mathbb{P}^{2}$ minus two lines.

Note that the case where $\operatorname{gcd}(w(f), w(g)) \neq 1$ turns out to be very special, and is completely described in [4]. Hence, solving Problem 2 for $X=\mathbb{P}\left(a_{1}, a_{2}, a_{3}\right)$ is equivalent to describing homogeneous locally nilpotent derivations of $B$. Since that class of derivations is not well understood, and corresponds to a class of $G_{a}$-actions on $\mathbb{A}^{3}$ which ought to be understood, there is ample reason to study affine rulings.

## Organization of the text

Fix a surface $X$ satisfying ( $\dagger$ ).
Section 1 contains generalities about affine rulings of $X$.

[^0]Section 2 defines a process which is used to modify affine rulings of $X$ (i.e., applying it to an affine ruling of $X$ produces a different affine ruling of $X$ ). The process makes its first appearance in the proof of Theorem 2.1, where it is shown that every non-basic affine ruling of $X$ can be "reduced" to a simpler one; this reduction process is in fact a special case of the modification process.

Some preparation is necessary before defining the modification process: 2.2 defines the notion of an " $X$-immersion"; then 2.3-2.8 show that each $X$-immersion determines an affine ruling of $X$, that each affine ruling can be obtained in this way, and that this can be turned into a bijective correspondence, modulo appropriate adjustments.

Given an $X$-immersion $I, 2.9$ defines a set $\Pi(I)$ and a new $X$-immersion $I * \pi$ for each $\pi \in \Pi(I)$. This operation $*$ is the modification process which was announced; it acts on $X$-immersions, so it indirectly modifies affine rulings via the correspondence mentioned in the preceding paragraph. Discussion 2.14 summarizes the results of section 2. In particular, it states that all affine rulings of $X$ can be constructed from the basic ones by using the $*$ operation; and consequently the solution of Problem 1 consists of two parts:
(1a) Make a list of all basic affine rulings of $X$.
(1b) For each $X$-immersion $I$, describe the set $\Pi(I)$.
Problem (1b) is essentially a problem in the theory of weighted graphs, independent of the surface, and is completely solved in sections 3 and 4: section 3 does the graph theory and section 4 states the consequences for $\Pi(I)$. This paper does not solve Problem (1a), which is highly dependent on the surface $X$; [6] solves it for the weighted projective planes.

In contrast with sections 2 and 4, where rulings are described by saying that they can be constructed from basic ones by using the modification process, section 5 gives direct information on affine rulings. The main result of that section is Theorem 5.13; it is complemented by several other (more practical) statements, notably 5.17, 5.22, 5.23, 5.25, 5.34, 5.40.

## Conventions

All curves and surfaces considered in this paper are assumed to be algebraic varieties over an algebraically closed field $\mathbf{k}$ of characteristic zero. In particular, curves and surfaces are irreducible and reduced.

If $f: X \rightarrow Y$ is a birational morphism of surfaces then the center of $f$ (denoted center $(f)$ ) is the set of points $y \in Y$ such that $f^{-1}(y)$ contains more than one point.

Let $S$ be a smooth complete surface. If $D$ is a divisor of $S$ then, by a component of $D$, we always mean an irreducible (or prime) component of $D$. If $D$ and $D^{\prime}$ are divisors of $S$ then $D \cdot D^{\prime}$ denotes their intersection number and $D^{2}=D \cdot D$. If $C \subset S$
is a smooth rational curve and $C^{2}=r$, we call $C$ an $r$-curve; by an $r$-component of a divisor $D$, we mean a component of $D$ which is an $r$-curve. A reduced effective divisor $D$ of $S$ has strong normal crossings if: (i) each component of $D$ is a smooth curve; (ii) if $D_{i}$ and $D_{j}$ are distinct components of $D$ then $D_{i} \cdot D_{j} \leq 1$; and (iii) if $D_{i}, D_{j}$ and $D_{k}$ are distinct components of $D$ then $D_{i} \cap D_{j} \cap D_{k}$ is empty.

Except for the graph $\mathbb{L}(X)$ of 2.14 , every graph considered in this paper is a weighted graph, i.e., a graph in which each vertex is assigned an integer (called its weight). Every weighted graph in this paper is a finite undirected graph such that no edge connects a vertex to itself and at most one edge joins any given pair of vertices.

If $S$ is a smooth complete surface and $D$ a divisor of $S$ with strong normal crossings, the dual graph of $(D, S)$ is the weighted graph $\mathcal{G}=\mathcal{G}(D, S)$ whose vertices are the components of $D$; distinct vertices $D_{i}$ and $D_{j}$ are joined by an edge if $D_{i} \cap D_{j} \neq \emptyset$; and the weight of a vertex $D_{i}$ is $D_{i}^{2}$. We assume familiarity with this idea, as well as with the basic theory of weighted graphs (their blowing-up, blowingdown and equivalence); the relevant definitions can be found in various sources, for instance [17], [16], or the appendix of [2] (see also the beginning of section 3, in this paper). Let $D_{1}, \ldots, D_{n}$ be the distinct components of $D$. We say that $D_{j}$ is a neighbor of $D_{i}$ if $i \neq j$ and $D_{i} \cap D_{j} \neq \emptyset$ (i.e., if the vertices $D_{i}, D_{j}$ of $\mathcal{G}$ are neighbors); the number of neighbors of $D_{i}$ is called its branching number; if this number is greater than or equal to 3 , we say that $D_{i}$ is a branching component of $D$ (or that the vertex $D_{i}$ is a branch point of $\mathcal{G}$ ). We say that $\mathcal{G}$ is a linear chain (or a linear tree) if it is a tree without branch points; an admissible chain is a linear chain in which every weight is strictly less than -1 ; note that the empty graph is an admissible chain. We say that $D$ is a tree (or a linear chain, or an admissible chain, etc) if $\mathcal{G}$ has the corresponding property.

Let $X$ and $X^{*}$ be complete normal surfaces, $\beta$ a birational isomorphism between them (either $X \xrightarrow{\beta} X^{*}$ or $X \stackrel{\beta}{\leftarrow} X^{*}$ ) and $\Lambda$ a one-dimensional linear system on $X$ without fixed components. In this situation, we will often use the fact that $\Lambda$ and $\beta$ determine, in a natural way, a one-dimensional linear system $\Lambda^{*}$ on $X^{*}$ without fixed components. The tacit understanding is that, for suitably chosen rational maps $X \xrightarrow{\lambda} \mathbb{P}^{1}$ and $X^{*} \xrightarrow{\lambda^{*}} \mathbb{P}^{1}$ determining $\Lambda$ and $\Lambda^{*}$ respectively, $\beta, \lambda$ and $\lambda^{*}$ form a commutative diagram.

The set of nonnegative (resp. positive) integers is denoted $\mathbb{N}$ (resp. $\mathbb{Z}^{+}$).

## 1. Preliminaries on affine rulings

1.1. Let $X$ be a complete normal rational surface. An "affine ruling" of $X$ is usually defined to be a morphism $p: U \rightarrow \Gamma$ where $\Gamma$ is a curve, $U$ is a nonempty open subset of $X$ isomorphic to $\Gamma \times \mathbb{A}^{1}$ and $p$ is the projection $\Gamma \times \mathbb{A}^{1} \rightarrow \Gamma$. Since $\Gamma \times \mathbb{A}^{1}$ is normal and rational, $\Gamma$ is an open subset of $\mathbb{P}^{1}$ and $U$ is contained in the
smooth locus of $X$. The morphism $p$ extends to a rational map $X \rightarrow \mathbb{P}^{1}$ which, in turn, determines a unique linear system $\Lambda$ on $X$ without fixed components. Since we do not want to distinguish between rulings which determine the same linear system $\Lambda$, we adopt the viewpoint that $\Lambda$ itself is the affine ruling:

Definition. Let $\Lambda$ be a one-dimensional linear system on $X$ without fixed components. We say that $\Lambda$ is an affine ruling of $X$ if there exist nonempty open subsets $U \subset X$ and $\Gamma \subseteq \mathbb{P}^{1}$ such that $U \cong \Gamma \times \mathbb{A}^{1}$ and such that the projection morphism $\Gamma \times \mathbb{A}^{1} \rightarrow \Gamma$ determines $\Lambda$.

If $\Lambda$ is an affine ruling of $X$ then the general member $C$ of $\Lambda$ satisfies $C \cap U \cong$ $\mathbb{A}^{1}$; it follows:

- the general member of $\Lambda$ is irreducible and reduced;
- $\quad \Lambda$ has at most one base point on $X$.

In the special case where $X$ is smooth and $\operatorname{Bs}(\Lambda)=\emptyset$, the general member $C$ of $\Lambda$ satisfies $C \cong \mathbb{P}^{1}$ and $C^{2}=0$; so 1.2 applies to this situation.
1.2. Let $X$ be a smooth, complete rational surface and $C$ a curve on $X$ satisfying $C \cong \mathbb{P}^{1}$ and $C^{2}=0$. Then the following facts are well-known (see 2.7.1 of [11] or Lemma 2.2 of [12], p. 115):
(1) The Riemann-Roch Theorem for $X$ implies that the complete linear system $\Lambda=$ $|C|$ has dimension one; since $\operatorname{Bs}(\Lambda)=\emptyset, \Lambda$ gives rise to a morphism $\lambda: X \rightarrow \mathbb{P}^{1}$.
(2) There exists an open subset $\Gamma \neq \emptyset$ of $\mathbb{P}^{1}$ such that $\lambda^{-1}(\Gamma) \cong \Gamma \times \mathbb{P}^{1}$ and such that the composition $\lambda^{-1}(\Gamma) \cong \Gamma \times \mathbb{P}^{1} \rightarrow \Gamma$ is the restriction of $\lambda$ (i.e., $\Lambda$ is a $\mathbb{P}^{1}$-ruling of $X)$.
(3) There exists an irreducible curve $H \subset X$ such that $H \cdot \Lambda=1$; such a curve $H$ is called a section of $\Lambda$ (or $\lambda$ ). If $H$ is a section then $H \cong \mathbb{P}^{1}$ and, given $\Gamma$ satisfying (2) and $\Gamma \neq \mathbb{P}^{1}$, we have $\lambda^{-1}(\Gamma) \backslash H \cong \Gamma \times \mathbb{A}^{1}$ and the composition $\lambda^{-1}(\Gamma) \backslash H \cong$ $\Gamma \times \mathbb{A}^{1} \rightarrow \Gamma$ is the restriction of $\lambda$ (so $\Lambda$ is also an affine ruling of $X$ ).
(4) If $U$ is any open subset of $X$ isomorphic to $\Gamma \times \mathbb{A}^{1}$ for some open subset $\Gamma \neq \emptyset$ of $\mathbb{P}^{1}$, and if the composition $U \cong \Gamma \times \mathbb{A}^{1} \rightarrow \Gamma$ is compatible with $\Lambda$, then $U=$ $X \backslash \operatorname{supp}\left(H+C_{1}+\cdots+C_{r}\right)$ for some section $H$ of $\Lambda$ and for some curves $C_{1}, \ldots, C_{r}$ where each $C_{i}$ is contained in some member of $\Lambda$.
Let $H$ be a section of $\Lambda$, let $m=-H^{2}$ and, for each reducible ${ }^{2}$ member $F$ of $\Lambda$, let $F^{\circ}$ be the unique irreducible component of $F$ which meets $H$ ( $F^{\circ}$ is an integral curve and occurs in $F$ with multiplicity one).
(5) For each reducible member $F$ of $\Lambda$, if $F^{\sharp}$ denotes the reduced effective divisor such that $\operatorname{supp}(F)=\operatorname{supp}\left(F^{\sharp}\right)$ then $F^{\sharp}$ has strong normal crossings and is a tree of projective lines. Moreover, $F^{\sharp}$ can be shrunk until only $F^{\circ}$ remains ( $F^{\circ}$ itself is not

[^1]shrunk) and, after that contraction, $\left(F^{\circ}\right)^{2}=0$. Note the following consequence: if $C$ is an irreducible component of $F$ and $C$ is branching in $\operatorname{supp}(F+H)$ then $C^{2}<-1$.
(6) If all reducible members are shrunk as described in (5), then one obtains the ruled surface $\mathbb{F}_{m}$. This shrinking process is a birational morphism $\sigma: X \rightarrow \mathbb{F}_{m}$ which maps the members of $\Lambda$ (resp. $H$ ) to the members (resp. the negative section) of the ruling of $\mathbb{F}_{m}$.
The shrinking processes described in (5) and (6) are uniquely determined by the choice of a section $H$.

Notation 1.3. If $X$ is a complete normal rational surface and $\Lambda$ is an affine ruling of $X$, let $X_{s}$ be the smooth locus of $X$ and $X^{\prime}=X_{s} \backslash \operatorname{Bs}(\Lambda)$. We write $(\bar{X}, \bar{\Lambda}) \succ$ $(X, \Lambda)$ to indicate that $\bar{X}$ is a smooth and complete surface containing $X^{\prime}$ as an open subset, the complement of $X^{\prime}$ in $\bar{X}$ is the support of a reduced effective divisor with strong normal crossings, $\bar{\Lambda}$ is a base point free affine ruling of $\bar{X}$ and $\left.\bar{\Lambda}\right|_{X^{\prime}}$ is equal to $\left.\Lambda\right|_{X^{\prime}}$.

Lemma 1.4. Let $X$ be a complete normal rational surface and $\Lambda$ an affine ruling of $X$ and suppose that $(\bar{X}, \bar{\Lambda}) \succ(X, \Lambda)$. Let $D$ be the divisor of $\bar{X}$ with strong normal crossings and whose complement is $X^{\prime}$. Then:
(1) Each connected component of $D$ is a tree of projective lines.
(2) At most one irreducible component of $D$ is a section of $\bar{\Lambda}$.
(3) Every irreducible component of $D$ which is not a section of $\bar{\Lambda}$ is contained in a member of $\bar{\Lambda}$.

Proof. Consider an open subset $U \subset X$ isomorphic to $\Gamma \times \mathbb{A}^{1}$ (for some open subset $\Gamma \neq \emptyset$ of $\mathbb{P}^{1}$ ) and such that the composition $U \cong \Gamma \times \mathbb{A}^{1} \rightarrow \Gamma$ is compatible with $\Lambda$; note that $U \subseteq X^{\prime}$. Since the complement of $\Gamma \times \mathbb{A}^{1}$ in $\mathbb{P}^{1} \times \mathbb{P}^{1}$ is a tree of projective lines, and since $\operatorname{supp}(D)$ is contained in $\bar{X} \backslash U$, it easily follows that assertion (1) holds. By part (4) of 1.2 we have $\bar{X} \backslash U=\operatorname{supp}\left(H+C_{1}+\cdots+C_{r}\right)$, for some section $H$ of $\bar{\Lambda}$ and for some curves $C_{1}, \ldots, C_{r}$, where each $C_{i}$ is contained in some member of $\bar{\Lambda}$; since $\operatorname{supp}(D) \subseteq \operatorname{supp}\left(H+C_{1}+\cdots+C_{r}\right)$, (2) and (3) hold.

Proposition 1.5. Let $X$ be a complete normal rational surface and $\Lambda$ an affine ruling of $X$. Let $X^{\prime}=X_{s} \backslash \operatorname{Bs}(\Lambda)$.
(1) There exists a unique pair $(X, \Lambda)^{\sim}=(\tilde{X}, \tilde{\Lambda})$ satisfying $(\tilde{X}, \tilde{\Lambda}) \succ(X, \Lambda)$ and the following condition:
(*) Every irreducible component $C$ of $\tilde{X} \backslash X^{\prime}$ satisfies $C^{2} \leq-1$, and if equality holds then $C$ is a section of $\tilde{\Lambda}$.
(2) Every irreducible component of $\tilde{X} \backslash X^{\prime}$ which is not a section of $\tilde{\Lambda}$ is contained in some reducible member of $\tilde{\Lambda}$.
(3) Every member of $\tilde{\Lambda}$ meets $X^{\prime}$.
(4) If $(\bar{X}, \bar{\Lambda})$ is any pair satisfying $(\bar{X}, \bar{\Lambda}) \succ(X, \Lambda)$, then there exists a birational morphism $\bar{X} \rightarrow \tilde{X}$ which restricts to an isomorphism from $X^{\prime} \subset \bar{X}$ to $X^{\prime} \subset \tilde{X}$.

Proof. We begin by proving (4): assume that ( $\tilde{X}, \tilde{\Lambda}$ ) is any pair satisfying $(\tilde{X}, \tilde{\Lambda}) \succ(X, \Lambda)$ and condition (*), and let $(\bar{X}, \bar{\Lambda})$ be as in assertion (4). There exists a smooth complete surface $S$ and two birational morphisms, $\tilde{\pi}: S \rightarrow \tilde{X}$ and $\bar{\pi}: S \rightarrow \bar{X}$, such that if we regard $\tilde{\pi}$ (resp. $\bar{\pi}$ ) as a composition of monoidal transformations then each one of these is centered at a point infinitely near $\tilde{X} \backslash X^{\prime}$ (resp. $\bar{X} \backslash X^{\prime}$ ). We also assume that ( $S, \tilde{\pi}, \bar{\pi}$ ) is minimal, i.e., that the total number of monoidal transformations in $\tilde{\pi}$ and $\bar{\pi}$ is minimal. It suffices to show that $\bar{\pi}$ is an isomorphism.

Assume that $\bar{\pi}$ is not an isomorphism and consider a curve $\Gamma \subset S$ which is first to shrink, in the contraction process going from $S$ to $\bar{X}$. By minimality of ( $S, \tilde{\pi}, \bar{\pi}$ ), $\Gamma$ is not in the exceptional locus of $\tilde{\pi}$; thus it is the strict transform of some component $C$ of $\tilde{X} \backslash X^{\prime}$, where $C^{2} \geq-1$. Since ( $\left.\tilde{X}, \tilde{\Lambda}\right)$ satisfies $\left(^{*}\right), C$ must be a section of $\tilde{\Lambda}$. Since $\bar{\pi}(\Gamma)$ is a point, it follows that $\bar{\Lambda}$ has a base point, contradicting $(\bar{X}, \bar{\Lambda}) \succ$ $(X, \Lambda)$. Hence, $\bar{\pi}$ is an isomorphism and (4) is proved.

Note that (4) implies, in particular, that if ( $\tilde{X}, \tilde{\Lambda}$ ) exists then it is unique (up to isomorphism). So, to finish the proof, there remains to construct a pair ( $\tilde{X}, \tilde{\Lambda}$ ) satisfying ( $1-3$ ).

Consider the minimal resolution of singularities $\hat{X} \rightarrow X$ of $X$ and let $\hat{E}$ be the inverse image of the singular points. Then $\hat{X}$ is a smooth complete surface, $\hat{E}$ is a reduced effective divisor of $\hat{X}$ with strong normal crossings and $\hat{X} \backslash \operatorname{supp}(\hat{E}) \rightarrow X_{s}$ is an isomorphism. Arguing as in the proof of 1.4 , we see that each connected component of $\hat{E}$ is a tree of projective lines. Moreover, every irreducible component $E$ of $\hat{E}$ satisfies $E^{2} \leq-1$, and if $E^{2}=-1$ then $E$ is branching in $\hat{E}$.

Then $\Lambda$ determines an affine ruling $\hat{\Lambda}$ of $\hat{X}$. Let $\rho: \tilde{X} \rightarrow \hat{X}$ be the minimal resolution of the base points of $\hat{\Lambda}$ and $\tilde{\Lambda}$ the corresponding base point free linear system on $\tilde{X}$. It is clear that $(\tilde{X}, \tilde{\Lambda}) \succ(X, \Lambda)$; we shall now argue that (*) holds. Let $D$ be the divisor of $\tilde{X}$ with strong normal crossings such that $\tilde{X} \backslash X^{\prime}=\operatorname{supp}(D)$ and consider a component $C$ of $D$.

Since $D$ is the union of the strict transform of $\hat{E}$ and of the exceptional locus of $\rho$, it is clear that $C^{2} \leq-1$.

Assume that $C$ is not a section of $\tilde{\Lambda}$. Then Lemma 1.4 implies that $C$ is contained in some member $F$ of $\tilde{\Lambda}$; since $F^{2}=0$ and $C^{2}<0, F$ must have reducible support, which proves assertion (2) of the Proposition. There remains to show that $C^{2}<-1$. Assume the contrary; then $C^{2}=-1$ and, by $1.2, C$ is not branching in $\operatorname{supp}(H+F)$ for any section $H$ of $\tilde{\Lambda}$.

Suppose that $C$ is the strict transform of some component $E$ of $\hat{E}$. Then $E^{2} \geq$ -1 in $\hat{X}$; by the properties of $\hat{E}, E^{2}=-1$ and $E$ is branching in $\hat{E}$. Consider three distinct neighbors $E_{i}(i=1,2,3)$ of $E$ in $\hat{E}$. Since $E^{2}=C^{2}$, we see that the strict transform $C_{i}$ of $E_{i}$ meets $C$ in $\tilde{X}$ (for all $i=1,2,3$ ). Since $C_{i}$ is a component of
$\tilde{X} \backslash X^{\prime}$, Lemma 1.4 implies that $C_{i}$ is a section of $\tilde{\Lambda}$ or is contained in some member $F_{i}$ of $\tilde{\Lambda}$; in the latter case, $C_{i} \cap C \neq \emptyset$ implies that $F_{i}=F$. Since at most one $C_{i}$ can be a section of $\tilde{\Lambda}$ it follows that, for a suitable section $H, C_{1}, C_{2}, C_{3}$ and $C$ are all contained in $\operatorname{supp}(H+F)$. This contradicts the fact that $C$ is not branching in $\operatorname{supp}(H+$ $F)$, so $C$ is not the strict transform of a component of $\hat{E}$.

Thus $C$ is in the exceptional locus of $\rho$ (and $\hat{\Lambda}$ has a base point). Write $\rho=$ $\rho_{r} \circ \cdots \circ \rho_{1}$, where $\rho_{i}: X_{i} \rightarrow X_{i-1}$ is a monoidal transformation ( $r \geq 1, X_{0}=\hat{X}$, $X_{r}=\tilde{X}$ ), and note that the exceptional curve $H \subset \tilde{X}$ of $\rho_{r}$ is a section of $\tilde{\Lambda}$. By (1.1), there is a unique base point on $X_{i-1}(1 \leq i \leq r)$; it follows that the center of $\rho_{i}$ lies on the exceptional curve of $\rho_{i-1}$ for each $i>1$, and consequently $C^{2}=-1$ implies $C=H$. This contradicts our assumption that $C$ is not a section of $\tilde{\Lambda}$, so we proved that $C^{2}<-1$.

To prove (3), suppose that $F \in \tilde{\Lambda}$ satisfies $\operatorname{supp}(F) \subseteq \operatorname{supp}(D)$. Then each component $C$ of $F$ satisfies $C^{2}<-1$ because $C \subseteq \operatorname{supp}(D)$ and $C$ is not a section. This contradicts the fact (1.2) that $F$ contracts to a 0 -curve (or is a 0 -curve).

Definition 1.6. Suppose that $X$ is a complete normal rational surface and that $\Lambda$ is an affine ruling of $X$. Let $X^{\prime}=X_{s} \backslash \operatorname{Bs}(\Lambda)$ and consider $(X, \Lambda)^{\sim}=(\tilde{X}, \tilde{\Lambda})$.

For each member $F$ of $\Lambda$, let $\tilde{F}$ be the unique element of $\tilde{\Lambda}$ such that $\tilde{F} \cap X^{\prime}=$ $F \cap X^{\prime}$; then $F \mapsto \tilde{F}$ defines a bijection $\Lambda \rightarrow \tilde{\Lambda}$ (because $\left.\tilde{\Lambda}\right|_{X^{\prime}}=\left.\Lambda\right|_{X^{\prime}}$ and, by 1.5, each member of $\tilde{\Lambda}$ meets $X^{\prime}$ ).
1.7. Let $X$ be a complete normal rational surface and $\Lambda$ an affine ruling of $X$. In this paragraph, we relate the rank of $\operatorname{Pic}\left(X_{s}\right)$ to some numbers determined by the pair ( $\tilde{X}, \tilde{\Lambda})$ of Proposition 1.5.

Let $D$ be the divisor of $\tilde{X}$ with strong normal crossings such that $\tilde{X} \backslash X^{\prime}=$ $\operatorname{supp}(D)$. Proposition 1.5 implies that, for a suitable choice of a section $H$ of $\tilde{\Lambda}$, every component $C$ of $D$ satisfies
(i) $C^{2} \leq-1$
and one of
(ii) $C=H$
(iii) $C$ is contained in some reducible member of $\tilde{\Lambda}$ and $C^{2}<-1$.

Let $m=-H^{2}$ and let $F_{1}, \ldots, F_{s}$ be the reducible members of $\tilde{\Lambda}$. For each $i$, we can write $F_{i}=F_{i}^{\circ}+F_{i}^{\star}$ where $F_{i}^{\circ}$ is an integral curve, $F_{i}^{\circ} \cdot H=1, F_{i}^{\star}$ is effective and $F_{i}^{\star} \cdot H=0$. By 1.2, $F_{i}^{\star}$ can be shrunk to a point and, if we do this for all $i=1, \ldots, s$, we obtain the ruled surface $\mathbb{F}_{m}$. Since $\operatorname{Pic}\left(\mathbb{F}_{m}\right)$ is freely generated by a section and a fibre, it follows that
$\operatorname{Pic}(\tilde{X})$ is freely generated by $H$, a general member $F$ of $\tilde{\Lambda}$ and all components of $F_{1}^{\star}, \ldots, F_{s}^{\star}$.

We write $F_{i}^{\star}=F_{i}^{\prime}+F_{i}^{\prime \prime}$, where $F_{i}^{\prime}$ and $F_{i}^{\prime \prime}$ are effective, $F_{i}^{\prime}$ contains the components of
$F_{i}^{\star}$ which meet $X^{\prime}$ and $F_{i}^{\prime \prime}$ contains those included in $\tilde{X} \backslash X^{\prime}$. We claim that $F_{i}^{\prime} \neq 0$. In fact, consider a component $C$ of $F_{i}^{\star}$ satisfying $C^{2}=-1$ (such a $C$ exists, since $F_{i}^{\star}$ is nonzero and shrinks to a point). Since $C$ satisfies neither (ii) nor (iii), it is not a component of $D$, so $C$ is contained in $F_{i}^{\prime}$. Hence, $F_{i}^{\prime} \neq 0$ for all $i$.

Observe that

$$
\begin{equation*}
\tilde{X} \backslash X^{\prime}=\operatorname{supp}\left(\delta H+\sum_{i=1}^{s}\left(F_{i}^{\prime \prime}+\delta_{i} F_{i}^{\circ}\right)\right) \tag{2}
\end{equation*}
$$

where

$$
\delta=\left\{\begin{array}{ll}
1, & \text { if } H \cap X^{\prime}=\emptyset, \\
0, & \text { if } H \cap X^{\prime} \neq \emptyset,
\end{array} \quad \text { and } \quad \delta_{i}= \begin{cases}1, & \text { if } F_{i}^{\circ} \cap X^{\prime}=\emptyset \\
0, & \text { if } F_{i}^{\circ} \cap X^{\prime} \neq \emptyset\end{cases}\right.
$$

In view of (1), (2) and the fact that, for each $i, F$ is linearly equivalent to $F_{i}=$ $F_{i}^{\circ}+F_{i}^{\prime}+F_{i}^{\prime \prime}$, we obtain that $\operatorname{Pic}\left(X^{\prime}\right)$ is the abelian group generated by $H, F$ and all components of $F_{1}^{\prime}, \ldots, F_{s}^{\prime}$, with relations:

$$
\begin{array}{ll}
F=F_{i}^{\prime} & \left(\text { for each i such that } \delta_{i}=1\right) \text { and } \\
H=0 & (\text { if } \delta=1) \tag{3}
\end{array}
$$

Note that $\operatorname{Pic}\left(X_{s}\right)=\operatorname{Pic}\left(X^{\prime}\right)$ and let $k_{i} \geq 1$ be the number of components of $F_{i}^{\prime}$. We conclude that

$$
\begin{equation*}
\operatorname{rank}\left(\operatorname{Pic} X_{s}\right)=(2-\delta)+\sum_{i=1}^{s}\left(k_{i}-\delta_{i}\right) \tag{4}
\end{equation*}
$$

where $1 \leq 2-\delta \leq 2$ and, for all $i, k_{i}-\delta_{i} \geq 0$.

## SURFACES SATISFYING THE CONDITION ( $\dagger$ )

From now-on, we restrict ourselves to the case where $X$ satisfies the condition $(\dagger)$ defined in the introduction.

Proposition 1.8. Suppose that $X$ satisfies $(\dagger)$, let $\Lambda$ be an affine ruling of $X$ and consider the pair $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$.
(1) $\Lambda$ has one base point on $X$ and exactly one irreducible component $H$ of $\tilde{X} \backslash X^{\prime}$ is a section of $\tilde{\Lambda}$.
(2) Every member $F$ of $\tilde{\Lambda}$ has a unique irreducible component $C_{F}$ which meets $X^{\prime}$. Consequently, every member of $\Lambda$ has irreducible support.
(3) If $F$ is reducible then $C_{F}^{2}=-1$ and $C_{F}$ is the only component of $F$ with this property. Moreover, $C_{F}$ does not meet $H$, is not branching in $\operatorname{supp}(F+H)$ and the multiplicity of $C_{F}$ in $F$ is strictly greater than 1.
(4) Under the bijection $\Lambda \rightarrow \tilde{\Lambda}$ defined in 1.6, the multiple members of $\Lambda$ correspond to the reducible members of $\tilde{\Lambda}$. If $M=\nu C \in \Lambda$, where $C \subset X$ is a curve and $\nu \in \mathbb{N}$, then $v$ is equal to the multiplicity of $C_{\tilde{M}}$ in $\tilde{M}$.
(5) Let $M_{i}=v_{i} C_{i}(1 \leq i \leq s)$ be the multiple members of $\Lambda$, where $C_{i} \subset X$ is a curve and $\nu_{i}>1$ is an integer, and let $M$ be any member of $\Lambda$. Then $\operatorname{Pic}\left(X_{s}\right)$ is the abelian group given by $s+1$ generators $M, C_{1}, \ldots, C_{s}$ and relations $\nu_{i} C_{i}=M$ for $i=1, \ldots, s$. In particular, $\operatorname{Pic}\left(X_{s}\right)=\mathbb{Z}$ if and only if $s<2$ or $\nu_{1}, \ldots, \nu_{s}$ are pairwise relatively prime.

Proof. Let $H$ be a section of $\tilde{\Lambda}$ satisfying conditions (i-iiii) of 1.7; let the notations $F_{i}, F_{i}^{\circ}, F_{i}^{\star}, F_{i}^{\prime}$ and $F_{i}^{\prime \prime}$ be as in 1.7.

We have $1=(2-\delta)+\sum_{i=1}^{s}\left(k_{i}-\delta_{i}\right)$ by equation (4), where $2-\delta \geq 1$ and $k_{i}-\delta_{i} \geq 0$ for all $i$; thus $\delta=1$ and $k_{i}=1=\delta_{i}$ for all $i=1, \ldots, s$. Since $\delta=1, H \cap X^{\prime}=\emptyset$ and assertion (1) is proved.

Let $F \in \tilde{\Lambda}$. If $F$ is irreducible then $F^{2}=0$ implies that $F \cap X^{\prime} \neq \emptyset$, by condition (i) of 1.7. If $F=F_{i}$ for some $i$, then $F_{i}^{\circ} \cap X^{\prime}=\emptyset$ (because $\delta_{i}=1$ ) and $F_{i}^{\prime}$ has irreducible support (because $k_{i}=1$ ). Assertion (2) follows.

We have $F_{i}^{\prime}=\nu_{i} C_{F_{i}}$ for some $\nu_{i} \geq 1$. In 1.7, when we proved that $F_{i}^{\prime} \neq 0$, we actually showed that at least one component $C$ of $F_{i}^{\prime}$ satisfies $C^{2}=-1$; thus $C_{F_{i}}^{2}=$ -1. Conversely, if $C$ is any component of $F_{i}$ such that $C^{2}=-1$, then $C \cap X^{\prime} \neq \emptyset$ (otherwise conditions (i-iii) of 1.7 would be violated), so $C=C_{F_{i}}$. Since $F_{i}^{\prime}$ does not meet $H, C_{F_{i}}$ does not meet $H ; C_{F_{i}}$ is not branching in $\operatorname{supp}(F+H)$ because, in the contraction of $F_{i}$ to a 0-curve, $C_{F_{i}}$ is the first component to shrink. By part (6) of Lemma 2.2 of [12], $C_{F_{i}}$ must be a multiple component of $F_{i}$. So (3) holds.

In part (4), the assertion about $v$ is trivial and the correspondence between multiple members of $\Lambda$ and reducible members of $\tilde{\Lambda}$ is essentially the fact that $C_{F_{i}}$ is a multiple component of $F_{i}$ (preceding paragraph).

Since $\delta=1$ and $\delta_{i}=1$ for all $i$, and in view of (3) of $1.7, \operatorname{Pic}\left(X_{s}\right)$ is generated by $F, C_{F_{1}}, \ldots, C_{F_{s}}$, with relations $v_{i} C_{F_{i}}=F$ for $i=1, \ldots, s$. This, together with (4), implies (5).
1.9. Suppose that $X$ satisfies $(\dagger)$, let $\Lambda$ be an affine ruling of $X$ and consider the pair $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$. By 1.2, each reducible member of $\tilde{\Lambda}$ can be shrunk to a 0 curve and the shrinking is uniquely determined by the choice of a section of $\tilde{\Lambda}$. From now-on, whenever we shrink reducible members of $\tilde{\Lambda}$ to 0 -curves, we tacitely assume that the shrinking is the one which is determined by the unique section of $\tilde{\Lambda}$ contained in $\tilde{X} \backslash X^{\prime}$ (see Proposition 1.8).
1.10. The following notations and remarks are useful. Suppose that $X$ satisfies $(\dagger)$, let $\Lambda$ be an affine ruling of $X$, consider $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$ and let $D$ be the divisor of $\tilde{X}$ with strong normal crossings such that $\tilde{X} \backslash X^{\prime}=\operatorname{supp}(D)$.

By Proposition 1.8, $\tilde{\Lambda}$ has a unique section $H$ contained in $D$ and, if $F$ is a reducible member of $\tilde{\Lambda}, F$ has a unique component $C_{F}$ which meets $X^{\prime}$ and the multiplicity $v$ of $C_{F}$ in $F$ is strictly greater than 1 ; moreover, $\operatorname{supp}\left(F-\nu C_{F}\right)$ has either one or two connected components and exactly one of those components meets $H$. Let us denote those connected components by $F^{u}$ and $F^{\ell}$, where $F^{u}$ is the one which meets $H$ and $F^{\ell}$ is allowed to be empty. We regard $F^{u}$ and $F^{\ell}$ either as sets or as reduced effective divisors; we have $F^{u} \neq \emptyset$ and, recalling how the morphism $\tilde{X} \rightarrow \mathbb{F}_{m}$ contracts $F$ (see 1.2, 1.9), we see that $F^{\ell}$ is either empty or an admissible chain. Finally, let $D_{0}$ denote the connected component of $D$ which contains $H$; thus $D_{0}=H+F_{1}^{u}+\cdots+F_{s}^{u}$, where $F_{1}, \ldots, F_{s}$ are the reducible members of $\tilde{\Lambda}$, and $D=D_{0}+F_{1}^{\ell}+\cdots+F_{s}^{\ell}$.

As explained in 1.1 , our definition of "affine ruling" is slightly different from the standard one. The following gives the exact relation between the two definitions:

Proposition 1.11. Suppose that $X$ satisfies ( $\dagger$ ) and that $\Lambda$ is an affine ruling of $X$. For an open subset $U$ of $X$, the following are equivalent:
(1) There exists an isomorphism $U \cong \Gamma \times \mathbb{A}^{1}$, for some open subset $\Gamma \neq \emptyset$ of $\mathbb{P}^{1}$, such that the composition $U \cong \Gamma \times \mathbb{A}^{1} \rightarrow \Gamma$ is compatible with $\Lambda$.
(2) $U=X \backslash \operatorname{supp}\left(M_{1}+\cdots+M_{p}\right)$, for some nonempty subset $\left\{M_{1}, \ldots, M_{p}\right\}$ of $\Lambda$ containing in particular all multiple members.
Moreover, if these conditions hold (and $M_{1}, \ldots, M_{p}$ are distinct) then $U$ is isomorphic to $\left(\mathbb{P}^{1}-p\right.$ points $) \times \mathbb{A}^{1}$ (or equivalently to $\mathbb{P}^{2}$ minus $p$ lines meeting at a point $)$.

Some graph theory is needed for proving the above result. Given $q \in \mathbb{N}$, let $\mathcal{S}_{q}$ be the weighted tree consisting of $q+1$ vertices $v_{0}, v_{1}, \ldots, v_{q}$, all of weight 0 , and of the $q$ edges $\left\{v_{0}, v_{i}\right\}, i=1, \ldots, q$. Note that $\operatorname{det}\left(\mathcal{S}_{1}\right)=-1$ and that $\operatorname{det}\left(\mathcal{S}_{q}\right)=0$ for all $q \neq 1$ (see 3.15 for the determinant of a weighted graph). Note that if $q \geq 1$ and $\mathcal{S}$ is identical to $\mathcal{S}_{q}$ except for the weight of $v_{0}$, then $\mathcal{S}$ is equivalent to $\mathcal{S}_{q}$. Note, also, that if $\mathcal{S}_{p}$ and $\mathcal{S}_{q}$ are equivalent then $p=q$.

Lemma 1.12. Let $p \geq 1$ and $r \geq 0$ be integers, $\mathcal{G}$ a weighted tree, $v$ a vertex of $\mathcal{G}, \mathcal{A}_{1}, \ldots, \mathcal{A}_{p}, \mathcal{B}_{1}, \ldots, \mathcal{B}_{r}$ the branches of $\mathcal{G}$ at $v$, where each $\mathcal{A}_{i}$ consists of a single vertex of weight 0 and, in each $\mathcal{B}_{i}$, every weight is strictly less than -1 .
(1) If $\mathcal{G}$ is equivalent to $\mathcal{S}_{q}$ for some $q \in \mathbb{N}$, then $r=0$ and $p=q$.
(2) If $\mathcal{G}$ is equivalent to a linear chain $\Gamma$ of the form


$$
\begin{equation*}
\left(q \geq 0, \omega_{i} \leq-2 \text { and } x \in \mathbb{Z}\right) \tag{*}
\end{equation*}
$$

then $\mathcal{G}$ itself has the form (*).

Proof. Let us say, temporarily, that a weighted tree $\mathcal{T}$ satisfies the condition (NN) if it has a branch point $b$ such that: (i) at least one branch of $\mathcal{T}$ at $b$ has all its weights strictly less than -1 ; and (ii) every branch of $\mathcal{T}$ at $b$ containing a weight $\geq-1$ contains a nonnegative weight. Then we leave it to the reader to verify the following fact:

If a weighted tree $\mathcal{T}$ satisfies (NN) then so does every minimal weighted tree equivalent to $\mathcal{T}$.

Note that $\mathcal{S}_{q}$ is minimal and does not satisfy ( NN ); also, $\Gamma$ contracts to a minimal chain which does not satisfy (NN). Since $\mathcal{G}$ is equivalent to $\mathcal{S}_{q}$ or $\Gamma$, it follows from (5) that $\mathcal{G}$ does not satisfy (NN). We claim:
(6) If $r \neq 0$ then $\mathcal{G}$ is of the form $(*)$ and $\operatorname{det}(\mathcal{G}) \leq-2$.

Indeed, suppose that $r \neq 0$; if either $v$ or some vertex of some $\mathcal{B}_{i}$ is a branch point of $\mathcal{G}$, then $\mathcal{G}$ satisfies (NN), a contradiction. So $\mathcal{G}$ is a linear chain. In particular, $p+r \leq 2$, so $p=1=r$ and $\mathcal{G}$ is of the form $(*)$. We have $\operatorname{det}(\mathcal{G})=-\operatorname{det}\left(\mathcal{B}_{1}\right)$ by 3.18, and $\operatorname{det}\left(\mathcal{B}_{1}\right) \geq 2$ by 3.19 ; so (6) holds.

To prove assertion (1), suppose that $\mathcal{G}$ is equivalent to $\mathcal{S}_{q}$. Then $\operatorname{det}(\mathcal{G})=$ $\operatorname{det}\left(\mathcal{S}_{q}\right) \geq-1$, so $r=0$ by (6). Since $r=0$ and $p>0, \mathcal{G}$ is equivalent to $\mathcal{S}_{p}$, so $p=q$.

To prove (2), suppose that $\mathcal{G}$ is equivalent to $\Gamma$. By (6), we may assume that $r=$ 0 . Then $\mathcal{G}$ is equivalent to $\mathcal{S}_{p}$, so $\operatorname{det}\left(\mathcal{S}_{p}\right)=\operatorname{det}(\Gamma)=-\operatorname{det}\left(\Gamma^{\prime}\right) \leq-1$ by 3.18 and 3.19, where $\Gamma^{\prime}$ is the admissible chain with weights $\omega_{1}, \ldots, \omega_{q}$. So $p=1$ (and $r=0$ ) and consequently $\mathcal{G}$ is of the form (*) (with $q=0$ ).

Proof of Proposition 1.11. We shall prove that (1) implies (2) and leave the rest to the reader. Suppose that $U$ satisfies condition (1) and let $q \in \mathbb{N}$ be such that $\Gamma=$ $\mathbb{P}^{1}-q$ points. Regard $U$ as an open subset of $\mathbb{P}^{1} \times \mathbb{P}^{1}$; then the complement of $U$ is a divisor $W$ with strong normal crossings and whose dual graph is $\mathcal{S}_{q}$. Note that $U$ is connected at infinity. We also observe that $U \subset X^{\prime}$, where $X^{\prime}=X_{s} \backslash \operatorname{Bs}(\Lambda)$; the inclusion is strict because the complement of $U$ in $X$ has pure dimension one (the intersection matrix of $W$ is not negative definite, so $W$ cannot be shrunk to a normal point).

Consider $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$ and recall that the open subset $X^{\prime}$ of $X$ can be embedded in $\tilde{X}$ as the complement of a divisor $D$ of $\tilde{X}$ with strong normal crossings. Since $U \subset X^{\prime}$ (strictly) and $\tilde{X} \backslash U$ has pure dimension one,

$$
\begin{equation*}
\tilde{X} \backslash U=\operatorname{supp}\left(D+C_{1}+\cdots+C_{p}\right) \quad(p>0) \tag{7}
\end{equation*}
$$

for some distinct curves $C_{1}, \ldots, C_{p}$ not contained in $D$. By Proposition 1.8, some component $H$ of $D$ is a section of $\tilde{\Lambda}$; thus part 4 of 1.2 implies that each $C_{i}$ is contained in a member of $\tilde{\Lambda}$. Since every member $F$ of $\tilde{\Lambda}$ has a unique component $C_{F}$
which meets $X^{\prime}$ (Proposition 1.8), we have $C_{i}=C_{G_{i}}(1 \leq i \leq p)$ for some distinct $G_{1}, \ldots, G_{p} \in \tilde{\Lambda}$. Using (7), we have (in $X$ ) $U=X^{\prime} \backslash \operatorname{supp}\left(M_{1}+\cdots+M_{p}\right)$ where $M_{i} \in \Lambda$ corresponds to $G_{i} \in \tilde{\Lambda}$ under the bijection $\Lambda \rightarrow \tilde{\Lambda}$ defined in 1.6. Since $X \backslash U$ has pure dimension one,

$$
\begin{equation*}
U=X \backslash \operatorname{supp}\left(M_{1}+\cdots+M_{p}\right) . \tag{8}
\end{equation*}
$$

Suppose that the reducible members $F_{1}, \ldots, F_{s}$ of $\tilde{\Lambda}$ have been labeled in such a way that

$$
\left\{F_{1}, \ldots, F_{s}\right\} \backslash\left\{G_{1}, \ldots, G_{p}\right\}=\left\{F_{1}, \ldots, F_{r}\right\} \quad \text { (where } 0 \leq r \leq s \text { ). }
$$

Since $U$ is connected at infinity, we may write (using (7) and 1.10):

$$
\begin{equation*}
\tilde{X} \backslash U=\operatorname{supp}\left(H+F_{1}^{u}+\cdots+F_{r}^{u}+G_{1}+\cdots+G_{p}\right) \tag{9}
\end{equation*}
$$

Let $\gamma: \tilde{X} \rightarrow S$ (where $S$ is smooth) be the shrinking of $G_{1}, \ldots, G_{p}$ to 0 -curves (see 1.2 and 1.9). Then $U \xrightarrow{\gamma} \gamma(U)$ is an isomorphism and $S \backslash \gamma(U)=\operatorname{supp}\left(D^{\prime}\right)$, where $D^{\prime}$ is a divisor of $S$ with strong normal crossings. By (9), the dual graph $\mathcal{G}$ of ( $S, D^{\prime}$ ) is a tree with $p+r$ branches at $\gamma(H)$ : $p$ branches $\gamma\left(G_{i}\right)$ consisting of a single vertex of weight zero and $r$ branches $\gamma\left(F_{i}^{u}\right)$ in which every weight is strictly less than -1 . Thus $\mathcal{G}$ satisfies the hypothesis of Lemma 1.12 and part 1 of that result gives $r=0$, so $\left\{F_{1}, \ldots, F_{s}\right\} \subseteq\left\{G_{1}, \ldots, G_{p}\right\}$. From this and (8), it follows that $U$ satisfies condition (2).
1.13. Let $X$ be a complete normal rational surface.

Given an affine ruling $\Lambda$ of $X$, consider $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$ and the divisor $D$ of $\tilde{X}$ with strong normal crossings such that $\operatorname{supp} D=\tilde{X} \backslash X^{\prime} ;$ let $\mathcal{G}(\Lambda)=\mathcal{G}(D, \tilde{X})$ (the dual graph of $D$ in $\tilde{X}$ ).

Then the equivalence class of the weighted graph $\mathcal{G}(\Lambda)$ depends only on $X$ and has a unique minimal element, say $\mathcal{E}_{X}$. Indeed, let $\hat{X}$ and $\hat{E}$ be as in the proof of Proposition 1.5, and let $\mathcal{E}_{X}$ be the dual graph of $\hat{E}$ in $\hat{X}$; then the weighted graph $\mathcal{E}_{X}$ is the only minimal element of its equivalence class and $\mathcal{G}(\Lambda)$ contracts to $\mathcal{E}_{X}$.

Definition 1.14. Let $X$ be a complete normal rational surface and $\Lambda$ an affine ruling of $X$. Define $\beta(\Lambda) \in \mathbb{N}$ by:

$$
\beta(\Lambda)=\text { number of branch points of } \mathcal{G}(\Lambda)-\text { number of branch points of } \mathcal{E}_{X},
$$

where $\mathcal{G}(\Lambda)$ and $\mathcal{E}_{X}$ are as in 1.13. If $\beta(\Lambda)=0$, we say that $\Lambda$ is basic.
Remark. In 1.13 and 1.14 , if $X$ satisfies ( $\ddagger$ ) (which includes the case where $X$ is smooth), then $\mathcal{E}_{X}$ has no branch point and, consequently, $\Lambda$ is basic if and only if the divisor $D$ has no branching component.

Theorem 1.15. Suppose that $X$ satisfies ( $\dagger$ ). Then:
(1) at most one singular point of $X$ is not a cyclic quotient singularity. Let $\Lambda$ be an affine ruling of $X$ and assume that at least one of the following conditions holds:
(i) $X$ satisfies ( $\ddagger$ ); or
(ii) $\beta(\Lambda)>0$.

Let $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$. Then the following hold:
(2) $\tilde{\Lambda}$ has at most two reducible members and one of them contains all branching components of $\tilde{X} \backslash X^{\prime}$.
(3) $\operatorname{Sing}(X) \cup \operatorname{Bs}(\Lambda)$ contains at most three points.
(4) $\Lambda$ has at most two multiple members. Moreover, if $\left\{F_{1}, F_{2}\right\}$ is a subset of $\Lambda$ containing all multiple members (where $F_{1} \neq F_{2}$ ), and if $F_{i}=v_{i} C_{i}$ (where $C_{i}$ is a curve and $\left.v_{i} \geq 1, i=1,2\right)$, then:
(5) $X \backslash\left(C_{1} \cup C_{2}\right)$ is isomorphic to $\mathbb{P}^{2}$ minus two lines.
(6) $\operatorname{Pic}\left(X_{s}\right) \cong \mathbb{Z} \oplus \mathbb{Z} / d \mathbb{Z}$, where $d=\operatorname{gcd}\left(\nu_{1}, \nu_{2}\right)$.

Proof. Let $\hat{X}, \hat{E}, \hat{\Lambda}$ and $\rho: \tilde{X} \rightarrow \hat{X}$ be as in the proof of Proposition 1.5; let the notation be as in 1.10.

First, it is clear that the connected components of $\tilde{X} \backslash X^{\prime}$ are $D_{0}$ and the nonempty $F_{i}^{\ell}$; in particular, there are at most $s+1$ such components and, taking images under $\tilde{X} \rightarrow \hat{X} \rightarrow X$, we get that $\operatorname{Sing}(X) \cup \operatorname{Bs}(\Lambda)$ contains at most $s+1$ points.

If $\hat{\Lambda}$ has a base point, denote it by $P \in \hat{X}$ and observe that $\rho^{-1}(P)$ is connected and that $H \subseteq \rho^{-1}(P) \subseteq \operatorname{supp}(D)$; thus $\rho^{-1}(P) \subseteq D_{0}$ and consequently the restriction of $\rho$ to the open set $\tilde{X} \backslash D_{0}$ is an isomorphism. Of course, this is also the case if $\hat{\Lambda}$ does not have a base point ( $\rho$ is the identity map). Since $F_{i}^{\ell}$ is contained in that open set, 1.10 implies:

$$
\rho\left(F_{i}^{\ell}\right) \text { is either empty or an admissible chain (for each } i=1, \ldots, s \text { ). }
$$

On the other hand, the connected components of $\hat{E}$ are among those of $\hat{X} \backslash X^{\prime}$, and these are $\rho\left(D_{0}\right)$ and the nonempty $\rho\left(F_{i}^{\ell}\right)$. So at most one connected component of $\hat{E}$ is not a $\rho\left(F_{i}^{\ell}\right)$; consequently, at most one connected component of $\hat{E}$ is not an admissible chain, i.e., (1) holds.

Recall that $F_{i}^{u}$ meets $H$ for all $i=1, \ldots, s$, so the branching number of $H$ in $D_{0}$ is precisely $s$. Assuming that (i) or (ii) holds, we will now show that $s \leq 2$ and that assertion (2) of the Theorem holds. For this, we may assume that $D$ (or equivalently $D_{0}$ ) has a branching component.

Note that $\rho\left(D_{0}\right)$ is either a point or a connected component of $\hat{E}$. Thus, under assumption (i), $D_{0}$ contracts to an admissible chain or to a single point; since we assumed that $D_{0}$ has a branching component, it follows that $\beta(\Lambda)>0$. Hence, we may assume that (ii) holds.

Then $D$ is not minimal, i.e., it has a component $C$ which is not branching in $D$ and which satisfies $C^{2}=-1$; since $C \neq H$ implies $C^{2}<-1$, we must have $C=H$, so
$H$ is not branching in $D$ and $s \leq 2$. In particular, $\operatorname{supp}\left(F_{1}+F_{2}\right)$ contains every branching component of $D$. If each of $F_{1}, F_{2}$ contains a branching component of $D$ then (since $C^{2}<-1$ for all components $C$ of $F_{1}+F_{2}$ ) no contraction of $D$ can decrease the number of branching components-contradicting the assumption that $\beta(\Lambda)>0$. This proves assertion (2) of the Theorem. The other assertions easily follow from (2) and results 1.8 and 1.11 .

Corollary 1.16. If $X$ satisfies ( $\dagger$ ) then at most one singular point of $X$ is not a cyclic quotient singularity. If $X$ satisfies ( $\ddagger$ ) then $X$ has at most three singular points.
1.17. We prove the following statement, which was claimed without proof in the introduction: Given $X$ satisfying ( $\ddagger$ ) and a curve $C$ on $X$, the following are equivalent:
(i) $\bar{\kappa}\left(X_{s} \backslash C\right)=-\infty$;
(ii) there exists at least one affine ruling $\Lambda$ of $X$ such that $n C \in \Lambda$ for some $n>0$.

Condition (ii) clearly implies (i). If (i) is satisfied then we have to show that $U=$ $X_{s} \backslash C$ is affine-ruled (then 1.11 implies (ii)). Consider $\tilde{X} \rightarrow \hat{X} \rightarrow X$, where $\hat{X} \rightarrow X$ is the minimal resolution of singularities of $X$ and $\tilde{X} \rightarrow X$ is further blowing-up so that the inverse image $\tilde{C}$ of $C$ has normal crossings. Then the complement of $U$ in $\tilde{X}$ is a divisor $D$ with normal crossings and every connected component of $D$ other than $\tilde{C}$ is a linear chain. Since the divisor class group of $X\left(\cong\right.$ Pic $\left.X_{s}\right)$ has rank 1 , any two curves on $X$ meet. Hence, if $E \subset \tilde{X}$ is a curve meeting $U$ which is shrunk in making ( $\tilde{X}, D$ ) almost minimal, $E$ meets $\tilde{C}$. Hence, on the almost minimal model, the boundary divisor again has at most one non-linear component. Since the connected component of the boundary containing $C$ is not contractible, [15] implies that $U$ is affine-ruled.

## 2. Modification of affine rulings

In the proof of the following theorem, we consider an arbitrary affine ruling $\Lambda$ satisfying $\beta(\Lambda)>0$ and "reduce" it to an affine ruling $\Lambda^{\prime}$ such that $\beta\left(\Lambda^{\prime}\right)<\beta(\Lambda)$ (see Definition 1.14 for $\beta$ ). We will see later that this reduction process is an instance of a more general modification process.

Theorem 2.1. If $X$ satisfies ( $\dagger$ ) then it admits a basic affine ruling.
Proof. Suppose that $\Lambda$ is an affine ruling of $X$ satisfying $\beta(\Lambda)>0$. Consider $X^{\prime}=X_{s} \backslash \operatorname{Bs}(\Lambda),(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$, let $D$ be the reduced effective divisor of $\tilde{X}$ such that $\tilde{X} \backslash X^{\prime}=\operatorname{supp}(D), H$ the unique section of $\tilde{\Lambda}$ contained in $D$ and $\tilde{X} \xrightarrow{\rho} \hat{X} \rightarrow X$ as in the proof of Proposition 1.5.

Since $\beta(\Lambda)>0$, at least one component $C$ of $D$ satisfies:
$C$ is branching in $D$ and $\rho(C)$ is either a point of $\hat{X}$ or a curve not branching in $\rho(D)$.

By Theorem 1.15, $\tilde{\Lambda}$ has at most two reducible members and one of them, say $\tilde{F}$, contains all branching components of $D$; in particular, $C \subset \operatorname{supp}(\tilde{F})$. Recall from Proposition 1.8 that $\tilde{F}$ has a unique component $C_{\tilde{F}}$ which meets $X^{\prime}$. Consider the connected components $\Gamma_{1}$ and $\Gamma_{2}$ of $\operatorname{supp}\left(D+C_{\tilde{F}}\right)$, where $\Gamma_{1}$ contains $H$ and $\Gamma_{2}$ is either empty or an admissible chain of projective lines. Explicitely, if $\tilde{F}$ is the only reducible member of $\tilde{\Lambda}$ then $\Gamma_{1}=\operatorname{supp}(H+\tilde{F})$ and $\Gamma_{2}=\emptyset$; if $\tilde{\Lambda}$ has two reducible members, say $G_{1}=\tilde{F}$ and $G_{2}$, then $\Gamma_{1}=\operatorname{supp}\left(H+G_{1}+G_{2}^{u}\right)$ and $\Gamma_{2}=G_{2}^{\ell}$ (see 1.10 for the notations $G_{2}^{u}$ and $G_{2}^{\ell}$ and note that $G_{2}^{\ell}$ may be empty).

Consider the birational morphism $m: \tilde{X} \rightarrow S$ which shrinks $\tilde{F}$ to a 0 -curve (see $1.2,1.9$ ) and regard it as a composition $\tilde{X}=S_{n} \xrightarrow{m_{n}} \cdots \xrightarrow{m_{1}} S_{0}=S$ of monoidal transformations. Since the exceptional locus of $m$ has only one ( -1 )-component (namely, $C_{\tilde{F}}$, the center of $m_{i}$ is on the exceptional curve of $m_{i-1}$ for each $i>1$. It follows, in particular, that the unique component of $\tilde{F}$ which meets $H$ is not branching in $D$, so $C$ is not that component and $m(C)$ is a point. Another consequence is that $\Gamma_{1}$ has precisely three branches at $C$, say $\mathcal{B}, \mathcal{B}^{u}$ and $\mathcal{B}^{\ell}$, where $\mathcal{B}$ contains $C_{\tilde{F}}, \mathcal{B}^{u}$ contains $H$ and every component of $\mathcal{B}^{\ell}$ has self-intersection strictly less than -1 .

Since $m(C)$ is a point, we may factor $m$ as $\tilde{X} \rightarrow \tilde{S} \rightarrow S$, where the image of $C$ in $\tilde{S}$ is a ( -1 )-curve and is the first curve to be shrunk by $\tilde{S} \rightarrow S$. Then it is easy to see that $\tilde{X} \rightarrow \tilde{S}$ is the shrinking of $\mathcal{B}$.

On the other hand, our choice of $C$ (condition (10), above) allows us to factor $\rho$ as $\tilde{X} \xrightarrow{\alpha} U \rightarrow \hat{X}$, where $\bar{C}=\alpha(C)$ is a curve, but is not branching in $\bar{D}=\alpha(D)$; then one sees that $\alpha$ is the shrinking of $\mathcal{B}^{u}$. So we may consider a commutative diagram of smooth complete surfaces and birational morphisms:

$$
\begin{equation*}
\tilde{X} \underset{\left(\mathcal{B}^{u}\right)}{\alpha} U \stackrel{\beta}{\rightleftarrows} \Omega \tag{11}
\end{equation*}
$$


where the labels (" $\mathcal{B}$ " or " $\mathcal{B}$ "") indicate what set is shrunk by each morphism-only the left square is being defined at this time. Let $v: \tilde{X} \rightarrow S_{+}^{\prime}$ be the composition of these maps.

Let $x$ be the self-intersection number of $v(C)$ in $S_{+}^{\prime}$. Since the image of $C$ in $\tilde{S}$ has self-intersection -1 and $\tilde{S} \xrightarrow{\gamma} S_{+}^{\prime}$ increases that number by at least one, we have $x \geq 0$. The dual graph of $v\left(\Gamma_{1}\right)$ in $S_{+}^{\prime}$ is:

where $q>0, \omega_{i} \leq-2$ and $x \geq 0$.
Let $P_{+}^{\prime} \in S_{+}^{\prime}$ be the unique point of $v(C)$ which also belongs to another component of $v\left(\Gamma_{1}\right)$ and consider the birational morphism $S^{\prime} \xrightarrow{\sigma} S_{+}^{\prime}$ obtained by blowing-up $x$ times at $P_{+}^{\prime}$, in such a way that the dual graph of $\sigma^{-1}\left(\nu\left(\Gamma_{1}\right)\right)$ is:

where the 0 -curve $C^{\prime}$ is the strict transform of $\nu(C)$. Since the morphism $U \rightarrow S_{+}^{\prime}$ is isomorphic in a neighborhood of $P_{+}^{\prime}$, the same sequence of $x$ blowings-up can be performed at the level of $U$; this defines a birational morphism $\Omega \xrightarrow{\beta} U$ and completes the definition of the above commutative diagram (11).

Note that each surface $Y$ considered in this argument comes equipped with a birational transformation, say $\tau_{Y}: S^{\prime} \rightarrow Y$; consequently, the complete linear system $\left|C^{\prime}\right|$ on $S^{\prime}$ (a $\mathbb{P}^{1}$-ruling of $S^{\prime}$, by 1.2 ) determines a linear system (without fixed components) on each one of these surfaces. In particular, we will consider the linear systems $\Lambda^{\prime}$ on $X$ and $\Lambda^{*}$ on $\Omega$ defined in this way. Clearly, $\Lambda^{*}$ is a $\mathbb{P}^{1}$-ruling of $\Omega$.

We claim that $\Lambda^{\prime}$ is an affine ruling of $X$. For $i=1,2$, let $\Gamma_{i}^{\prime}=\sigma^{-1}\left(\nu\left(\Gamma_{i}\right)\right) \subset S^{\prime}$. Then the birational transformation $\tau_{X}: S^{\prime} \rightarrow X$ restricts to an isomorphism $\mu^{\prime}$ going from the open subset $W^{\prime}=S^{\prime} \backslash\left(\Gamma_{1}^{\prime} \cup \Gamma_{2}^{\prime}\right)$ of $S^{\prime}$ to the open subset $X_{s} \backslash \operatorname{supp}(F)$ of $X$ (where $F$ is the member of $\Lambda$ which corresponds to $\tilde{F}$ under the bijection $\Lambda \rightarrow \tilde{\Lambda}$ defined in Definition 1.6). Then $\Lambda^{\prime}$ is the affine ruling of $X$ determined by the $X$ immersion ( $S^{\prime}, \mu^{\prime}$ ) (see 2.3 for details).

We now argue that $\Lambda$ and $\Lambda^{\prime}$ have the same base point on $X$. Let $D^{*}=\beta^{-1}(\bar{D})$ and let $C^{*}$ be the strict transform of $\bar{C}$ with respect to $\beta$. Note that $\alpha(H)$ is a point of $\bar{C}$ and that the image of $\bar{C}$ under $U \rightarrow \hat{X} \rightarrow X$ is a point (because the image of $D$ under $\tilde{X} \rightarrow \hat{X} \rightarrow X$ is a finite set of points); thus the base point of $\Lambda$ is the image of $\bar{C}$ under $U \rightarrow \hat{X} \rightarrow X$. On the other hand, consider the component $H^{\prime}$ of $\Gamma_{1}^{\prime}$ which is a section of $\left|C^{\prime}\right|$ (if $x>0$ (resp. $x=0$ ) then $H^{\prime}$ is the neighbor of the vertex of weight 0 in the graph (13) (resp. (12))); then the strict transform $H^{*}$ of $H^{\prime}$ (with respect to $\Omega \rightarrow S^{\prime}$ ) is a section of $\Lambda^{*}$ and satisfies $H^{*} \cap C^{*} \neq \emptyset$. Since $H^{*}$ and $C^{*}$ are components of $D^{*}$ and, under $\Omega \rightarrow U \rightarrow \hat{X} \rightarrow X, D^{*}$ is mapped to a finite set of points, we deduce that the image of $H^{*}$ in $X$ coincides with that of $\bar{C}$; so $\Lambda$ and $\Lambda^{\prime}$ have the same base point.

The morphisms $\tilde{X} \xrightarrow{\alpha} U \stackrel{\beta}{\longleftarrow} \Omega$ give an isomorphism $\tilde{X} \backslash \operatorname{supp}(D) \cong \Omega \backslash \operatorname{supp}\left(D^{*}\right) ;$ it follows that the birational morphism $\Omega \xrightarrow{\beta} U \rightarrow \hat{X} \rightarrow X$ restricts to an isomorphism from $\Omega \backslash \operatorname{supp}\left(D^{*}\right)$ to $X_{s} \backslash \operatorname{Bs}(\Lambda)$, which is equal to $X_{s} \backslash \operatorname{Bs}\left(\Lambda^{\prime}\right)$ by the preceding paragraph. Since $D^{*}$ is a reduced effective divisor with s.n.c., $\left(\Omega, \Lambda^{*}\right) \succ\left(X, \Lambda^{\prime}\right)$. Noting that the number of branching components of $D^{*}$ is strictly less than that of $D$, and taking into account assertion (4) of Proposition 1.5, we conclude that $\beta\left(\Lambda^{\prime}\right)<\beta(\Lambda)$.

## FORMALIZATION OF THE REDUCTION PROCESS

Definition 2.2. Suppose that $X$ is a complete normal rational surface. An $X$ immersion is a pair $(S, \mu)$ where:
(1) $S$ is a smooth complete surface and $\mu$ is an isomorphism from an open subset $W$ of $S$ to an open subset of $X$.
(2) $S \backslash W$ is nonempty and is the support of a divisor (of $S$ ) with strong normal crossings.
(3) Exactly one of the connected components of $S \backslash W$ is a linear chain of projective lines with dual graph:

where $q \geq 0, \omega_{i} \leq-2$ and $x$ is any integer. We call this connected component the main component of $(S, \mu)$ and often denote it by $\Gamma$. We stress that $\Gamma$ has at least two irreducible components, corresponding to the vertices of weights 0 and $x$ in the above picture.
(4) If $C$ is an irreducible component of $S \backslash W$ which is not in the main component $\Gamma$, then $C^{2} \leq-1$ and if equality holds then $C$ is branching in $S \backslash W$.
By $\operatorname{dom} \mu$ we mean the open set $W$; by the zero-component of $(S, \mu)$, we mean the component of $\Gamma$ which corresponds to the pending vertex of weight 0 in (3). ${ }^{3}$ The neighbor of the zero-component (neighbor in the graph (3)) is called the section of $(S, \mu)$. If $x=-1$ in (3), we say that $(S, \mu)$ is in standard form.

Remark. Let the assumptions and notations as in Definition 2.2. Then $C \cong \mathbb{P}^{1}$ for every irreducible component $C$ of $S \backslash W$. This follows from 2.3, below: $C \subset$ $\operatorname{supp}\left(\Sigma+Z_{1}+\cdots+Z_{n}\right)$.
2.3. Let $X$ be a complete normal rational surface. We claim that each $X$ immersion determines an affine ruling of $X$.

To see this, let $(S, \mu)$ be an $X$-immersion; let $W=\operatorname{dom} \mu$ and let $\Gamma, Z$ and $\Sigma$ be the main component, zero-component and section of $(S, \mu)$ respectively. By 1.2, the complete linear system $|Z|$ is a $\mathbb{P}^{1}$-ruling of $S$; also, $\Sigma$ is a section of $|Z|$. Every irreducible component $C$ of $S \backslash W$ other than $\Sigma$ satisfies $C \cdot Z=0$, so is contained in some member of $|Z|$. Consequently, we can choose a finite subset $\left\{Z_{1}, \ldots, Z_{n}\right\}$ of $|Z|$ such that the open set

$$
W_{0}=S \backslash \operatorname{supp}\left(\Sigma+Z_{1}+\cdots+Z_{n}\right)
$$

is contained in $W$. Enlarging the set $\left\{Z_{1}, \ldots, Z_{n}\right\}$ if necessary, we may arrange that

[^2]the morphism $S \rightarrow \mathbb{P}^{1}$ induced by $|Z|$ restricts to a projection map $W_{0}=\Gamma \times \mathbb{A}^{1} \rightarrow \Gamma$. It follows that, if we let $\Lambda$ denote the linear system on $X$ (without fixed components) determined by $|Z|$ via $\mu$, then $\Lambda$ is an affine ruling of $X$.

We will describe the image of $\mu$ in the case where $X$ satisfies ( $\dagger$ ); to do it, we need:

Definition 2.4. Suppose that $X$ satisfies ( $\dagger$ ), let $\Lambda$ be an affine ruling of $X$ and consider the pair $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$. Let $\tilde{\Lambda}_{*}$ be the set of members $\tilde{F}$ of $\tilde{\Lambda}$ which satisfy:
(1) at most one member of $\tilde{\Lambda} \backslash\{\tilde{F}\}$ is reducible; and
(2) all branching components of $\tilde{X} \backslash X^{\prime}$ are in $\tilde{F}$.

We also define $\Lambda_{*}=\left\{F \in \Lambda \mid \tilde{F} \in \tilde{\Lambda}_{*}\right\}$, where $F \mapsto \tilde{F}$ is the bijection $\Lambda \rightarrow \tilde{\Lambda}$ of Definition 1.6.
2.5. Note that, in Definition $2.4, \Lambda_{*} \neq \emptyset$ if and only if $\tilde{\Lambda}_{*} \neq \emptyset$, if and only if $\tilde{\Lambda}$ has at most two reducible members and some member contains all branching components of $\tilde{X} \backslash X^{\prime}$. In particular, Theorem 1.15 implies:
(1) If $X$ satisfies ( $\ddagger$ ) then $\Lambda_{*}$ is nonempty.
(2) If $\beta(\Lambda)>0$, then $\Lambda_{*}$ has exactly one element.

Lemma and definition 2.6. Suppose that $X$ satisfies ( $\dagger$ ), let $(S, \mu)$ be an $X$ immersion and let $Z$ and $\Gamma$ be the zero-component and the main component of ( $S, \mu$ ) respectively.
(1) The complete linear system $|Z|$ on $S$ determines (via $\mu$ ) an affine ruling $\Lambda$ of $X$. Moreover, there is a unique $F \in \Lambda_{*}$ such that $\operatorname{im} \mu=X_{s} \backslash \operatorname{supp}(F)$. In this context, we say that $(S, \mu)$ determines $(\Lambda, F)$.
(2) $S \backslash \operatorname{dom}(\mu)$ has at most two connected components and, if it has two, the component other than $\Gamma$ is an admissible chain.

Proof. In view of 2.3, the proof of assertion (1) will be complete if we can show that $\operatorname{im} \mu=X_{s} \backslash \operatorname{supp}(F)$ for some $F \in \Lambda_{*}$.

Let $W=\operatorname{dom} \mu$ and $X^{\prime}=X_{s} \backslash \operatorname{Bs}(\Lambda)$; since $W$ is smooth and $|Z|$ is base point free, $\mu(W) \subseteq X^{\prime}$. If $\mu(W)=X^{\prime}$ then $(S,|Z|) \succ(X, \Lambda)$ and, by part (4) of Proposition 1.5, there exists a birational morphism $S \rightarrow \tilde{X}$ which restricts to an isomorphism $W \rightarrow X^{\prime}$. Let $C \subset \tilde{X}$ be the image of $Z$ under $S \rightarrow \tilde{X}$; then $C$ is a component of $D$ satisfying $C^{2} \geq 0$, which is absurd. Hence $\mu(W) \subset X^{\prime}$ (strictly).

Consider $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$ and let the notations of 1.10 be in effect (in particular $D, H$ and, given $F \in \tilde{\Lambda}, C_{F}, F^{u}$ and $F^{\ell}$ ). Regard $\mu(W)$ and $X^{\prime}$ as open subsets of $\tilde{X}$. Observe that, in $S$, no connected component of $S \backslash W$ can be shrunk to a smooth
point; thus $\tilde{X} \backslash \mu(W)$ has pure dimension one, so

$$
\tilde{X} \backslash \mu(W)=\operatorname{supp}\left(D+C_{1}+\cdots+C_{p}\right) \quad(p>0)
$$

where the $C_{i}$ are distinct curves not contained in $D$.
In 2.3 we noted that the morphism $S \rightarrow \mathbb{P}^{1}$ determined by $|Z|$ restricts to a projection map $W_{0}=\Gamma \times \mathbb{A}^{1} \rightarrow \Gamma$, where $W_{0} \subseteq W$. Then, as in the proof of 1.11, 1.2 implies that $C_{i}=C_{G_{i}}(1 \leq i \leq p)$ for some distinct $G_{1}, \ldots, G_{p} \in \tilde{\Lambda}$. Let $F_{1}, \ldots, F_{r}$ ( $r \geq 0$ ) denote the reducible members of $\tilde{\Lambda} \backslash\left\{G_{1}, \ldots, G_{p}\right\}$. Define

$$
\begin{align*}
G^{\circ} & =\operatorname{supp}\left(H+G_{1}+\cdots+G_{p}+F_{1}^{u}+\cdots+F_{r}^{u}\right),  \tag{14}\\
G & =\tilde{X} \backslash \mu(W)=\operatorname{supp}\left(G^{\circ}+F_{1}^{\ell}+\cdots+F_{r}^{\ell}\right) \tag{15}
\end{align*}
$$

and note that $G^{\circ}$ and the nonempty $F_{i}^{\ell}$ are the connected components of $G$.
Let $\mathcal{G}$ be the dual graph of $G$ and $\mathcal{G}^{\circ}$ the dual graph of $G^{\circ}$ in $\tilde{X}$ (so $\mathcal{G}^{\circ}$ is a connected component of $\mathcal{G}$ ); let $\mathcal{Q}$ be the dual graph of $S \backslash W$ in $S$ and let $\mathcal{Q}^{\circ}$ be the dual graph of $\Gamma$ (so $\mathcal{Q}^{\circ}$ is a connected component of $\mathcal{Q}$ ). Clearly, $\mathcal{Q}$ and $\mathcal{G}$ are equivalent weighted graphs. Because no connected component of $\mathcal{G}$ or $\mathcal{Q}$ is equivalent to the empty graph, the connected components of $\mathcal{Q}$ correspond bijectively to those of $\mathcal{G}$, in such a way that each component of $\mathcal{Q}$ is equivalent to the corresponding component of $\mathcal{G}$. We claim that $\mathcal{Q}^{\circ}$ corresponds to $\mathcal{G}^{\circ}$ under that bijection. Indeed, $\mathcal{Q}^{\circ}$ corresponds (and so is equivalent) to some connected component $\mathcal{G}^{\prime}$ of $\mathcal{G}$; if $\mathcal{G}^{\prime} \neq \mathcal{G}^{\circ}$ then $\mathcal{G}^{\prime}$ must be the dual graph of $F_{i}^{\ell}$ for some $i$, so every weight in $\mathcal{G}^{\prime}$ is strictly less than -1 and $\mathcal{G}^{\prime}$ is the unique minimal element of its equivalence class; consequently, $\mathcal{Q}^{\circ}$ contracts to $\mathcal{G}^{\prime}$. This is absurd, because any contraction of $\mathcal{Q}^{\circ}$ contains a nonnegative weight. So $\mathcal{G}^{\circ}$ is equivalent to $\mathcal{Q}^{\circ}$, which is of the form ( $*$ ) described in Lemma 1.12.

By 1.2 (and 1.9), each $G_{j}$ can be contracted to a 0 -curve. Let $\overline{\mathcal{G}^{\circ}}$ be the weighted graph obtained from $\mathcal{G}^{\circ}$ by contracting all $G_{j}$ to 0 -curves; in view of (14), $\overline{\mathcal{G}^{\circ}}$ has $p+r$ branches at $H: p$ branches consisting of a single vertex of weight zero and $r$ branches in which every vertex has weight strictly less than -1 . Thus part (2) of Lemma 1.12 implies that $\overline{\mathcal{G}^{\circ}}$ is of the form $(*)$. So $p=1, r \leq 1$ and, if $r=1, H+F_{1}^{u}$ is a linear chain. So (15) simplifies to:

$$
\tilde{X} \backslash \mu(W)= \begin{cases}\operatorname{supp}\left(H+G_{1}\right), & \text { if } r=0 ;  \tag{16}\\ \operatorname{supp}\left(H+G_{1}+F_{1}^{u}+F_{1}^{\ell}\right), & \text { if } r=1 .\end{cases}
$$

Since $r$ is the number of reducible members of $\tilde{\Lambda} \backslash\left\{G_{1}, \ldots, G_{p}\right\}=\tilde{\Lambda} \backslash\left\{G_{1}\right\}$, we have:

At most one member of $\tilde{\Lambda} \backslash\left\{G_{1}\right\}$ is reducible.
Regarding (16), we observe: $H$ is not branching in $\tilde{X} \backslash \mu(W)$; if $F_{1}^{\ell}$ is nonempty, then it is an admissible chain and a connected component of $\tilde{X} \backslash \mu(W)$; if $r=1$ then $H+$
$F_{1}^{u}$ is a linear chain. Thus all branching components of $\tilde{X} \backslash \mu(W)$ are in $G_{1}$ and in particular:
(18) All branching components of $\tilde{X} \backslash X^{\prime}$ are in $G_{1}$.

By (17) and (18), we obtain $G_{1} \in \tilde{\Lambda}_{*}$ and consequently $M_{1} \in \Lambda_{*}$.
In $\tilde{X}$ we have $\mu(W)=\tilde{X} \backslash \operatorname{supp}\left(D+C_{1}\right)=\tilde{X} \backslash \operatorname{supp}\left(D+G_{1}\right)=X^{\prime} \backslash \operatorname{supp}\left(G_{1}\right)$ so, in $X, \mu(W)=X^{\prime} \backslash \operatorname{supp}\left(M_{1}\right)=X_{s} \backslash \operatorname{supp}\left(M_{1}\right)$. This proves assertion (1). Assertion (2) follows from (16) and the argument concerning the connected components of $\mathcal{G}$ and $\mathcal{Q}$.

Definition 2.7. Let $X$ be a complete normal rational surface.
(1) Let $(S, \mu)$ be an $X$-immersion, with zero-component $Z$ and section $\Sigma$, and let $W=\operatorname{dom} \mu$. If $P$ is a point of $Z$, we define an $X$-immersion $\left(S^{\prime}, \mu^{\prime}\right)=\operatorname{elm}_{P}(S, \mu)$ as follows: let $\pi: \tilde{S} \rightarrow S$ be the blowing-up of $S$ at $P, \tilde{Z}$ the strict transform of $Z$ on $\tilde{S}$ and $\pi^{\prime}: \tilde{S} \rightarrow S^{\prime}$ the contraction of $\tilde{Z}$. Let $W^{\prime}=\pi^{\prime}\left(\pi^{-1}(W)\right.$ ), consider the isomorphism $\theta: W^{\prime} \rightarrow W$ obtained by restricting $\pi \circ\left(\pi^{\prime}\right)^{-1}$ and define $\mu^{\prime}=\mu \circ \theta$. We say that $\left(S^{\prime}, \mu^{\prime}\right)$ is obtained from $(S, \mu)$ by an elementary transformation. We distinguish two types of elementary transformations: $\operatorname{elm}_{P}$ is of sprouting type (resp. of subdivisional type) if $P \in Z \backslash \Sigma$ (resp. $\{P\}=Z \cap \Sigma$ ). Note that, if ( $\left.S^{\prime}, \mu^{\prime}\right)=\operatorname{lom}_{P}(S, \mu)$, then $(S, \mu)=\operatorname{elm}_{Q}\left(S^{\prime}, \mu^{\prime}\right)$ for a suitable choice of a point $Q$; here, $\operatorname{elm}_{P}$ and $\operatorname{elm}_{Q}$ are of distinct types.
(2) Two $X$-immersions are equivalent if one can be obtained from the other by a sequence of elementary transformations.
(3) Given $X$-immersions $(S, \mu)$ and $\left(S^{\prime}, \mu^{\prime}\right)$, we write $\left(S^{\prime}, \mu^{\prime}\right) \leq(S, \mu)$ to indicate that ( $S^{\prime}, \mu^{\prime}$ ) is produced by performing on ( $S, \mu$ ) a sequence of elementary transformations of subdivisional type.

Proposition 2.8. Suppose that $X$ satisfies $(\dagger)$.
(1) If $\Lambda$ is an affine ruling of $X$ and $F \in \Lambda_{*}$ then there exists an $X$-immersion $(S, \mu)$ which determines $(\Lambda, F)$ (as in 2.6).
(2) Let $(S, \mu)$ and ( $\left.S^{\prime}, \mu^{\prime}\right)$ be $X$-immersions determining pairs $(\Lambda, F)$ and $\left(\Lambda^{\prime}, F^{\prime}\right)$ respectively. Then:

$$
(\Lambda, F)=\left(\Lambda^{\prime}, F^{\prime}\right) \text { if and only if }(S, \mu) \text { is equivalent to }\left(S^{\prime}, \mu^{\prime}\right) .
$$

Proof. Let $\Lambda$ be an affine ruling of $X$ and $F \in \Lambda_{*}$; let $V=X_{s} \backslash \operatorname{supp}(F)$. Consider $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$ and $\tilde{F} \in \tilde{\Lambda}_{*}$ (recall the bijection $\Lambda \rightarrow \tilde{\Lambda}, F \mapsto \tilde{F}$, defined in 1.6). Since $V \subseteq X^{\prime}=X_{s} \backslash \operatorname{Bs}(\Lambda)$ and $X^{\prime}$ can be viewed as a subset of $\tilde{X}$, we may write $V=X^{\prime} \backslash \operatorname{supp}(\tilde{F})$. Let $S$ be the surface obtained from $\tilde{X}$ by shrinking $\tilde{F}$ to a 0 -curve (see $1.2,1.9$ ) and let $m: \tilde{X} \rightarrow S$ be the corresponding morphism. Let $W=m(V)$ and let $\mu: W \rightarrow V$ be the restriction of $m^{-1}$. Then $(S, \mu)$ is an $X$-immersion and
determines the pair $(\Lambda, F)$; so (1) is proved.
The fact that equivalent $X$-immersions determine the same pair $(\Lambda, F)$ is quite clear. Conversely, suppose that $(S, \mu)$ and $\left(S^{\prime}, \mu^{\prime}\right)$ are $X$-immersions determining the same pair $(\Lambda, F)$; we will show that ( $S, \mu$ ) and ( $S^{\prime}, \mu^{\prime}$ ) are equivalent.

For $(S, \mu)$, we use the notations $W, Z$ and $\Sigma$ as in 2.7 ; for ( $S^{\prime}, \mu^{\prime}$ ), we use $W^{\prime}$, $Z^{\prime}$ and $\Sigma^{\prime}$. Since $\mu$ and $\mu^{\prime}$ have the same image $X_{s} \backslash \operatorname{supp}(F)$, they determine a birational isomorphism $S \rightarrow S^{\prime}$ which restricts to an isomorphism $W \rightarrow W^{\prime}$. So there exists a smooth complete surface $\Omega$ and two birational morphisms, $\pi: \Omega \rightarrow S$ and $\pi^{\prime}: \Omega \rightarrow S^{\prime}$, such that if we regard $\pi$ (resp. $\pi^{\prime}$ ) as a composition of monoidal transformations then each one of these is centered at a point infinitely near $S \backslash W$ (resp. $S^{\prime} \backslash W^{\prime}$ ). We also assume that $\left(\Omega, \pi, \pi^{\prime}\right)$ is minimal, i.e., that the total number of monoidal transformations in $\pi$ and $\pi^{\prime}$ is minimal. We denote this number by $N\left((S, \mu),\left(S^{\prime}, \mu^{\prime}\right)\right)$. Since we assumed that $(S, \mu)$ and ( $\left.S^{\prime}, \mu^{\prime}\right)$ determine the same $\Lambda$, it follows that $\tilde{\Sigma}=\tilde{\Sigma}^{\prime}$, where $\tilde{\Sigma}$ (resp. $\tilde{\Sigma}^{\prime}$ ) is the strict transform of $\Sigma$ (resp. $\Sigma^{\prime}$ ) on $\Omega$.

If $\pi$ is an isomorphism then $S^{\prime}$ is obtained from $S$ by contracting some irreducible components of $S \backslash W$; since no component of $S \backslash W$ is a ( -1 )-curve except possibly $\Sigma$, and since $\pi^{\prime}$ does not shrink $\Sigma$ (for $\tilde{\Sigma}=\tilde{\Sigma}^{\prime}$ ), $\pi^{\prime}$ must then be an isomorphism and we are done in this case.

Suppose that $\pi$ is not an isomorphism; by the above paragraph (with $\pi$ and $\pi^{\prime}$ interchanged), $\pi^{\prime}$ is not an isomorphism and we may consider a curve $\tilde{C} \subset \Omega$ which is first to be shrunk by $\pi^{\prime}$. By minimality of $\left(\Omega, \pi, \pi^{\prime}\right), \tilde{C}$ is the strict transform of some component $C$ of $S \backslash W$ satisfying $C^{2} \geq-1$. Since $\pi^{\prime}$ does not shrink $\tilde{\Sigma}$, we must have $C=Z$. Thus exactly one of the monoidal transformations making-up $\pi$ has a center $P$ which is a point of $Z$. It follows that

$$
N\left(\operatorname{elm}_{P}(S, \mu),\left(S^{\prime}, \mu^{\prime}\right)\right)<N\left((S, \mu),\left(S^{\prime}, \mu^{\prime}\right)\right)
$$

and we are done by induction.

Definition 2.9. Suppose that $X$ is a complete normal rational surface and that $(S, \mu)$ is an $X$-immersion. In this paragraph, we define a set $\Pi(S, \mu)$ of birational morphisms and, given $\pi \in \Pi(S, \mu)$, an $X$-immersion $(S, \mu) * \pi$ determined by ( $S, \mu$ ) and $\pi$.

Let $W=\operatorname{dom} \mu$ and let $\Gamma, Z$ and $\Sigma$ be the main component, zero-component and section of $(S, \mu)$ respectively.

Let $\Pi(S, \mu)$ be the set of birational morphisms $\pi: \tilde{S} \rightarrow S$, with $\tilde{S}$ smooth and complete, satisfying:
(1) the exceptional locus of $\pi$ has a unique ( -1 )-component, which we denote $E$;
(2) $\pi(E)$ is a point of $Z \backslash \Sigma$;
(3) $\pi^{-1}(\Gamma)$ is a linear chain and $E$ has two neighbors in it;
(4) one of the two branches of $\pi^{-1}(\Gamma)$ at $E$ can be shrunk to a smooth point (this
must be the branch containing the strict transforms of $Z$ and $\Sigma$; moreover, the first curve to shrink is either $Z$ or $\Sigma$ ).
Given a point $P$ of $Z \backslash \Sigma$, we also define

$$
\left.\Pi_{P}(S, \mu)=\{\pi \in \Pi(S, \mu) \mid \pi \text { is centered at } P \text { (i.e., } \pi(E)=P)\right\} .
$$

Given $\pi \in \Pi(S, \mu)$, let $\gamma: \tilde{S} \rightarrow S_{+}^{\prime}$ be the birational morphism (with $S_{+}^{\prime}$ smooth) whose exceptional locus is the branch of $\pi^{-1}(\Gamma)$ at $E$ containing the strict transforms of $Z$ and $\Sigma$. Note that $\gamma$ is uniquely determined by $\pi$ and that its exceptional locus has exactly one $(-1)$-component. Moreover, $\gamma(E)$ is a curve whose self-intersection number is nonnegative; $\gamma(Z)$ is a point of $\gamma(E)$ and $\gamma(E)$ is the only irreducible component of $\gamma\left(\pi^{-1}(\Gamma)\right)$ containing that point; $\gamma\left(\pi^{-1}(\Gamma)\right)$ is a linear chain with dual graph


Consider the birational morphism $\sigma: S^{\prime} \rightarrow S_{+}^{\prime}$ defined as follows:
(a) If $x=0$, let $S^{\prime}=S_{+}^{\prime}$ and let $\sigma$ be the identity map.
(b) If $x>0$, let $P_{+}^{\prime} \in S_{+}^{\prime}$ be the unique point of $\gamma(E)$ which also belongs to another irreducible component of $\gamma\left(\pi^{-1}(\Gamma)\right)$; define $\sigma$ by blowing-up $x$ times at $P_{+}^{\prime}$, in such a way that the dual graph of $\sigma^{-1}\left(\gamma\left(\pi^{-1}(\Gamma)\right)\right)$ in $S^{\prime}$ is:

where the 0 -curve is the strict transform of $\gamma(E)$.
Then let $W^{\prime}=\sigma^{-1}\left(\gamma\left(\pi^{-1}(W)\right)\right)$ and let $\mu^{\prime}$ be the composite

$$
W^{\prime} \xrightarrow{\sigma} \gamma\left(\pi^{-1}(W)\right) \xrightarrow{\gamma^{-1}} \pi^{-1}(W) \xrightarrow{\pi} W \xrightarrow{\mu} \mu(W) .
$$

Then $\left(S^{\prime}, \mu^{\prime}\right)$ is an $X$-immersion, determined by $(S, \mu)$ and $\pi$. We write $\left(S^{\prime}, \mu^{\prime}\right)=$ $(S, \mu) * \pi$ and, informally, think of $\left(S^{\prime}, \mu^{\prime}\right)$ as the result of $\pi$ "acting" on $(S, \mu)$. Note that $\mu$ and $\mu^{\prime}$ have the same image.

Definition 2.10. Suppose that $X$ satisfies ( $\dagger$ ).
(1) Let $\mathcal{C}$ be an equivalence class of $X$-immersions. Then $\mathcal{C}$ determines a pair $(\Lambda, F)$ which, in turn, determines $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$ and $\tilde{F} \in \tilde{\Lambda}_{*}$. As shown in the first paragraph of the proof of Proposition 2.8, contracting $\tilde{F}$ to a 0 -curve gives rise to an $X$ immersion ( $S, \mu$ ) which determines $(\Lambda, F)$. We call $(S, \mu)$ the distinguished element of $\mathcal{C}$.
(2) Suppose that $\Lambda$ is an affine ruling of $X$ such that $\beta(\Lambda)>0$. Then (by 2.5) $\Lambda_{*}$ has exactly one element, say $F$, and we may consider the distinguished element ( $S, \mu$ )
of the equivalence class of $X$-immersions which determine $(\Lambda, F)$. We call $(S, \mu)$ the standard $X$-immersion associated to $\Lambda$ (it is an $X$-immersion in standard form). Note that $(S, \mu)$ comes equipped with a birational morphism $m: \tilde{X} \rightarrow S$ (the contraction of $\tilde{F}$ to a 0 -curve).

Corollary 2.11 (Reduction Theorem). Suppose that $X$ satisfies $(\dagger)$ and that $\Lambda$ is an affine ruling of $X$ such that $\beta(\Lambda)>0$. Consider the unique element $F$ of $\Lambda_{*}$, the standard $X$-immersion $(S, \mu)$ associated to $\Lambda$ and the center $P \in S$ of the birational morphism $m: \tilde{X} \rightarrow S$ (the contraction of $\tilde{F}$ to a 0 -curve). Then, for some $\pi \in \Pi_{P}(S, \mu)$, the pair $\left(\Lambda^{\prime}, F^{\prime}\right)$ determined by the $X$-immersion $(S, \mu) * \pi$ satisfies

$$
\beta\left(\Lambda^{\prime}\right)=\beta(\Lambda)-1 \quad \text { and } \quad \operatorname{supp}\left(F^{\prime}\right)=\operatorname{supp}(F) .
$$

Remark. In the conclusion of Corollary 2.11, we can replace "for some $\pi \in$ $\Pi_{P}(S, \mu)$ " by "for every $\pi \in \Pi_{P}(S, \mu)$ ". This is because of part (4) of Lemma 4.4, which also implies that $\left(\Lambda^{\prime}, F^{\prime}\right)$ is uniquely determined by $\Lambda$, i.e., is independent of the choice of $\pi \in \Pi_{P}(S, \mu)$.

Proof of 2.11. Let $C$ be the branching component of $D$ which is closest to $H$ (notations $D, H$, etc as in the proof of 2.1 ); then it is easy to see that $C$ satisfies the condition (10) of the proof. As in the proof of 2.1, factor $m$ as $\tilde{X} \rightarrow \tilde{S} \xrightarrow{\pi} S$ and consider $S \stackrel{\pi}{\longleftarrow} \tilde{S} \xrightarrow{\gamma} S_{+}^{\prime} \stackrel{\sigma}{\longleftarrow} S^{\prime}$. Then it is quite clear that $\pi \in \Pi_{P}(S, \mu)$ and that the $X$-immersion ( $S^{\prime}, \mu^{\prime}$ ) constructed in the proof is exactly $(S, \mu) * \pi$. Then ( $S^{\prime}, \mu^{\prime}$ ) determines a pair ( $\Lambda^{\prime}, F^{\prime}$ ) and the proof of 2.1 shows that $\beta\left(\Lambda^{\prime}\right)<\beta(\Lambda)$. Actually, we have $\beta\left(\Lambda^{\prime}\right)=\beta(\Lambda)-1$ because of how we chose $C$.

Lemma 2.12. Let $X$ be a complete normal rational surface. Suppose that $I$ is an $X$-immersion, that $\pi \in \Pi_{P}(I)$ and let $J=I * \pi$ (where $P$ is a point of the 0 component but not of the section of $I$ ). Let $I^{-}$(resp. $J^{-}$) denote the $X$-immersion obtained from I (resp. J) by performing one subdivisional elementary transformation.
(1) $I^{-} * \pi^{\prime}=J$ for some $\pi^{\prime} \in \Pi_{P^{-}}\left(I^{-}\right)$, where $P^{-}$is the point, on the 0 -component of $I^{-}$, which is the image of the strict transform of the 0 -component of $I$.
(2) $I * \pi^{\prime}=J^{-}$, for some $\pi^{\prime} \in \Pi_{P}(I)$.
(3) If $I^{\prime} \leq I$ and $J^{\prime} \leq J$ then there exists $\pi^{\prime} \in \Pi\left(I^{\prime}\right)$ satisfying $I^{\prime} * \pi^{\prime}=J^{\prime}$.
(4) There exist $I^{\prime} \leq I, J^{\prime} \leq J, \pi^{\prime} \in \Pi\left(I^{\prime}\right)$ and $\pi^{\prime \prime} \in \Pi\left(J^{\prime}\right)$ satisfying

$$
I^{\prime} * \pi^{\prime}=J^{\prime} \text { and } J^{\prime} * \pi^{\prime \prime}=I^{\prime}
$$

Proof. Write $I=(S, \mu)$ and let $Z$ and $\Sigma$ be the 0 -component and section of $I$; write $J=\left(S^{\prime}, \mu^{\prime}\right)=I * \pi$ and consider $S \stackrel{\pi}{\leftarrow} \tilde{S} \stackrel{\gamma}{\rightarrow} S_{+}^{\prime} \stackrel{\sigma}{\leftarrow} S^{\prime}$, as in Definition 2.9.

To prove (1), consider the point $\{Q\}=Z \cap \Sigma$, write $(T, \nu)=I^{-}=\operatorname{elm}_{Q}(S, \mu)$ and
consider the commutative diagram

where $\alpha$ is the blowing-up at $Q$ and $\beta$ contracts the strict transform of $Z$. Then $\pi^{\prime}=$ $\beta \circ \pi_{Y} \in \Pi_{P-}\left(I^{-}\right)$and $I^{-} * \pi^{\prime}=J$ (where $P^{-}$is the center of $\beta$ ).

To prove (2), let $Z^{\prime}$ and $\Sigma^{\prime}$ be the 0 -component and section of $J$, consider the point $\{Q\}=Z^{\prime} \cap \Sigma^{\prime}$, write $\operatorname{elm}_{Q}(J)=J^{-}=(T, \nu)$, let $\alpha: Y \rightarrow S^{\prime}$ be the blowing-up of $S^{\prime}$ at $Q$ and $\beta: Y \rightarrow T$ the contraction of the strict transform of $Z^{\prime}$. Consider the commutative diagram


Then $\pi^{\prime}=\pi \circ u \circ \tilde{u} \in \Pi_{P}(I)$ and $I * \pi^{\prime}=J^{-}$.
Assertion (3) follows immediately from (1) and (2). To prove (4), consider the sections $\Sigma$ and $\Sigma^{\prime}$ of $I$ and $J$ respectively. In view of (3), we may assume that $\Sigma^{2}<-1$ and $\left(\Sigma^{\prime}\right)^{2}<-1$. Then, in the diagram $S \stackrel{\pi}{\leftarrow} \tilde{S} \xrightarrow{\gamma} S_{+}^{\prime} \stackrel{\sigma}{\leftarrow} S^{\prime}$, the map $\sigma$ is the identity map, $\gamma \in \Pi(J)$ and $J * \gamma=I$.

Proposition 2.13. Let $X$ be a complete normal rational surface and suppose that $(S, \mu)$ and $(T, \nu)$ are $X$-immersions. Then the condition $\operatorname{im} \mu=\operatorname{im} \nu$ is equivalent to the existence of a sequence $\left\{\left(S_{j}^{*}, \mu_{j}^{*}\right)\right\}_{j=0}^{n}$ of $X$-immersions satisfying:
(1) $\left(S_{0}^{*}, \mu_{0}^{*}\right) \leq(S, \mu)$ and $\left(S_{n}^{*}, \mu_{n}^{*}\right) \leq(T, \nu)$;
(2) for all $j=1, \ldots, n$, we have

$$
\left(S_{j-1}^{*}, \mu_{j-1}^{*}\right) * \pi \text { is equivalent to }\left(S_{j}^{*}, \mu_{j}^{*}\right), \text { for some } \pi \in \Pi\left(S_{j-1}^{*}, \mu_{j-1}^{*}\right)
$$

For the proof, we will need the following notations. Given $X$-immersions $I_{j}=$ $\left(S_{j}, \mu_{j}\right)(j=1,2)$ such that $\operatorname{im} \mu_{1}=\operatorname{im} \mu_{2}$, let $\mathcal{D}\left(I_{1}, I_{2}\right)$ denote the set of triples ( $\Omega, \pi_{1}, \pi_{2}$ ) satisfying:
(1) $\Omega$ is a smooth complete surface and $\pi_{1}: \Omega \rightarrow S_{1}$ and $\pi_{2}: \Omega \rightarrow S_{2}$ are birational morphisms;
(2) $\pi_{j}$ is centered at points of $S_{j} \backslash \operatorname{dom} \mu_{j}(j=1,2)$ and $\pi_{1}^{-1}\left(S_{1} \backslash \operatorname{dom} \mu_{1}\right)=\pi_{2}^{-1}\left(S_{2} \backslash\right.$ dom $\mu_{2}$ );
(3) the birational transformations $\pi_{2} \pi_{1}^{-1}$ and $\mu_{2}^{-1} \mu_{1}$, from $S_{1}$ to $S_{2}$, are equal.

Note that $\mathcal{D}\left(I_{1}, I_{2}\right)$ is nonempty (because $\operatorname{im} \mu_{1}=\operatorname{im} \mu_{2}$ ) and that, given any $\left(\Omega, \pi_{1}, \pi_{2}\right) \in \mathcal{D}\left(I_{1}, I_{2}\right)$, if one of $\pi_{1}, \pi_{2}$ is an isomorphism then both $\pi_{1}, \pi_{2}$ are.

Given a birational morphism $f: U \rightarrow V$ of smooth complete surfaces, let $N(f) \geq$ 0 be the number of monoidal transformations in $f$.

Given $D=\left(\Omega, \pi_{1}, \pi_{2}\right) \in \mathcal{D}\left(I_{1}, I_{2}\right)$, let $N(D)=N\left(\pi_{1}\right)+N\left(\pi_{2}\right)$. Also, let $N\left(I_{1}, I_{2}\right)=$ $\min _{D \in \mathcal{D}\left(I_{1}, I_{2}\right)} N(D)$.

Proof of Proposition 2.13. Clearly, the existence of the sequence implies $\operatorname{im} \mu=$ $\operatorname{im} \nu$.

For the converse, let $I=(S, \mu)$ and $J=(T, \nu)$ be $X$-immersions such that im $\mu=$ im $v$ and consider the set

$$
\mathcal{I}_{(I, J)}=\left\{\left(I_{1}, I_{2}\right) \mid I_{1} \leq(S, \mu), \quad I_{2} \leq(T, \nu), \quad \Sigma_{1}^{2}<-1, \quad \Sigma_{2}^{2}<-1\right\},
$$

where $I_{j}$ is an $X$-immersion and $\Sigma_{j}$ is its section. We proceed by induction on the natural number $d(I, J)$ defined by

$$
d(I, J)=\min \left\{N\left(I_{1}, I_{2}\right) \mid\left(I_{1}, I_{2}\right) \in \mathcal{I}_{(I, J)}\right\} .
$$

If $d(I, J)=0$ then $I$ and $J$ are equivalent; then it is easy to see that there exists an $X$-immersion $\left(S_{0}^{*}, \mu_{0}^{*}\right)$ satisfying both $\left(S_{0}^{*}, \mu_{0}^{*}\right) \leq(S, \mu)$ and $\left(S_{0}^{*}, \mu_{0}^{*}\right) \leq(T, \nu)$, so we are done in this case. From now-on, we assume that $d(I, J)>0$.

Choose $\left(I_{1}, I_{2}\right) \in \mathcal{I}_{(I, J)}$ such that $N\left(I_{1}, I_{2}\right)=d(I, J)$; write $I_{j}=\left(S_{j}, \mu_{j}\right)$, $D_{j}=S_{j} \backslash \operatorname{dom} \mu_{j}$ and let $Z_{j}$ and $\Sigma_{j}$ be the 0 -component and section of $I_{j}$. Choose $\left(\Omega, \pi_{1}, \pi_{2}\right) \in \mathcal{D}\left(I_{1}, I_{2}\right)$ such that $N\left(\Omega, \pi_{1}, \pi_{2}\right)=N\left(I_{1}, I_{2}\right)$.

$$
S_{1} \stackrel{\pi_{1}}{\rightleftarrows} \Omega \xrightarrow{\pi_{2}} S_{2}
$$

We claim that
(i) Neither of $\pi_{1}, \pi_{2}$ is an isomorphism.
(ii) For each $j=1,2$, the exceptional locus of $\pi_{j}$ has a unique ( -1 )-component, say $E_{j} \subset \Omega$, and $\pi_{j}$ is centered at a point of $Z_{j} ;$ also, $\pi_{2}\left(E_{1}\right)=Z_{2}$ and $\pi_{1}\left(E_{2}\right)=Z_{1}$. Moreover, we claim that $\left(I_{1}, I_{2}\right)$ and $\left(\Omega, \pi_{1}, \pi_{2}\right)$ can be chosen in such a way that the following conditions hold:
(iii) $\pi_{j}$ is centered at a point of $Z_{j} \backslash \Sigma_{j}$ (for $j=1,2$ );
(iv) $E_{j}$ has two neighbors in $\pi_{j}^{-1}\left(\Gamma_{j}\right)$ (for each $j=1,2$ ), where $\Gamma_{j} \subseteq D_{j}$ is the main component of ( $S_{j}, \mu_{j}$ ).
If (i) is false then, as pointed out just before the proof, both $\pi_{1}, \pi_{2}$ are isomorphisms; this contradicts $d(I, J)>0$, so (i) holds.

By (i), the exceptional locus of $\pi_{1}$ has at least one ( -1 )-component; let $E_{1} \subset \Omega$ be such a component. Since $N\left(\Omega, \pi_{1}, \pi_{2}\right)=N\left(I_{1}, I_{2}\right), \pi_{2}$ does not contract $E_{1}$. So, $\pi_{2}\left(E_{1}\right)$ is a non-branching component of $D_{2}$ satisfying $\pi_{2}\left(E_{1}\right)^{2} \geq-1$ and consequently $\pi_{2}\left(E_{1}\right)=Z_{2}$. In particular, $E_{1}$ is unique.

If the center of $\pi_{1}$ is not on $Z_{1}$ then the strict transform $\tilde{Z}_{1} \subset \Omega$ of $Z_{1}$ satisfies $\tilde{Z}_{1}^{2}=0$. Thus $\pi_{2}\left(\tilde{Z}_{1}\right)$ is a component of $D_{2}$ with nonnegative self-intersection number
and, consequently, $\pi_{2}\left(\tilde{Z}_{1}\right)=Z_{2}$. This is impossible, because $\pi_{2}\left(E_{1}\right)=Z_{2}$ and $E_{1} \neq \tilde{Z}_{1}$. Thus the center of $\pi_{1}$ is on $Z_{1}$. Then (ii) follows by symmetry in $\pi_{1}$ and $\pi_{2}$.

For each $j=1,2$, let $P_{j} \in Z_{j}$ be the center of $\pi_{j}$; define an $X$-immersion $I_{j}^{\prime}=$ $\left(S_{j}^{\prime}, \mu_{j}^{\prime}\right) \leq I_{j}$ and a morphism $\pi_{j}^{\prime}: \Omega \rightarrow S_{j}^{\prime}$ as follows:

- If $P_{j} \in Z_{j} \backslash \Sigma_{j}$, let $I_{j}^{\prime}=I_{j}$ and $\pi_{j}^{\prime}=\pi_{j}$;
- if $P_{j} \in Z_{j} \cap \Sigma_{j}$, let $I_{j}^{\prime}=\operatorname{elm}_{P_{j}}\left(I_{j}\right)$ and consider

where $\alpha_{j}$ is the blowing-up of $S_{j}$ at $P_{j}, \beta_{j}$ is the contraction of the strict transform of $Z_{j}$ relative to $\alpha_{j}$ and $\pi_{j}^{-}$is defined by $\pi_{j}=\alpha_{j} \circ \pi_{j}^{-}$. Then set $\pi_{j}^{\prime}=\beta_{j} \circ \pi_{j}^{-}$.

Then $\left(I_{1}^{\prime}, I_{2}^{\prime}\right) \in \mathcal{I}_{(I, J)},\left(\Omega, \pi_{1}^{\prime}, \pi_{2}^{\prime}\right) \in \mathcal{D}\left(I_{1}^{\prime}, I_{2}^{\prime}\right)$ and $N\left(\Omega, \pi_{1}^{\prime}, \pi_{2}^{\prime}\right)=N\left(\Omega, \pi_{1}, \pi_{2}\right)=$ $d(I, J)$. Moreover, the center of $\pi_{j}^{\prime}$ is a point of $Z_{j}^{\prime} \backslash \Sigma_{j}^{\prime}$ (for each $j=1,2$ ), where $Z_{j}^{\prime}$ and $\Sigma_{j}^{\prime}$ are the 0 -component and section of $I_{j}^{\prime}$ respectively. In other words, we may simply assume that $\left(I_{1}, I_{2}\right)$ and $\left(\Omega, \pi_{1}, \pi_{2}\right)$ have been chosen in such a way that (iii) holds. Finally, (iv) follows immediately from (i-iii).

We proved that there exists $\left(I_{1}, I_{2}\right) \in \mathcal{I}_{(I, J)}$ and $\left(\Omega, \pi_{1}, \pi_{2}\right) \in \mathcal{D}\left(I_{1}, I_{2}\right)$ satisfying $N\left(\Omega, \pi_{1}, \pi_{2}\right)=N\left(I_{1}, I_{2}\right)=d(I, J)$ and conditions (i-iv). We will now show that $d\left(I_{1} *\right.$ $\left.\pi, I_{2}\right)<d(I, J)$ for some $\pi \in \Pi\left(I_{1}\right)$, which will complete the proof.

If $\pi_{1}^{-1}\left(\Gamma_{1}\right)$ is a linear chain then $\pi_{1} \in \Pi\left(I_{1}\right)$ and $I_{1} * \pi_{1}=I_{2}$, so we are done in this case.

Assume that $\pi_{1}^{-1}\left(\Gamma_{1}\right)$ is not a linear chain and consider the branching component $C$ of $\pi_{1}^{-1}\left(\Gamma_{1}\right)$ which is closest to the strict transform $\tilde{Z}_{1} \subset \Omega$ of $Z_{1}$. Note that $C$ is contained in the exceptional locus of $\pi_{1}$, for otherwise we would have $C=\tilde{Z}_{1}$, but $\tilde{Z}_{1}$ is not branching in $\pi_{1}^{-1}\left(\Gamma_{1}\right)$ (because $Z_{1}$ has one neighbor in $\Gamma_{1}$ and the center of $\pi_{1}$ is one point). Also, $\pi_{1}^{-1}\left(\Gamma_{1}\right)$ has exactly three branches at $C$, say $\mathcal{B}, \mathcal{B}^{u}$ and $\mathcal{B}^{\ell}$, where $\mathcal{B}^{u}$ contains $\tilde{Z}_{1}$ and $\mathcal{B}$ contains $E_{1}$. Note that $\tilde{Z}_{1}$ and $E_{1}$ are the only ( -1 )-components of $\pi_{1}^{-1}\left(\Gamma_{1}\right)$ and that all other components have self-intersection strictly less than -1 .

Since $\pi_{2}\left(\pi_{1}^{-1}\left(\Gamma_{1}\right)\right)$ is the linear chain $\Gamma_{2}$, we know that $\mathcal{B}^{u}$ can be shrunk to a point, i.e., we may factor $\pi_{2}$ as $\Omega \xrightarrow{\alpha} U \xrightarrow{\beta} S_{2}$, where $\alpha$ is the contraction of $\mathcal{B}^{u}$. We may also factor $\pi_{1}$ as $\Omega \xrightarrow{\alpha_{1}} \tilde{S}_{1} \xrightarrow{\pi} S_{1}$, in such a way that $\alpha_{1}(C)$ is a $(-1)$-curve on $\tilde{S}_{1}$; then $\alpha_{1}$ is the contraction of $\mathcal{B}$ to a point and $\alpha_{1}(C)$ is the only ( -1 )-component of the exceptional locus of $\pi$. This gives the first of the following commutative diagrams
of smooth complete surfaces and birational morphisms:

$$
\Omega \xrightarrow[\alpha]{\left(\mathcal{B}^{u}\right)} U \xrightarrow[\beta]{\longrightarrow} S_{2}
$$



$$
\Omega^{\prime} \xrightarrow[\alpha^{\prime}]{ } U \xrightarrow[\beta]{\longrightarrow} S_{2}
$$

$$
\begin{equation*}
\tilde{S}_{1} \xrightarrow[\gamma]{\left(\mathcal{B}^{u}\right)} S_{+}^{\prime} \tag{19}
\end{equation*}
$$

$$
\downarrow \pi
$$

$$
\begin{aligned}
&(\mathcal{B )}) \\
& \downarrow_{\pi_{1}^{\prime}} \\
&{ }^{(\mathcal{B})} \downarrow \\
& S^{\prime} \\
& \\
& \\
& \hline
\end{aligned} S_{+}^{\prime}
$$

$S_{1}$
where the labels $(\mathcal{B})$ and $\left(\mathcal{B}^{u}\right)$ indicate which set is contracted by each morphism.
Note that $\pi \in \Pi\left(I_{1}\right)$, so we may consider the $X$-immersion $I_{1}^{\prime}=\left(S^{\prime}, \mu^{\prime}\right)=I_{1} * \pi$. Recall, from Definition 2.9, that the construction of $I_{1} * \pi$ involves a birational morphism $\sigma: S^{\prime} \rightarrow S_{+}^{\prime}$ which is the composition of $x$ monoidal transformations, where $x \geq 0$ is the self-intersection number of the curve $\left(\gamma \circ \alpha_{1}\right)(C) \subset S_{+}^{\prime}$. Let $\alpha^{\prime}: \Omega^{\prime} \rightarrow U$ consist of the "same" $x$ monoidal transformations as $\sigma$, but performed at the level of $U$. This gives the second diagram in (19).

Let $\pi_{2}^{\prime}=\beta \circ \alpha^{\prime}: \Omega^{\prime} \rightarrow S_{2}$, then $\left(\Omega^{\prime}, \pi_{1}^{\prime}, \pi_{2}^{\prime}\right)$ belongs to $\mathcal{D}\left(I_{1}^{\prime}, I_{2}\right)$ but not necessarely to $\mathcal{I}_{\left(I_{1}^{\prime}, I_{2}\right)}$. Note that the section $\Sigma_{1}^{\prime}$ of $I_{1}^{\prime}$ satisfies $\left(\Sigma_{1}^{\prime}\right)^{2} \leq-1$ and let $I_{1}^{\prime \prime}$ be the $X$-immersion obtained from $I_{1}^{\prime}$ by performing one subdivisional elementary transformation. Then $\left(I_{1}^{\prime \prime}, I_{2}\right) \in \mathcal{I}_{\left(I_{1}^{\prime}, I_{2}\right)}$, so

$$
\begin{equation*}
d\left(I_{1}^{\prime}, I_{2}\right) \leq N\left(I_{1}^{\prime \prime}, I_{2}\right) \leq N\left(I_{1}^{\prime}, I_{2}\right)+2 \leq N\left(\Omega^{\prime}, \pi_{1}^{\prime}, \pi_{2}^{\prime}\right)+2 \tag{20}
\end{equation*}
$$

We have $d(I, J)=N\left(\Omega, \pi_{1}, \pi_{2}\right)=|\mathcal{B}|+N(\pi)+\left|\mathcal{B}^{u}\right|+N(\beta)$ and $N\left(\Omega^{\prime}, \pi_{1}^{\prime}, \pi_{2}^{\prime}\right)=$ $N\left(\pi_{1}^{\prime}\right)+N\left(\alpha^{\prime}\right)+N(\beta)=|\mathcal{B}|+x+N(\beta)$, where $|\mathcal{B}|$ and $\left|\mathcal{B}^{u}\right|$ denote the numbers of irreducible components of $\mathcal{B}$ and $\mathcal{B}^{u}$. So

$$
d(I, J)-N\left(\Omega^{\prime}, \pi_{1}^{\prime}, \pi_{2}^{\prime}\right)=N(\pi)+\left|\mathcal{B}^{u}\right|-x .
$$

Note that the self-intersection numbers of $\alpha_{1}(C) \subset \tilde{S}_{1}$ and $\gamma\left(\alpha_{1}(C)\right) \subset S_{+}^{\prime}$ are -1 and $x$ respectively, so $\gamma$ increases that number by $x+1$. Since $N(\gamma)=\left|\mathcal{B}^{u}\right|$, we must have $x+1 \leq\left|\mathcal{B}^{u}\right|$, so

$$
d(I, J)-N\left(\Omega^{\prime}, \pi_{1}^{\prime}, \pi_{2}^{\prime}\right)>N(\pi)
$$

and, by (20),

$$
d\left(I_{1}^{\prime}, I_{2}\right)<d(I, J)-N(\pi)+2 .
$$

It is easy to see that $N(\pi) \geq 3$, so $d\left(I_{1}^{\prime}, I_{2}\right)<d(I, J)$ and we are done.

## Conclusion

2.14. Given a surface $X$ satisfying $(\dagger)$, consider the directed graph $\mathbb{L}(X)$ whose vertices are the affine rulings of $X$ and where, given vertices $\Lambda$ and $\Lambda^{\prime}$, we draw an arrow $\Lambda \rightarrow \Lambda^{\prime}$ if the following condition holds: There exists an $X$-immersion $I$ and an element $\pi$ of $\Pi(I)$ such that (i) $I$ determines $(\Lambda, F)$ for some $F \in \Lambda_{*}$ and (ii) $I * \pi$ determines $\left(\Lambda^{\prime}, F^{\prime}\right)$ for some $F^{\prime} \in \Lambda_{*}^{\prime}$.

Part (4) of Lemma 2.12 implies that if there is an arrow $\Lambda \rightarrow \Lambda^{\prime}$ then there is also an arrow $\Lambda \leftarrow \Lambda^{\prime}$. Corollary 2.11 implies that each connected component of $\mathbb{L}(X)$ contains a basic affine ruling. Thus, if we want to describe all affine rulings of $X$, we have to solve the following two problems:
(1) Make a list of all basic rulings of $X$.
(2) Describe the set $\Pi(S, \mu)$, for each $X$-immersion $(S, \mu) .^{4}$

Each one of these problems is nontrivial. The first one is highly dependent on the surface $X$; [6] solves it for the weighted projective planes (so in particular for $\mathbb{P}^{2}$ ). The second problem turns out to be independent of the surface and is completely solved in sections 3 and 4 (see in particular Corollary 4.4).

Remarks. Let $X$ be a surface satisfying ( $\dagger$ ).
(1) One can show ${ }^{5}$ that an affine ruling $\Lambda$ is an isolated vertex of $\mathbb{L}(X)$ if and only if $\Lambda_{*}=\emptyset$. Thus, if we make the additional assumption that $X$ satisfies $(\ddagger)$, then no vertex of $\mathbb{L}(X)$ is isolated (see 2.5 ).
(2) Let us temporarily agree that, given affine rulings $\Lambda$ and $\Lambda^{\prime}$ of $X$, the phrase " $\Lambda$ and $\Lambda^{\prime}$ have a common member" means that there exists a curve $C \subset X$ and positive integers $n$ and $n^{\prime}$ satisfying $n C \in \Lambda_{*}$ and $n^{\prime} C^{\prime} \in \Lambda_{*}^{\prime}$. Then Proposition 2.13 implies: Two affine rulings $\Lambda$ and $\Lambda^{\prime}$ of $X$ are in the same connected component of $\mathbb{L}(X)$ if and only if there exists a sequence $\left\{\Lambda_{i}\right\}_{i=0}^{n}$ of affine rulings of $X$ such that $\Lambda_{0}=\Lambda$, $\Lambda_{n}=\Lambda^{\prime}$ and, for each $i<n, \Lambda_{i}$ and $\Lambda_{i+1}$ have a common member.

## 3. Contraction of weighted trees

We assume familiarity with weighted graphs, their blowing-up and blowing-down. We stress that, in weighted graphs, we do not allow multiple edges between a given pair of vertices. The empty weighted graph is denoted $\boldsymbol{\emptyset}$. A weighted tree without branch points is called a linear weighted tree or a linear chain.
3.1. Given weighted graphs $\mathcal{G}$ and $\mathcal{G}^{\prime}$, the symbol $\mathcal{G} \leftarrow \mathcal{G}^{\prime}$ indicates that $\mathcal{G}^{\prime}$ is obtained from $\mathcal{G}$ by blowing-up once. In that case, if $V$ (resp. $V^{\prime}$ ) denotes the set of

[^3]vertices of $\mathcal{G}$ (resp. $\mathcal{G}^{\prime}$ ) then $V$ can be viewed as a subset of $V^{\prime}$ and $V^{\prime} \backslash V$ contains a single vertex, say $e$. We call $e$ the vertex created by $\mathcal{G} \leftarrow \mathcal{G}^{\prime}$; we also say that $\mathcal{G}$ is the blowing-down of $\mathcal{G}^{\prime}$ at $e$. If $e$ has one neighbor $v$ in $\mathcal{G}^{\prime}$, then $v$ can be viewed as a vertex of $\mathcal{G}$ and $\mathcal{G} \leftarrow \mathcal{G}^{\prime}$ is called the blowing-up of $\mathcal{G}$ at the vertex $v$. If $e$ has two neighbors $u$ and $v$ in $\mathcal{G}^{\prime}$, then $\{u, v\}$ is an edge of $\mathcal{G}$ and $\mathcal{G} \leftarrow \mathcal{G}^{\prime}$ is called the blowing-up of $\mathcal{G}$ at the edge $\{u, v\} .{ }^{6}$ Also, if $\mathcal{G}$ is any weighted graph and $\mathcal{G}^{\prime}$ is the weighted graph obtained from $\mathcal{G}$ by adding an isolated vertex of weight -1 , then we regard $\mathcal{G}^{\prime}$ as a blowing-up of $\mathcal{G}$.
3.2. Two weighted graphs are equivalent if one can be obtained from the other by means of a finite sequence of blowings-up and blowings-down. We will use the symbol " $\sim$ " for equivalence of weighted graphs (and " $\approx$ " for equivalence of weighted pairs, Definition 3.9).

## BLOWING-UP ACCORDING TO A TABLEAU

3.3. Let $\mathcal{G}_{0}$ be a weighted graph, $e_{0}$ a vertex of $\mathcal{G}_{0}$ and $c \geq p>0$ integers. By blowing-up $\mathcal{G}_{0}$ at $e_{0}$ according to $\binom{p}{c}$, we mean producing the sequence $\mathcal{G}_{0} \leftarrow \cdots \leftarrow$ $\mathcal{G}_{n}$ defined as follows.
(1) Let $\mathcal{G}_{0} \leftarrow \mathcal{G}_{1}$ be the blowing-up at $e_{0}$ and let $e_{1}$ be the vertex of $\mathcal{G}_{1}$ so created. Define $\left(\begin{array}{c}u_{1} \\ v_{1} \\ v_{1}\end{array} x_{1}, y_{1}\right)=\left(\begin{array}{ccc}e_{1} & p \\ e_{0} & c-p\end{array}\right)$.
(2) If $i \geq 1$ is such that $\mathcal{G}_{i}, e_{i}$ and $\left(\begin{array}{l}u_{i} \\ v_{i} \\ v_{i}\end{array} y_{i}\right)$ have been defined, then:
(a) If $y_{i}=0$ then we set $n=i$ and stop.
(b) If $y_{i} \neq 0$ then let $\mathcal{G}_{i+1}$ be the blowing-up of $\mathcal{G}_{i}$ at the edge $\left\{u_{i}, v_{i}\right\}$, let $e_{i+1}$ be the vertex of $\mathcal{G}_{i+1}$ so created and define

$$
\left(\begin{array}{ll}
u_{i+1} & x_{i+1} \\
v_{i+1} & y_{i+1}
\end{array}\right)= \begin{cases}\left(\begin{array}{cc}
e_{i+1} & x_{i} \\
v_{i} & y_{i}-x_{i}
\end{array}\right) & \text { if } x_{i} \leq y_{i} \\
\left(\begin{array}{cc}
u_{i} & x_{i}-y_{i} \\
e_{i+1} & y_{i}
\end{array}\right) & \text { if } x_{i}>y_{i}\end{cases}
$$

Remark. In 3.3 we have $n \geq 1$, with equality if and only if $p=c$. Of the $n$ blowings-up in $\mathcal{G}_{0} \leftarrow \cdots \leftarrow \mathcal{G}_{n}$, only $\mathcal{G}_{0} \leftarrow \mathcal{G}_{1}$ is a blowing-up at a vertex.

Definition 3.4. Let $\mathcal{G}_{0}$ be a weighted graph, $e_{0}$ a vertex of $\mathcal{G}_{0}$ and

$$
T=\left(\begin{array}{ccc}
p_{1} & \cdots & p_{k} \\
c_{1} & \cdots & c_{k}
\end{array}\right)
$$

[^4]a matrix such that $p_{i} \leq c_{i}$ are positive integers for all $i$.
We define the sequence $\mathcal{G}_{0} \leftarrow \cdots \leftarrow \mathcal{G}_{n}$ obtained by blowing-up $\mathcal{G}_{0}$ at $e_{0}$ according to $T$ by induction on $k$ :

- If $k=0$ (i.e., $T$ is the empty matrix), then $n=0$ (no blowing-up is performed).
- If $k=1$, then $\mathcal{G}_{0} \leftarrow \cdots \leftarrow \mathcal{G}_{n}$ is defined in 3.3.
- If $k>1$, then $\mathcal{G}_{0} \leftarrow \cdots \leftarrow \mathcal{G}_{n}$ is

$$
\mathcal{G}_{0} \leftarrow \cdots \leftarrow \mathcal{G}_{m-1} \leftarrow \mathcal{G}_{m} \leftarrow \mathcal{G}_{m+1} \leftarrow \cdots \leftarrow \mathcal{G}_{n},
$$

where $\mathcal{G}_{0} \leftarrow \cdots \leftarrow \mathcal{G}_{m}$ is the sequence obtained by blowing-up $\mathcal{G}_{0}$ at $e_{0}$ according to $\binom{p_{1}}{c_{1}}$ and $\mathcal{G}_{m} \leftarrow \cdots \leftarrow \mathcal{G}_{n}$ is obtained by blowing-up $\mathcal{G}_{m}$ at $e_{m}$ according to $\left(\begin{array}{ccc}p_{2} & \cdots & p_{k} \\ c_{2} & c_{k}\end{array}\right)$ (where $e_{m}$ is the vertex of $\mathcal{G}_{m}$ created by $\mathcal{G}_{m-1} \leftarrow \mathcal{G}_{m}$ ).

Definition 3.5. A tableau is a matrix $T=\left(\begin{array}{ccc}p_{1} & \ldots & p_{k} \\ c_{1} & \ldots & c_{k}\end{array}\right)$ whose entries are integers satisfying $c_{i} \geq p_{i} \geq 1$ and $\operatorname{gcd}\left(p_{i}, c_{i}\right)=1$ for all $i=1, \ldots, k$. We allow $k=0$, in which case we say that $T$ is the empty tableau and write $T=\mathbf{1}$. The set of all tableaux is denoted $\mathcal{T}$. Given $T \in \mathcal{T}$, let $h(T)$ denote the number of columns of $T$ which are different from $\binom{1}{1}$.
3.6. Let $T^{\prime}=\left(\begin{array}{ccc}p_{1}^{\prime} & \ldots & p_{k}^{\prime} \\ c_{1}^{\prime} & \ldots & c_{k}^{\prime}\end{array}\right)$ and $T^{\prime \prime}=\left(\begin{array}{ccc}p_{1}^{\prime \prime} . . & p_{k}^{\prime \prime} \\ c_{1}^{\prime \prime} & \ldots & c_{k}^{\prime \prime}\end{array}\right)$ be two $2 \times k$ matrices as in 3.4. We say that $T^{\prime}$ and $T^{\prime \prime}$ are equivalent if there exists a $k$-tuple $\left(r_{1}, \ldots, r_{k}\right)$ of positive rational numbers satisfying $\binom{p_{i}^{\prime}}{c_{i}^{\prime}}=r_{i}\binom{p_{c_{i}^{\prime \prime}}^{\prime}}{c_{i}}$ for all $i=1, \ldots, k$. If this is the case then, given a weighted graph $\mathcal{G}_{0}$ and a vertex $e_{0}$ of $\mathcal{G}_{0}$, blowing-up $\mathcal{G}_{0}$ at $e_{0}$ according to $T^{\prime}$ or $T^{\prime \prime}$ gives the same sequence $\mathcal{G}_{0} \leftarrow \cdots \leftarrow \mathcal{G}_{n}$.

Clearly, each matrix $T^{\prime}$ as above is equivalent to a unique tableau $T \in \mathcal{T}$ (see 3.5). Also, every Hamburger-Noether tableau

$$
\mathrm{HN}=\left(\begin{array}{cccc}
p_{1} & \cdots & p_{k-1} & p_{k} \\
c_{1} & \cdots & c_{k-1} & c_{k} \\
\alpha_{1} & \cdots & \alpha_{k-1} & \alpha_{k}
\end{array}\right) \quad \text { (as in the appendix of [11]) }
$$

determines a unique tableau

$$
\overline{\mathrm{HN}}=\left(\begin{array}{ccc}
\bar{p}_{1} & \cdots & \bar{p}_{k-1} \\
\bar{c}_{1} & \bar{p}_{k} \\
\bar{c}_{k-1} & \bar{c}_{k}
\end{array}\right) \in \mathcal{T} \quad \text { where } \quad\left(\bar{p}_{i}, \bar{c}_{i}\right)=\left(\frac{p_{i}}{\operatorname{gcd}\left(p_{i}, c_{i}\right)}, \frac{c_{i}}{\operatorname{gcd}\left(p_{i}, c_{i}\right)}\right) .
$$

3.7. Consider an arbitrary sequence $S: \mathcal{G}_{0} \leftarrow \cdots \leftarrow \mathcal{G}_{n}$ of blowings-up of weighted graphs and, for $i=1, \ldots n$, let $e_{i}$ be the vertex of $\mathcal{G}_{i}$ created by $\mathcal{G}_{i-1} \leftarrow \mathcal{G}_{i}$. Suppose that $S$ satisfies the two conditions:
(1) If $n \geq 1$ then $\mathcal{G}_{0} \leftarrow \mathcal{G}_{1}$ is the blowing-up at a vertex $e_{0}$; and
(2) if $n \geq 2$ then, for each $i=1, \ldots, n-1, \mathcal{G}_{i} \leftarrow \mathcal{G}_{i+1}$ is the blowing-up at the vertex $e_{i}$, or at an edge incident to $e_{i}$.

Then there exists a unique tableau $T \in \mathcal{T}$ such that $S$ is the blowing-up of $\mathcal{G}_{0}$ at $e_{0}$ according to $T$. Moreover, those two conditions are necessary for the existence of $T$.

## WEIGHTED PAIRS

Definition 3.8. If $\mathcal{G}$ is a nonempty weighted graph and $v$ a vertex of $\mathcal{G}$ then we say that $(\mathcal{G}, v)$ is a weighted pair.

Definition 3.9. Let $(\mathcal{G}, v)$ and $\left(\mathcal{G}^{\prime}, v^{\prime}\right)$ be weighted pairs.
Let us say, provisionally, that $\left(\mathcal{G}^{\prime}, v^{\prime}\right)$ is an elementary contraction of $(\mathcal{G}, v)$ if $\mathcal{G}^{\prime}$ is the blowing-down of $\mathcal{G}$ at some vertex $e \neq v$ and if the canonical inclusion $V^{\prime} \hookrightarrow V$ maps $v^{\prime}$ to $v$ (where $V$ and $V^{\prime}$ are the sets of vertices of $\mathcal{G}$ and $\mathcal{G}^{\prime}$ respectively).

We say that $(\mathcal{G}, v)$ is equivalent to $\left(\mathcal{G}^{\prime}, v^{\prime}\right)$, written $(\mathcal{G}, v) \approx\left(\mathcal{G}^{\prime}, v^{\prime}\right)$, if there exists a sequence $\left(\mathcal{G}_{0}, v_{0}\right), \ldots,\left(\mathcal{G}_{n}, v_{n}\right)$ of weighted pairs satisfying $\left(\mathcal{G}_{0}, v_{0}\right)=(\mathcal{G}, v)$, $\left(\mathcal{G}_{n}, v_{n}\right)=\left(\mathcal{G}^{\prime}, v^{\prime}\right)$ and such that, for each $i=1, \ldots, n$, one of the following holds:
(1) $\left(\mathcal{G}_{i}, v_{i}\right)$ is an elementary contraction of $\left(\mathcal{G}_{i-1}, v_{i-1}\right)$; or
(2) $\left(\mathcal{G}_{i-1}, v_{i-1}\right)$ is an elementary contraction of $\left(\mathcal{G}_{i}, v_{i}\right)$.

In the special case where condition (1) holds for all $i=1, \ldots, n$, we say that $(\mathcal{G}, v)$ contracts to $\left(\mathcal{G}^{\prime}, v^{\prime}\right)$ and we write $(\mathcal{G}, v) \geq\left(\mathcal{G}^{\prime}, v^{\prime}\right)$.

When $(\mathcal{G}, v) \approx\left(\mathcal{G}^{\prime}, v^{\prime}\right)$, we sometimes identify $v$ with $v^{\prime}$.
Definition 3.10. A weighted pair $(\mathcal{G}, v)$ is called a linear pair if $\mathcal{G}$ is a linear weighted tree and $v$ has at most one neighbor in $\mathcal{G}$.

Definition 3.11. A weighted pair $(\mathcal{L}, w)$ satisfies the condition ( 0 ) if $\mathcal{L}$ is a tree of the form

and if $w$ is the vertex of weight 0 . (Remark: Because $w$ is uniquely determined by $\mathcal{L}$, we will often use the symbol $\mathcal{L}$ to represent the pair $(\mathcal{L}, w)$. For instance, we will write $(\mathcal{G}, v) \approx \mathcal{L}$, or we will say that the "weighted pair $(\mathcal{G}, v)$ is equivalent to a tree $\mathcal{L}$ satisfying the condition (0)", when we mean that $(\mathcal{G}, v) \approx(\mathcal{L}, w))$.

If $\mathcal{L}$ satisfies the condition (0), with notation as above, we define the transpose of $\mathcal{L}$ by


We also define $\mathcal{L}^{t^{i}}(i \geq 0)$ the obvious way: $\mathcal{L}^{t^{0}}=\mathcal{L}$ and $\mathcal{L}^{t^{i+1}}=\left(\mathcal{L}^{t^{i}}\right)^{t}$.
In the special case where either $m=0$ or $\omega_{i}=-2$ for all $i$, we say that $\mathcal{L}$ is degenerate.

We now state one of the main results of this paper. In condition (2) of the theorem, $M(\mathcal{L}) \cdot\binom{1}{1}$ is the product of the $2 \times 2$ matrix $M(\mathcal{L})$ (defined in 3.21 , below) with the column $\binom{1}{\nu}$. For the proof, see Theorem 3.32.

Theorem 3.12. Let $\left(\mathcal{G}_{0}, e_{0}\right)$ be a weighted pair and $\binom{p}{c} \in \mathcal{T},\binom{p}{c} \neq\binom{ 1}{1}$. Consider the blowing-up $\mathcal{G}_{0} \leftarrow \cdots \leftarrow \mathcal{G}_{n}$ of $\mathcal{G}_{0}$ at $e_{0}$ according to $\binom{p}{c}$ and let $e_{n}$ be the vertex of $\mathcal{G}_{n}$ created by $\mathcal{G}_{n-1} \leftarrow \mathcal{G}_{n}$. Then the following are equivalent:
(1) $\left(\mathcal{G}_{n}, e_{n}\right)$ is equivalent to a linear pair;
(2) $\left(\mathcal{G}_{0}, e_{0}\right)$ is equivalent to a tree $\mathcal{L}$ satisfying the condition (0) and $\binom{p}{c}=M(\mathcal{L}) \cdot\binom{1}{v}$ for some integer $v \geq 0$.
Moreover, suppose that these conditions are satisfied, let $\mathcal{G}_{n} \leftarrow \cdots \leftarrow \mathcal{G}_{n+\nu}$ be the blowing-up of $\mathcal{G}_{n}$ at $e_{n}$ according to the $2 \times v$ tableau $\left(\begin{array}{ccc}1 & \ldots & 1 \\ 1 & \cdots & 1\end{array}\right)$ and let $e_{n+v}$ be the vertex created by $\mathcal{G}_{n+\nu-1} \leftarrow \mathcal{G}_{n+\nu}$. Then $\left(\mathcal{G}_{n+v}, e_{n+v}\right)$ is equivalent to $\mathcal{L}^{t}$.

## Preliminaries to the proof of Theorem 3.12

Notation 3.13 (Blowing-up as an action). Define a binary operation on the set $\mathcal{T}$ of tableaux (see 3.5) by $\left(\begin{array}{ccc}p_{1} & \ldots & p_{k} \\ c_{1} & \ldots & c_{k}\end{array}\right)\left(\begin{array}{ccccc}p_{k+1} & \ldots & p_{\ell} \\ c_{k+1} & \ldots & c_{\ell}\end{array}\right)=\left(\begin{array}{ccccccccc}p_{1} & \ldots & p_{k} & p_{k+1} & \ldots & p_{\ell} \\ c_{1} & \ldots & c_{k} & c_{k+1} & \ldots & c_{\ell}\end{array}\right)$. Then $\mathcal{T}$ is actually the free monoid on the set of columns $\binom{p}{c}$ where $p \leq c$ are relatively prime positive integers.

Let $\left(\mathcal{G}_{0}, e_{0}\right)$ be a weighted pair and $T \in \mathcal{T}$ a tableau, consider the blowing-up $\mathcal{G}_{0} \leftarrow \cdots \leftarrow \mathcal{G}_{n}$ of $\mathcal{G}_{0}$ at $e_{0}$ according to $T$ and let $e_{n}$ be the vertex of $\mathcal{G}_{n}$ created by $\mathcal{G}_{n-1} \leftarrow \mathcal{G}_{n}$. Then we will write $\left(\mathcal{G}_{0}, e_{0}\right) T=\left(\mathcal{G}_{n}, e_{n}\right)$. Hence, blowing-up is a right action of $\mathcal{T}$ on the set of weighted pairs.
3.14. Let $(\mathcal{G}, v)$ and $\left(\mathcal{G}^{\prime}, v^{\prime}\right)$ be weighted pairs and $T \in \mathcal{T}$ a tableau. If $(\mathcal{G}, v) \approx$ $\left(\mathcal{G}^{\prime}, v^{\prime}\right)$, then $(\mathcal{G}, v) T \approx\left(\mathcal{G}^{\prime}, v^{\prime}\right) T$. Hence, blowing-up is also a right action of $\mathcal{T}$ on the set of equivalence classes of weighted pairs.
3.15. Let $\mathcal{G}$ be a weighted graph, $v_{1}, \ldots, v_{n}$ its vertices and $\omega_{i}$ the weight of $v_{i}$. Recall that one defines the determinant of $\mathcal{G}$ by $\operatorname{det}(\mathcal{G})=\operatorname{det}(-A)$, where $A$ denotes the "intersection matrix" of $\mathcal{G}$, i.e., the $n \times n$ matrix with entries $A_{i i}=\omega_{i}$ and, if $i \neq j, A_{i j}=1$ (resp. 0 ) if $v_{i}, v_{j}$ are neighbors (resp. are not neighbors). Then $\operatorname{det}(\mathcal{G})$ is independent of the ordering of the vertices and, if $\mathcal{G}$ and $\mathcal{G}^{\prime}$ are equivalent weighted graphs, $\operatorname{det}(\mathcal{G})=\operatorname{det}\left(\mathcal{G}^{\prime}\right)$.
3.16 ([11], A.14). Let $\mathcal{G}$ be a weighted tree, $v$ a vertex of weight $\Omega(v)$ in $\mathcal{G}$, $\mathcal{G}_{1}, \ldots, \mathcal{G}_{n}$ the branches of $\mathcal{G}$ at $v$ and $v_{i}$ the vertex of $\mathcal{G}_{i}$ which is a neighbor of $v$ in
$\mathcal{G}$. If $d_{i}=\operatorname{det} \mathcal{G}_{i}$ and $d_{i}^{\prime}=\operatorname{det}\left(\mathcal{G}_{i}-\left\{v_{i}\right\}\right)$, then

$$
\operatorname{det} \mathcal{G}=-\Omega(v) \cdot d_{1} \cdots d_{n}-\sum_{i=1}^{n} d_{1} \cdots d_{i-1} d_{i}^{\prime} d_{i+1} \cdots d_{n}
$$

Notation 3.17. Let $\mathcal{G}$ be a linear weighted tree $\dot{v_{1}} \cdots \dot{v}_{n}$ and $v=v_{1}$. Then the following abbreviation is very convenient:

$$
\operatorname{det}_{i}(\mathcal{G}, v)= \begin{cases}\operatorname{det} \mathcal{G}, & \text { if } i=0 \\ \operatorname{det}\left(\mathcal{G}-\left\{v_{1}, \ldots, v_{i}\right\}\right), & \text { if } 0<i<n \\ 1, & \text { if } i=n \\ 0, & \text { if } i>n\end{cases}
$$

3.18. Let the notation be as in 3.17 and let $\Omega\left(v_{j}\right)$ be the weight of $v_{j}$. Then, by 3.16 ,

$$
\operatorname{det}_{i}(\mathcal{G}, v)=-\Omega\left(v_{i+1}\right) \operatorname{det}_{i+1}(\mathcal{G}, v)-\operatorname{det}_{i+2}(\mathcal{G}, v) \quad(0 \leq i<n) .
$$

In particular, if $\Omega\left(v_{1}\right)=0$ then $\operatorname{det}_{2}(\mathcal{G}, v)=-\operatorname{det} \mathcal{G}$.
3.19. Recall that an admissible chain is a linear tree in which every weight is at most -2 . Using 3.18 , it is easy to see that every admissible chain has a strictly positive determinant; note, also, that $\varnothing$ is the only admissible chain with determinant 1. We also recall the following fact, which follows easily from 3.16 and 3.18:

Let $\mathcal{G}$ be a linear weighted tree and e a vertex of $\mathcal{G}$. Suppose that all weights in $\mathcal{G}$ are strictly negative, and that $e$ is the only vertex of weight -1 . Then:

- If e has two neighbors and both of them have weight -2 , then $\operatorname{det}(\mathcal{G}) \leq 0$.
- If $\operatorname{det}(\mathcal{G})>0$ then $\mathcal{G}$ contracts to an admissible chain.
- If $\operatorname{det}(\mathcal{G})=1$ then $\mathcal{G}$ contracts to $\boldsymbol{\emptyset}$.

Notation 3.20 ([11], A.16). Given relatively prime positive integers $a$ and $b$, define $\binom{a}{b}^{*}=\binom{x}{y}$, where $x$ and $y$ are the unique nonnegative integers which satisfy

$$
\left|\begin{array}{ll}
x & a \\
y & b
\end{array}\right|=-1 \quad \text { and } \quad x<a \text { or } y<b
$$

Definition 3.21. Given a weighted tree $\mathcal{L}$ satisfying the condition ( 0 ), we shall now define a $2 \times 2$ matrix $M(\mathcal{L})$, and a subset $\mathcal{T}(\mathcal{L})$ of $\mathcal{T}$. Let $v$ denote the vertex of weight 0 in $\mathcal{L}$ and consider the relatively prime integers $r_{0}>r_{1} \geq 0$ given by
$r_{0}=\operatorname{det}_{2}(\mathcal{L}, v)$ and $r_{1}=\operatorname{det}_{3}(\mathcal{L}, v)$ (see 3.17 and 3.20 for notations). Then define

$$
M(\mathcal{L})=\left(\begin{array}{cc}
x & r_{0}-r_{1} \\
y & r_{0}
\end{array}\right), \quad \text { where } \quad\binom{x}{y}=\binom{r_{0}-r_{1}}{r_{0}}^{*} .
$$

Note that $\mathcal{L}$ is completely determined by the second column of $M(\mathcal{L})$.
If $\mathcal{L}$ is nondegenerate (resp. degenerate) then, for each integer $v \geq 0$ (resp. $v>0$ ), let $T_{v}$ temporarily denote the $2 \times(\nu+1)$ matrix $\left(\begin{array}{ccc}p & 1 & \ldots \\ c & 1 & 1 \\ c\end{array}\right)$, where $\binom{p}{c}=M(\mathcal{L}) \cdot\binom{1}{v}$. Then $T_{\nu} \in \mathcal{T}$ and the first column of $T_{v}$ is not $\binom{1}{1}$. Define

$$
\mathcal{T}(\mathcal{L})=\left\{T_{v} \mid v \geq 0 \quad(\text { resp. } v>0)\right\} .
$$

Given $k \in \mathbb{N}$, we also define $\mathcal{T}_{k}(\mathcal{L})=\left\{T \in \mathcal{T} \left\lvert\, T\binom{1}{1}^{k} \in \mathcal{T}(\mathcal{L})\right.\right\}$ (so $\mathcal{T}_{0}(\mathcal{L})=\mathcal{T}(\mathcal{L})$ ). Here, $T\binom{1}{1}^{k}$ is a product in the monoid $\mathcal{T}$.

In the following statement, we abbreviate $\operatorname{det}\left(\stackrel{\omega_{1}}{\bullet} \cdots \xrightarrow{\omega_{m}}\right)$ by $\operatorname{det}\left(\omega_{1}, \ldots, \omega_{m}\right)$.
Lemma 3.22. Let $\omega_{1}, \ldots, \omega_{m} \leq-2$ be integers (where $m \geq 1$ ) and define

$$
\begin{aligned}
b & =\operatorname{det}\left(\omega_{1}, \ldots, \omega_{m}\right), \\
a & =\operatorname{det}\left(\omega_{2}, \ldots, \omega_{m}\right), \quad a^{\prime}=\operatorname{det}\left(\omega_{1}, \ldots, \omega_{m-1}\right) \quad\left(a=1=a^{\prime} \text { if } m=1\right) \\
a^{\prime \prime} & =\operatorname{det}\left(\omega_{2}, \ldots, \omega_{m-1}\right) \quad\left(a^{\prime \prime}=0 \text { if } m=1\right) .
\end{aligned}
$$

Then:
(1) $\binom{a}{b}^{*}=\binom{a^{\prime \prime}}{a^{\prime}},\binom{b-a}{b}^{*}=\binom{b-a-a^{\prime}+a^{\prime \prime}}{b-a^{\prime}}$ and $\binom{b-a^{\prime}}{b}^{*}=\binom{b-a-a^{\prime}+a^{\prime \prime}}{b-a}$;
(2) $\operatorname{det}\left(\omega_{1}, \ldots, \omega_{m-1}\right)=b-y$ and $\operatorname{det}\left(\omega_{2}, \ldots, \omega_{m-1}\right)=a+x-y$, where $\binom{x}{y}=\binom{b-a}{b}^{*}$.

Proof. Lemma 3.6 of [7] gives $a a^{\prime}-b a^{\prime \prime}=1,0 \leq a^{\prime \prime}<\min \left(a, a^{\prime}\right)$ and $\max \left(a, a^{\prime}\right)<b$; this gives $\binom{a}{b}^{*}=\binom{a^{\prime \prime}}{a^{\prime}}$ and it also follows that $\left|\begin{array}{c}b-a-a^{\prime}+a^{\prime \prime} \\ b-a^{\prime}\end{array} b_{b} \begin{array}{l}a \\ b\end{array}\right|=-1$. Since $b, b-a$ and $b-a^{\prime}$ are positive integers, $\left(b-a-a^{\prime}+a^{\prime \prime}\right) b=(b-a)\left(b-a^{\prime}\right)-1 \geq 0$, so $b-a-a^{\prime}+a^{\prime \prime} \geq 0$ and we obtain the second equality of assertion (1). The third equality follows from the second by symmetry, i.e., by interchanging $a$ and $a^{\prime}$. Assertion (2) follows from (1).

Lemma 3.23. Let $c>p>0$ be relatively prime integers, let $\mathcal{G}$ be the weighted graph which consists of a single vertex $v$ of weight zero, and let $\left(\mathcal{G}^{\prime}, v^{\prime}\right)=(\mathcal{G}, v)\binom{p}{c}$.

Then $\mathcal{G}^{\prime}$ has two branches at $v^{\prime}$, with determinants of subtrees as follows:
$\left(\mathcal{G}^{\prime}, v^{\prime}\right):$

where we define $\binom{p^{\prime \prime}}{p^{\prime}}=\binom{p}{c}^{*}$.
Moreover, if we let $\left(\mathcal{G}^{\prime \prime}, v^{\prime \prime}\right)=\left(\mathcal{G}^{\prime}, v^{\prime}\right)\binom{1}{N}($ with $N \geq 1)$ then the connected component of $\mathcal{G}^{\prime \prime} \backslash\left\{v^{\prime \prime}\right\}$ containing $v$ and $v^{\prime}$ is as follows:


Proof. This follows from Lemma 3.22 and from A.18.2 and A.18.3 of [11].

Lemma 3.24. If $\mathcal{L}$ is a tree satisfying the condition (0) then $M\left(\mathcal{L}^{t}\right)=M(\mathcal{L})^{t}$.
Proof. Use the notation of 3.11 for $\mathcal{L}$. If $m \geq 1$, the result follows from Lemma 3.22; if $m=0$, it is trivial.

We recall two properties of weigthed graphs ${ }^{7}$ and state them in the language of weighted pairs. First, if a weighted graph is equivalent to a linear weighted graph, then it contracts to a linear weighted graph. For weighted pairs, one has:
3.25. If a weighted pair is equivalent to a linear pair, then it contracts to a linear pair.

For the second property, consider a sequence $\mathcal{G}_{0} \leftarrow \cdots \leftarrow \mathcal{G}_{n}$ of blowings-up of weighted graphs satisfying the two conditions of 3.7 and such that ( $\mathcal{G}_{n}, e_{n}$ ) contracts to a linear pair; then $\left(\mathcal{G}_{j}, e_{j}\right)$ contracts to a linear pair, for every $j<n$ satisfying:

[^5]$\mathcal{G}_{j} \leftarrow \mathcal{G}_{j+1}$ is a blowing-up at a vertex. This can be conveniently expressed as part (1) of:
3.26. Let $(\mathcal{G}, v)$ be a weighted pair.
(1) If there exists $T \in \mathcal{T}$ such that $(\mathcal{G}, v) T$ contracts to a linear pair, then $(\mathcal{G}, v)$ contracts to a linear pair.
(2) $(\mathcal{G}, v)$ contracts to a linear pair if and only if $(\mathcal{G}, v)\binom{1}{1}$ contracts to a linear pair.

Definition 3.27. A weighted pair $(\mathcal{L}, w)$ satisfies the condition (+) if $\mathcal{L}$ is a tree of the form

$$
\stackrel{\omega_{1}}{\bullet} \quad \cdots \stackrel{\omega_{m}}{\bullet} \quad\left(\alpha>0, m \geq 0, \omega_{i} \in \mathbb{Z}, \omega_{i} \leq-2\right)
$$

and if $w$ is the vertex of positive weight.
3.28. If $(\mathcal{G}, v) \approx(\mathcal{L}, w)$ are weighted pairs and $(\mathcal{L}, w)$ satisfies the condition $(+)$, then $(\mathcal{G}, v) \geq(\mathcal{L}, w)$.

The (straightforward) proof of 3.28 is left to the reader. Statement 3.29 follows immediately from 3.28 :
3.29. Let $\mathcal{C}$ be an equivalence class of weighted pairs. Then:
(1) The class $\mathcal{C}$ contains a pair satisfying the condition (0) if and only if it contains one satisfying the condition (+).
(2) The class $\mathcal{C}$ contains at most one pair satisfying the condition (0) and at most one pair satisfying the condition (+).

Definition 3.30. A weighted pair $(\mathcal{G}, v)$ is contractible if it is equivalent to some pair $(\mathcal{L}, w)$ which satisfies the condition ( 0 ). Then $\mathcal{L}$ is unique (by 3.29) and we say that $(\mathcal{G}, v)$ is of type $\mathcal{L}$.

Lemma 3.31. If $(\mathcal{G}, e)$ is any weighted pair then at most one integer $r \geq 0$ is such that $(\mathcal{G}, e)\binom{1}{1}^{r}$ is contractible.

Proof. It suffices to show that, if $r>0$ and $(\mathcal{L}, w)$ satisfies $(+)$, then $\left(\mathcal{L}^{\prime}, w^{\prime}\right)=$ $(\mathcal{L}, w)\binom{1}{1}^{r}$ does not contract to a pair which satisfies the condition (+). But this is trivial.

## Main result

Except for notation, the following is exactly the same as Theorem 3.12.

Theorem 3.32. Let $(\mathcal{G}, e)$ be a weighted pair and $\binom{p}{c} \in \mathcal{T},\binom{p}{c} \neq\binom{ 1}{1}$. Then the following are equivalent:
(1) $(\mathcal{G}, e)\binom{p}{c}$ contracts to some linear pair;
(2) $(\mathcal{G}, e)$ is contractible, and $\binom{p}{c}=M(\mathcal{L}) \cdot\binom{1}{v}$ for some integer $v \geq 0$, where $\mathcal{L}$ is the type of $(\mathcal{G}, e)$.
Moreover, if these conditions are satisfied then $(\mathcal{G}, e)\binom{p}{c}\binom{1}{1}^{\nu}$ is equivalent to $\mathcal{L}^{t}$.
Proof. If condition (1) holds then, by $3.26,(\mathcal{G}, e)$ contracts to a linear pair $(\mathcal{M}, e)$ which has no vertex of weight -1 , except possibly $e$ :


We claim that $(\mathcal{M}, e)$ satisfies the condition $(+)$. Indeed, let $\left(\mathcal{M}^{\prime}, e^{\prime}\right)=(\mathcal{M}, e)\binom{p}{c}$ :


Because $p \neq c$, we know that $\mathcal{M}^{\prime}$ has two branches $\mathcal{C}$ and $\mathcal{C}^{\prime}$ at $e^{\prime}$; let $\mathcal{C}$ be the one which contains $e$. Since ( $\mathcal{M}^{\prime}, e^{\prime}$ ) contracts to a linear pair (by 3.14 and 3.25), and since every vertex of $\mathcal{C}^{\prime}$ has weight at most $-2, \mathcal{C}$ must be equivalent to the empty graph. This implies that all $\alpha_{i}$ are negative, so $\alpha_{i} \leq-2$ for all $i$. Another consequence is that $\alpha^{\prime}=-1$, because all vertices of $\mathcal{C}$ other than $e$ have weight at most -2 . We also have $\alpha^{\prime} \leq \alpha-2$, because $\binom{p}{c}$ produces at least two blowings-up, the first blowingup is at the vertex $e$ and the second one is at an edge incident to $e$. We conclude that $\alpha>0$, so $\mathcal{M}$ satisfies the condition ( + ).

In view of 3.29 , condition (1) implies that $(\mathcal{G}, e)$ is equivalent to a pair $(\mathcal{L}, e)$ satisfying the condition (0); thus, in order to prove that conditions (1) and (2) are equivalent, we may assume that $(\mathcal{G}, e) \approx(\mathcal{L}, e)$ :

$$
(\mathcal{G}, e) \approx(\mathcal{L}, e)=\stackrel{0}{\stackrel{0}{e}} \quad-\cdot \quad \stackrel{\omega_{1}}{\bullet} \cdots \stackrel{\omega_{m}}{\longrightarrow} \quad\left(m \geq 0 \text { and } \omega_{i} \leq-2\right) .
$$

Consider the integers $r_{0}=\operatorname{det}_{2}(\mathcal{L}, e)$ and $r_{1}=\operatorname{det}_{3}(\mathcal{L}, e)$ used in the definition of $M(\mathcal{L})$. Write $\left(\mathcal{L}^{\prime}, e^{\prime}\right)=(\mathcal{L}, e)\binom{p}{c}$, then:

where the numbers under the braces represent the determinants of the indicated subtrees of $\mathcal{L}^{\prime}$ (in particular, the $p$ and $c$ in the left part of the picture are obtained from 3.23). Note that the extra assumptions made for drawing this picture (e.g., $m>1$ ) have no effect on the following argument.

Let $\mathcal{B}$ and $\mathcal{B}^{\prime}$ be the two branches of $\mathcal{L}^{\prime}$ at $e^{\prime}$, where $\mathcal{B}$ is the one containing $e$; then, by $3.16, \operatorname{det} \mathcal{B}=c r_{0}-p r_{0}-c r_{1}$. Now condition (1) of the Theorem is equivalent to $\operatorname{det} \mathcal{B}=1$, hence to

$$
\left|\begin{array}{cc}
p & r_{0}-r_{1}  \tag{22}\\
c & r_{0}
\end{array}\right|=-1 .
$$

This holds if and only if $\binom{p}{c}-v\binom{r_{0}-r_{1}}{r_{0}}=\binom{r_{0}-r_{1}}{r_{0}}^{*}$ for some $v \geq 0$, and this is equivalent to condition (2) of the Theorem. Hence, conditions (1) and (2) of the Theorem are equivalent.

Assume that conditions (1) and (2) hold; continuing with the same notation, there remains to prove that $\left(\mathcal{L}^{\prime}, e^{\prime}\right)\binom{1}{1}^{\nu}$ is equivalent to $\mathcal{L}^{t}$.

The pair ( $\mathcal{L}^{\prime}, e^{\prime}$ ) contracts to a linear pair ( $\mathcal{L}^{\prime \prime}, e^{\prime}$ ):
where $\mathcal{L}^{\prime \prime}-\left\{e^{\prime}\right\}$ is identical to the branch $\mathcal{B}^{\prime}$ of $\mathcal{L}^{\prime}$ at $e^{\prime}$. Since $\mathcal{B}^{\prime}$ is nonempty and every weight in it is at most -2 , we have

$$
\begin{equation*}
c=\operatorname{det}_{1}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right)>\operatorname{det}_{2}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right)>\operatorname{det}_{3}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right) \geq 0 \tag{24}
\end{equation*}
$$

where the equality comes from the fact that $\operatorname{det} \mathcal{B}^{\prime}=c$ (see the picture at line (21)).
By 3.18, $\operatorname{det} \mathcal{L}=-\operatorname{det}_{2}(\mathcal{L}, e)=-r_{0}$, so

$$
\begin{equation*}
\operatorname{det} \mathcal{H}=-r_{0}, \quad \text { for each weighted graph } \mathcal{H} \text { equivalent to } \mathcal{L} . \tag{25}
\end{equation*}
$$

So we have $-r_{0}=\operatorname{det} \mathcal{L}^{\prime \prime}=-\alpha \operatorname{det}_{1}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right)-\operatorname{det}_{2}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right)$, i.e.,

$$
\begin{equation*}
r_{0}=\alpha c+\operatorname{det}_{2}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right) \tag{26}
\end{equation*}
$$

We have to separate two cases.
CASE $\alpha>0$. Since $\operatorname{det}_{2}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right)>0$, we have $c<r_{0}$ by equation (26). From this and equation (22), we deduce that $\binom{p}{c}=\binom{r_{0}-r_{1}}{r_{0}}^{*}$ and hence that $\nu=0$. So we have to show that $\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right)$ is equivalent to $\mathcal{L}^{t}$.

Observe that $\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right) \approx\left(\mathcal{L}^{(3)}, e^{\prime}\right)$, where

satisfies the condition (0) $\mathcal{L}^{(3)}$ is obtained from $\mathcal{L}^{\prime \prime}$ by blowing-up $\alpha$ times). We have $\operatorname{det}_{2}\left(\mathcal{L}^{(3)}, e^{\prime}\right)=-\operatorname{det} \mathcal{L}^{(3)}$ by 3.18 , and since $\operatorname{det} \mathcal{L}^{(3)}=-r_{0}$ by equation (25),

$$
\begin{equation*}
\operatorname{det}_{2}\left(\mathcal{L}^{(3)}, e^{\prime}\right)=r_{0} \tag{27}
\end{equation*}
$$

We have $\operatorname{det}_{1}\left(\mathcal{L}^{(3)}, e^{\prime}\right)=1 \cdot \operatorname{det}_{2}\left(\mathcal{L}^{(3)}, e^{\prime}\right)-\operatorname{det}_{3}\left(\mathcal{L}^{(3)}, e^{\prime}\right)=r_{0}-\operatorname{det}_{3}\left(\mathcal{L}^{(3)}, e^{\prime}\right)$. Since the weighted trees $\mathcal{L}^{(3)}-\left\{e^{\prime}\right\}$ and $\mathcal{L}^{\prime \prime}-\left\{e^{\prime}\right\}$ are equivalent, we also have $\operatorname{det}_{1}\left(\mathcal{L}^{(3)}, e^{\prime}\right)=$ $\operatorname{det}_{1}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right)=c$. Thus

$$
\begin{equation*}
\operatorname{det}_{3}\left(\mathcal{L}^{(3)}, e^{\prime}\right)=r_{0}-c . \tag{28}
\end{equation*}
$$

From equations (27) and (28), we obtain that the second column of $M\left(\mathcal{L}^{(3)}\right)$ is $\binom{c}{r_{0}}$, which is identical to the second column of $M(\mathcal{L})^{t}=M\left(\mathcal{L}^{t}\right)$ (by 3.24). Hence, $\mathcal{L}^{(3)}=$ $\mathcal{L}^{t}$, i.e., $(\mathcal{G}, e)\binom{p}{c}\binom{1}{1}^{\nu}$ is equivalent to $\mathcal{L}^{t}$ in this case.

CASE $\alpha=0$. This time we have $\operatorname{det}_{2}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right)=r_{0}$ by (26), and $c=\operatorname{det}_{1}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right)=$ $q \operatorname{det}_{2}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right)-\operatorname{det}_{3}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right) ;$ so, if we write $\rho=\operatorname{det}_{3}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right)$,

$$
c=q r_{0}-\rho \quad\left(q \geq 2, \quad 0 \leq \rho<r_{0}\right) .
$$

In particular we have $c>r_{0}$, so $v>0$. Since $\alpha=0$ and $v>0$, the pair $\left(\mathcal{L}^{(3)}, e^{\prime \prime}\right)=$ $\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right)\binom{1}{1}^{\nu}$ looks like this:

and $\left(\mathcal{L}^{(3)}, e^{\prime \prime}\right) \geq\left(\mathcal{L}^{(4)}, e^{\prime \prime}\right)$, where

$$
\left(\mathcal{L}^{(4)}, e^{\prime \prime}\right): \quad \cdots \xrightarrow[u]{\stackrel{v-q}{u}} \quad{ }_{e^{\prime \prime}}^{0}
$$

On the other hand, if we write $M(\mathcal{L})=\left(\begin{array}{cc}x \\ y & r_{0}-r_{1} \\ r_{0}\end{array}\right)$ then by definition of $v$ we have $c=$ $v r_{0}+y=(v+1) r_{0}-\left(r_{0}-y\right)$ with $v+1 \geq 2$ and $0 \leq r_{0}-y<r_{0}$. So $q=v+1$ and $\rho=r_{0}-y$. In particular, $\left(\mathcal{L}^{(4)}, e^{\prime \prime}\right)$ satisfies the condition (0).

Since $\mathcal{L}^{(4)}-\left\{e^{\prime \prime}, u\right\}$ is identical to $\mathcal{L}^{\prime \prime}-\left\{e^{\prime}, u\right\}$, we have

$$
\operatorname{det}_{i}\left(\mathcal{L}^{(4)}, e^{\prime \prime}\right)=\operatorname{det}_{i}\left(\mathcal{L}^{\prime \prime}, e^{\prime}\right) \quad(\text { all } i \geq 2)
$$

so, in particular,

$$
\operatorname{det}_{2}\left(\mathcal{L}^{(4)}, e^{\prime \prime}\right)=r_{0} \quad \text { and } \quad \operatorname{det}_{3}\left(\mathcal{L}^{(4)}, e^{\prime \prime}\right)=\rho=r_{0}-y
$$

So the second column of $M\left(\mathcal{L}^{(4)}\right)$ is $\binom{y}{r_{0}}$, which is identical to the second column of $M(\mathcal{L})^{t}=M\left(\mathcal{L}^{t}\right)$. Hence, $\mathcal{L}^{(4)}=\mathcal{L}^{t}$, i.e., $(\mathcal{G}, e)\binom{p}{c}\binom{1}{1}^{v}$ is equivalent to $\mathcal{L}^{t}$.

We now give some corollaries to Theorem 3.32. See Definition 3.21 for $\mathcal{T}(\mathcal{L})$ and $\mathcal{T}_{k}(\mathcal{L})$.

Corollary 3.33. Let $(\mathcal{G}, e)$ be a weighted pair and $T=\binom{p}{c}\binom{1}{1}^{r} \in \mathcal{T}$, where $r \geq 0$ and $\binom{p}{c} \neq\binom{ 1}{1}$. Then the following are equivalent:
(1) $(\mathcal{G}, e) T$ is contractible;
(2) $(\mathcal{G}, e)$ is contractible of type $\mathcal{L}$ and $T \in \mathcal{T}(\mathcal{L})$.

Moreover, if these conditions hold then $(\mathcal{G}, e) T$ is equivalent to $\mathcal{L}^{t}$.
Remark. By definition of $\mathcal{T}(\mathcal{L})$ (3.21), $r=0$ can occur if and only if $\mathcal{L}$ is nondegenerate.

Proof. Suppose that condition (1) holds. Then, in particular, $(\mathcal{G}, e)\binom{p}{c}\binom{1}{1}^{r}$ contracts to a linear pair, so $(\mathcal{G}, e)\binom{p}{c}$ contracts to a linear pair by 3.26. By Theorem 3.32, we obtain that $(\mathcal{G}, e)$ is equivalent to a tree $\mathcal{L}$ which satisfies the condition ( 0 ), that $\binom{p}{c}=M(\mathcal{L}) \cdot\binom{1}{v}$ for some $v \geq 0$ and that $(\mathcal{G}, e)\binom{p}{c}\binom{1}{1}^{v}$ is equivalent to $\mathcal{L}^{t}$. By Lemma 3.31 we get $r=v$; hence, $T \in \mathcal{T}(\mathcal{L})$ and $(\mathcal{G}, e) T$ is equivalent to $\mathcal{L}^{t}$.

The proof that (2) implies (1) is left to the reader.
Notation 3.34. $\mathcal{T}^{\#}=\mathcal{T} \backslash\binom{1}{1} \mathcal{T}$
Hence, $\mathcal{T}^{\#}$ contains the empty tableau, and all nonempty tableaux whose first column is not $\binom{1}{1}$. This is a submonoid of $\mathcal{T}$ with the property that each $T \in \mathcal{T}^{\#}$ has a unique factorization into irreducibles: $T=T_{r} \cdots T_{1}, T_{i} \in \mathcal{T}^{\#}, h\left(T_{i}\right)=1$. Note also that $\mathcal{T}_{k}(\mathcal{L}) \subset \mathcal{T}^{\#}$, for any $k \in \mathbb{N}$ and $\mathcal{L}$ satisfying the condition (0).

Iterating Corollary 3.33 gives:
Corollary 3.35. Let $A$ be a weighted pair and $T \in \mathcal{T}^{\#}$. If $T=T_{r} \cdots T_{1}$ is the irreducible factorization of $T$ in $\mathcal{T}^{\#}$, then the following are equivalent:
(1) $A T$ is contractible of type $\mathcal{L}$,
(2) $A$ is contractible of type $\mathcal{L}^{t^{r}}$, and $T_{i} \in \mathcal{T}\left(\mathcal{L}^{t^{i}}\right)$ for all $i=1, \ldots, r$.

Definition 3.36. A weighted pair $P=(\mathcal{G}, v)$ is pseudo-linear if $v$ has exactly one neighbor $v^{\prime}$ in $\mathcal{G}$ and the connected component $\Gamma$ of $\mathcal{G}$ containing $v$ has the form:

$$
\Gamma: \begin{array}{lllll}
0 & x & \omega_{1} & \cdots & \omega_{n} \\
v & v^{\prime} & \bullet & \cdots & \bullet
\end{array} \quad\left(n \geq 0, x, \omega_{i} \in \mathbb{Z}, x \leq-1, \omega_{i} \leq-2\right)
$$

We also say that $P$ is pseudo-linear of type $(-1-x, \mathcal{L})$, where $\mathcal{L}$ is the weighted pair (satisfying the condition (0)) obtained from the above picture by replacing the " $x$ " by a " -1 ". If $P$ is pseudo-linear, with $\Gamma$ as in the above picture, let $P^{t}$ be the weighted pair obtained from $P$ by changing the weights in $\Gamma$, so as to obtain

and by leaving the other connected components unchanged. Note that if $P$ is pseudolinear of type $(k, \mathcal{L})$, then $P^{t}$ is pseudo-linear of type $\left(k, \mathcal{L}^{t}\right)$.

If $P$ is pseudo-linear of type $(k, \mathcal{L})$ then any weighted pair equivalent to $P$ is said to be pseudo-contractible, or pseudo-contractible of type $(k, \mathcal{L})$ (note that the type is well-defined). If a weighted pair $P$ is pseudo-contractible of type $(k, \mathcal{L})$, then $k \in \mathbb{N}$; if $k>0$, then $P\binom{1}{1}$ is pseudo-contractible of type $(k-1, \mathcal{L})$.

As an immediate consequence of Corollary 3.35, we have:

Corollary 3.37. Let $P$ be a weighted pair and $T \in \mathcal{T}^{\#} \backslash\{\mathbf{1}\}$. If $T=T_{r} \cdots T_{1}$ is the irreducible factorization of $T$ in $\mathcal{T}^{\#}$, then the following are equivalent:
(1) $P T$ is pseudo-contractible of type $(k, \mathcal{L})$,
(2) $P$ is pseudo-contractible of type $\left(0, \mathcal{L}^{t^{r}}\right)$ and

$$
T_{i} \in \begin{cases}\mathcal{T}_{k}\left(\mathcal{L}^{t}\right), & \text { if } i=1 \\ \mathcal{T}\left(\mathcal{L}^{i}\right), & \text { for all } i=2, \ldots, r\end{cases}
$$

## 4. Description of the set $\Pi(S, \mu)$

4.1. Let $f: X \rightarrow Y$ be a birational morphism of smooth complete surfaces and $D$ a nonzero divisor of $Y$ with strong normal crossings. We say that $\overline{\mathrm{HN}}(f, D)$ is defined if the following condition holds:

If center $(f) \cap \operatorname{supp}(D)$ is nonempty then it is a single point $P, P$ belongs to exactly one component $Z$ of $D$ and $f^{-1}(P)$ contains exactly one $(-1)$-curve. If this condition holds, then we define $\overline{\mathrm{HN}}(f, D) \in \mathcal{T}$ as follows.

- If center $(f) \cap \operatorname{supp}(D)=\emptyset$, define $\overline{\mathrm{HN}}(f, D)=\mathbf{1}$ (the empty tableau).
- If center $(f) \cap \operatorname{supp}(D)=\{P\}$, let $E \subset X$ denote the unique $(-1)$-curve in $f^{-1}(P)$ and choose local coordinates $(\xi, \eta)$ of $Y$ at $P$ such that $\xi$ is a local equation of $Z$. Consider the finite Hamburger-Noether tableau

$$
\mathrm{HN}=\mathrm{HN}(E ; \xi, \eta)=\mathrm{HN}(f ; \xi, \eta)=\left(\begin{array}{cccc}
p_{1} & \cdots & p_{k-1} & p_{k} \\
c_{1} & \cdots & c_{k-1} & c_{k} \\
\alpha_{1} & \cdots & \alpha_{k-1} & \alpha_{k}
\end{array}\right)
$$

as defined in the appendix of [11]. Then HN uniquely determines a tableau $\overline{\mathrm{HN}} \in \mathcal{T}$ (3.6) and $\overline{\mathrm{HN}}$ is independent of the choice of $(\xi, \eta)$. We define $\overline{\mathrm{HN}}(f, D)=\overline{\mathrm{HN}}$. Note that $\overline{\mathrm{HN}}(f, D)=\overline{\mathrm{HN}}(f, Z)$.

We state two important properties of $\overline{\mathrm{HN}}(f, D)$. Recall that $\mathcal{G}(D, Y)$ denotes the dual graph of $D$ in $Y$.
(1) Consider the weighted pairs $R=(\mathcal{G}(D, Y), Z)$ and $R^{\prime}=\left(\mathcal{G}\left(f^{-1}(D), X\right), E\right)$, where we regard $f^{-1}(D)$ as a reduced effective divisor (with strong normal crossings) of $X$,
and where

$$
E= \begin{cases}f^{-1}(Z), & \text { if } \operatorname{center}(f) \cap \operatorname{supp}(D)=\emptyset \\ \text { the }(-1) \text {-curve in } f^{-1}(P), & \text { if } \operatorname{center}(f) \cap \operatorname{supp}(D)=\{P\} .\end{cases}
$$

Then $R^{\prime}=R \overline{\mathrm{HN}}(f, D)$.
(2) (a) Suppose that $f$ factors as $X \xrightarrow{\alpha} S \xrightarrow{\beta} Y$ and that center $(\alpha) \cap \beta^{-1}(D)$, if nonempty, belongs to a unique component of $\beta^{-1}(D)$. Then

$$
\overline{\mathrm{HN}}(\beta \circ \alpha, D)=\overline{\mathrm{HN}}(\beta, D) \overline{\mathrm{HN}}\left(\alpha, \beta^{-1}(D)\right) .
$$

(b) Conversely, given any factorization $\overline{\mathrm{HN}}(f, D)=B A$ with $A, B \in \mathcal{T}$, there is an essentially unique way to factor $f$ as $X \xrightarrow{\alpha} S \xrightarrow{\beta} Y$ such that center $(\alpha) \cap$ $\beta^{-1}(D)$, if nonempty, belongs to a unique component of $\beta^{-1}(D), \overline{\mathrm{HN}}(\beta, D)=B$ and $\overline{\mathrm{HN}}\left(\alpha, \beta^{-1}(D)\right)=A$.

Definition 4.2. Suppose that $X$ is a complete normal rational surface and that $I=(S, \mu)$ is an $X$-immersion. Let $\Gamma, Z$ and $\Sigma$ be the main component, 0 -component and section of $I$, respectively, and let $D$ be the divisor of $S$, with strong normal crossings, with support $S \backslash \operatorname{dom} \mu$. We define two weighted pairs determined by $I$ :

$$
\mathcal{P}(I)=(\mathcal{G}(D, S), Z) \text { and } \mathcal{L}(I)=(\mathcal{G}(\Gamma, S), Z) .
$$

Note that $\mathcal{L}(I)$ is the connected component of $\mathcal{P}(I)$ containing the distinguished vertex; also, if $\Sigma^{2}<0$ then $\mathcal{P}(I)$ and $\mathcal{L}(I)$ are pseudo-linear of type $\left(-1-\Sigma^{2}, \mathcal{L}\right)$ (Definition 3.36), where $\mathcal{L}$ is the weighted pair obtained from $\mathcal{L}(I)$ by replacing the weight of $\Sigma$ by " -1 ".
4.3. Suppose that $X$ is a complete normal rational surface and that $I=(S, \mu)$ is an $X$-immersion. Let $\Gamma, Z$ and $\Sigma$ be the main component, 0 -component and section of $I$, respectively.

Given any morphism $\pi: \tilde{S} \rightarrow S$ satisfying conditions (1) and (2) of 2.9 , the tableau $\overline{\mathrm{HN}}(\pi, \Gamma)$ (see 4.1) contains enough information to decide whether $\pi$ also satisfies conditions (3) and (4) of 2.9. Indeed,

$$
\mathcal{L}(I) \overline{\mathrm{HN}}(\pi, \Gamma)=\left(\mathcal{G}\left(\pi^{-1}(\Gamma), \tilde{S}\right), E\right),
$$

and we immediately see that conditions (3) and (4) are equivalent to
( $3^{\prime}$ ) $\overline{\mathrm{HN}}(\pi, \Gamma)=\binom{1}{1}^{r}\binom{p}{c}$, for some $r \geq 0$ and $\binom{p}{c} \neq\binom{ 1}{1}$;
(4) $\mathcal{L}(I)\binom{1}{1}^{r}\binom{p}{c}$ contracts to some linear pair.

Hence, Theorem 3.32 allows us to give a complete description of $\Pi_{P}(S, \mu)$ (see 2.9). In particular, if condition (4') holds then $\mathcal{L}(I)\binom{1}{1}^{r}$ is contractible. Note that this
implies $r=-1-\Sigma^{2}$; it also follows that there exists an $X$-immersion $\left(S_{0}, \mu_{0}\right)$ in standard form, obtained from $(S, \mu)$ via a sequence of $r$ elementary transformations of sprouting type, and there exists $\pi_{0} \in \Pi\left(S_{0}, \mu_{0}\right)$, such that $(S, \mu) * \pi=\left(S_{0}, \mu_{0}\right) * \pi_{0}$.

Corollary 4.4. Suppose that $X$ is a complete normal rational surface and that $(S, \mu)$ is an $X$-immersion; let $\Gamma, Z$ and $\Sigma$ be the main component, 0 -component and section of $(S, \mu)$ respectively. Let $P \in Z \backslash \Sigma$.
(1) $\Pi_{P}(S, \mu) \neq \emptyset$ if and only if $\Sigma^{2}<0$.
(2) If $\pi \in \Pi(S, \mu)$ then there exists an $X$-immersion $\left(S_{0}, \mu_{0}\right) \geq(S, \mu)$ in standard form satisfying $(S, \mu) * \pi=\left(S_{0}, \mu_{0}\right) * \pi_{0}$ for some $\pi_{0} \in \Pi\left(S_{0}, \mu_{0}\right)$.
(3) Suppose that $I=(S, \mu)$ is in standard form. If $\mathcal{L}(I)$ is non-degenerate (resp. degenerate) then

$$
\left.\Pi_{P}(S, \mu)=\left\{\pi_{v} \mid v \in \mathbb{N} \text { (resp. } v \in \mathbb{N} \backslash\{0\}\right)\right\}
$$

where $\pi_{v}: \tilde{S}_{v} \rightarrow S$ is the unique birational morphism which is centered at $P$, whose exceptional locus has a unique ( -1 -component, and which satisfies

$$
\overline{\mathrm{HN}}\left(\pi_{\nu}, \Gamma\right)=M(\mathcal{L}(I)) \cdot\binom{1}{v} .
$$

Moreover, (i) the section $\Sigma_{v}$ of the $X$-immersion $(S, \mu) * \pi_{v}$ satisfies $\Sigma_{v}^{2}=-1-v$; and (ii) if $I^{\prime}$ is an $X$-immersion equivalent to $(S, \mu) * \pi_{\nu}$ and in standard form then $\mathcal{L}\left(I^{\prime}\right)$ is the transpose of $\mathcal{L}(I)$.
(4) Suppose that $(S, \mu)$ is in standard form. Given any $\pi, \pi^{\prime} \in \Pi_{P}(S, \mu)$, the $X$ immersions $(S, \mu) * \pi$ and $(S, \mu) * \pi^{\prime}$ are equivalent.

Proof. Assertion (3) is a direct consequence of Theorem 3.32. Assertion (2) was pointed out in 4.3 and the "only if" part of (1) follows from (2). Observe that (3) implies, in particular, that $\Pi_{P}(I)$ is nonempty whenever $I$ is in standard form; the "if" part of (1) easily follows from this and part (1) of Lemma 2.12.

In view of (3), it suffices to prove (4) in the special case where $\pi=\pi_{\nu}$ and $\pi^{\prime}=\pi_{\nu+1}$. Write $J=I * \pi_{\nu}$ and consider the $X$-immersion $J^{-}$obtained from $J$ by performing one elementary transformation of subdivisional type. By part (2) of Lemma 2.12, there exists $\pi^{\prime \prime} \in \Pi_{P}(I)$ such that $I * \pi^{\prime \prime}=J^{-}$. By (3), the section of $J$ has self-intersection $-1-v$, so that of $J^{-}$has self-intersection $-1-(\nu+1)$. Again by part (3), we have $\pi^{\prime \prime}=\pi_{n}$ for some $n$ and the section of $J^{-}=I * \pi^{\prime \prime}$ has self-intersection $-1-n$. Hence, $n=v+1$. Consequently, $I * \pi_{\nu+1}=I * \pi^{\prime \prime}=J^{-}$is equivalent to $J=I * \pi_{\nu}$.

## 5. Description of affine rulings by discrete data

See 5.3, below, for an introduction to this section.
5.1. Let $X$ be a surface satisfying $(\dagger)$ and $\Lambda$ an affine ruling of $X$.

Let $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$ be as in Proposition 1.5 and recall that $X^{\prime}=X_{s} \backslash \operatorname{Bs}(\Lambda) \subset X$ is embedded in $\tilde{X}$ as the complement of a divisor $D$ with strong normal crossings, and that exactly one component $H$ of $D$ is a section of $\tilde{\Lambda}$ (see 1.8). Let $m=-H^{2} \geq 1$. In view of 1.2 and convention 1.9 , there is a unique birational morphism $\pi: \tilde{X} \rightarrow \mathbb{F}_{m}$ which contracts each reducible member of $\tilde{\Lambda}$ to a 0 -curve and whose exceptional locus is disjoint from $H$.

Assume that $\Lambda_{*}$ is nonempty. Each choice of an element $F \in \Lambda_{*}$ determines a factorization

$$
\tilde{X} \xrightarrow{\pi_{2}} S \xrightarrow{\pi_{1}} \mathbb{F}_{m},
$$

of $\pi$, where:

- $\pi_{2}$ is the contraction of $\tilde{F}$ to a 0 -curve, where $\tilde{F} \in \tilde{\Lambda}_{*}$ is the image of $F \in$ $\Lambda_{*}$ under the bijection $\Lambda \rightarrow \tilde{\Lambda}$ of 1.6. (Note that $\pi_{2}$ is the identity map when $\tilde{F}$ is irreducible, or equivalently when $F$ is a reduced member of $\Lambda$.)
- If some member of $\tilde{\Lambda} \backslash\{\tilde{F}\}$ is reducible then it is unique (by definition of $\Lambda_{*}$ ) and we denote it by $\tilde{G}$; if there is no such member, let $\tilde{G}$ be any member of $\tilde{\Lambda} \backslash\{\tilde{F}\}$. Let $\pi_{1}$ be the contraction of $\tilde{G}$ (or rather, of $\pi_{2}(\tilde{G})$ ) to a 0 -curve. (This gives $\pi_{1}=$ id when every member of $\tilde{\Lambda} \backslash\{\tilde{F}\}$ is irreducible.)
We will sometimes refer to $\pi_{1}$ and $\pi_{2}$ as the pair of morphisms determined by ( $\Lambda, F$ ).
Regard $D_{2}=\pi_{2}(\operatorname{supp}(\tilde{F}+D))$ as a reduced effective divisor of $S$ (with strong normal crossings) and observe that it has no branching component (because $F \in \Lambda_{*}$ ); note that $Z_{2}=\pi_{2}(\operatorname{supp} \tilde{F})$ and $\Sigma_{2}=\pi_{2}(H)$ are respectively a 0 -component and a $(-m)$-component of $D_{2}$. The curve $\Sigma_{1}=\pi_{1}\left(\Sigma_{2}\right)=\pi_{1}\left(\pi_{2}(H)\right) \subset \mathbb{F}_{m}$ is the negative section of the standard ruling of $\mathbb{F}_{m}$; also, $Z_{1}=\pi_{1}\left(\pi_{2}(\operatorname{supp} \tilde{G})\right)$ and $\pi_{1}\left(Z_{2}\right)$ are distinct members of that ruling and $D_{1}=Z_{1}+\Sigma_{1}+\pi_{1}\left(Z_{2}\right)$ is a divisor of $\mathbb{F}_{m}$ with strong normal crossings.

For each $i \in\{1,2\}$, the exceptional locus of $\pi_{i}$ contains at most one ( -1 )-curve and, if $\pi_{i} \neq \mathrm{id}$, the center $P_{i}$ of $\pi_{i}$ is a single point and belongs to $Z_{i} \backslash \operatorname{supp}\left(D_{i}-Z_{i}\right)$. Thus we may consider $T_{i}=\overline{\mathrm{HN}}\left(\pi_{i}, D_{i}\right) \in \mathcal{T}$, as defined in 4.1. In this way, $(\Lambda, F)$ determines a unique triple $\left(m, T_{1}, T_{2}\right) \in \mathbb{Z}^{+} \times \mathcal{T} \times \mathcal{T}$, which we call the discrete part of $(\Lambda, F)$ (or of $(X, \Lambda, F)$ ).

Definition 5.2. (1) Given a triple $(X, \Lambda, F)$, where $X$ is a surface satisfying ( $\dagger$ ), $\Lambda$ is an affine ruling of $X$ and $F \in \Lambda_{*}$, the discrete part of $(X, \Lambda, F)$ is the triple $\left(m, T_{1}, T_{2}\right)$ defined in 5.1. The notation is $\operatorname{disc}(X, \Lambda, F)=\left(m, T_{1}, T_{2}\right)$. We sometimes call $\left(m, T_{1}, T_{2}\right)$ the discrete part of $(\Lambda, F)$.
(2) Given a surface $X$ satisfying ( $\dagger$ ), $\mathbb{T}(X)$ denotes the set of $\operatorname{disc}(X, \Lambda, F)$ such that $\Lambda$ is an affine ruling of $X$ and $F \in \Lambda_{*} ; \mathbb{T}_{0}(X) \subseteq \mathbb{T}(X)$ denotes the set of $\operatorname{disc}(X, \Lambda, F)$ such that $\Lambda$ is a basic affine ruling of $X$ and $F \in \Lambda_{*}$.
5.3. Let $X$ be a surface satisfying $(\dagger)$. Can a description of the set $\mathbb{T}(X)$ be regarded as a solution to Problem 1 for $X$ ? There are two difficulties:
(D1) $X$ may admit affine rulings $\Lambda$ such that $\Lambda_{*}=\emptyset$, and $\mathbb{T}(X)$ contains no information about such rulings.
Note that if we assume that all basic affine rulings of $X$ are known then, in particular, all $\Lambda$ satisfying $\Lambda_{*}=\emptyset$ are known (see 2.5); this is why (D1) did not cause problems in sections 2 and 4 . In this section, however, (D1) can only be resolved by assuming that $X$ satisfies ( $\ddagger$ ), in which case all $\Lambda$ satisfy $\Lambda_{*} \neq \emptyset$ (by 2.5 again).
(D2) Given $\tau=\left(m, T_{1}, T_{2}\right) \in \mathbb{T}(X)$, we need a method for constructing all ( $\left.\Lambda, F\right)$ (on $X)$ such that $\operatorname{disc}(X, \Lambda, F)=\tau$.
Paragraph 5.29, below, describes a method for constructing all ( $X^{\prime}, \Lambda^{\prime}, F^{\prime}$ ) such that $\operatorname{disc}\left(X^{\prime}, \Lambda^{\prime}, F^{\prime}\right)=\tau$, and this is good enough for (D2) if one can prove that all such $X^{\prime}$ are isomorphic to $X$. Thus Corollary 5.32 implies that, if $X$ satisfies ( $\ddagger$ ), describing $\mathbb{T}(X)$ does solve Problem 1 for $X$.

Some of the results of this section (5.17, 5.22, 5.23,5.39) describe $\mathbb{T}(X)$ in terms of $\mathbb{T}_{0}(X)$, or in terms of the subset min $\mathbb{T}(X)$ of $\mathbb{T}_{0}(X)$. So, given $X$ satisfying $(\ddagger)$,
this section reduces Problem 1 to the problem of describing $\mathbb{T}_{0}(X)$ or $\min \mathbb{T}(X)$.
Definition 5.4. (1) Let $n \geq 1$. By a weighted $n$-tuple, we mean an ordered $n$ tuple $S=\left(\mathcal{G}, v_{1}, \ldots, v_{n-1}\right)$ where $\mathcal{G}$ is a weighted graph and $v_{1}, \ldots, v_{n-1}$ are distinct vertices of $\mathcal{G}$ (when $n=1, S$ is a weighted graph; when $n=2$, it is a weighted pair 3.8).
(2) Let $S$ be a weighted $n$-tuple, with $n \geq 2$. Given $T \in \mathcal{T}$, we define a weighted $n$ tuple $S T$ and a weighted ( $n-1$ )-tuple $S \ominus T$ as follows. Write $S=\left(\mathcal{G}, v_{1}, \ldots, v_{n-1}\right)$ and let $\left(\mathcal{G}^{\prime}, e\right)$ denote the weighted pair $\left(\mathcal{G}, v_{1}\right) T$. Note that $v_{2}, \ldots, v_{n-1}$ can be regarded as vertices of $\mathcal{G}^{\prime} \backslash\{e\}$. Then we define

$$
S T=\left(\mathcal{G}^{\prime}, e, v_{2}, \ldots, v_{n-1}\right) \quad \text { and } \quad S \ominus T=\left(\mathcal{G}^{\prime} \backslash\{e\}, v_{2}, \ldots, v_{n-1}\right) .
$$

Remarks. Let $S=\left(\mathcal{G}, v_{1}, \ldots, v_{n-1}\right)$ be a weighted $n$-tuple.
(1) When $n=2$, the definition of $S T$ given in 5.4 agrees with the one given in section 3.
(2) The above definition gives $S \mathbf{1}=S$ and $S \ominus \mathbf{1}=\left(\mathcal{G} \backslash\left\{v_{1}\right\}, v_{2}, \ldots, v_{n-1}\right)$ (where $\mathbf{1}$ is the empty tableau). So, given $T, T^{\prime} \in \mathcal{T}, S \ominus T=S T \ominus \mathbf{1}$ and $S \ominus\left(T T^{\prime}\right)=(S T) \ominus T^{\prime}$. (3) Let $P$ and $P^{\prime}$ be weighted pairs and $T \in \mathcal{T}$. If $P \approx P^{\prime}$ then, by 3.14, $P \ominus T \sim$ $P^{\prime} \ominus T$ (where " $\approx$ " (resp. " $\sim$ ") means equivalence of weighted pairs (resp. weighted graphs)).

Notation 5.5. Given $x \in \mathbb{Z}$, let $\mathcal{G}_{(x)}$ denote the weighted triple $\left(\mathcal{G}, v_{1}, v_{2}\right)$, where $\mathcal{G}$ is the weighted graph

5.6. Consider the weighted pair $S$ consisting of a single vertex of weight zero. For any $T \in \mathcal{T}$, the condition
$S \ominus T$ has no branch point and every weight in it is strictly less than -1
holds if and only if one of the following holds:
(1) $T=\mathbf{1}$;
(2) $T=\binom{p}{c}$, where $\binom{p}{c} \neq\binom{ 1}{1}$;
(3) $T=\left(\begin{array}{cc}p & 1 \\ c & N\end{array}\right)$, where $\binom{p}{c} \neq\binom{ 1}{1}$ and $N \geq 1$.
5.7. Let $x$ be a negative integer and $T_{1}, T_{2} \in \mathcal{T}$.
(1) The condition

$$
\mathcal{G}_{(x)} \ominus T_{1} \text { is pseudo-linear }
$$

holds if and only if $T_{1}$ satisfies one of conditions (1-3) of 5.6. Moreover, if $\mathcal{G}_{(x)} \ominus T_{1}$ is pseudo-linear then it has at most two connected components and the one which does not contain the distinguished vertex is an admissible chain.
(2) The condition
$\left(\mathcal{G}_{(x)} \ominus T_{1}\right) \ominus T_{2}$ has no branch point and every weight in it, except possibly that of the middle vertex of $\mathcal{G}_{(x)}$, is strictly less than -1 , holds if and only if each of $T_{1}, T_{2}$ satisfies one of conditions (1-3) of 5.6.

Proof. To prove (1), write $\mathcal{G}_{(x)}=\left(\mathcal{G}, v_{1}, v_{2}\right)$ and consider the weighted pair $S=\left(\left\{v_{1}\right\}, v_{1}\right)$ (a single vertex of weight 0 ). We may regard $S \ominus T$ as the graph obtained from the weighted pair $P=\mathcal{G}_{(x)} \ominus T$ by deleting the distinguished vertex (i.e., $v_{2}$ ), its unique neighbor and all edges incident to these two vertices. Note that $P$ has at most two connected components, say $\mathcal{L}$ and $A$, where $\mathcal{L}$ contains the distinguished vertex and $A$ is a (possibly empty) admissible chain. If $P$ is pseudo-linear, $S \ominus T$ has no branch point (otherwise $\mathcal{L}$ would have one) and every weight in $S \ominus T$ is strictly less than -1 ; thus (by 5.6) $T$ satisfies one of conditions (1-3) of 5.6. The converse is equally trivial, as is assertion (2).

Notation 5.8. Given $T \in \mathcal{T}$ satisfying one of the conditions (1-3) of 5.6 , we define $\check{T} \in \mathcal{T}$ as follows:

$$
\check{T}=\left\{\begin{array}{ll}
\mathbf{1}, & \text { if } T \text { satisfies 5.6.1; } \\
\binom{p^{\prime}}{c} & \text { if } T \text { satisfies 5.6.2, where } p^{\prime} \text { is given by }\binom{p^{\prime \prime}}{p^{\prime}}=\binom{p}{c}^{*} \text { (see 3.20); } \\
c & N
\end{array}\right), \text { if } T \text { satisfies 5.6.3. } . ~ l
$$

Note that if $T$ satisfies condition 5.6.i (where $i \in\{1,2,3\}$ ) then so does $\check{T}$. If $s$ is a positive integer, write $T^{(s)}=\left(T^{(v(s-1))}\right)^{2}$, where $T^{(0)}=T$. Note that $T^{(2)}=T$.

Lemma 5.9. Let $\left(m, T_{1}, T_{2}\right)$ be the discrete part of $(X, \Lambda, F)$, where $X$ is a surface satisfying $(\dagger), \Lambda$ is an affine ruling of $X$ and $F \in \Lambda_{*}$.
(1) The weighted pair $P=\mathcal{G}_{(-m)} \ominus T_{1}$ is isomorphic to $\mathcal{P}(I)$ (see 4.2), where I is the distinguished element of the equivalence class of $X$-immersions determining ( $\Lambda, F$ ). In particular, $P$ is pseudo-linear of type $(m-1, \mathcal{L})$ for some $\mathcal{L}$; moreover, $P$ has at most one connected component $A$ other than the one containing the distinguished vertex, and $A$ is an admissible chain.
(2) There is an isomorphism of weighted graphs $\left(\mathcal{G}_{(-m)} \ominus T_{1}\right) \ominus T_{2} \rightarrow \mathcal{G}(\Lambda)$ which maps the middle vertex of $\mathcal{G}_{(-m)}$ to the vertex $H$ of $\mathcal{G}(\Lambda)$ (see 1.13 for the definition of $\mathcal{G}(\Lambda)$; $H$ denotes the unique component of $\tilde{X} \backslash X^{\prime}$ which is a section of $\left.\tilde{\Lambda}\right)$.

Proof. Let the notation ( $S, D_{2}$, etc.) be as in 5.1. By definition of $I$ (2.10), we have $I=(S, \mu)$ for some $\mu$ and, moreover, $S \backslash \operatorname{dom} \mu=\operatorname{supp}\left(D_{2}\right)$. So we have $\mathcal{P}(I)=$ $\left(\mathcal{G}\left(D_{2}, S\right), Z_{2}\right)$. For each $i=1,2$, let

$$
E_{i}= \begin{cases}\text { the unique }(-1) \text {-curve in } \pi_{i}^{-1}\left(P_{i}\right), & \text { if } \pi_{i} \neq \mathrm{id}, \\ Z_{i}, & \text { if } \pi_{i}=\mathrm{id}\end{cases}
$$

Consider the weighted triple $\left(\mathcal{G}\left(D_{1}, \mathbb{F}_{m}\right), Z_{1}, \pi_{1}\left(Z_{2}\right)\right)=\mathcal{G}_{(-m)}$. Since $\pi_{1}^{-1}\left(\operatorname{supp}\left(D_{1}\right)\right)=$ $\operatorname{supp}\left(D_{2}\right) \cup E_{1}$ and $E_{1}$ is not a component of $D_{2}$, we have $\mathcal{G}_{(-m)} \ominus T_{1}=\mathcal{P}(I)$ and (1) holds. Since $\pi_{2}^{-1}\left(D_{2}\right)=\operatorname{supp}(D) \cup E_{2}$ and $E_{2}$ is not a component of $D, \mathcal{P}(I) \ominus T_{2}=$ $\mathcal{G}(D, \tilde{X})=\mathcal{G}(\Lambda)$.

Notation 5.10. (1) Let $\mathbb{T}$ be the set of triples $\left(m, T_{1}, T_{2}\right) \in \mathbb{Z}^{+} \times \mathcal{T} \times \mathcal{T}$ such that $T_{2} \in \mathcal{T}^{\#}$ (Notation 3.34) and $T_{1}$ satisfies one of the conditions (1-3) of 5.6.
(2) Let $\mathbb{T}(\dagger)$ be the set of $\left(m, T_{1}, T_{2}\right) \in \mathbb{T}$ such that the intersection matrix (see 3.15) of the weighted graph $\left(\mathcal{G}_{(-m)} \ominus T_{1}\right) \ominus T_{2}$ is negative definite.

The following says, in particular, that $\mathbb{T}(X) \subseteq \mathbb{T}(\dagger)$ for each $X$ satisfying ( $\dagger$ ).
Lemma 5.11. Let $\left(m, T_{1}, T_{2}\right)$ be the discrete part of $(X, \Lambda, F)$, where $X$ is a surface satisfying ( $\dagger$ ), $\Lambda$ is an affine ruling of $X$ and $F \in \Lambda_{*}$. Then $\left(m, T_{1}, T_{2}\right) \in \mathbb{T}(\dagger)$ and the following are equivalent:
(1) $X$ satisfies ( $\ddagger$ ) and $\Lambda$ is basic;
(2) $T_{2}$ satisfies one of the conditions (1-3) of 5.6.

Proof. By 5.7 and part (1) of Lemma 5.9, $T_{1}$ satisfies one of the conditions (13) of 5.6 .

By part (2) of Lemma 5.9, every vertex of $\left(\mathcal{G}_{(-m)} \ominus T_{1}\right) \ominus T_{2}$, except possibly the middle vertex of $\mathcal{G}_{(-m)}$, has weight strictly less than -1 . Write $\mathcal{G}_{(-m)}=\left(\mathcal{G}, v_{1}, v_{2}\right)$ and note that the distinguished vertex $v_{2}$ of the weighted pair $\mathcal{G}_{(-m)} \ominus T_{1}$ has weight 0 . If $T_{2} \notin \mathcal{T}^{\#}$ then $T_{2} \neq \mathbf{1}$ and the first column of $T_{2}$ is $\binom{1}{1}$, so the weight of $v_{2}$ in $\left(\mathcal{G}_{(-m)} \ominus T_{1}\right) \ominus T_{2}$ is -1 , contradicting the above observation. Hence, $T_{2} \in \mathcal{T}^{\#}$.

Let $\hat{X} \rightarrow X$ be the minimal resolution of singularities of $X$ and let $\hat{E} \subset \hat{X}$ be the exceptional locus; since $X$ is normal, the divisor $\hat{E}$ has a negative definite intersection matrix; since $\left(\mathcal{G}_{(-m)} \ominus T_{1}\right) \ominus T_{2} \cong \mathcal{G}(\Lambda)$ by Lemma 5.9, and $\mathcal{G}(\Lambda)$ contracts to $\mathcal{G}(\hat{E}, \hat{X})$, we get $\left(m, T_{1}, T_{2}\right) \in \mathbb{T}(\dagger)$.

By 5.7, $\left(\mathcal{G}_{(-m)} \ominus T_{1}\right) \ominus T_{2}$ (hence $\left.\mathcal{G}(\Lambda)\right)$ has no branch point if and only if $T_{2}$ satisfies one of the conditions (1-3) of 5.6. Hence, (1) and (2) are equivalent.

Definition 5.12. Given $\left(n, T_{1}, T_{2}\right),\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right) \in \mathbb{T}$, write $\left(n, T_{1}, T_{2}\right) \equiv\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right)$ to indicate that

$$
\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) T_{2} \approx\left(\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}\right) T_{2}^{\prime}
$$

(equivalence of weighted pairs). Note that " $\equiv$ " is an equivalence relation on the set $\mathbb{T}$.
Theorem 5.13. Let $\tau, \tau^{\prime} \in \mathbb{T}$ be such that $\tau \equiv \tau^{\prime}$. Suppose that $\tau=$ $\operatorname{disc}(X, \Lambda, F)$, where $X$ is a surface satisfying $(\dagger), \Lambda$ is an affine ruling of $X$ and $F \in \Lambda_{*}$. Then there exist an affine ruling $\Lambda^{\prime}$ of $X$ and $F^{\prime} \in \Lambda_{*}^{\prime}$ such that $\tau^{\prime}=$ $\operatorname{disc}\left(X, \Lambda^{\prime}, F^{\prime}\right)$ and $\operatorname{supp}\left(F^{\prime}\right)=\operatorname{supp}(F)$.

In view of 2.14, the above result relates the viewpoint of this section with the operation "*" of sections 2 and 4. See also Proposition 5.23.

The proof requires 5.14 and 5.15:

Lemma 5.14. If $P$ is a pseudo-linear weighted pair then:
(1) At most one pair $(x, T) \in \mathbb{Z} \times \mathcal{T}$ satisfies $\mathcal{G}_{(x)} \ominus T=P$.
(2) Suppose that $\mathcal{G}_{(x)} \ominus T=P$. Then $T$ satisfies one of conditions (1-3) of 5.6 and $\mathcal{G}_{(x)} \ominus \check{T}=\left(\mathcal{G}_{(x)} \ominus T\right)^{t}=P^{t}$.

Proof. Write $P=(\mathcal{G}, v)$. We may assume that $P=\mathcal{G}_{(x)} \ominus T$ for some $(x, T)$. Then $v$ has a unique neighbor $v^{\prime}$ in $\mathcal{G}$, and the weight of $v^{\prime}$ is $x$; hence, $x$ is uniquely determined. By 5.7, $T$ satisfies one of conditions (1-3) of 5.6 (which proves part of assertion (2)). Note also that $\mathcal{G}$ has either one or two connected components; we say that $\mathcal{G}$ has two connected components, $\mathcal{L}$ and $A$, where $\mathcal{L}$ contains $v$ and $A$ is a (possibly empty) admissible chain. Moreover, $\mathcal{L}$ is as follows:

$$
\mathcal{L}: \quad \begin{array}{llll}
0 & x & \omega_{1} & \cdots \\
\stackrel{v^{\prime}}{\bullet} & y_{1} & \cdots & \omega_{n}
\end{array} \quad\left(n \geq 0, \omega_{i} \in \mathbb{Z}, \omega_{i} \leq-2\right) .
$$

We now show that $T$ is unique. If $n=0$ (resp. $n=1$ ) then $T$ must be $\mathbf{1}$ (resp. $\binom{1}{-\omega_{1}}$ ), so we may assume that $n \geq 2$. We consider two cases.

If $A$ is nonempty and contains a weight other than -2 , then $T=\binom{p}{c}$ for some $p, c$ satisfying $1<p<c$. Then Lemma 3.23 implies that $c=\operatorname{det}\left(\omega_{1}, \ldots, \omega_{n}\right)$ and $p=\operatorname{det}\left(\omega_{2}, \ldots, \omega_{n}\right)$, so $T$ is unique (notation as in Lemma 3.22).

Before treating the second case, let us observe that at most one $i \in\{1, \ldots, n\}$ can satisfy $\operatorname{det}\left(\omega_{1}, \ldots, \omega_{i-1}\right)=\operatorname{det}\left(\omega_{i+1}, \ldots, \omega_{n}\right)$, because the left-hand-side is a strictly increasing function of $i$, while the right-hand-side is strictly decreasing.

If $A$ is a chain of $N-1$ vertices of weight -2 (where $N \geq 1$ ), then $T=\left(\begin{array}{ll}p & 1 \\ c & N\end{array}\right)$, for some $\binom{p}{c} \neq\binom{ 1}{1}$. Consider the vertex $e$ (of weight -1 ) which is deleted from $\mathcal{G}_{(x)} T$ in order to define $\mathcal{G}_{(x)} \ominus T$; then $e$ has a unique neighbor among $\left\{y_{1}, \ldots, y_{n}\right\}$, say $y_{j}$. By Lemma 3.23 applied to the first column $\binom{p}{c}$ of $T$, we have $\operatorname{det}\left(\omega_{1}, \ldots, \omega_{j-1}\right)=$ $c=\operatorname{det}\left(\omega_{j+1}, \ldots, \omega_{n}\right)$ and $\operatorname{det}\left(\omega_{2}, \ldots, \omega_{j-1}\right)=p$. So $j$ must be the unique $i$ of the preceding paragraph; since $j$ is uniquely determined, so are $c=\operatorname{det}\left(\omega_{1}, \ldots, \omega_{j-1}\right)$ and $p=\operatorname{det}\left(\omega_{2}, \ldots, \omega_{j-1}\right)$. This proves assertion (1).

Assertion (2) is obtained from the following observation, which is a consequence of Lemma 3.23: Let $(\mathcal{G}, v)$ be the weighted pair consisting of a single vertex of weight 0 , let $\binom{p}{c} \in \mathcal{T},\binom{p}{c} \neq\binom{ 1}{1}$, and consider the weighted pair $\left(\mathcal{G}^{\prime}, v^{\prime}\right)=(\mathcal{G}, v)\binom{p}{c}$. Use the following notation for the weights in $\left(\mathcal{G}^{\prime}, v^{\prime}\right)$ :


If we define $p^{\prime}$ by $\binom{p^{\prime \prime}}{p^{\prime}}=\binom{p}{c}^{*}$, then:

$$
(\mathcal{G}, v)\binom{p^{\prime}}{c}: \quad \stackrel{\omega_{n}}{\bullet} \quad \cdots \quad \begin{array}{cccc}
\omega_{1} & -1 & a_{m} \\
\bullet & \cdots & \bullet & a_{1}
\end{array}
$$

On the other hand,

5.15. Let $S$ be a smooth complete surface, $D$ a divisor of $S$ with strong normal crossings and such that each component of $D$ is rational, and $\mathcal{G}=\mathcal{G}(D, S)$, the dual graph of $D$ in $S$. Let also $\mathcal{G}^{\prime}$ be a weighted graph.
(1) Suppose that $\mathcal{G}$ can be contracted to $\mathcal{G}^{\prime}$. Let $v_{1}, \ldots, v_{n}$ be the vertices of $\mathcal{G}$ which disappear in that process and let $D_{1}, \ldots, D_{n}$ be the corresponding components of $D$. Then there is an essentially unique birational morphism $\pi: S \rightarrow S^{\prime}$ whose exceptional locus is $D_{1} \cup \cdots \cup D_{n}$ (where $S^{\prime}$ is a smooth complete surface). Then the divisor $D^{\prime}=$ $\pi(D)$ of $S^{\prime}$ with strong normal crossings has dual graph $\mathcal{G}^{\prime}$.
(2) Suppose that $\mathcal{G}^{\prime}$ can be contracted to $\mathcal{G}$. Then there exists a (not necessarely
unique) birational morphism $\pi: S^{\prime} \rightarrow S$ (where $S^{\prime}$ is a smooth complete surface) such that the divisor $D^{\prime}=\pi^{-1}(D)$ of $S^{\prime}$, with strong normal crossings, satisfies $\mathcal{G}\left(D^{\prime}, S^{\prime}\right)=$ $\mathcal{G}^{\prime}$. The exceptional locus of $\pi$ consists of the components of $D^{\prime}$ corresponding to the vertices of $\mathcal{G}^{\prime}$ which disappear in the contraction to $\mathcal{G}$.
In case (1) (resp. (2)), we call $\pi$ simply "the (resp. a) birational morphism corresponding to $\mathcal{G} \geq \mathcal{G}^{\prime}$ (resp. $\mathcal{G}^{\prime} \geq \mathcal{G}$ )"; it is tacitely assumed that the above conditions are satisfied. Similar remarks hold if both $\mathcal{G}$ and $\mathcal{G}^{\prime}$ are weighted pairs; in this case, we have the additional information that $\pi$ does not shrink the curve which corresponds to the distinguished vertex.

Proof of Theorem 5.13. Write $\tau=\left(n, T_{1}, T_{2}\right)$ and $\tau^{\prime}=\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right)$.
Consider $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$ and recall that $X \backslash(\operatorname{Sing} X \cup \operatorname{Bs} \Lambda)$ is embedded in $\tilde{X}$ as the complement of a divisor $D$ with strong normal crossings. Also, consider the curve $C=C_{\tilde{F}}$ in $\tilde{X}$ (notation as in 1.8), where $\tilde{F} \in \tilde{\Lambda}$ corresponds to $F \in \Lambda$ via the bijection $\Lambda \rightarrow \tilde{\Lambda}$ (Definition 1.6). Then we have $\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) T_{2}=(\mathcal{G}(D+C, \tilde{X}), C)$.

Since $\tau \equiv \tau^{\prime}$, we have $(\mathcal{G}(D+C, \tilde{X}), C) \approx\left(\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}\right) T_{2}^{\prime}$; this can be written as $(\mathcal{G}(D+C, \tilde{X}), C) \leq \mathcal{P} \geq\left(\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}\right) T_{2}^{\prime}$, where $\mathcal{P}$ is a weighted pair and the inequalities indicate contractions of weighted pairs. In view of 5.15 we may consider a diagram $\tilde{X} \stackrel{\omega}{\leftarrow} \Omega \xrightarrow{\omega^{\prime}} \tilde{Y}$, where $\Omega$ and $\tilde{Y}$ are smooth complete surfaces and $\omega$ and $\omega^{\prime}$ are birational morphisms corresponding to $(\mathcal{G}(D+C, \tilde{X}), C) \leq \mathcal{P}$ and $\mathcal{P} \geq\left(\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}\right) T_{2}^{\prime}$ respectively. Define $D^{\prime}=\omega^{\prime}\left(\omega^{-1} D\right)$ and $C^{\prime}=\omega^{\prime}(\tilde{C})$, where $\tilde{C} \subset \Omega$ is the strict transform of $C$. Then $D^{\prime}+C^{\prime}$ is a divisor of $\tilde{Y}$ with strong normal crossings and $\left(\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}\right) T_{2}^{\prime}=\left(\mathcal{G}\left(D^{\prime}+C^{\prime}, \tilde{Y}\right), C^{\prime}\right)$. Moreover, $\tilde{X} \leftarrow \Omega \rightarrow \tilde{Y}$ gives an isomorphism $\tilde{Y} \backslash \operatorname{supp} D^{\prime} \rightarrow \tilde{X} \backslash \operatorname{supp} D$ which maps $C^{\prime}$ onto $C$.

Since $\left(\mathcal{G}\left(D^{\prime}+C^{\prime}, \tilde{Y}\right), C^{\prime}\right)=\left(\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}\right) T_{2}^{\prime}$, the weighted graph $\mathcal{G}\left(D^{\prime}+C^{\prime}, \tilde{Y}\right)$ contracts to the underlying weighted graph of $\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}$; by 5.15 again, this contraction gives a birational morphism $\pi_{2}^{\prime}: \tilde{Y} \rightarrow S^{\prime}$, where $S^{\prime}$ is smooth. Consider the divisor $M^{\prime}=\pi_{2}^{\prime}\left(D^{\prime}+C^{\prime}\right)$ of $S^{\prime}$ (with strong normal crossings); then $\mathcal{G}\left(M^{\prime}, S^{\prime}\right)$ is the underlying weighted graph of $\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}$, i.e., $\left(\mathcal{G}\left(M^{\prime}, S^{\prime}\right), Z^{\prime}\right)=\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}$ for some component $Z^{\prime}$ of $M^{\prime}$. By 5.7, $\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}$ is pseudo-linear, has at most two connected componenents, and the connected component which does not contain the distinguished vertex is an admissible chain. Thus we obtain an $X$-immersion ( $S^{\prime}, \mu^{\prime}$ ), where $\mu^{\prime}: S^{\prime} \backslash \operatorname{supp}\left(M^{\prime}\right) \rightarrow X_{s} \backslash \operatorname{supp}(F)$ is the isomorphism determined by $\pi_{2}^{\prime}, \omega^{\prime}, \omega$ and $\tilde{X} \backslash \operatorname{supp}(D+C) \cong X_{s} \backslash \operatorname{supp}(F)$. The $X$-immersion $\left(S^{\prime}, \mu^{\prime}\right)$ determines an affine ruling $\Lambda^{\prime}$ of $X$ and an element $F^{\prime}$ of $\Lambda_{*}^{\prime}$ satisfying $\operatorname{supp}\left(F^{\prime}\right)=\operatorname{supp}(F)$ (because the image of $\mu^{\prime}$ is $X_{s} \backslash \operatorname{supp}(F)$ ). Also, $Z^{\prime}$ is the 0 -component of ( $S^{\prime}, \mu^{\prime}$ ) and let $\Sigma^{\prime}$ be the section of ( $S^{\prime}, \mu^{\prime}$ ). Since the unique neighbor of the distinguished vertex of $\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}$ has weight $-m$, we have $\left(\Sigma^{\prime}\right)^{2}=-m$. Note that $\left(X, \Lambda^{\prime}\right)^{\sim}=\left(\tilde{Y},\left|Z^{\prime}\right|^{\sim}\right)$, where $\left|Z^{\prime}\right|^{\sim}$ denotes the strict transform of $\left|Z^{\prime}\right|$. Also, $\overline{\mathrm{HN}}\left(\pi_{2}^{\prime}, M^{\prime}\right)$ is defined and is equal to $T_{2}^{\prime}$.

Let $\pi_{1}^{\prime}: S^{\prime} \rightarrow \mathbb{F}_{m}$ be the unique birational morphism which contracts each reducible member of $\left|Z^{\prime}\right|$ to a 0 -curve and whose exceptional locus is disjoint from $\Sigma^{\prime}$
(see 1.2).
We claim that $N\left(\pi_{1}^{\prime}\right)=m^{\prime}-2$, where $m^{\prime}$ is the number of irreducible components of $M^{\prime}$ and $N\left(\pi_{1}^{\prime}\right)$ is (as usual) the number of irreducible components in the exceptional locus of $\pi_{1}^{\prime}$. To see this, let $\pi$ be the composition $\Omega \xrightarrow{\omega} \tilde{X} \xrightarrow{\pi_{2}} S \xrightarrow{\pi_{1}} \mathbb{F}_{n}$, where $\pi_{1}$ and $\pi_{2}$ are the two morphisms determined by $(X, \Lambda, F)$ as in 5.1. Then $N(\pi)=|\mathcal{P}|-2$ and consequently $N\left(\pi^{\prime}\right)=|\mathcal{P}|-2$, where $\pi^{\prime}$ is the composition $\Omega \xrightarrow{\omega^{\prime}} \tilde{Y} \xrightarrow{\pi_{2}^{\prime}} S^{\prime} \xrightarrow{\pi_{1}^{\prime}} \mathbb{F}_{m}$. Since $\omega^{\prime}$ and $\pi_{2}^{\prime}$ correspond to contractions of graphs, it follows that $N\left(\pi_{1}^{\prime}\right)=\left|\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}\right|-2$, from which the claim follows.

Note that $\pi_{1}^{\prime}\left(\Sigma^{\prime}\right)$ is the negative section of the standard ruling $\Lambda_{m}$ of $\mathbb{F}_{m}$ and that $\pi_{1}^{\prime}\left(Z^{\prime}\right)$ is a member of $\Lambda_{m}$. We claim that, for some member $L$ of $\Lambda_{m}$ other than $\pi_{1}^{\prime}\left(Z^{\prime}\right)$,
(30) $\pi_{1}^{\prime}\left(M^{\prime}\right) \subseteq L+\pi_{1}^{\prime}\left(\Sigma^{\prime}\right)+\pi_{1}^{\prime}\left(Z^{\prime}\right), \quad \operatorname{center}\left(\pi_{1}^{\prime}\right) \subset L \backslash \pi_{1}^{\prime}\left(\Sigma^{\prime}\right) \quad$ and $\overline{\mathrm{HN}}\left(\pi_{1}^{\prime}, L\right)=T_{1}^{\prime}$.

The verification of this splits into two cases.
If $m^{\prime}=2$ then $\left|\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}\right|=2$, so $T_{1}^{\prime}$ is the empty tableau. On the other hand, $N\left(\pi_{1}^{\prime}\right)=m^{\prime}-2=0$ implies that $\pi_{1}^{\prime}$ is an isomorphism. If we let $L$ be any member of $\Lambda_{m}$ other than $\pi_{1}^{\prime}\left(Z^{\prime}\right)$, then (30) holds.

If $m^{\prime}>2$ then $\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}$ has more than 2 vertices, so the middle vertex of $\mathcal{G}_{(-m)}$ has exactly two neighbors, i.e., $\Sigma^{\prime}$ has two neighbors $Z^{\prime}$ and $Z^{\prime \prime}$ in $M^{\prime}$. Let $M_{1}^{\prime}, \ldots, M_{m^{\prime}-3}^{\prime}$ be the components of $M^{\prime}$ other than $Z^{\prime}, \Sigma^{\prime}$ and $Z^{\prime \prime}$; since each $M_{i}^{\prime}$ is contained in a member of the ruling $\left|Z^{\prime}\right|$ (because $M_{i}^{\prime} \cap Z^{\prime}=\emptyset$ ) and is disjoint from $\Sigma^{\prime}$, each $M_{i}^{\prime}$ is shrunk by $\pi_{1}^{\prime}$. Since $N\left(\pi_{1}^{\prime}\right)=m^{\prime}-2$, the exceptional locus of $\pi_{1}^{\prime}$ is $E \cup M_{1}^{\prime} \cup \cdots \cup M_{m^{\prime}-3}^{\prime}$, for some curve $E$ not contained in $M_{1}^{\prime} \cup \cdots \cup M_{m^{\prime}-3}^{\prime}$. Since $\left(M_{i}^{\prime}\right)^{2}<-1$ for all $i, E$ is the unique ( -1 )-component of the exceptional locus of $\pi_{1}^{\prime}$. Let $L=\pi_{1}^{\prime}\left(Z^{\prime \prime}\right)$ and note that $L$ is a member of $\Lambda_{m}$ other than $\pi_{1}^{\prime}\left(Z^{\prime}\right)$ and satisfying:

$$
\pi_{1}^{\prime}\left(M^{\prime}\right)=L+\pi_{1}^{\prime}\left(\Sigma^{\prime}\right)+\pi_{1}^{\prime}\left(Z^{\prime}\right) \quad \text { and } \quad \operatorname{center}\left(\pi_{1}^{\prime}\right) \subset L \backslash \pi_{1}^{\prime}\left(\Sigma^{\prime}\right) .
$$

Using $\mathcal{G}_{(-m)}=\left(\mathcal{G}\left(\pi_{1}^{\prime}\left(M^{\prime}\right), \mathbb{F}_{m}\right), L, \pi_{1}^{\prime}\left(Z^{\prime}\right)\right)$, we obtain

$$
\begin{aligned}
\mathcal{G}_{(-m)} \ominus \overline{\operatorname{HN}}\left(\pi_{1}^{\prime}, L\right) & =\left(\mathcal{G}\left(\pi_{1}^{\prime}\left(M^{\prime}\right), \mathbb{F}_{m}\right), L, \pi_{1}^{\prime}\left(Z^{\prime}\right)\right) \ominus \overline{\operatorname{HN}}\left(\pi_{1}^{\prime}, L\right) \\
& =\left(\mathcal{G}\left(M^{\prime}, S^{\prime}\right), Z^{\prime}\right)=\mathcal{G}_{(-m)} \ominus T_{1}^{\prime},
\end{aligned}
$$

so $\overline{\mathrm{HN}}\left(\pi_{1}^{\prime}, L\right)=T_{1}^{\prime}$ by Lemma 5.14 and (30) holds in this case too.
We conclude that $\pi_{1}^{\prime}$ and $\pi_{2}^{\prime}$ are the two morphisms determined by ( $X, \Lambda^{\prime}, F^{\prime}$ ) (5.1) and that $\operatorname{disc}\left(X, \Lambda^{\prime}, F^{\prime}\right)=\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right)$.

The following order relation is useful for describing $\mathbb{T}(X)$ explicitely:
Definition 5.16. We define a transitive relation $>$ on the set $\mathbb{T}$ by declaring that $\left(n, T_{1}, T_{2}\right)>\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right)$ if $n=1$ and the following holds:

Let $\mathcal{L}$ be the weighted pair such that $\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}$ is pseudo-linear of type $(m-1, \mathcal{L})$. Then there exist an integer $s \geq 1$ and tableaux $X_{1}, \ldots, X_{s}$ such that $T_{1}=\left(T_{1}^{\prime}\right)^{(s)}$, $T_{2}=X_{s} \cdots X_{1} T_{2}^{\prime}$ and $X_{i} \in \mathcal{T}_{k_{i}}\left(\mathcal{L}^{t^{i}}\right)$, where $k_{1}=m-1$ and $k_{i}=0$ for all $i>1$. We define the symbols $<, \geq$ and $\leq$ the usual way. (See Definition 3.21 for $\mathcal{T}_{k}(\mathcal{L})$.)

Remark. There cannot be an infinite descending sequence $\tau_{1}>\tau_{2}>\cdots$ in $\mathbb{T}$. Indeed, if $\left(n, T_{1}, T_{2}\right)>\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right)$ then the number of columns of $T_{2}^{\prime}$ is strictly less than that of $T_{2}$.

Note that, if $\tau^{\prime} \in \mathbb{T}$ is given, we may explicitely describe all $\tau \in \mathbb{T}$ satisfying $\tau>\tau^{\prime}$ (this is done in 5.39, below). Thus the following (see also Corollary 5.22) describes $\mathbb{T}(X)$ in terms of $\mathbb{T}_{0}(X)$ :

Corollary 5.17. Let $X$ be a surface satisfying ( $\dagger$ ).
(1) If $\tau, \tau^{\prime} \in \mathbb{T}$ are such that $\tau>\tau^{\prime}$, then $\tau \in \mathbb{T}(X) \Longleftrightarrow \tau^{\prime} \in \mathbb{T}(X)$.
(2) Given any $\tau \in \mathbb{T}(X) \backslash \mathbb{T}_{0}(X)$, there exists $\tau^{\prime} \in \mathbb{T}_{0}(X)$ such that $\tau>\tau^{\prime}$.

Although 5.17 is essentially a corollary of Theorem 5.13, its proof requires some preparation.

Lemma 5.18. Let $\left(n, T_{1}, T_{2}\right),\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right) \in \mathbb{T}$.
(1) If $\left(n, T_{1}, T_{2}\right)>\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right)$ then $\left(n, T_{1}, T_{2}\right) \equiv\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right)$.
(2) If $\left(n, T_{1}, T_{2}\right) \equiv\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right)$ then $\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) \ominus T_{2}$ and $\left(\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}\right) \ominus T_{2}^{\prime}$ are equivalent weighted graphs and consequently

$$
\left(n, T_{1}, T_{2}\right) \in \mathbb{T}(\dagger) \Longleftrightarrow\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right) \in \mathbb{T}(\dagger)
$$

Proof. Suppose that $\left(n, T_{1}, T_{2}\right)>\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right)$. Recall that $n=1$ and let the notations $\left(\mathcal{L}, s, X_{1}, \ldots, X_{s}, k_{i}\right)$ be as in Definition 5.16. Since $\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}$ is pseudolinear of type $(m-1, \mathcal{L})$, it follows that $\mathcal{G}_{(-1)} \ominus T_{1}^{\prime}$ is pseudo-linear of type $(0, \mathcal{L})$, so $\mathcal{G}_{(-1)} \ominus\left(T_{1}^{\prime}\right)^{(s s)}=\left(\mathcal{G}_{(-1)} \ominus T_{1}^{\prime}\right)^{t^{s}}$ is pseudo-linear of type $\left(0, \mathcal{L}^{t^{s}}\right)$. By Corollary 3.37, $\left(\mathcal{G}_{(-1)} \ominus\left(T_{1}^{\prime}\right)^{(s)}\right) X_{s} \cdots X_{1}$ is pseudo-contractible of type $(m-1, \mathcal{L})$.

Let $P \asymp P^{\prime}$ mean, temporarily, that the weighted pairs $P$ and $P^{\prime}$ are the same outside of the connected component containing the distinguished vertex. Then

$$
\mathcal{G}_{(-m)} \ominus T_{1}^{\prime} \asymp \mathcal{G}_{(-1)} \ominus T_{1}^{\prime} \asymp \mathcal{G}_{(-1)} \ominus\left(T_{1}^{\prime}\right)^{(s)} \asymp\left(\mathcal{G}_{(-1)} \ominus\left(T_{1}^{\prime}\right)^{(s s)}\right) X_{s} \cdots X_{1},
$$

where the second " $\nearrow$ " follows from part (2) of 5.14 and the other two are obvious. Thus the weighted pairs $\left(\mathcal{G}_{(-1)} \ominus\left(T_{1}^{\prime}\right)^{(s s)}\right) X_{s} \cdots X_{1}$ and $\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}$ are pseudocontractible of the same type, and identical outside of the connected component containing the distinguished vertex; it follows that

$$
\begin{equation*}
\left(\mathcal{G}_{(-1)} \ominus\left(T_{1}^{\prime}\right)^{(w s)}\right) X_{s} \cdots X_{1} \approx \mathcal{G}_{(-m)} \ominus T_{1}^{\prime} \tag{31}
\end{equation*}
$$

and consequently

$$
\left(\mathcal{G}_{(-1)} \ominus T_{1}\right) T_{2}=\left(\mathcal{G}_{(-1)} \ominus\left(T_{1}^{\prime}\right)^{(v s)}\right) X_{s} \cdots X_{1} T_{2}^{\prime} \approx\left(\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}\right) T_{2}^{\prime},
$$

which proves assertion (1). If $\left(n, T_{1}, T_{2}\right) \equiv\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right)$ then $\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) T_{2} \approx\left(\mathcal{G}_{(-m)} \ominus\right.$ $\left.T_{1}^{\prime}\right) T_{2}^{\prime}$, so

$$
\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) \ominus T_{2}=\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) T_{2} \ominus \mathbf{1} \sim\left(\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}\right) T_{2}^{\prime} \ominus \mathbf{1}=\left(\mathcal{G}_{(-m)} \ominus T_{1}^{\prime}\right) \ominus T_{2}^{\prime}
$$

and (2) holds.
Lemma 5.19. Let $\tau=\left(n, T_{1}, T_{2}\right) \in \mathbb{T}(\dagger)$. If the weighted graph $\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) \ominus T_{2}$ can be contracted to a weighted graph whose number of branch points is strictly less than that of $\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) \ominus T_{2}$, then $\tau>\tau^{\prime}$ for some $\tau^{\prime} \in \mathbb{T}(\dagger)$.

Proof. Let $\mathcal{L}$ be the weighted pair such that $P=\mathcal{G}_{(-n)} \ominus T_{1}$ is pseudo-linear of type ( $n-1, \mathcal{L}^{t}$ ). Note that $T_{2} \in \mathcal{T}^{\#}$ but that, since $\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) \ominus T_{2}$ has a branch point, $T_{2}$ satisfies none of the conditions of 5.6 (this follows from 5.7). Thus, if we write $T_{2}=C T$ with $C, T \in \mathcal{T}$ and $C$ a single column, we have $C \neq\binom{ 1}{1}, T \neq \mathbf{1}$ and if $T$ is a single column then it is not of the form $\binom{1}{k}$. Consider the weighted pair $P C=(\mathcal{H}, e)$ and regard $e$ as a vertex of $\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) \ominus T_{2}=(\mathcal{H}, e) \ominus T$. Then $\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) \ominus T_{2}$ has three branches at $e$, say $\mathcal{B}, \mathcal{B}^{\prime}$ and $\mathcal{B}^{\prime \prime}$, where $\mathcal{B}$ contains the vertices of $P, \mathcal{B} \cup \mathcal{B}^{\prime}$ contains no branch point of $\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) \ominus T_{2}$ and every weight in $\mathcal{B} \cup \mathcal{B}^{\prime} \cup \mathcal{B}^{\prime \prime}$ is strictly less than -1 , except possibly the middle vertex of $\mathcal{G}_{(-n)}$ (which belongs to $\mathcal{B}$ and has weight $-n$ ). Since $\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) \ominus T_{2}$ contracts to a graph with less branch points, $n=1$ (so $P$ is of type $\left(0, \mathcal{L}^{t}\right)$ ) and $\mathcal{B}$ shrinks. In other words, the connected component $\mathcal{L}^{t} C$ of $P C$ (regard $\mathcal{L}^{t} C$ as a weighted pair) contracts to a linear pair. By 3.32, $\mathcal{L}^{t} C\binom{1}{1}^{v}$ is contractible of type $\mathcal{L}$ and $C\binom{1}{1}^{v} \in \mathcal{T}\left(\mathcal{L}^{t}\right)$, for some $v \in \mathbb{N}$; so $P C$ is pseudo-contractible of type ( $\nu, \mathcal{L}$ ).

We may write $T_{2}=X_{1} T_{2}^{\prime}$ with $X_{1}=C\binom{1}{1}^{\ell} \in \mathcal{T}$ (some $\ell \in \mathbb{N}$ ) and $T_{2}^{\prime} \in \mathcal{T}^{\#}$. If $\ell>v$ then $P X_{1}$ contracts to a weighted pair $(W, w)$ which contains a vertex $v \neq w$ of nonnegative weight; then $\left(\mathcal{G}_{(-1)} \ominus T_{1}\right) \ominus T_{2}=P X_{1} \ominus T_{2}^{\prime}$ contracts to a weighted graph containing a nonnegative weight, contradicting the fact that its intersection matrix is negative definite. So $\ell \leq v$ and consequently $X_{1} \in \mathcal{T}_{v-\ell}\left(\mathcal{L}^{t}\right)$, which we rewrite as $X_{1} \in \mathcal{T}_{m-1}\left(\mathcal{L}^{t}\right)$, where $m \geq 1$. It is then clear that the triple $\tau^{\prime}=\left(m, \check{T}_{1}, T_{2}^{\prime}\right)$ belongs to $\mathbb{T}$ and satisfies $\tau>\tau^{\prime}$. By Lemma 5.18, $\tau^{\prime} \in \mathbb{T}(\dagger)$.

Proof of Corollary 5.17. Since $\tau>\tau^{\prime}$ implies $\tau \equiv \tau^{\prime}$ by Lemma 5.18, assertion (1) of 5.17 follows from 5.13. Also, (2) follows from (1): If $\tau \in \mathbb{T}(X) \backslash \mathbb{T}_{0}(X)$ then Lemma 5.19 implies that $\tau$ is not minimal in $(\mathbb{T}(\dagger),<)$; since there is no infinite descending sequence in $(\mathbb{T}(\dagger),<)$, we may therefore choose a minimal $\tau^{\prime}$ in $(\mathbb{T}(\dagger),<)$ such that $\tau>\tau^{\prime}$; then (1) implies $\tau^{\prime} \in \mathbb{T}(X)$, so $\tau^{\prime} \in \mathbb{T}_{0}(X)$ by Lemma 5.19.

Notation 5.20. (1) Given $\tau \in \mathbb{T}$, define

$$
[\tau, \infty)=\left\{\tau^{\prime} \in \mathbb{T} \mid \tau^{\prime} \geq \tau\right\} \quad \text { and } \quad(-\infty, \tau]=\left\{\tau^{\prime} \in \mathbb{T} \mid \tau^{\prime} \leq \tau\right\}
$$

(2) If $X$ satisfies $(\dagger), \min (\mathbb{T}(X))=\{\tau \in \mathbb{T}(X) \mid \tau$ is a minimal element of $(\mathbb{T}(\dagger),<)\}$.

Remark. By Lemma 5.19, $\min (\mathbb{T}(X)) \subseteq \mathbb{T}_{0}(X)$.
Lemma 5.21. Given $\tau \in \mathbb{T}$, the set $(-\infty, \tau]$ is totally ordered and finite. Consequently, if $\tau$ is not minimal in $\mathbb{T}$ then there exists exactly one $\tau^{-} \in \mathbb{T}$ such that

$$
\tau>\tau^{-} \text {and no } \tau^{*} \in \mathbb{T} \text { satisfies } \tau>\tau^{*}>\tau^{-}
$$

We call $\tau^{-}$the immediate predecessor of $\tau$.
Proof. We show that if $\tau, \tau^{\prime}, \tau^{\prime \prime} \in \mathbb{T}$ satisfy $\tau>\tau^{\prime}$ and $\tau>\tau^{\prime \prime}$, then $\tau^{\prime} \geq \tau^{\prime \prime}$ or $\tau^{\prime} \leq \tau^{\prime \prime}$. Write $\tau=\left(1, T_{1}, T_{2}\right), \tau^{\prime}=\left(m^{\prime}, T_{1}^{\prime}, T_{2}^{\prime}\right)$ and $\tau^{\prime \prime}=\left(m^{\prime \prime}, T_{1}^{\prime \prime}, T_{2}^{\prime \prime}\right)$ and let $q^{\prime}$ (resp. $q^{\prime \prime}$ ) be the number of columns of $T_{2}^{\prime}$ (resp. $T_{2}^{\prime \prime}$ ). We may assume that $q^{\prime} \leq q^{\prime \prime}$.

Since $\tau>\tau^{\prime}$, we have $T_{2}=X_{s} \cdots X_{1} T_{2}^{\prime}$ (notation as in Definition 5.16) and similarly $\tau>\tau^{\prime \prime}$ gives $T_{2}=Y_{r} \cdots Y_{1} T_{2}^{\prime \prime}$. Thus $T_{2}^{\prime}$ (resp. $T_{2}^{\prime \prime}$ ) consists of the rightmost $q^{\prime}$ (resp. $q^{\prime \prime}$ ) columns of $T_{2}$; since $q^{\prime} \leq q^{\prime \prime}$, it follows that $T_{2}^{\prime \prime}=W T_{2}^{\prime}$ for some $W \in \mathcal{T}$ (and $T_{2}^{\prime \prime} \in \mathcal{T}^{\#}$ implies $W \in \mathcal{T}^{\#}$ ). So $X_{s} \cdots X_{1} T_{2}^{\prime}=T_{2}=Y_{r} \cdots Y_{1} W T_{2}^{\prime}$ and consequently $X_{s} \cdots X_{1}=Y_{r} \cdots Y_{1} W$. Since the $X_{i}$ are irreducible elements of the monoid $\mathcal{T}^{\#}$, it follows that $W=X_{j} \cdots X_{1}$ (some $j \geq 0$ ) by unique factorization in $\mathcal{T}^{\#}$ (see 3.34). Thus $T_{2}^{\prime \prime}=X_{j} \cdots X_{1} T_{2}^{\prime}$ and it follows that $\tau^{\prime \prime}>\tau^{\prime}$ or $\tau^{\prime \prime}=\tau^{\prime}$. This shows that $(-\infty, \tau]$ is totally ordered; the other assertions are trivial.

Corollary 5.22. If $X$ satisfies $(\dagger)$ then $\{[\tau, \infty) \mid \tau \in \min (\mathbb{T}(X))\}$ is a partition of $\mathbb{T}(X)$.

Remark. [ $\tau, \infty$ ) is described explicitely in 5.39 , below.
Proof. By Corollary 5.17, the union of the sets $[\tau, \infty)$ is $\mathbb{T}(X)$. If $\tau^{\prime}, \tau^{\prime \prime} \in$ $\min (\mathbb{T}(X))$ are such that $\left[\tau^{\prime}, \infty\right) \cap\left[\tau^{\prime \prime}, \infty\right) \neq \emptyset$, then Lemma 5.21 implies $\tau^{\prime}=\tau^{\prime \prime}$.

In relation with the reduction process of Corollary 2.11, we give:

Proposition 5.23. Let $X$ be a surface satisfying ( $\dagger$ ), $\Lambda$ a non-basic affine ruling of $X$ and $F$ the unique element of $\Lambda_{*}$. Consider the pair $\left(\Lambda^{-}, F^{-}\right)$obtained from $(\Lambda, F)$ by means of the reduction process of Corollary 2.11, i.e., if $I=(S, \mu)$ is the distinguished $X$-immersion determining $(\Lambda, F), P \in S$ the center of the morphism
$\pi_{2}: \tilde{X} \rightarrow S$ which contracts $\tilde{F}$ to a 0 -curve and $\pi$ any ${ }^{8}$ element of $\Pi_{P}(I)$, then $\left(\Lambda^{-}, F^{-}\right)$is the pair determined by the $X$-immersion $I * \pi$. Let $\tau$ and $\tau^{-}$denote the discrete parts of $(\Lambda, F)$ and $\left(\Lambda^{-}, F^{-}\right)$respectively. Then $\tau^{-}$is the immediate predecessor of $\tau$ (see Lemma 5.21).

Proof. Write $\tau=\left(n, T_{1}, T_{2}\right)$. Since $\Lambda$ is non-basic, $\left(\mathcal{G}_{(-n)} \ominus T_{1}\right) \ominus T_{2} \cong \mathcal{G}(\Lambda)$ can be contracted to a weighted graph with less branch points; then the proof of Lemma 5.19 produces a $\tau^{\prime}=\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right)$ such that $\tau>\tau^{\prime}, T_{2}=X_{1} T_{2}^{\prime}$ and $X_{1}=C\binom{1}{1}^{\ell}$ (note that $\tau^{\prime}$ is the immediate predecessor of $\tau$ ). Since $\tau>\tau^{\prime}$ implies $\tau \equiv \tau^{\prime}$, the proof of Theorem 5.13 produces a pair $\left(\Lambda^{\prime}, F^{\prime}\right)$ whose discrete part is $\tau^{\prime}$. The factorization $T_{2}=C\binom{1}{1}^{\ell} T_{2}^{\prime}$ determines a factorization of $\pi_{2}$ as

$$
\begin{equation*}
\left.\tilde{X} \xrightarrow[T_{2}^{\prime}]{ } R^{\prime \prime} \xrightarrow[(1))^{l}\right]{ } \tilde{S} \xrightarrow[C]{\pi} S \tag{32}
\end{equation*}
$$

and $C \in \mathcal{T}_{v}\left(\mathcal{L}^{t}\right)$ implies that $\pi \in \Pi_{P}(S, \mu)$. Then we see that the $X$-immersion ( $S^{\prime}, \mu^{\prime}$ ) (in the proof of Theorem 5.13) is equivalent to $I * \pi$. Since ( $S^{\prime}, \mu^{\prime}$ ) determines ( $\Lambda^{\prime}, F^{\prime}$ ) and $I * \pi$ determines $\left(\Lambda^{-}, F^{-}\right)$, this means that $\left(\Lambda^{\prime}, F^{\prime}\right)=\left(\Lambda^{-}, F^{-}\right)$. So $\tau^{\prime}=\tau^{-}$and we are done.

## Surfaces Satisfying ( $\ddagger$ )

Notation 5.24.
(1) Consider triples $(X, \Lambda, F)$ where $X$ satisfies $(\dagger), \Lambda$ is an affine ruling of $X$ and $F \in \Lambda_{*}$. Two such triples are equivalent, $(X, \Lambda, F) \sim\left(X^{\prime}, \Lambda^{\prime}, F^{\prime}\right)$, when there exists an isomorphism $X \rightarrow X^{\prime}$ which transforms $\Lambda$ into $\Lambda^{\prime}$ and $F$ into $F^{\prime}$. If this is the case then $\operatorname{disc}(X, \Lambda, F)=\operatorname{disc}\left(X^{\prime}, \Lambda^{\prime}, F^{\prime}\right)$, so we may speak of the discrete part of the equivalence class $[X, \Lambda, F]$ of $(X, \Lambda, F)$. So we obtain a map disc : $\mathbb{S}(\dagger) \rightarrow \mathbb{T}(\dagger)$, where $\mathbb{S}(\dagger)$ denotes the set of equivalence classes $[X, \Lambda, F]$.
(2) We will also consider the restriction disc : $\mathbb{S}_{0}(\ddagger) \rightarrow \mathbb{T}_{0}(\ddagger)$ of the above map $\mathbb{S}(\dagger) \rightarrow \mathbb{T}(\dagger)$, where $\mathbb{S}_{0}(\ddagger)=\{[X, \Lambda, F] \in \mathbb{S}(\dagger) \mid X$ satisfies $(\ddagger)$ and $\Lambda$ is basic $\}$ and where $\mathbb{T}_{0}(\ddagger)$ is the set of ( $m, T_{1}, T_{2}$ ) $\in \mathbb{T}$ such that (i) each of $T_{1}, T_{2}$ satisfies one of conditions (1-3) of 5.6; and (ii) the weighted graph $\left(\mathcal{G}_{(-m)} \ominus T_{1}\right) \ominus T_{2}$ has a negative definite intersection matrix. (See Lemma 5.11 for the fact that disc maps $\mathbb{S}_{0}(\ddagger)$ in $\mathbb{T}_{0}(\ddagger$ ); see also 5.41.)
(3) Let $\overline{\mathbb{S}}(\ddagger$ ) be the set of isomorphism classes of surfaces satisfying ( $\ddagger$ ). The isomorphism class of $X$ is denoted $[X]$. Then $[X, \Lambda, F] \mapsto[X]$ defines a map $\mathbb{S}_{0}(\ddagger) \rightarrow \overline{\mathbb{S}}(\ddagger)$.

In particular, we will show:

[^6]Proposition 5.25. $\quad \mathbb{S}_{0}(\ddagger) \rightarrow \overline{\mathbb{S}}(\ddagger)$ and $\mathbb{S}(\dagger) \rightarrow \mathbb{T}(\dagger)$ are surjective and $\mathbb{S}_{0}(\ddagger) \rightarrow$ $\mathbb{T}_{0}(\ddagger)$ is bijective.

Proof that $\mathbb{S}_{0}(\ddagger) \rightarrow \overline{\mathbb{S}}(\ddagger)$ is surjective. If $X$ is any surface satisfying ( $\ddagger$ ), then $X$ admits a basic affine ruling $\Lambda$ by Theorem 2.1 and $\Lambda_{*} \neq \emptyset$ by 2.5 ; thus $[X]$ is in the image of $\mathbb{S}_{0}(\ddagger) \rightarrow \overline{\mathbb{S}}(\ddagger)$.

The proof of the other assertions requires some preparation.

Definition 5.26. Let $m$ be a positive integer, $\Lambda_{m}$ the standard ruling of $\mathbb{F}_{m}$ and $\Sigma_{m} \subset \mathbb{F}_{m}$ the negative section of $\Lambda_{m}\left(\Sigma_{m}^{2}=-m\right)$. Let $T_{1}, T_{2} \in \mathcal{T}$.
(1) By a blowing-up of $\mathbb{F}_{m}$ according to $\left(T_{1}, T_{2}\right)$, we mean a triple $\left(\pi, P_{1}, P_{2}\right)$ where $\pi: Y \rightarrow \mathbb{F}_{m}$ is a birational morphism (with $Y$ smooth and complete), $P_{1}, P_{2}$ are points of $\mathbb{F}_{m} \backslash \Sigma_{m}$ belonging to distinct members of $\Lambda_{m}\left(P_{i} \in Z_{i} \in \Lambda_{m}, Z_{1} \neq Z_{2}\right)$, center $(\pi) \subseteq\left\{P_{1}, P_{2}\right\}$ and, for each $i=1,2, \pi^{-1}\left(P_{i}\right)$ contains at most one ( -1 )-curve and $\overline{\mathrm{HN}}\left(\pi, Z_{i}\right)=T_{i}$.
(2) Let $\beta=\left(\pi, P_{1}, P_{2}\right)$ and $\beta^{\prime}=\left(\pi^{\prime}, P_{1}^{\prime}, P_{2}^{\prime}\right)$ be two blowings-up of $\mathbb{F}_{m}$ according to ( $T_{1}, T_{2}$ ). We say that $\beta$ is equivalent to $\beta^{\prime}$ if there exists a commutative diagram:

where the horizontal arrows are isomorphisms and, for each $i=1,2, \varphi\left(P_{i}\right)=P_{i}^{\prime}$.
Lemma 5.27. Let $\left(m, T_{1}, T_{2}\right) \in \mathbb{Z}^{+} \times \mathcal{T} \times \mathcal{T}$ be such that:
(i) Each $T_{i}$ satisfies one of conditions (1-3) of 5.6; and
(ii) if both $T_{i}$ are nonempty then $m c_{1} c_{2}-c_{1} p_{2}-c_{2} p_{1} \neq 0$, where $\binom{p_{i}}{c_{i}}$ is the first column of $T_{i}$.
Then any two blowings-up of $\mathbb{F}_{m}$ according to $\left(T_{1}, T_{2}\right)$ are equivalent.
Proof. Let $\beta=\left(\pi, P_{1}, P_{2}\right)$ and $\beta^{\prime}=\left(\pi^{\prime}, P_{1}^{\prime}, P_{2}^{\prime}\right)$ be two blowings-up of $\mathbb{F}_{m}$ according to $\left(T_{1}, T_{2}\right)$. Since there exists an automorphism ${ }^{9}$ of $\mathbb{F}_{m}$ which maps $P_{1}$ and $P_{2}$ to $P_{1}^{\prime}$ and $P_{2}^{\prime}$ respectively, we may assume that $\left(P_{1}, P_{2}\right)=\left(P_{1}^{\prime}, P_{2}^{\prime}\right)$. Let $Z_{i}$ be the member of $\Lambda_{m}$ containing $P_{i}\left(Z_{1} \neq Z_{2}\right)$ and choose a section $S$ of $\Lambda_{m}$ such that $S \cap$ $\Sigma_{m}=\emptyset$ and $P_{1}, P_{2} \in S$. We can write $\mathbb{F}_{m} \backslash \Sigma_{m}=\operatorname{Spec} \mathbf{k}\left[x_{1}, y_{1}\right] \cup \operatorname{Spec} \mathbf{k}\left[x_{2}, y_{2}\right]$, where $x_{i}, y_{i}$ are local equations at $P_{i}$ for $Z_{i}$ and $S$ respectively, $x_{2}=x_{1}^{-1}$ and $y_{2}=y_{1} x_{1}^{-m}$. Then the Hamburger-Noether tableaux $\mathrm{HN}_{i}=\mathrm{HN}\left(\pi ; x_{i}, y_{i}\right)$ and $\mathrm{HN}_{i}^{\prime}=\mathrm{HN}\left(\pi^{\prime} ; x_{i}, y_{i}\right)$ satisfy $\overline{\mathrm{HN}}_{i}=T_{i}=\overline{\mathrm{HN}}_{i}^{\prime}(i=1,2)$.

[^7]Note that, for each $(\sigma, \tau) \in\left(\mathbf{k}^{*}\right)^{2}, x_{1} \mapsto \sigma x_{1}, y_{1} \mapsto \tau y_{1}$ induces an automorphism $\varphi_{\sigma, \tau}$ of $\mathbb{F}_{m}$ which leaves $\left(P_{1}, P_{2}, S\right)$ unchanged.

Let $I=\left\{i \in\{1,2\} \mid T_{i}\right.$ has two columns $\}$ and, for each $i \in I$, define $\alpha_{i}, \alpha_{i}^{\prime} \in$ $\mathbf{k}^{*}$ by saying that $\binom{*}{\alpha_{i}}\left(\right.$ resp. $\left.\left(\begin{array}{c}* \\ \alpha_{i}^{*} \\ \alpha_{i}\end{array}\right)\right)$ is the first column of $\mathrm{HN}_{i}$ (resp. $\mathrm{HN}_{i}^{\prime}$ ). If the two sequences $\left\{\alpha_{i}\right\}_{i \in I}$ and $\left\{\alpha_{i}^{\prime}\right\}_{i \in I}$ are equal, then the assertion is trivial. So it suffices to show that the sequence $\left\{\alpha_{i}\right\}_{i \in I}$ can be transformed into the constant sequence with value 1 by composing $\pi$ with automorphisms $\varphi_{\sigma, \tau}$.

Let us study the following situation. Let $P_{i}, Z_{i}, S$ and $\left(x_{i}, y_{i}\right)$ be as above. Let $\alpha_{1}, \alpha_{2} \in \mathbf{k}^{*}$ and $p_{1}, c_{1}, p_{2}, c_{2} \in \mathbb{N}$ be such that $0<p_{i} \leq c_{i}$ are relatively prime $(i=1,2)$ and $m c_{1} c_{2}-c_{1} p_{2}-c_{2} p_{1} \neq 0$; consider a birational morphism $f: Y \rightarrow \mathbb{F}_{m}(Y$ smooth and complete) satisfying center $(f)=\left\{P_{1}, P_{2}\right\}$ and, for each $i=1,2, f^{-1}\left(P_{i}\right)$ contains a unique (-1)-curve $E_{i}$ and $\operatorname{HN}\left(f ; x_{i}, y_{i}\right)=\left(\begin{array}{c}p_{i} \\ c_{i} \\ \alpha_{i}\end{array}\right)$. For each $i=1,2$, the HNalgorithm of [11] produces a parameter $u_{i}$ for $E_{i} \cong \mathbb{P}^{1}$ and the condition $u_{i}=\alpha_{i}$ determines a point on $E_{i}$. Moreover, $u_{i}=y_{i}^{c_{i}} / x_{i}^{p_{i}}$ or $u_{i}=x_{i}^{p_{i}} / y_{i}^{c_{i}}$. We have $\varphi_{\sigma, \tau}\left(u_{i}\right)=$ $\sigma^{v_{i}} \tau^{\mu_{i}} u_{i}$, with $\left(\nu_{1}, \mu_{1}\right)= \pm\left(-p_{1}, c_{1}\right)$ and $\left(\nu_{2}, \mu_{2}\right)= \pm\left(p_{2}-m c_{2}, c_{2}\right)$. Since $\left|\begin{array}{l}\nu_{1} \\ \nu_{2} \mu_{1}\end{array}\right|=$ $\pm\left(m c_{1} c_{2}-c_{1} p_{2}-c_{2} p_{1}\right) \neq 0$, we may choose $(\sigma, \tau)$ such that

$$
\operatorname{HN}\left(\varphi_{\sigma, \tau} \circ f ; x_{1}, y_{1}\right)=\left(\begin{array}{c}
p_{1} \\
c_{1} \\
1
\end{array}\right) \quad \text { and } \quad \operatorname{HN}\left(\varphi_{\sigma, \tau} \circ f ; x_{2}, y_{2}\right)=\left(\begin{array}{c}
p_{2} \\
c_{2} \\
1
\end{array}\right) .
$$

Lemma 5.28. Let $\left(m, T_{1}, T_{2}\right) \in \mathbb{T}_{0}(\ddagger)$.
(1) If both $T_{i}$ are nonempty then $m c_{1} c_{2}-c_{1} p_{2}-c_{2} p_{1}>0$, where $\binom{p_{i}}{c_{i}}$ is the first column of $T_{i}$.
(2) The blowing-up of $\mathbb{F}_{m}$ according to $\left(T_{1}, T_{2}\right)$ is unique, up to equivalence.

Proof. If both $T_{i}$ are nonempty then let $\Gamma$ be the connected component of

$$
\left(\mathcal{G}_{(-m)} \ominus\binom{p_{1}}{c_{1}}\right) \ominus\binom{p_{2}}{c_{2}}
$$

containing the vertices of $\mathcal{G}_{(-m)}$. Since $\Gamma$ is a subgraph of $\left(\mathcal{G}_{(-m)} \ominus T_{1}\right) \ominus T_{2}$, it must have a negative definite intersection matrix. In particular, $\operatorname{det}(\Gamma)>0$. By 3.16 and Lemma 3.23, $\operatorname{det}(\Gamma)=m c_{1} c_{2}-c_{1} p_{2}-c_{2} p_{1}$. This proves (1), and (2) follows from (1) and Lemma 5.27.
5.29 (Proof of Proposition 5.25, continued). Given $\tau=\left(m, T_{1}, T_{2}\right) \in \mathbb{T}(\dagger)$, we describe a method for constructing all $(X, \Lambda, F)$ such that $\operatorname{disc}(X, \Lambda, F)=\tau$ (where $X$ satisfies ( $\dagger$ ), $\Lambda$ is an affine ruling of $X$ and $F \in \Lambda_{*}$ ). This will show, in particular, that disc : $\mathbb{S}(\dagger) \rightarrow \mathbb{T}(\dagger)$ is surjective.

Choose a blowing-up

$$
\left(\tilde{X} \xrightarrow{\pi} \mathbb{F}_{m}, P_{1}, P_{2}\right)
$$

of $\mathbb{F}_{m}$ according to $\left(T_{1}, T_{2}\right)$ and let $Z_{1}$ and $Z_{2}$ be the elements of $\Lambda_{m}$ satisfying $P_{i} \in$ $Z_{i}$. Recall that center $(\pi) \subseteq\left\{P_{1}, P_{2}\right\}$ and $\overline{\mathrm{HN}}\left(\pi, Z_{i}\right)=T_{i}$. For $i=1,2$, define

$$
E_{i}= \begin{cases}\pi^{-1}\left(Z_{i}\right), & \text { if } P_{i} \notin \operatorname{center}(\pi), \\ \text { the }(-1) \text {-curve in } \pi^{-1}\left(P_{i}\right), & \text { if } P_{i} \in \operatorname{center}(\pi)\end{cases}
$$

and let $D$ be the divisor of $\tilde{X}$ with strong normal crossings defined by $\pi^{-1}\left(\operatorname{supp}\left(Z_{1}+\right.\right.$ $\left.\left.\Sigma_{m}+Z_{2}\right)\right)=\operatorname{supp}\left(E_{1}+D+E_{2}\right)$ and $E_{1}, E_{2} \nsubseteq \operatorname{supp}(D)$. Then $\mathcal{G}(D, \tilde{X})=\left(\mathcal{G}_{(-m)} \ominus T_{1}\right) \ominus T_{2}$ and consequently $D$ has a negative definite intersection matrix (because $\tau \in \mathbb{T}(\dagger)$ ). So there exists a complete normal surface $X$ and a birational morphism $\tilde{X} \rightarrow X$ with exceptional locus $\operatorname{supp}(D)$. Note that $\Lambda_{m}$ determines an affine ruling $\Lambda$ of $X$, because $\mathbb{F}_{m} \leftarrow \tilde{X} \rightarrow X$ restrict to an isomorphism between $\mathbb{F}_{m} \backslash \operatorname{supp}\left(Z_{1}+\Sigma_{m}+Z_{2}\right)$ and an open subset of $X$. Moreover, if $\tilde{\Lambda}$ is the strict transform of $\Lambda_{m}$ with respect to $\pi$, then $(\tilde{X}, \tilde{\Lambda})=(X, \Lambda)^{\sim}$. Equation (4) of 1.7 implies that $\operatorname{Pic}\left(X_{s}\right)$ has rank 1 , so $X$ satisfies $(\dagger)$. Note that the image of $E_{i}$ under $\tilde{X} \rightarrow X$ is the support of some $F_{i} \in$ $\Lambda$; moreover, $F_{2} \in \Lambda_{*}$ and $\operatorname{disc}\left(X, \Lambda, F_{2}\right)=\tau$. It is clear, also, that $\left(X, \Lambda, F_{2}\right)$ is determined by the choice of the blowing-up $\left(\pi, P_{1}, P_{2}\right.$ ) and that every triple ( $X, \Lambda, F$ ) with discrete part $\tau$ can be obtained in this way, i.e., by choosing a suitable blowingup.
5.30 (End of proof of Proposition 5.25). We show that $\mathbb{S}_{0}(\ddagger) \rightarrow \mathbb{T}_{0}(\ddagger)$ is bijective. Given $\tau=\left(m, T_{1}, T_{2}\right) \in \mathbb{T}_{0}(\ddagger)$, consider a triple $\left(X, \Lambda, F_{2}\right)$ constructed as in 5.29. By Lemma $5.11, X$ satisfies ( $\ddagger$ ) and $\Lambda$ is basic, so $\left[X, \Lambda, F_{2}\right] \in \mathbb{S}_{0}(\ddagger)$. Also, uniqueness (Lemma 5.28) of the blowing-up ( $\pi, P_{1}, P_{2}$ ) up to equivalence implies uniqueness of ( $X, \Lambda, F_{2}$ ) up to equivalence; in other words, $\tau \mapsto\left[X, \Lambda, F_{2}\right]$ is a well-defined map $\mathbb{T}_{0}(\ddagger) \rightarrow \mathbb{S}_{0}(\ddagger)$, and this is the inverse of the "discrete part" map $\mathbb{S}_{0}(\ddagger) \rightarrow \mathbb{T}_{0}(\ddagger)$.

Corollary 5.31. There exists a surjective map $f: \mathbb{T}_{0}(\ddagger) \rightarrow \overline{\mathbb{S}}(\ddagger)$ satisfying:
Given $\tau \in \mathbb{T}_{0}(\ddagger)$ and $X$ satisfying $(\ddagger), f(\tau)=[X]$ if and only if there exists an affine ruling $\Lambda$ of $X$ and an $F \in \Lambda_{*}$ such that $\tau$ is the discrete part of $(\Lambda, F)$.

Remark. One interesting aspect of the surjection $f: \mathbb{T}_{0}(\ddagger) \rightarrow \overline{\mathbb{S}}(\ddagger)$ of Corollary 5.31 is that, given $\tau \in \mathbb{T}_{0}(\ddagger)$, we may construct, in a very explicit way, a surface $X$ such that $f(\tau)=[X]$ (the construction is carried out in 5.29). Since the elements of $\mathbb{T}_{0}(\ddagger)$ can be described explicitely (see 5.41 ), this gives an interesting description of the class of surfaces satisfying ( $\ddagger$ ).

Corollary 5.32. Let $X_{1}$ and $X_{2}$ be surfaces satisfying ( $\dagger$ ) and such that $\mathbb{T}\left(X_{1}\right) \cap$ $\mathbb{T}\left(X_{2}\right) \neq \emptyset$. Then:
(1) $\mathbb{T}_{0}\left(X_{1}\right) \cap \mathbb{T}_{0}\left(X_{2}\right) \neq \emptyset$.
(2) If at least one of $X_{1}, X_{2}$ satisfies ( $\ddagger$ ), then $X_{1} \cong X_{2}$.

Proof. Assertion (1) follows immediately from Corollary 5.17. To prove (2), assume that $X_{1}$ satisfies ( $\ddagger$ ) and consider $\tau=\left(m, T_{1}, T_{2}\right) \in \mathbb{T}_{0}\left(X_{1}\right) \cap \mathbb{T}_{0}\left(X_{2}\right)$. By Lemma 5.11, each of $T_{1}, T_{2}$ satisfies one of the conditions (1-3) of 5.6; since $\tau \in$ $\mathbb{T}\left(X_{2}\right)$, Lemma 5.11 implies that $X_{2}$ satisfies $(\ddagger)$. Then the surjection $f: \mathbb{T}_{0}(\ddagger) \rightarrow \overline{\mathbb{S}}(\ddagger)$ of Corollary 5.31 satisfies $f(\tau)=\left[X_{1}\right]$ and $f(\tau)=\left[X_{2}\right]$, so $\left[X_{1}\right]=\left[X_{2}\right]$.
5.33. Consider the equivalence relation " $\sim$ " on $\mathbb{T}$ which is generated by declaring that $\tau \sim \tau^{\prime}$ whenever $\tau<\tau^{\prime}$. Then $\tau \sim \tau^{\prime} \Longrightarrow \tau \equiv \tau^{\prime}$, but the converse does not hold. Indeed, Lemma 5.21 implies that $\mathbb{T} / \sim=\{[\tau, \infty) \mid \tau \in \min (\mathbb{T})\}$, so each equivalence class with respect to $\sim$ contains exactly one minimal element of $(\mathbb{T},<$ ). However, if $\tau=\left(1,\left(\begin{array}{ll}16 & 1 \\ 39 & 3\end{array}\right),\left(\begin{array}{ll}135 & 1 \\ 229 & 4\end{array}\right)\right)$ and $\tau^{\prime}=\left(1,\left(\begin{array}{ll}23 & 1 \\ 39 & 3\end{array}\right),\left(\begin{array}{ll}2 & 1 \\ 5 & 4\end{array}\right)\right)$ then $\tau$ and $\tau^{\prime}$ are distinct minimal elements of $(\mathbb{T},<)$ and $\tau \equiv \tau^{\prime}$.

Regarding the relation $\sim$ of 5.33 , we have the following:
Corollary 5.34. For $i=1,2$, let $X_{i}$ be a surface satisfying ( $\ddagger$ ), let $\Lambda_{i}$ be an affine ruling of $X_{i}$ and let $F_{i} \in\left(\Lambda_{i}\right)_{*}$. If $\operatorname{disc}\left(X_{1}, \Lambda_{1}, F_{1}\right) \sim \operatorname{disc}\left(X_{2}, \Lambda_{2}, F_{2}\right)$, then there exist $\left(\Lambda_{1}^{\prime}, F_{1}^{\prime}\right)$ and $\left(\Lambda_{2}^{\prime}, F_{2}^{\prime}\right)$ satisfying:
(1) For each $i, \Lambda_{i}^{\prime}$ is a basic affine ruling of $X_{i}, F_{i}^{\prime} \in\left(\Lambda_{i}^{\prime}\right)_{*}$ and $\operatorname{supp}\left(F_{i}^{\prime}\right)=\operatorname{supp}\left(F_{i}\right)$;
(2) there exists an isomorphism $X_{1} \rightarrow X_{2}$ which carries $\Lambda_{1}^{\prime}$ to $\Lambda_{2}^{\prime}$ and $F_{1}^{\prime}$ to $F_{2}^{\prime}$.

In particular, there exists an isomorphism $X_{1} \rightarrow X_{2}$ which maps $\operatorname{supp}\left(F_{1}\right)$ onto $\operatorname{supp}\left(F_{2}\right)$.

Proof. Let $\tau_{i} \in \mathbb{T}$ be the discrete part of $\left(\Lambda_{i}, F_{i}\right)$. Then Lemma 5.21 implies that there exists $\tau^{\prime} \in \mathbb{T}$ such that (for all i) $\tau_{i} \geq \tau^{\prime}$; clearly, $\tau^{\prime}$ may be chosen so that it is a minimal element of $\mathbb{T}$. By Theorem 5.13 , for each $i$ there exists an affine ruling $\Lambda_{i}^{\prime}$ of $X_{i}$ and $F_{i}^{\prime} \in\left(\Lambda_{i}^{\prime}\right)_{*}$ satisfying $\operatorname{supp}\left(F_{i}^{\prime}\right)=\operatorname{supp}\left(F_{i}\right)$ and such that the discrete part of $\left(X_{i}, \Lambda_{i}^{\prime}, F_{i}^{\prime}\right)$ is $\tau^{\prime}$. So (Lemma 5.19) $\Lambda_{i}^{\prime}$ is basic and the two elements [ $X_{1}, \Lambda_{1}^{\prime}, F_{1}^{\prime}$ ] and $\left[X_{2}, \Lambda_{2}^{\prime}, F_{2}^{\prime}\right]$ of $\mathbb{S}_{0}(\ddagger)$ have the same image (namely, $\tau^{\prime}$ ) under the bijective map $\mathbb{S}_{0}(\ddagger) \rightarrow \mathbb{T}_{0}(\ddagger)$. Hence, $\left[X_{1}, \Lambda_{1}^{\prime}, F_{1}^{\prime}\right]=\left[X_{2}, \Lambda_{2}^{\prime}, F_{2}^{\prime}\right]$.

## Multiplicities

Defintion 5.35. Given a tableau $T=\left(\begin{array}{ccc}p_{1} & \ldots & p_{k} \\ c_{1} & . . & c_{k}\end{array}\right) \in \mathcal{T}$, we define

$$
\mu(T)= \begin{cases}1, & \text { if } T=\mathbf{1} \\ c_{1} \cdots c_{k}, & \text { else } .\end{cases}
$$

Note that $\mu: \mathcal{T} \rightarrow \mathbb{N} \backslash\{0\}$ is a homomorphism of multiplicative monoids.
Remark. Given a finite Hamburger-Noether tableau $\mathrm{HN}=\left(\begin{array}{c}p_{1} \\ c_{1} \\ \alpha_{1}\end{array}\right)$, consider $T=$ $\overline{\mathrm{HN}} \in \mathcal{T}$ defined as in 3.6. Then $\mu(T)=c_{1}$ (or 1 , if HN is empty).

By the above remark and A.10.1 of [11], we have
5.36. Let $f: X \rightarrow Y$ be a birational morphism of smooth complete surfaces and $D$ a nonzero divisor of $Y$ with strong normal crossings. Assume that the exceptional locus of $f$ contains at most one ( -1 -curve and that the center of $f$, if nonempty, is a point $P$ belonging to exactly one component $Z$ of $D$. Let

$$
E= \begin{cases}f^{-1}(Z), & \text { if } f \text { is an isomorphism, } \\ \text { the }(-1) \text {-curve in } f^{-1}(P), & \text { if } f \text { is not an isomorphism. }\end{cases}
$$

Then the multiplicity of $E$ in the total transform of $D$ is equal to $\mu(\overline{\mathrm{HN}}(f, D))$.
The above statement and Proposition 1.8 give:
Corollary 5.37. Let $X$ be a surface satisfying ( $\dagger$ ). If ( $m, T_{1}, T_{2}$ ) is the discrete part of $(\Lambda, F)$, where $\Lambda$ is an affine ruling of $X$ and $F \in \Lambda_{*}$, and if $G \in \Lambda \backslash\{F\}$ is such that $\{F, G\}$ contains all multiple members of $\Lambda$ (such a $G$ exists, by definition of $\Lambda_{*}$ ), then

$$
F=\mu\left(T_{2}\right) C_{2} \quad \text { and } \quad G=\mu\left(T_{1}\right) C_{1},
$$

where $C_{1}, C_{2} \subset X$ are (irreducible) curves. Moreover, $\operatorname{Pic}\left(X_{s}\right) \cong \mathbb{Z} \oplus \mathbb{Z} / d \mathbb{Z}$, where $d=\operatorname{gcd}\left(\mu\left(T_{1}\right), \mu\left(T_{2}\right)\right)$.

Remark. If $X=\mathbb{P}^{2}$, or more generally a weighted projective plane $\mathbb{P}(a, b, c)$ where $a, b, c$ are pairwise relatively prime, then $\mu\left(T_{1}\right)=\operatorname{deg} C_{2}$ and $\mu\left(T_{2}\right)=\operatorname{deg} C_{1}$. (In view of the above result, this follows immediately from $\operatorname{gcd}\left(\operatorname{deg} C_{1}, \operatorname{deg} C_{2}\right)=1$, for which we refer to [5] or [6].)

See also Corollary 5.40.

## Some explicit computations

5.38. Let $m>0$ be an integer and suppose that $T \in \mathcal{T}$ satisfies one of conditions (1-3) of 5.6.
(1) Recall that $\mathcal{G}_{(-m)} \ominus T$ is pseudo-linear of type ( $m-1, \mathcal{L}$ ), where $\mathcal{L}$ is a weighted pair satisfying the condition (0), uniquely determined by $T$. Then Lemma 3.23 gives:

$$
M(\mathcal{L})= \begin{cases}\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right) & \text { if } T \text { satisfies 5.6.1, } \\
\left(\begin{array}{cc}
c-p-p^{\prime}+p^{\prime \prime} & c-p \\
c-p^{\prime} & c
\end{array}\right) & \text { if } T \text { satisfies 5.6.2, } \\
\left(\begin{array}{cc}
N p(c-p)-1 & N c^{2}-N c p-1 \\
N c p-1 & N c^{2}
\end{array}\right) & \text { if } T \text { satisfies 5.6.3, }\end{cases}
$$

where $p^{\prime}$ and $p^{\prime \prime}$ are defined by $\binom{p^{\prime \prime}}{p^{\prime}}=\binom{p}{c}^{*}$. Note that $\mathcal{L}$ is degenerate if and only if $T \in\left\{\mathbf{1},\left(\begin{array}{ll}1 & 1 \\ 2 & 1\end{array}\right)\right\} \cup\left\{\left.\binom{n}{n+1} \right\rvert\, n \geq 1\right\}$.
(2) The conditions

$$
M(\mathcal{L})=\left(\begin{array}{cc}
\dot{\gamma}(T) & \alpha_{1} \\
\alpha_{2} & \gamma(T)
\end{array}\right) \quad \text { and } \quad \alpha_{i+2}=\alpha_{i}(i \geq 1)
$$

define positive integers $\dot{\gamma}(T)$ and $\gamma(T)$ and an infinite sequence $\alpha(T)=\left(\alpha_{1}, \alpha_{2}, \ldots\right)$ of positive integers. Note that these are uniquely determined by $T$ and can be computed from (1). They satisfy $M\left(\mathcal{L}^{t^{i}}\right)=\left(\begin{array}{ccc}\dot{\gamma}(T) & \alpha_{i+1} \\ \alpha_{i} & \gamma(T)\end{array}\right)$ for all $i \geq 1$.
(3) A sequence $v=\left(v_{1}, \ldots, v_{s}\right)$ of natural numbers is said to be $(m, T)$-admissible if $s \geq 1$ and the following conditions hold:
(a) If $T \in\left\{\mathbf{1},\left(\begin{array}{ll}1 & 1 \\ 2 & 1\end{array}\right)\right\} \cup\left\{\left.\binom{n}{n+1} \right\rvert\, n \geq 1\right\}, v_{1} \geq \max (1, m-1)$ and $v_{i} \geq 1$ for all $i>1$.
(b) For all other $T, v_{1} \geq m-1$ and $v_{i} \geq 0$ for all $i>1$.
(4) Given an $(m, T)$-admissible sequence $v=\left(\nu_{1}, \ldots, v_{s}\right)$, consider the sequence of tableaux $\left(X_{1}, \ldots, X_{s}\right) \in \mathcal{T}^{s}$ given by $X_{i}=\binom{p_{i}}{c_{i}}\binom{1}{1}^{v_{i}-k_{i}}$, where $\binom{p_{i}}{c_{i}}$ is the matrix product $M\left(\mathcal{L}^{t^{i}}\right)\binom{1}{v_{i}}, k_{1}=m-1$ and $k_{i}=0$ for all $i \geq 1$. Then $X_{i} \in \mathcal{T}_{k_{i}}\left(\mathcal{L}^{t^{i}}\right)$ for all $i=1, \ldots, s$.
5.39. Let $\tau=\left(m, T_{1}, T_{2}\right) \in \mathbb{T}$. For each ( $m, T_{1}$ )-admissible sequence $v=$ $\left(v_{1}, \ldots, v_{s}\right)$, define $\tau_{v} \in \mathbb{T}$ by $\tau_{v}=\left(1,\left(T_{1}\right)^{(v s)}, X_{s} \cdots X_{1} T_{2}\right)$, where $\left(X_{1}, \ldots, X_{s}\right)$ is determined by $v$ and $\left(m, T_{1}\right)$ as in part 4 of 5.38. Then

$$
[\tau, \infty)=\{\tau\} \cup\left\{\tau_{v} \mid v \text { is an }\left(m, T_{1}\right) \text {-admissible sequence }\right\} .
$$

Corollary 5.40. Let $X$ be a surface satisfying $(\dagger)$ and suppose that $\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right) \in$ $\mathbb{T}(X)$. Let $\gamma=\gamma\left(T_{1}^{\prime}\right)$ and $\alpha\left(T_{1}^{\prime}\right)=\left(\alpha_{1}, \alpha_{2}, \ldots\right)$. Then the set

$$
\left\{\left(\mu\left(T_{1}\right), \mu\left(T_{2}\right)\right) \mid\left(1, T_{1}, T_{2}\right) \in \mathbb{T}(X) \text { and }\left(1, T_{1}, T_{2}\right)>\left(m, T_{1}^{\prime}, T_{2}^{\prime}\right)\right\}
$$

is equal to

$$
\left\{\left(\mu\left(T_{1}^{\prime}\right), \mu\left(T_{2}^{\prime}\right) \cdot \prod_{i=1}^{s}\left(\alpha_{i}+v_{i} \gamma\right)\right) \mid\left(v_{1}, \ldots, v_{s}\right) \text { is }\left(m, T_{1}^{\prime}\right) \text {-admissible }\right\} .
$$

5.41. We describe the elements of $\mathbb{T}_{0}(\ddagger)$. Consider a triple $\tau=\left(m, T_{1}, T_{2}\right)$ where $m$ is a positive integer and each $T_{i}$ is a tableau ( $T_{i} \in \mathcal{T}$ ) satisfying one of conditions (1-3) of 5.6 (each element of $\mathbb{T}_{0}(\ddagger$ ) is such a triple). Consider the connected component $\Gamma$ of $\left(\mathcal{G}_{(-m)} \ominus T_{1}\right) \ominus T_{2}$ containing the vertices of $\mathcal{G}_{(-m)}$. Then every connected component of $\left(\mathcal{G}_{(-m)} \ominus T_{1}\right) \ominus T_{2}$ is a linear chain and every vertex, except possibly the middle vertex of $\mathcal{G}_{(-m)}$ (which has weight $-m$ ), has weight strictly less than -1 . So $\tau \in \mathbb{T}_{0}(\ddagger) \Leftrightarrow \operatorname{det}(\Gamma)>0$ and in particular:

- If $m>1$ then $\tau \in \mathbb{T}_{0}(\ddagger)$;
- if $\mathbf{1} \in\left\{T_{1}, T_{2}\right\}$ then $\tau \in \mathbb{T}_{0}(\ddagger)$.

Assume that $m=1$ and that neither of $T_{1}, T_{2}$ is empty; then $T_{i}$ is either $\binom{p_{i}}{c_{i}}$ or $\left(\begin{array}{ll}p_{i} & 1 \\ c_{i} & x_{i}\end{array}\right)$ with $x_{i} \geq 1$. We may then compute $\operatorname{det}(\Gamma)$ in each case and conclude:
(1) If $T_{1}=\binom{p_{1}}{c_{1}}$ and $T_{2}=\binom{p_{2}}{c_{2}}, \tau \in \mathbb{T}_{0}(\ddagger) \Longleftrightarrow \Delta>0$;
(2) if $T_{i}=\binom{p_{i}}{c_{i}}$ and $T_{j}=\left(\begin{array}{cc}p_{j} & 1 \\ c_{j} & x_{j}\end{array}\right), \tau \in \mathbb{T}_{0}(\ddagger) \Longleftrightarrow \Delta c_{j} x_{j}-c_{i}>0$;
(3) if $T_{1}=\left(\begin{array}{cc}p_{1} & 1 \\ c_{1} & x_{1}\end{array}\right)$ and $T_{2}=\left(\begin{array}{cc}p_{2} & 1 \\ c_{2} & x_{2}\end{array}\right), \tau \in \mathbb{T}_{0}(\ddagger) \Longleftrightarrow \Delta c_{1} c_{2} x_{1} x_{2}-c_{1}^{2} x_{1}-c_{2}^{2} x_{2}>0$, where $\Delta=m c_{1} c_{2}-c_{1} p_{2}-c_{2} p_{1}=c_{1} c_{2}-c_{1} p_{2}-c_{2} p_{1}$.

## References

[1] D. Daigle: Birational endomorphisms of the affine plane, Ph. D. thesis, McGill University, Montréal, Canada, 1987.
[2] D. Daigle: Birational endomorphisms of the affine plane, J. Math. Kyoto Univ. 31 (1991), 329358.
[3] D. Daigle: On some properties of locally nilpotent derivations, J. Pure Appl. Algebra 114 (1997), 221-230.
[4] D. Daigle: Homogeneous locally nilpotent derivations of $k[x, y, z]$, J. Pure Appl. Algebra 128 (1998), 109-132.
[5] D. Daigle: On kernels of homogeneous locally nilpotent derivations of $k[x, y, z]$, Osaka J. Math, 37 (2000), 689-699.
[6] D. Daigle and P. Russell: On weighted projective planes and their affine rulings, Osaka J. Math. to appear.
[7] T. Fujita: On the topology of non-complete algebraic surfaces, J. Fac. Sci. Univ. Tokyo 29 (1982), 503-566.
[8] S. Iitaka: Geometry on complements of lines in $P^{2}$, Tokyo J. Math. 1 (1978), 1-19.
[9] S. Iitaka: On the homogeneous Lüroth theorem, Proc. Japan Acad. 55-A (1979), 88-91.
[10] H. Kashiwara: Fonctions rationnelles de type $(0,1)$ sur le plan projectif complexe, Osaka J. Math. 24 (1987), 521-577.
[11] M. Koras and P. Russell: $C^{*}$-actions on $C^{3}$ : The smooth locus of the quotient is not of hyperbolic type, CICMA report (Concordia, Laval, McGill), 1996.
[12] M. Miyanishi: Curves on rational and unirational surfaces, Tata Inst. Fund. Res. Lectures on Math. and Phys., vol. 60, Tata Inst. Fund. Res., Bombay, 1978.
[13] M. Miyanishi: Normal affine subalgebras of a polynomial ring, Algebraic and Topological Theories-to the memory of Dr. Takehiko MIYATA, Kinokuniya, 1985, 37-51.
[14] M. Miyanishi and T. Sugie: On a projective plane curve whose complement has logarithmic Kodaira dimension $-\infty$, Osaka J. Math. 18 (1981), 1-11.
[15] M. Miyanishi and S. Tsunoda: Logarithmic del Pezzo surfaces of rank one with noncontractible boundaries, Japan. J. Math. 19 (1984), 271-319.
[16] M. Miyanishi and S. Tsunoda: Non-complete algebraic surfaces with logarithmic Kodaira dimension $-\infty$ and with non-connected boundaries at infinity, Japan. J. Math. 10 (1984), 195242.
[17] A.R. Shastri: Divisors with finite local fundamental group on a surface, Algebraic Geometry, Bowdoin 1985, Proceedings of Symposia in Pure Mathematics, vol. 46, American Mathematical Society, 1987, 467-481.
[18] I. Wakabayashi: On the logarithmic Kodaira dimension of the complement of a curve in $P^{2}$, Proc. Japan Acad. (Ser. A) 54 (1978), 157-162.
[19] H. Yoshihara: On plane rational curves, Proc. Japan Acad. (Ser. A) 55 (1979), 152-155.

Daniel Daigle
Department of Mathematics and Statistics University of Ottawa
Ottawa, Canada K1N 6N5
e-mail: ddaigle@uottawa.ca
Peter Russell
Department of Mathematics and Statistics
McGill University
Montréal, Qc, Canada H3A 2K6
e-mail: russell@math.mcgill.ca


[^0]:    ${ }^{1}$ According to the definition of "affine ruling" adopted in 1.1, below, $\Lambda$ is a linear system of $X$, so it makes sense to write $n C \in \Lambda$.

[^1]:    ${ }^{2}$ Note that if $F \in \Lambda$ has irreducible support then the condition $F \cdot H=1$ implies that it is also reduced (i.e., $F$ is an integral curve).

[^2]:    ${ }^{3}$ Note that the pending vertex of weight 0 is not unique when $q=0$ and $x=0$; let us agree that an $X$-immersion always comes equipped with a choice of a zero-component.

[^3]:    ${ }^{4}$ The point would be in particular to describe explicitely how to increase $\beta(\Lambda)$. Section 5 includes a complete answer to this question, as the value of $\beta$ is easily determined by inspecting the data contained in the "discrete part".
    ${ }^{5}$ By part (1) of Proposition 2.8 and part (1) of Corollary 4.4.

[^4]:    ${ }^{6}$ In [7] and [11], a blowing-up at a vertex (resp. at an edge) is called "sprouting" (resp. "subdivisional").

[^5]:    ${ }^{7}$ The first of these two facts is proved in [1], I.4.13. We don't know a reference for the second one.

[^6]:    ${ }^{8}$ For the fact that $\left(\Lambda^{-}, F^{-}\right)$is independent of the choice of $\pi \in \Pi_{P}(I)$, see the last assertion of Corollary 4.4.

[^7]:    ${ }^{9}$ The automorphism preserves fibres and $\Sigma_{m}$, since $m>0$.

