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ON A CONJECTURE OF Y. NAKAI

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Assume that A is the affine ring of an algebraic variety defined over a characteristic zero field k. Denote by Der (A/k) the algebra of high order derivations of A to itself, and denote by der (A/k) the subalgebra of Der (A/k) which is generated by the first order derivations of A to itself. Recently Y. Nakai in his discussions of high order derivations (see [3]) asked the following question: is the condition der $(A/k)=$ Der (A/k) equivalent to the regularity of A? In this note we shall show that if A is the affine ring of a curve, then A is regular (i.e. the curve is non-singular) if and only if der $(A/k)=$ Der (A/k).

1. In all that follows, k will denote a field of characteristic zero. If A is a k algebra, then we shall denote by $\mathrm{Der}^n(A/k)$ the module of n-th order derivations over k, from A to itself. We shall denote by $D^n(A/k)$ the A module of n-th order differentials, thus $\mathrm{Der}^n(A/k)=\mathrm{Hom}_A(D^n(A/k), A)$. We shall denote by $\mathrm{der}^n(A/k)$ the submodule of $\mathrm{Der}^n(A/k)$ which consists of linear combinations of derivations of the form $\delta_1\cdots\delta_j$ where $1\leqq j\leqq n$ and $\delta_i\in\mathrm{Der}^1(A/k)$. (For a discussion of these concepts see [1] or [3].)

In [1] Theorem 16.11.2 it is shown that if A is differentiably lisse over k (hence in particular if A is regular), then each n-th order derivation is a combination of first order derivations. Thus to prove our assertion we need only prove the following theorem.

Theorem. *Suppose that k is a field of characteristic zero and suppose that A is the local ring of a point on an irreducible algebraic curve. If A is not regular, then there exists an integer N and an element* $w\in \mathrm{Der}^N(A/k)$ *such that w is not an element of der* (A/k).

Proof. Denote by $\bar{A}$ the integral closure of A in the quotient field Q of A. The ring $\bar{A}$ is semi local, thus $\bar{A}=B_1\cap\cdots\cap B_s$ where each B_i is a discrete rank one valuation ring with integer valued valuation v_i. Denote by $\bar{d}$ the canonical derivation from $\bar{A}$ to $D^1(\bar{A}/k)$. The restriction of $\bar{d}$ to A is a k derivation from

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A to $D^1(\bar{A}/k)$, therefore there exists a map σ from $D^1(A/k)$ to $D^1(\bar{A}/k)$ which is A linear such that if d is the canonical derivation from A to $D^1(A/k)$, then $\sigma d=\bar{d}$. Denote by K the kernel of this map. Because the formation of D^1 commutes with localization (see [1, 16.4.15.]) $Q\otimes_A D^1(A/k)=Q\otimes_{\bar{A}} D^1(\bar{A}/k)$. Thus $D^1(A/k)$ has rank one and therefore K consists entirely of torsion elements. We assert that Image $(\sigma)=D^1(A/k)/K$ cannot be free when A is not regular. To see this first note that $\mathrm{Hom}_A(D^1(A/k), A)=\mathrm{Hom}_A(D^1(A/k)/K, A)$. Thus if $D^1(A/k)/K$ is free, then $\mathrm{Der}^1(A/k)$ is free. However, Lipman [2] has shown that $\mathrm{Der}^1(A/k)$ is free if and only if A is regular.

From now on we shall assume that A is not regular. The $\bar{A}$ module $D^1(\bar{A}/k)$ is projective and hence free because $\bar{A}$ is regular and semi local. Let u be a basis for $D^1(\bar{A}/k)$. Thus Image $(\sigma)=Mu$, where M is a finitely generated A submodule of $\bar{A}$. Let φ denote a derivation from A to A. The derivation φ extends uniquely to a derivation of $\bar{A}$ to Q. Let δ denote the derivation from $\bar{A}$ to $\bar{A}$ which as an element of $\mathrm{Hom}_{\bar{A}}(D^1(\bar{A}/k), \bar{A})$ carries u to 1. Then $\varphi=\beta\delta$ where $\beta\in Q$. The fact that φ carries A to itself implies that $\beta M\leqq A$. We know that $\beta M\neq A$ because M is not free. Thus βM is contained in the radical m of A.

Suppose now that $x_1, \cdots, x_r$ are generators for m. The module M is then generated by the elements $\delta(x_1), \cdots, \delta(x_r)$. W wish to show that $v_i(\beta)>0$ for each i when $\beta\delta$ carries A to itself. Let t_i denote a generator for the maximal ideal of B_i. Because $B_i\otimes_{\bar{A}} D^1(\bar{A}/k)=D^1(B_i/k)$, and $D^1(B_i/k)$ has basis dt_i, $u=u_i dt_i$ where u_i is a unit in B_i. Thus $\delta(x_j)=u_i^{-1}\dfrac{d}{dt_i}(x_j)$ for $1\leqq j\leqq r$ $\left(\dfrac{d}{dt_i}\text{ is the dual of } dt_i\right)$, therefore $v_i(\delta x_j)=v_i\left(\dfrac{d}{dt_i}(x_j)\right)$. Clearly, $v_i\left(\dfrac{d}{dt_i}(x_j)\right)=v_i(x_j)-1$. As we have noted before $\beta\delta x_j$ is in m for each j. Hence $v_i(\beta\delta x_j)\geqq\min_j(v_i(x_j))$, thus $v_i(\beta)\geqq 1$.

Now denote by I the A submodule of Q which consists of the β such that $\beta\delta$ is a derivation of A to itself. The previous discussion has shown that I is contained in the radical of $\bar{A}$, and it is clearly finitely generated. Now if $\delta_1, \cdots, \delta_v$ are firsrt order derivations of A to itself, then $\delta_i=\beta_i\delta$ for some $\beta_i\in I$. Thus $\delta_1\cdots\delta_v=\sum_{j<v}y_j\delta^j+\beta_1\cdots\beta_v\delta^v$, and therefore each composition of v derivations of A to itself has as coefficient of δ^v an element of I^v. Then, if R is the radical of $\bar{A}$, $\beta_1\cdots\beta_v\in R^v$. Let C be the conductor of $\bar{A}$, hence there is an N such that $R^N\subseteq C$ and $R^N\neq C$. So, choose $y\in C$, $y\notin R^N$. Then $y\delta^N$ is an N-th order derivation from A to itself, but $y\delta^N$ is not in $\mathrm{der}(A/k)$ because y is not in $I^N\subseteq R^N$.

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